18.354J / 3541

Nonlinear dynamics: continuum systems

Course intro

18.354J/3541
Nonlinear dynamics: continuum systems

Spring 2020

Lectures:	MW 2:30-4.00	in	253-7826
Instructor:	Jörn Dunkel		
Contact:	dunkel@mit.edu		253-7826 (office phone)
Office Hours:	Thursdays 2:30-3:30 (2-381)		
Course website:	math.mit.edu/∼dunkel/Teach/18.354/		

https://math.mit.edu/~dunkel/Teach/18.354/

course info, psets, lecture notes, ….

GRADING

- 60%: Problem sets
- 10%: Project proposal
- 30%: Final project presentation + report

HOMEWORK - PROBLEM SETS

Homework will be assigned roughly every two-three weeks. We aim to have 4 problem sets in total. Each homework set may contain analytical and computational problems, and even the odd experiment. Assignments must be handed in at the start of class on the due date.

First unexcused late homework score will be multiplied by 3/4. No subsequent unexcused late homework is accepted.

You are welcome to discuss solution strategies and even solutions, but please write up the solution on your own. Be sure to support your answer by explaining important steps or listing any relevant theorems. Be as clear and concise as possible. I strongly encourage the computational problems to be written in MATHEMATICA, MATLAB or JULIA.

TEXTBOOKS

Although there are no textbooks which follow the precise spirit of this course, there are literally hundreds of textbooks where the topics we will cover are discussed. For most lectures, typed notes can be downloaded from the course webpage. Useful books for further reading include:

- M. Cross, *Pattern Formation and Dynamics in Non-equilibrium Systems*, Cambridge University Press (2009).
- A. Goriely, *The Mathematics and Mechanics of Biological Growth*, Springer (2017).
- B. Audoly & Y. Pomeau, *Elasticity & Geometry*, Oxford University Press (2010).
- D. J. Acheson, *Elementary Fluid Dynamics*, Oxford University Press (1990).

IMPORTANT DATES

- Wed Feb 26 - Problem Set 1 DUE
- Wed Mar 11 - Problem Set 2 DUE
- Wed Mar 18 - Proposal (1 page) for final project DUE
- Wed Apr 1 - Problem Set 3 DUE
- Wed Apr 22 - Problem Set 4 DUE
- May 3, 5 & 10 - Final project presentations
- May 10 - Final project report DUE

Note: The exact due dates for the P-sets may be subject to change

Course topic:

Physical Applied Mathematics

Q: What is Physical Applied Maths?

A:  PAM is like cooking...

Often the ingredients (*physical principles*) are already known but not the way (*mathematics/equations/couplings*) to turn them into a nice dinner

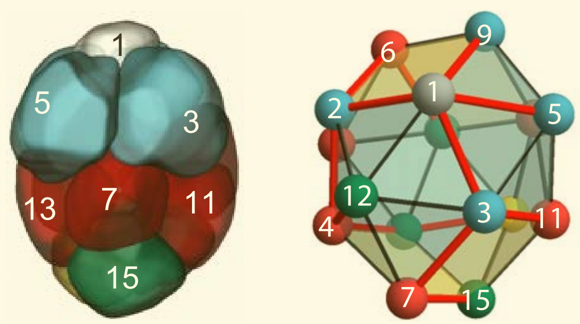
With some creativity, many new dishes (*novel phenomena*) can be created (*discovered/understood*)

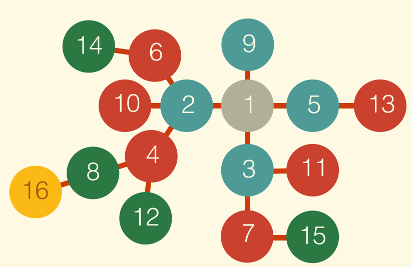
Why study Applied Maths?

- intellectual challenge
- obtain general understanding of physical phenomena and the world around us
- be able to make prediction about physical processes
- development of general tools to be applied to other fields

Discrete active matter

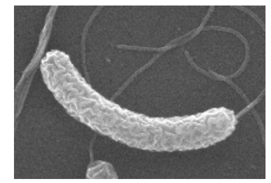
Oogenesis



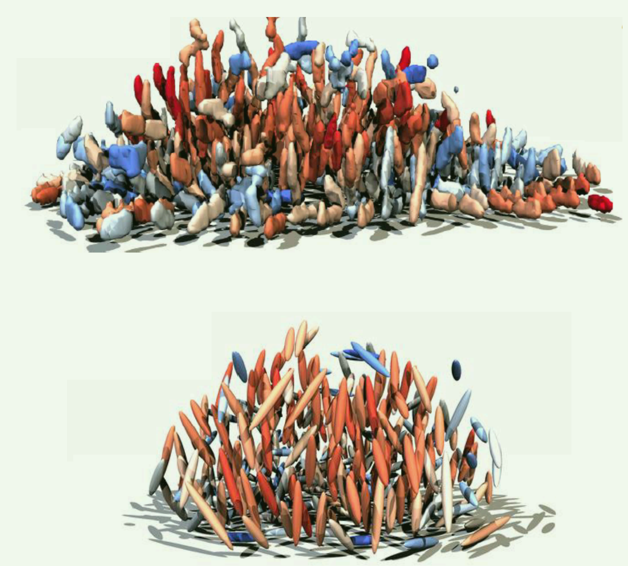


Shvartsman lab, Princeton

Nature Physics 2018

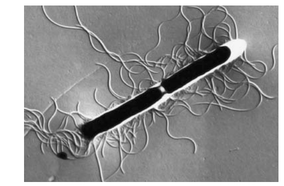


Bacterial biofilms

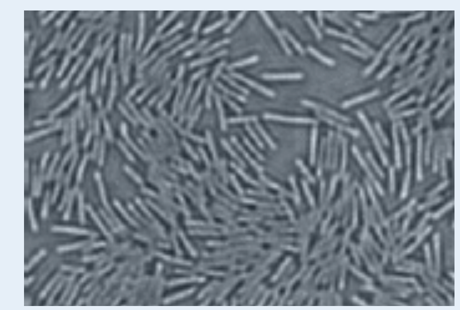


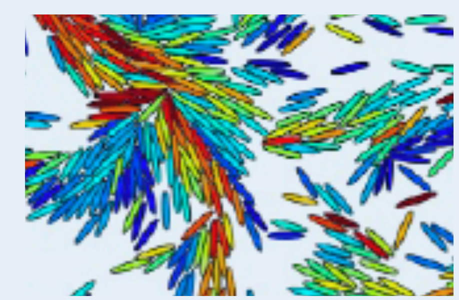
Drescher lab, MPI Marburg

PNAS 2016
Nature Physics 2019
Nature Microbiology 2019
Physical Review Letters 2019



Bacterial swarming



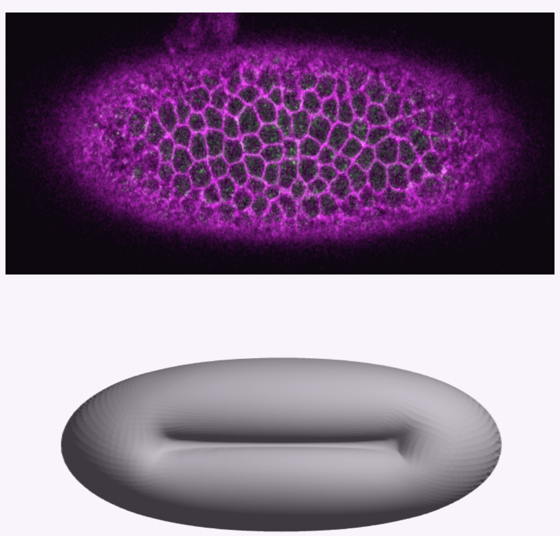


Drescher lab, MPI Marburg

PNAS 2019

Continuous biological (&) soft matter

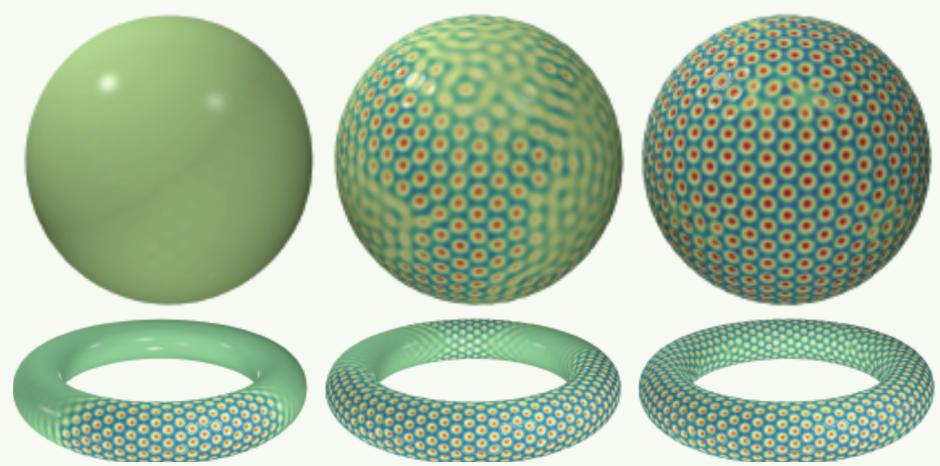
Morphogenesis



Martin lab, MIT Biology

Development 2017
Developmental Cell 2019

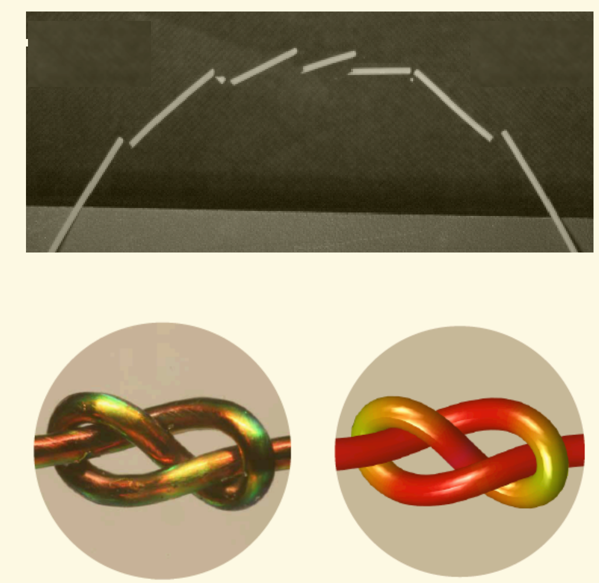
Curvature-controlled pattern formation



Reis lab, MIT MechE ⇒ EPFL

Nature Materials 2015
Physical Review Letters 2016

Thin structures



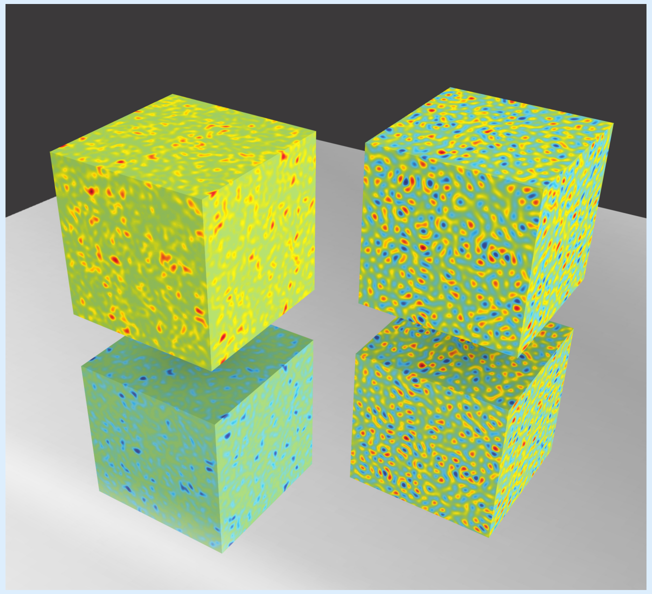
Kolle lab, MIT MechE

PNAS 2018
Science 2020

Continuous active matter

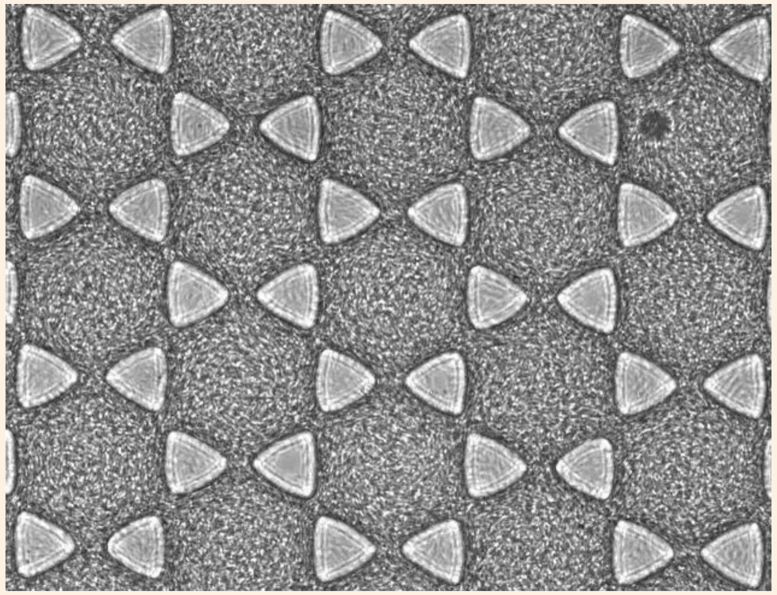
Active fluids

Energy transport in active fluids



PNAS 2012
Physical Review Letter 2013
PNAS 2017
J Fluid Mech 2018

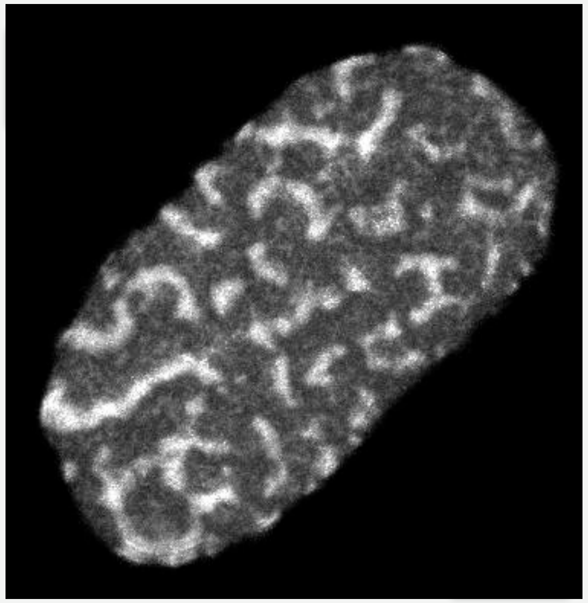
Geometric control of microbial suspensions



Goldstein lab, Cambridge

Physical Review Letters 2013
PNAS 2013
eLife 2014
Nature Physics 2016

Cell membranes



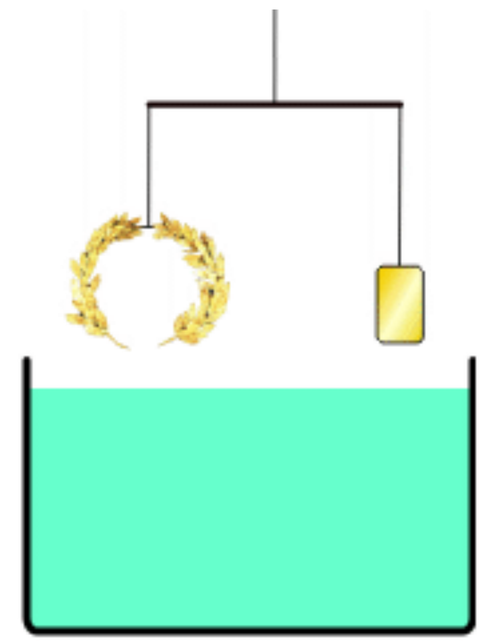
Fakhri lab, MIT Physics

Nature Physics 2020 (in press)

Historical backdrop … quiz:

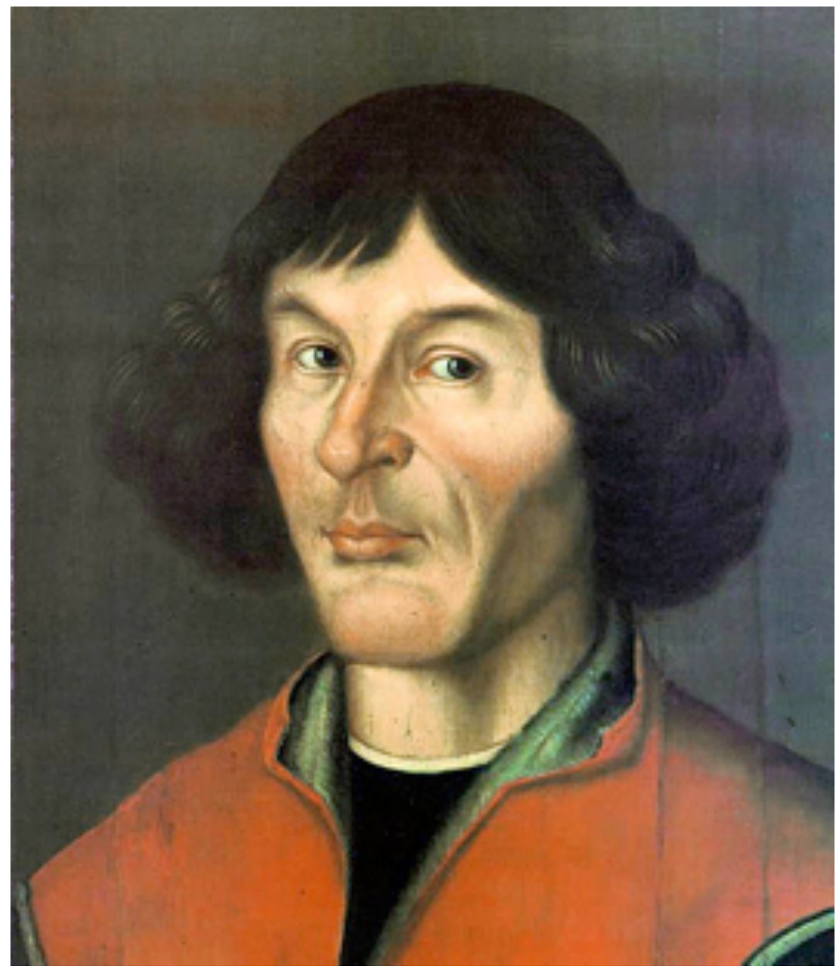
Some famous thought on Applied Math ?

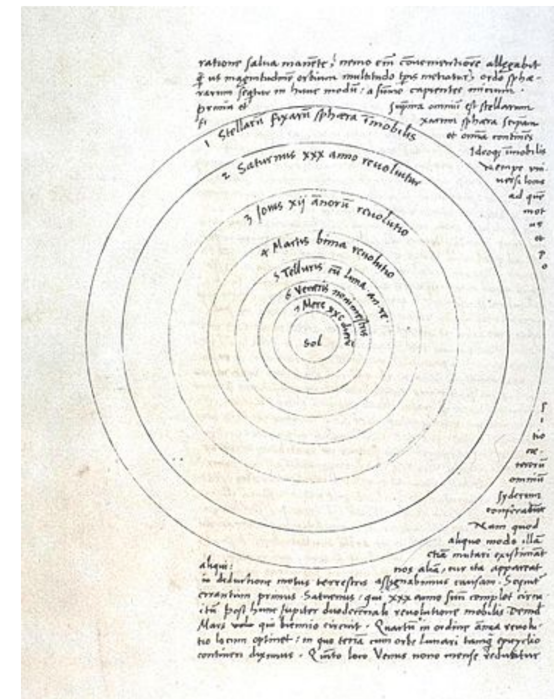
'*Eureka, Eureka*'



c. 287 BC - c. 212 BC

'Mathematic is written for mathematicians'

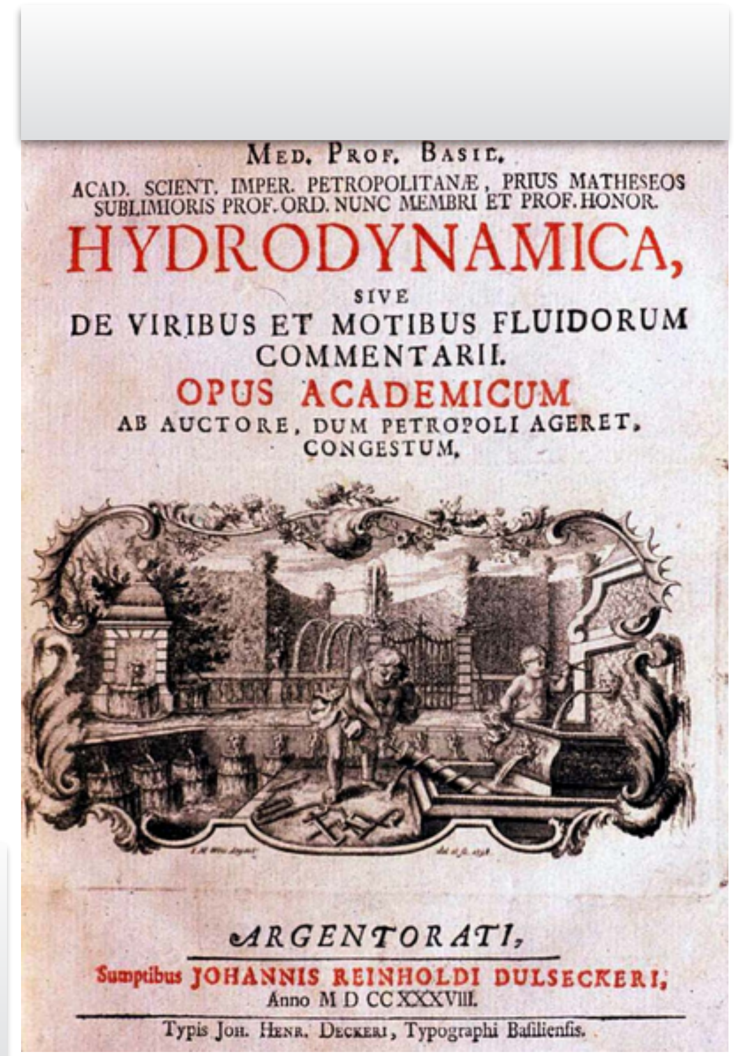




19 February 1473 – 24 May 1543

'It would be better for the true physics if there were no mathematicians on earth'

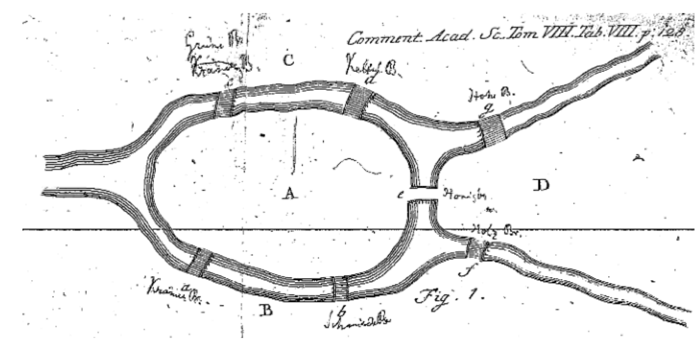
8 February 1700 – 17 March 1782



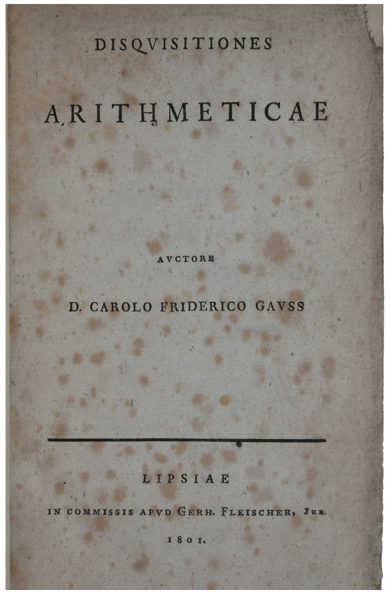
MED. PROF. BASIL.
ACAD. SCIENT. IMPER. PETROPOLITANÆ, PRIUS MATHESEOS SUBLIMIORIS PROF. ORD. NUNC MEMBRI ET PROF. HONOR.
HYDRODYNAMICA,
SIVE
DE VIRIBUS ET MOTIBUS FLUIDORUM COMMENTARII.
OPUS ACADEMICUM
AB AUCTORE, DUM PETROPOLI AGERET, CONGESTUM.

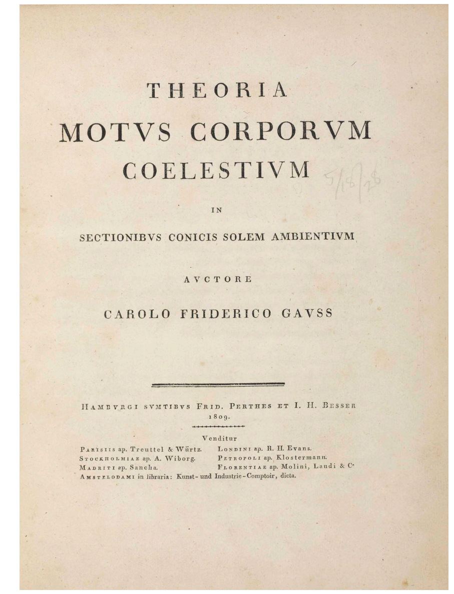
ARGENTORATI,
Sumptibus JOHANNIS REINHOLDI DULSECKERI,
Anno M D CC XXXVIII.
Typis JOH. HENR. DECKERI, Typographi Basiliensis.

'Now I will have less distraction'



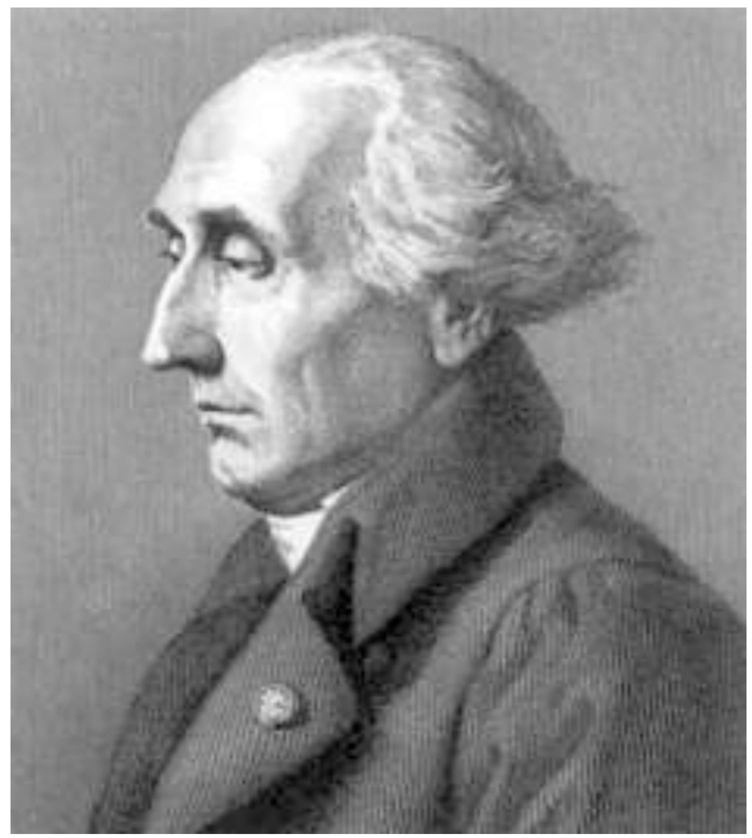
15 April 1707 – 18 September 1783

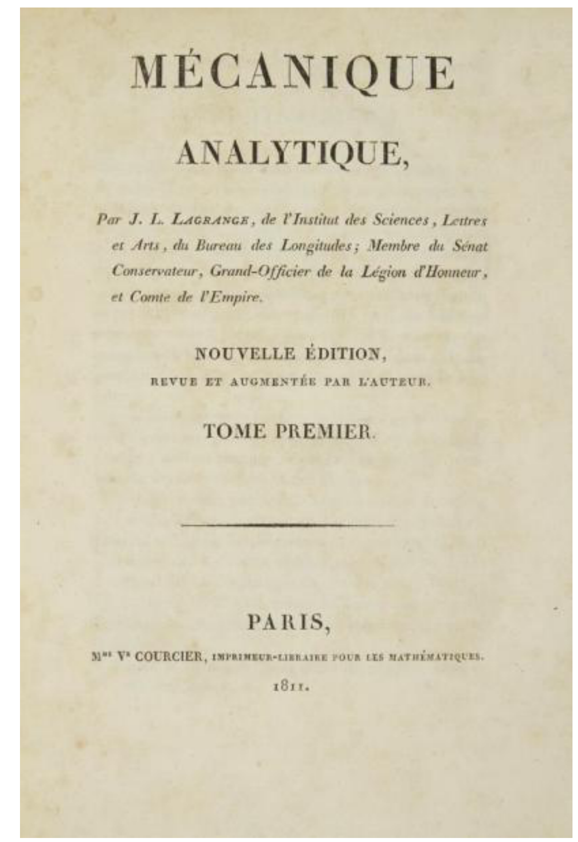




"If this subjet has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had +1, -1 and $\sqrt{-1}$, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say,of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity."

'I do not know'




MÉCANIQUE

ANALYTIQUE,

Par J. L. Lagrange, de l'Institut des Sciences, Lettres et Arts, du Bureau des Longitudes; Membre du Sénat Conservateur, Grand-Officier de la Légion d'Honneur, et Comte de l'Empire.

NOUVELLE ÉDITION,

REVUE ET AUGMENTÉE PAR L'AUTEUR.

TOME PREMIER.

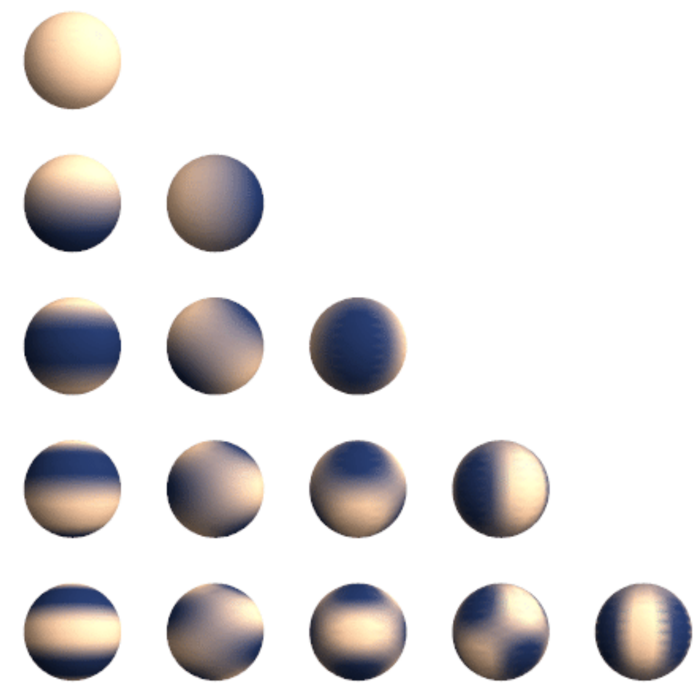
PARIS,

Mme Ve COURCIER, IMPRIMEUR-LIBRAIRE POUR LES MATHÉMATIQUES.

1811.

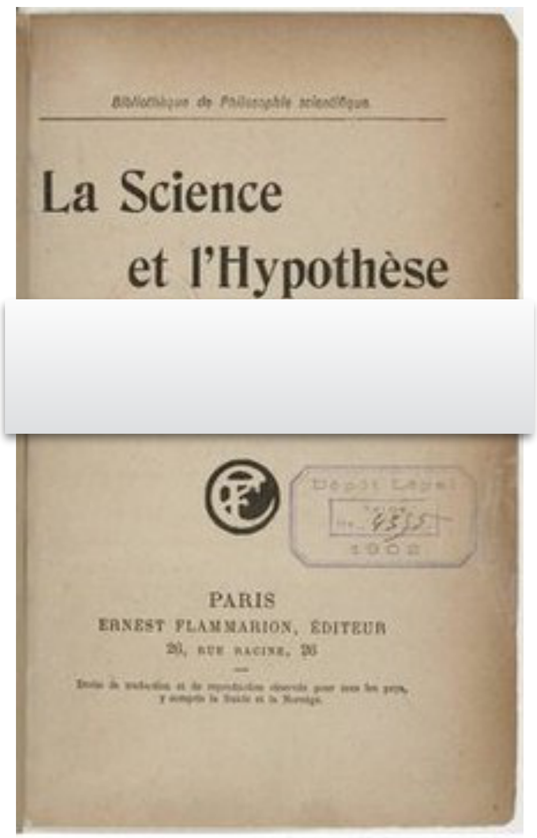
25 January 1736 - 10 April 1813

'*Nature laughs at the difficulties of integration*'



23 March 1749 – 5 March 1827

'*Mathematicians are born, not made*'


Bibliothèque de Philosophie scientifique.

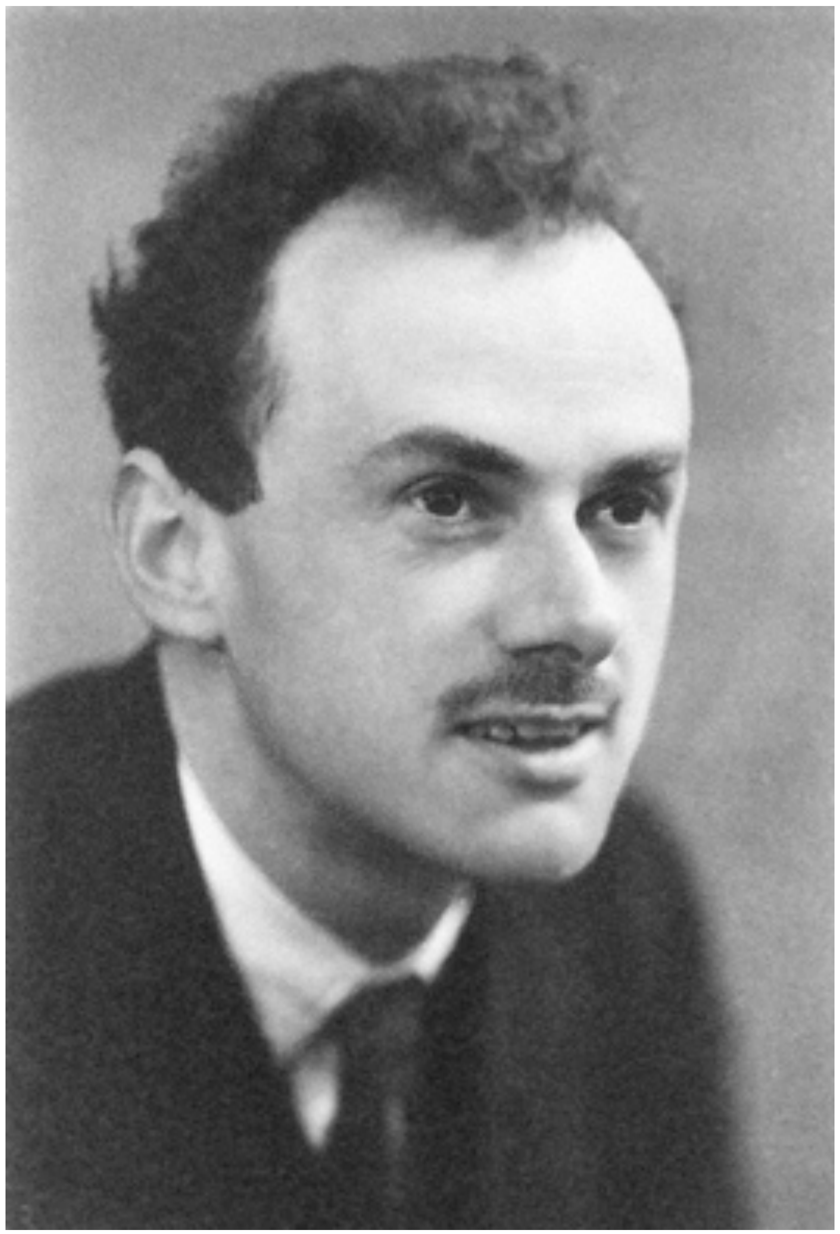
La Science
et l'Hypothèse

PARIS
ERNEST FLAMMARION, ÉDITEUR
26, RUE RACINE, 26

29 April 1854 – 17 July 1912

'Prediction is very difficult, especially about the future'

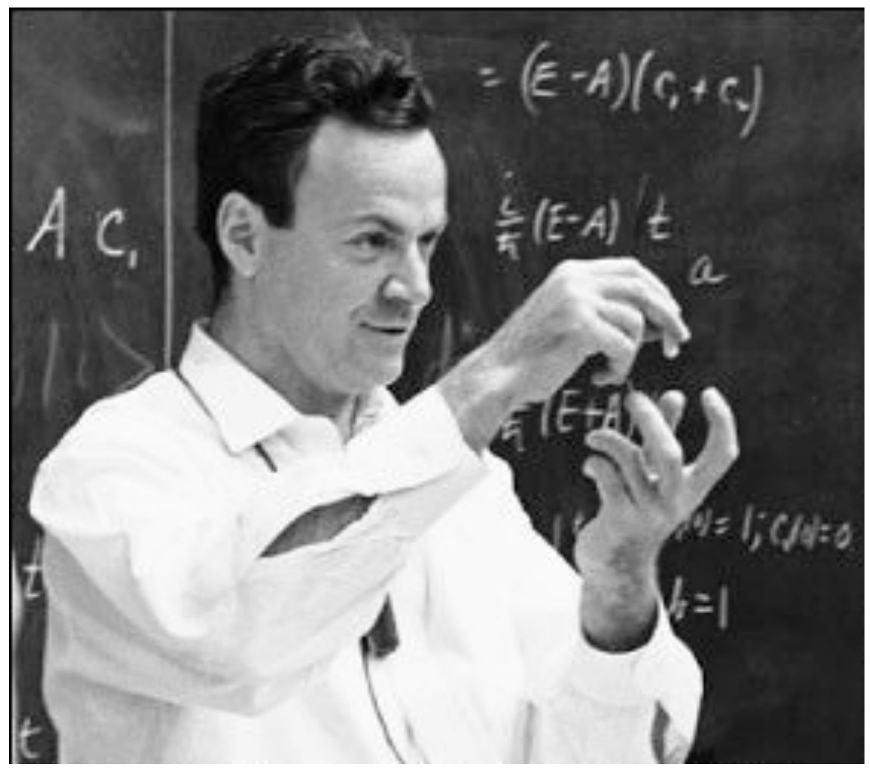
7 October 1885 – 18 November 1962



'*It is more important to have beauty in one's equations than to have them fit experiment*' and '*This result is too beautiful to be false*'

8 August 1902 – 20 October 1984

'To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature'



May 11, 1918 – February 15, 1988

BSc 1939

Course content overview

Concepts

- linear/nonlinear PDEs for scalar, vector and tensor fields
- model derivation/construction, dimensional analysis
- solution techniques, Fourier methods
- stability, perturbation theory
- calculus of variations

Examples

- Hamiltonian dynamics & Liouville equation
- Brownian motion (free and confined)
- quantum mechanics (linear and nonlinear)
- pattern formation theory (Turing)
- elasticity theory
- fluid dynamics (passive and active)

Dimensional analysis

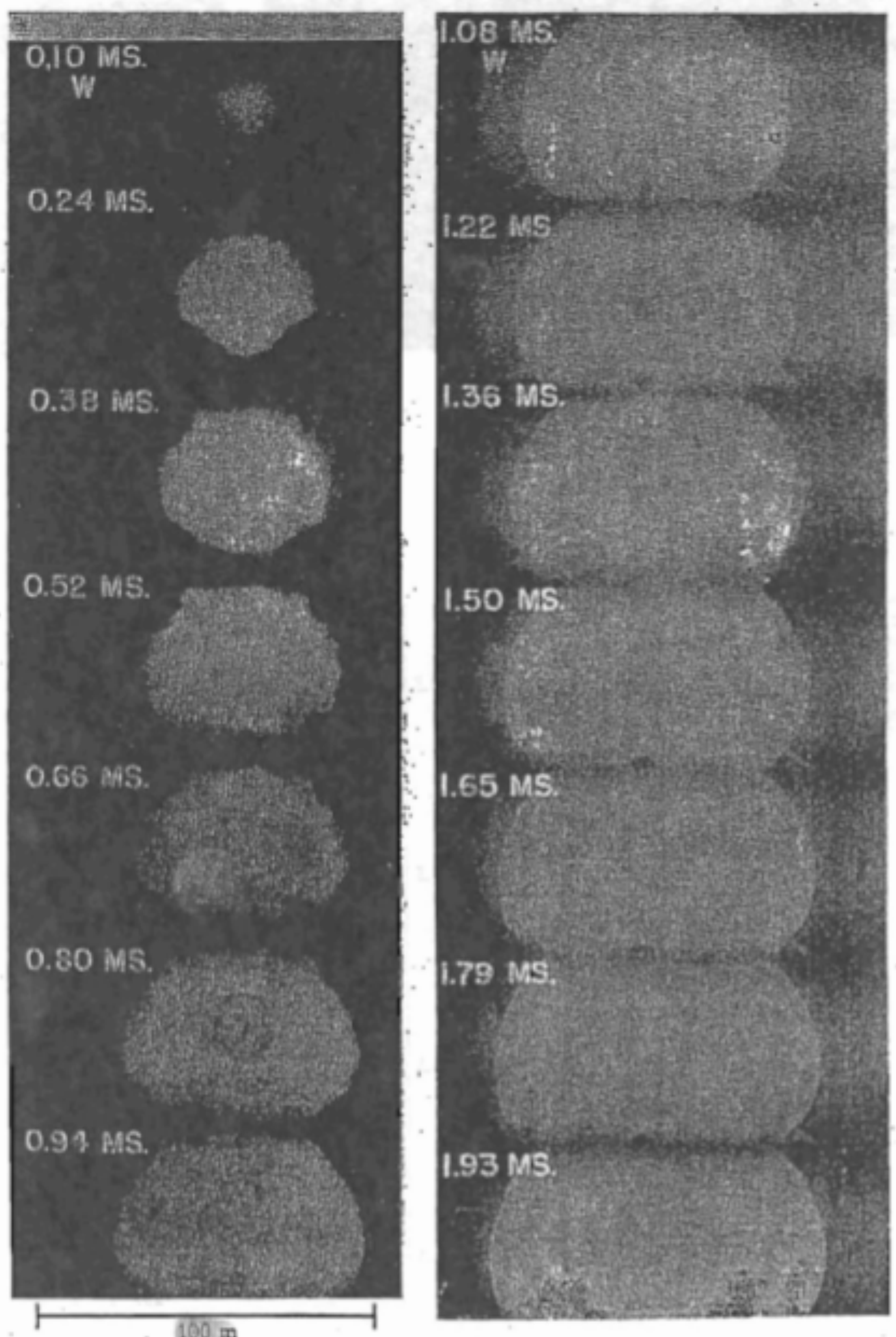
G I Taylor
1886-1975

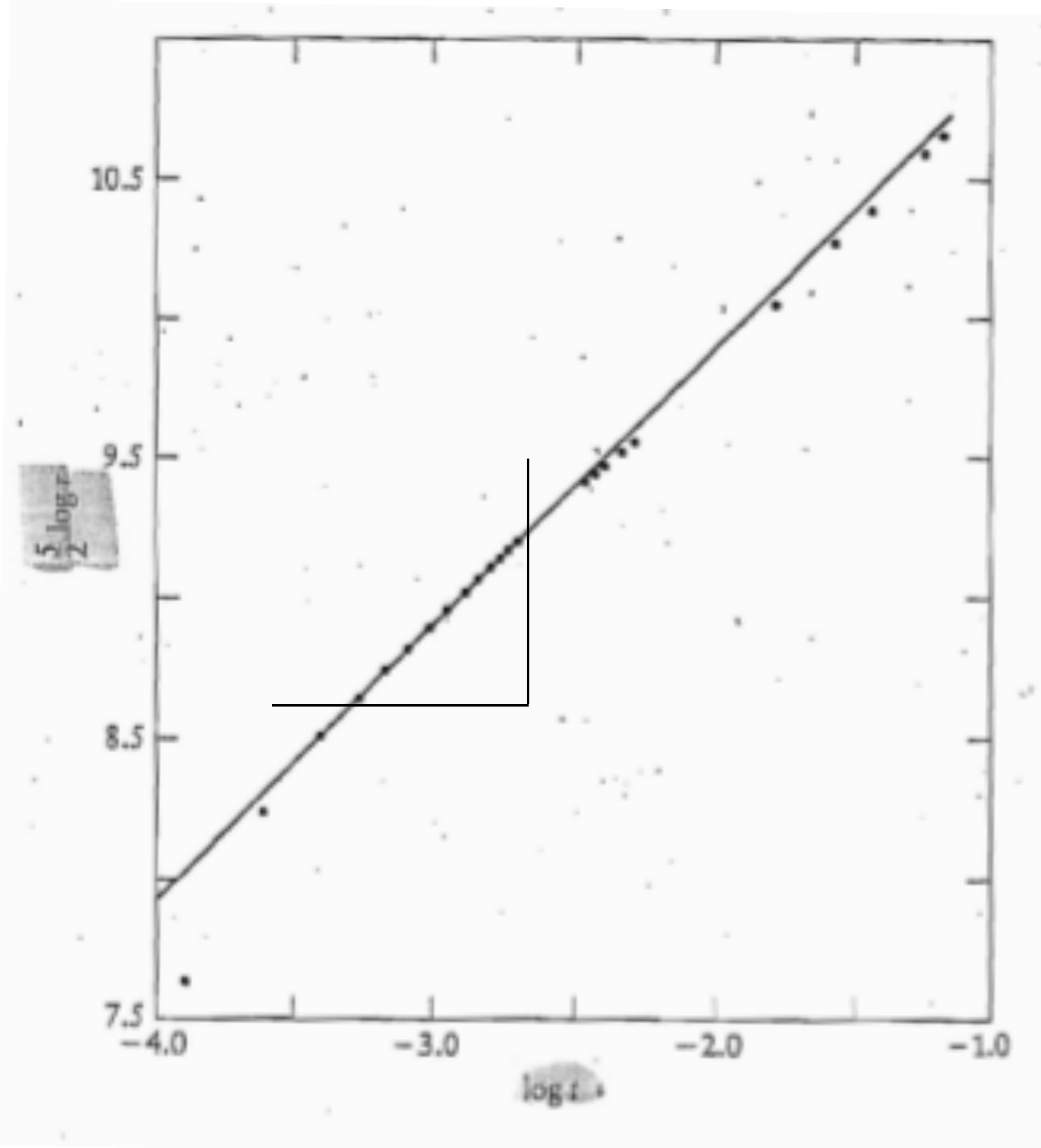
Trinity nuclear test, July 1945
Life Magazine, August 20, 1945

The formation of a blast wave by a very intense explosion.
II. The atomic explosion of 1945

By Sir Geoffrey Taylor, F.R.S.

(*Received* 10 *November* 1949)

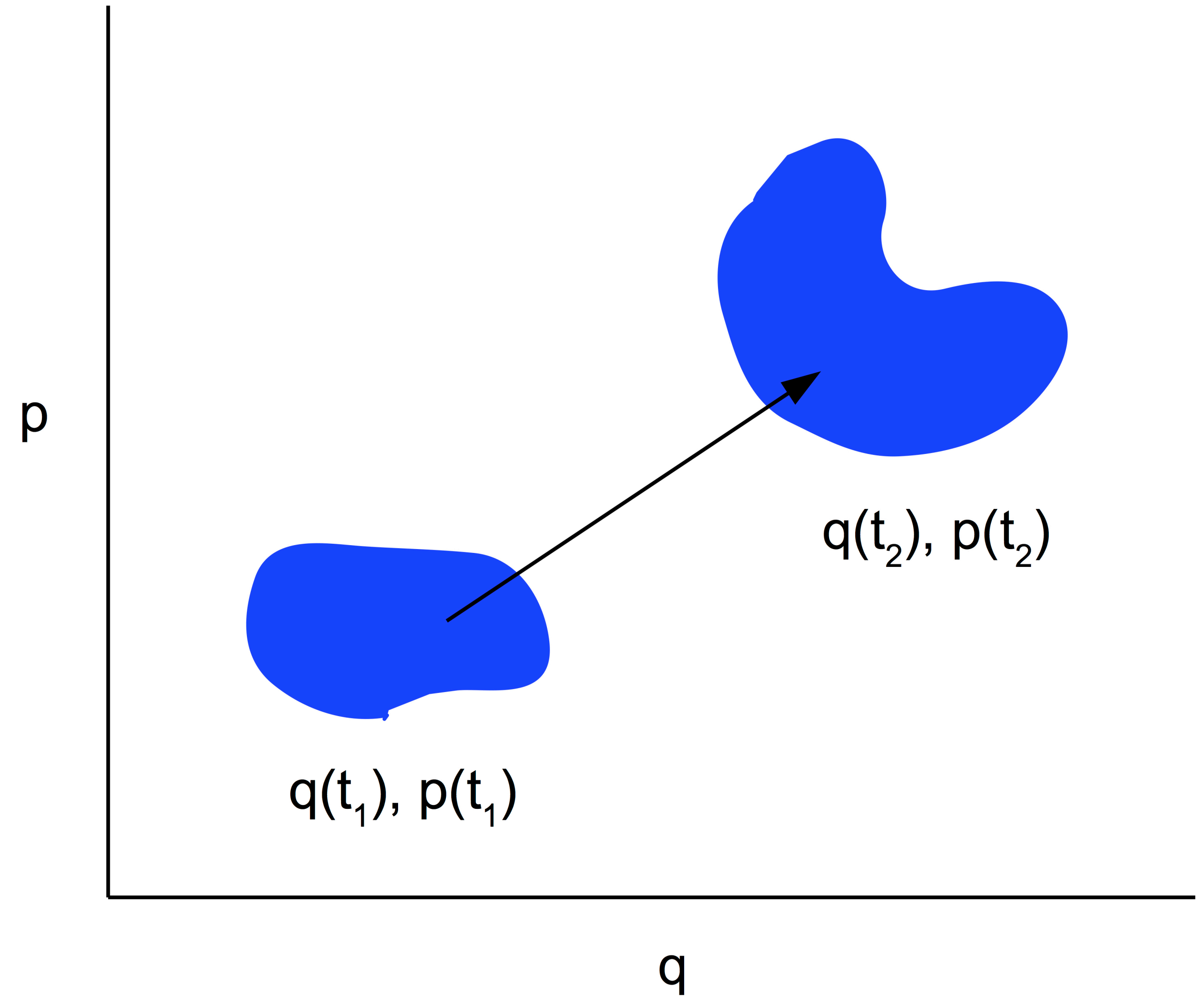




$$\ln R = \ln\left[c\left(\frac{E}{\rho}\right)^{1/5}\right] + \frac{2}{5}\ln t$$

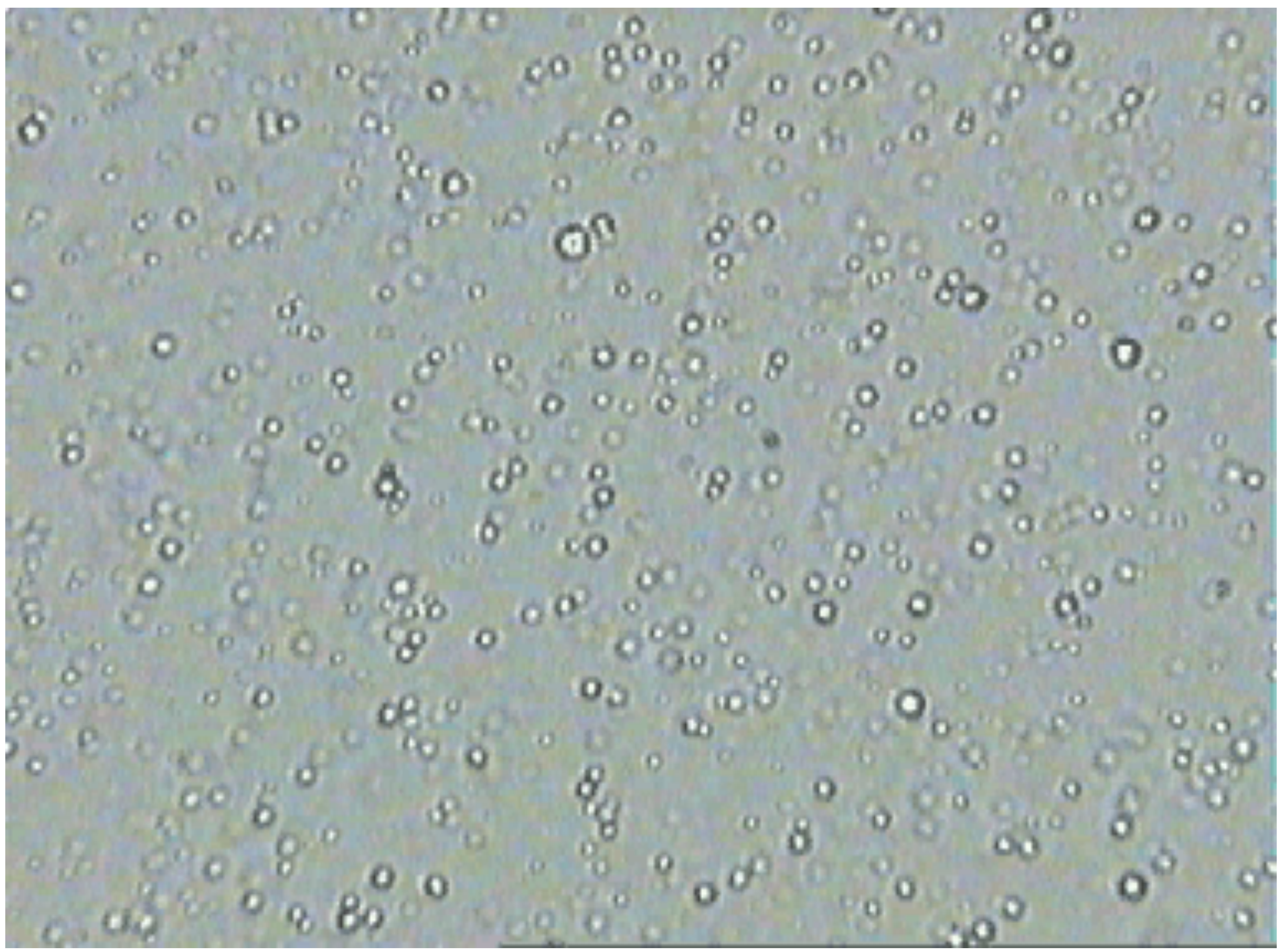
Hamiltonian dynamics & Liouville equation

$$H = \sum_i \frac{p_i^2}{2m_i} + U(x_1, \ldots, x_N)$$



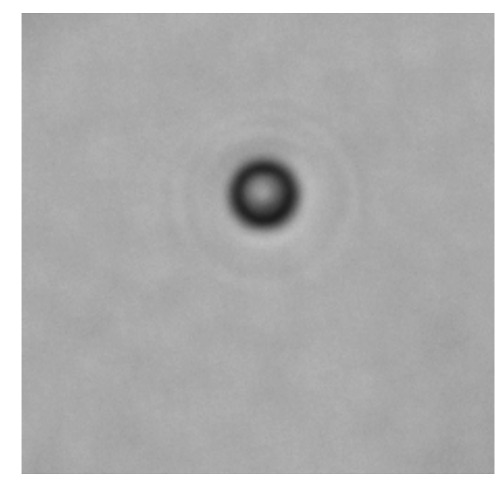
Brownian motion

Random walks & diffusion



David Walker

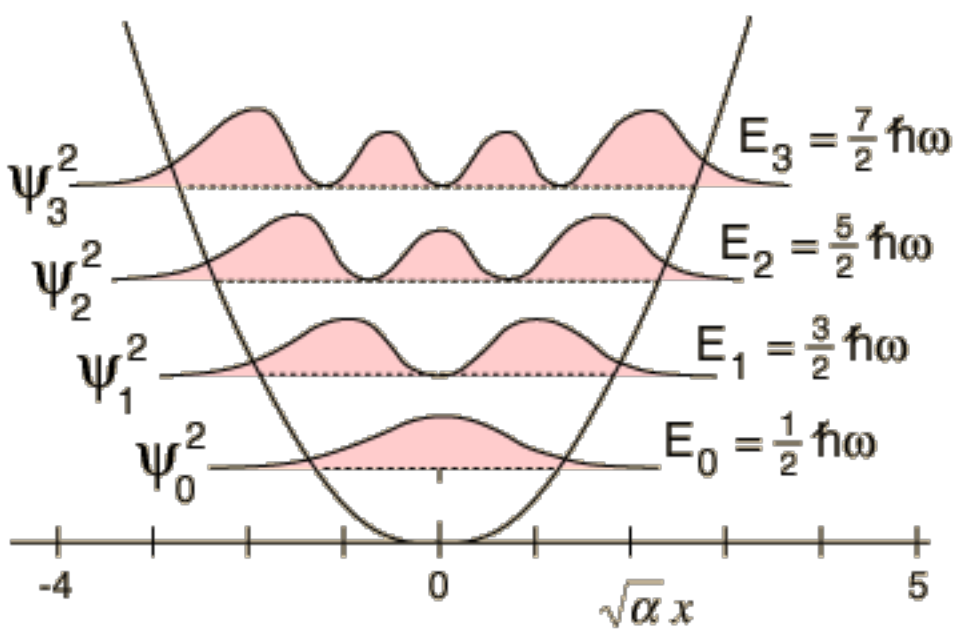
$$\frac{\partial \rho}{\partial t} = \alpha \frac{\partial^2 \rho}{\partial x^2}$$

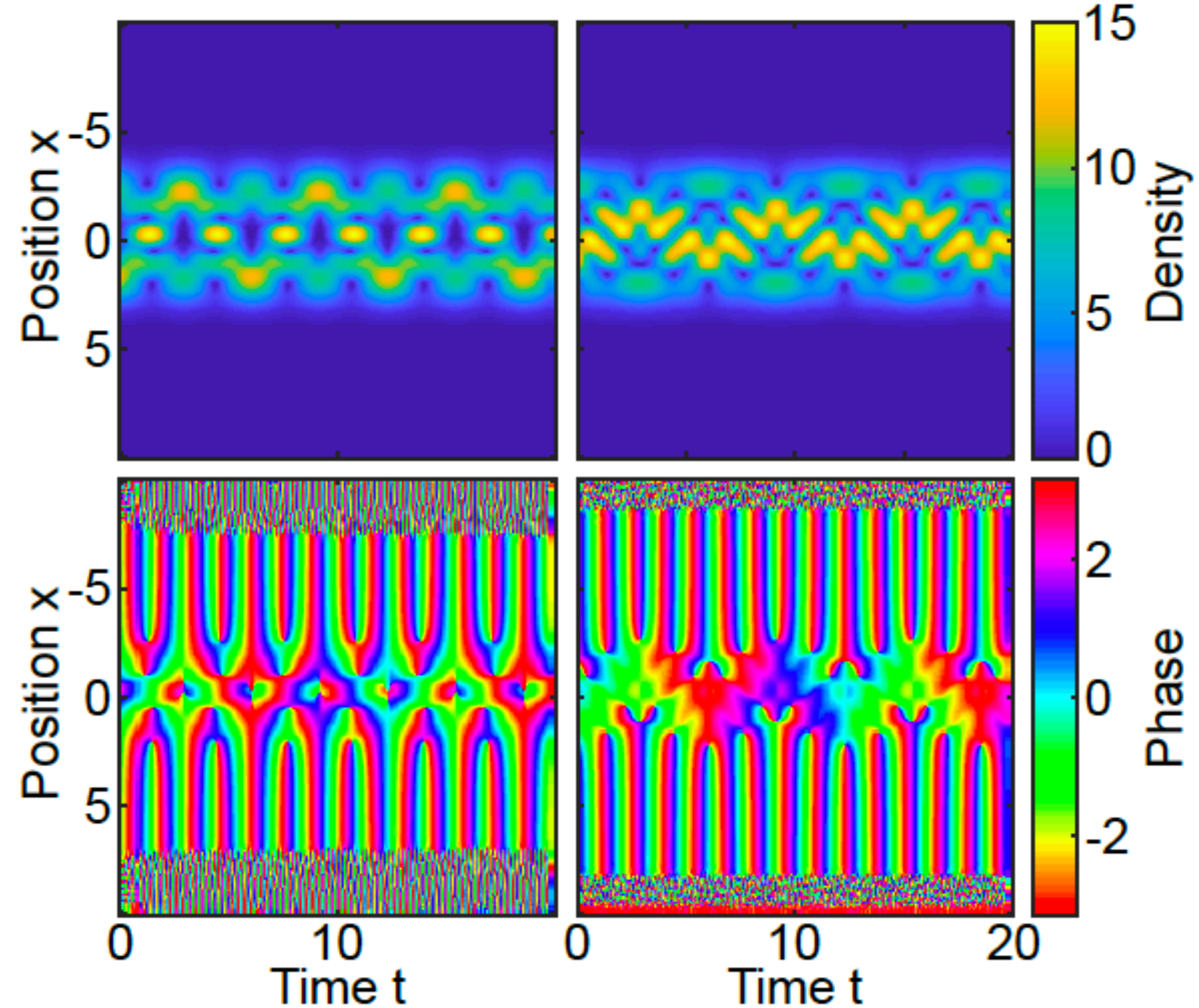


Mark Haw

Quantum mechanics

$$i\psi_t = -\psi_{xx} + \omega^2 x^2 \psi$$





Higher-order quantum hydrodynamics

NLS
GPE

$$i\hbar\partial_t\Psi(t,\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + g_0\left|\Psi(t,\mathbf{r})\right|^2\right)\Psi(t,\mathbf{r})$$

$$u = g_0\delta(\mathbf{x}-\mathbf{x}')$$

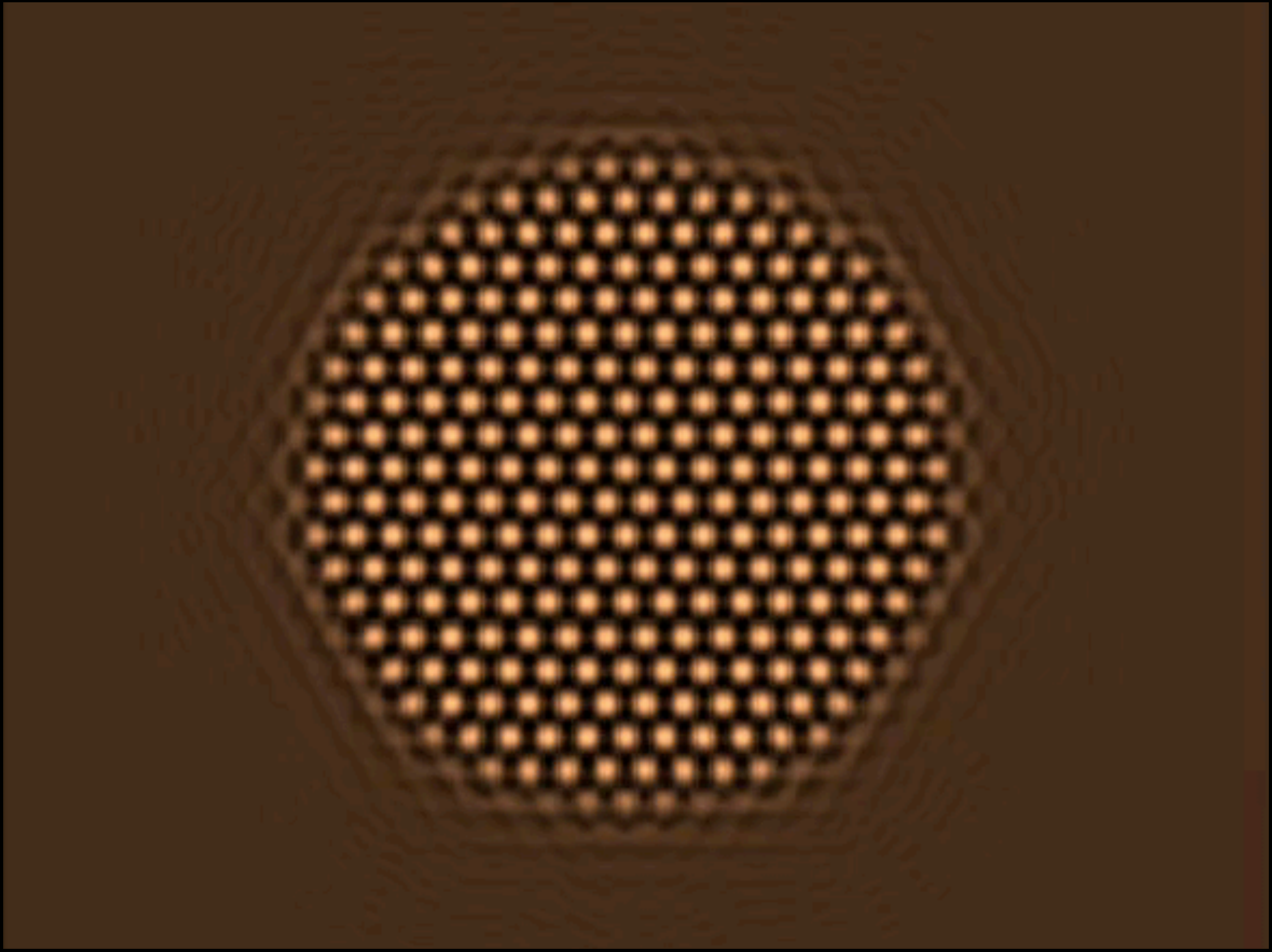
$$\langle u_{\text{int}}\rangle = \frac{1}{2}\int\int d\mathbf{r}_1\, d\mathbf{r}_2\left[\Psi^*(t,\mathbf{r}_2)\Psi^*(t,\mathbf{r}_1)u_{\text{int}}(|\mathbf{r}_2-\mathbf{r}_1|)\Psi(t,\mathbf{r}_1)\Psi(t,\mathbf{r}_2)\right]$$

$$\hat{u}_{\text{int}}(k) = \sum_{j=0}^{\infty} g_{2j}k^{2j}$$

$$i\hbar\partial_t\Psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + \left(\sum_{j=0}^{\infty}(-1)^j g_{2j}(\nabla^2)^j\left|\Psi\right|^2\right)\right]\Psi$$

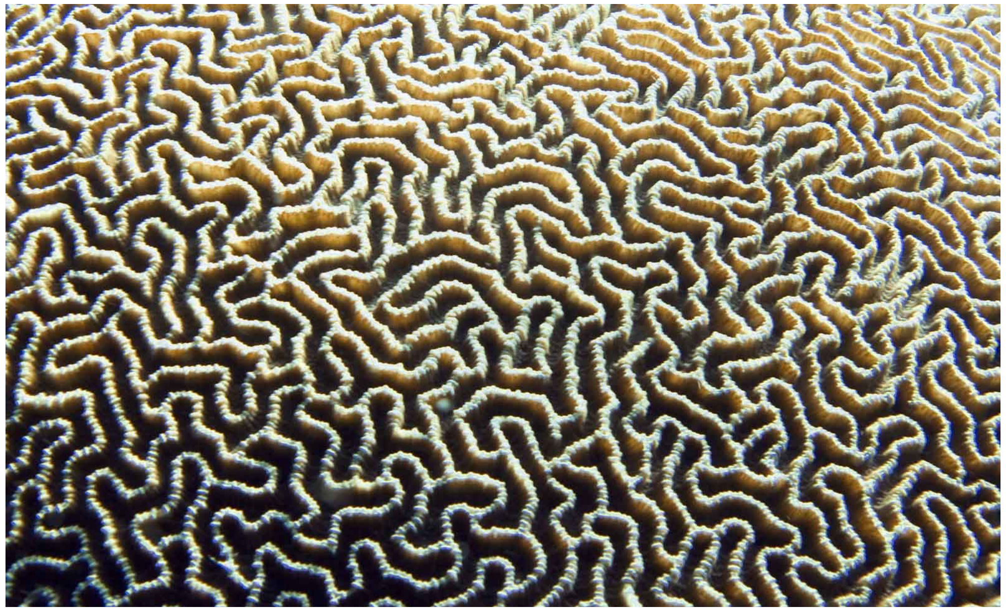
Analytically tractable theory of super-solids

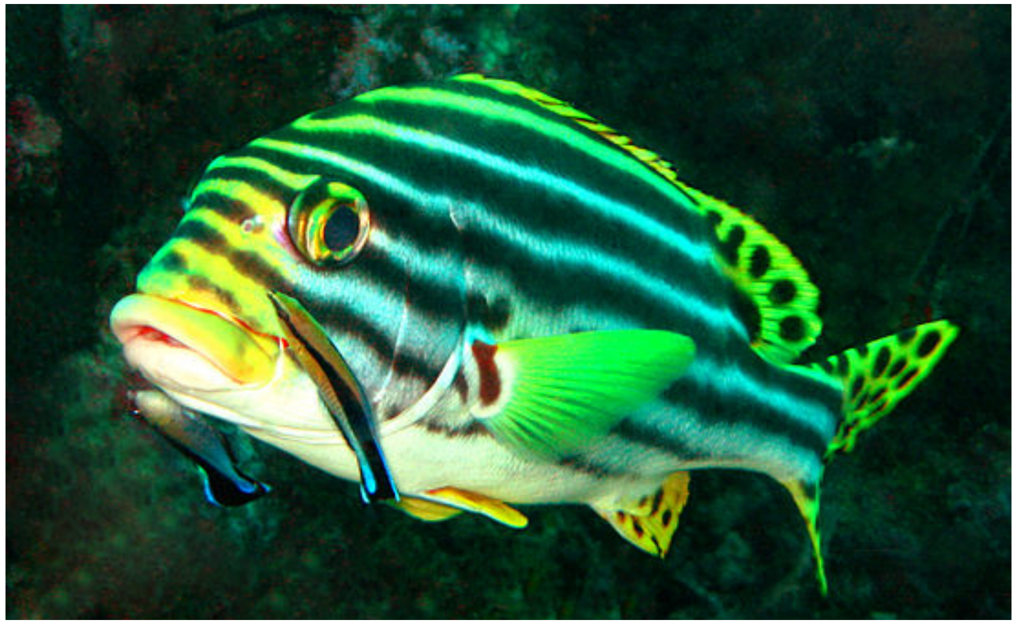
$|\Psi|^2$



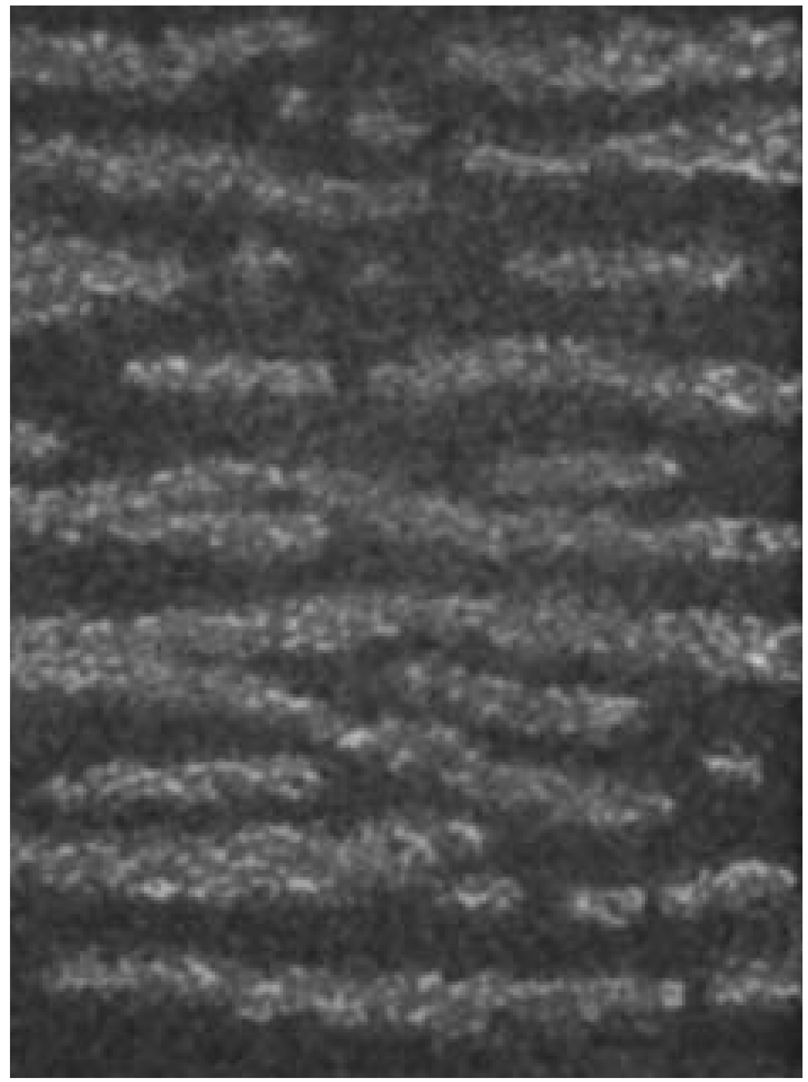
Heinonen & JD, *Physical Review A* 2019

Pattern formation





zebra *vs.* granular medium *vs.* fingerprint



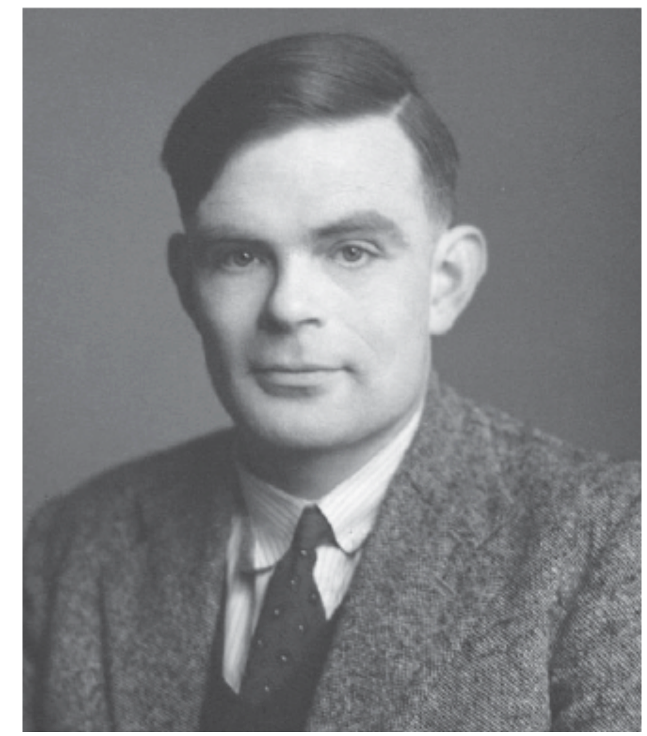
Mathematical theory of pattern formation

Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences, Vol. 237, No. 641. (Aug. 14, 1952), pp. 37-72.

THE CHEMICAL BASIS OF MORPHOGENESIS

BY A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)



The Royal Society

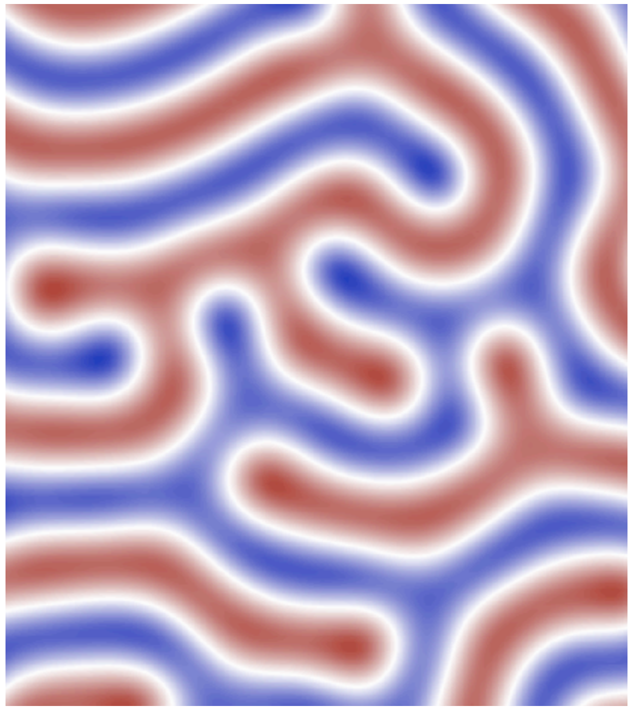
$$\partial_t \boldsymbol{c}(t, \boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{c}) + \mathbf{D} \cdot \boldsymbol{c}$$

local	non-local
reactions	diffusive interactions
(nonlinear)	(linear)

Swift-Hohenberg theory

$$\partial_t \psi = -U'(\psi) + \gamma_0 \nabla^2 \psi - \gamma_2 (\nabla^2)^2 \psi$$

$$U(\psi) = \frac{a}{2}\psi^2 + \frac{b}{3}\psi^3 + \frac{c}{4}\psi^4$$



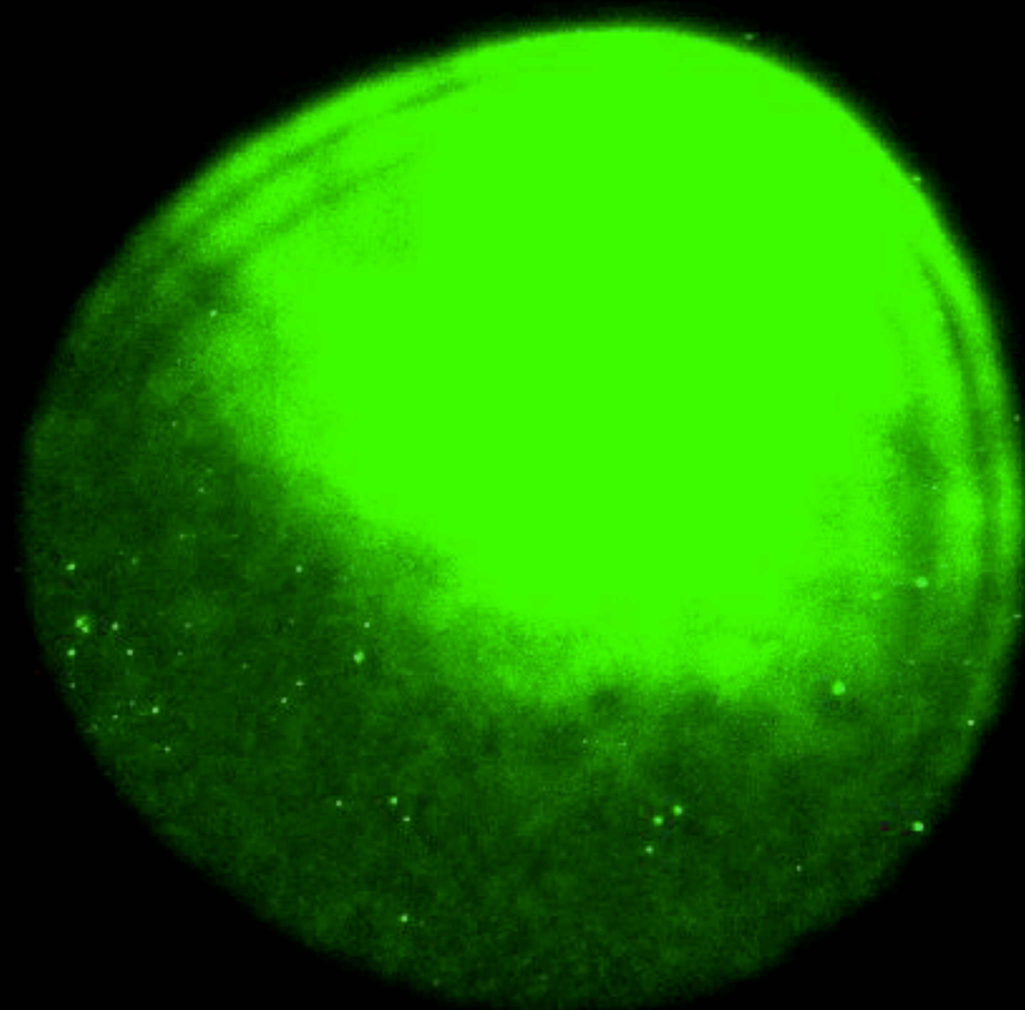
wikimedia

Beyond Turing

Starfish egg cell

Biochemical
pattern formation

Mechanical
Response



Fakhri lab
MIT Physics

Topological turbulence in the membrane of a living cell

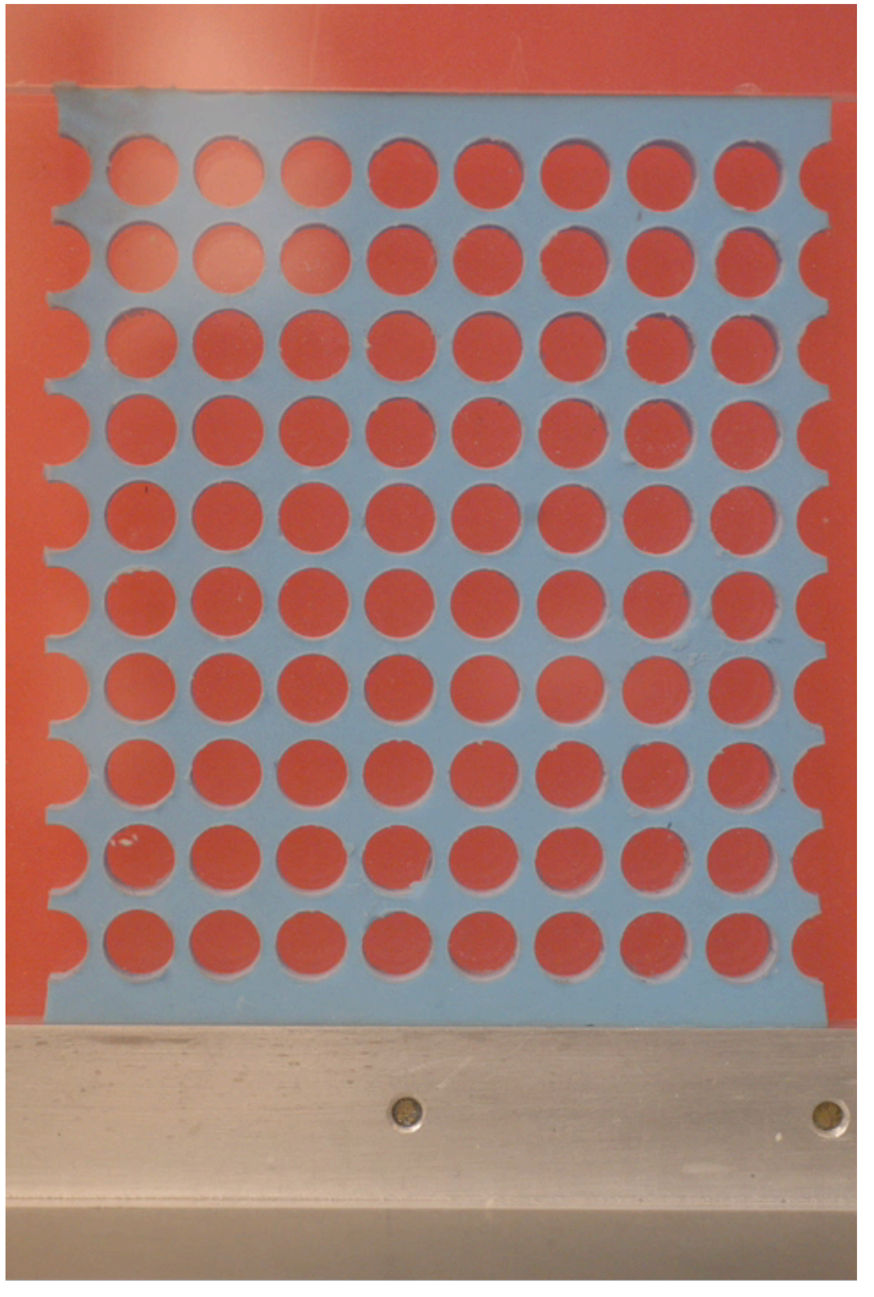
Tan*, Liu*, Miller*, Tekant, JD & Fakhri, Nature Physics (in press)

Calculus of variations

$$\begin{aligned}
\frac{\delta I[Y]}{\delta Y} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{I[f(x)+\epsilon\delta(x-y)]-I[f(x)]\right\} \\
&= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial Y}\delta(x-y)+\frac{\partial f}{\partial Y'}\delta'(x-y)\right]dx \\
&= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial Y}-\frac{d}{dx}\frac{\partial f}{\partial Y'}\right]\delta(x-y)dx.
\end{aligned}$$

$$0 = \frac{\partial f}{\partial Y}-\frac{d}{dx}\frac{\partial f}{\partial Y'}$$

Elasticity



Elasticity

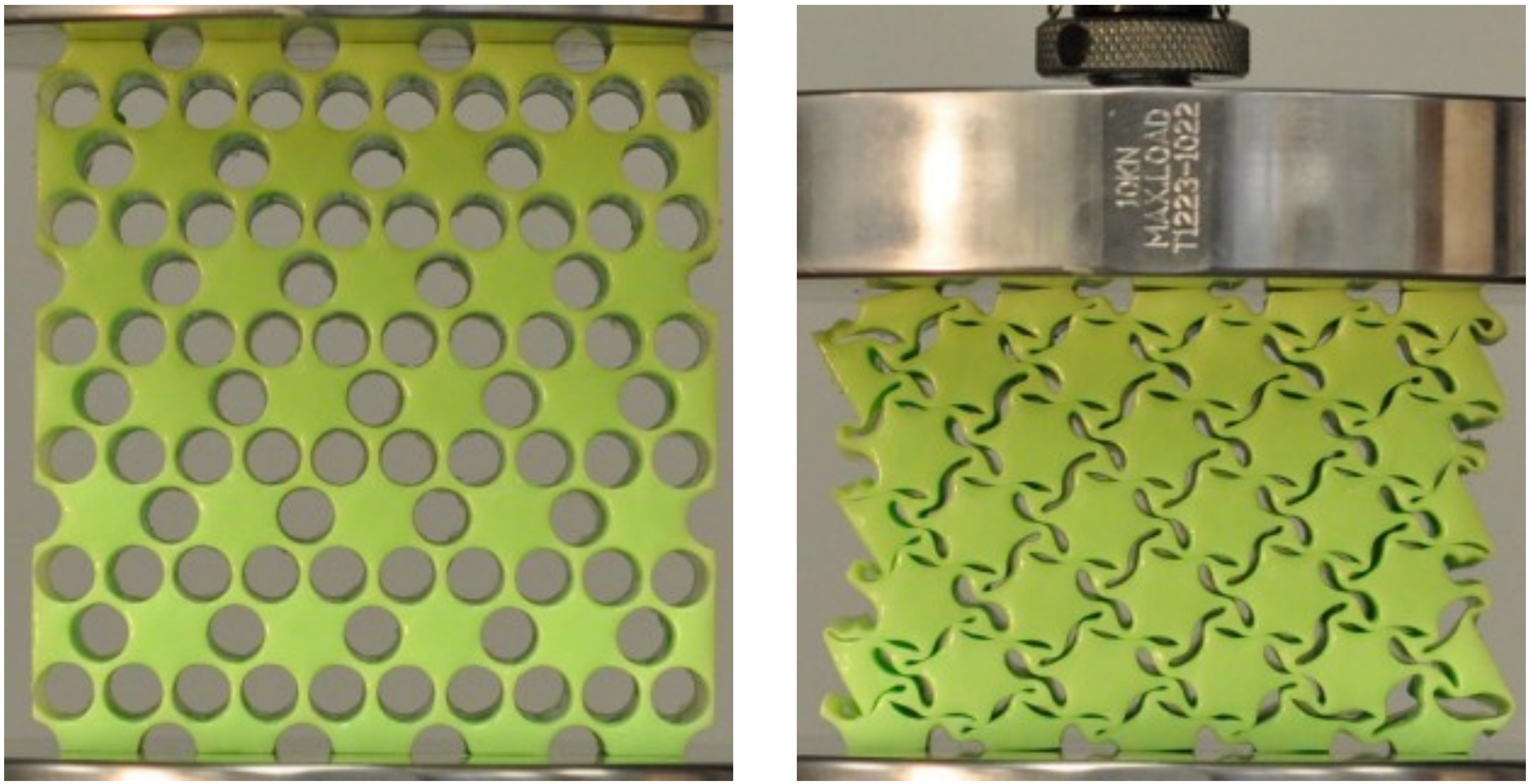
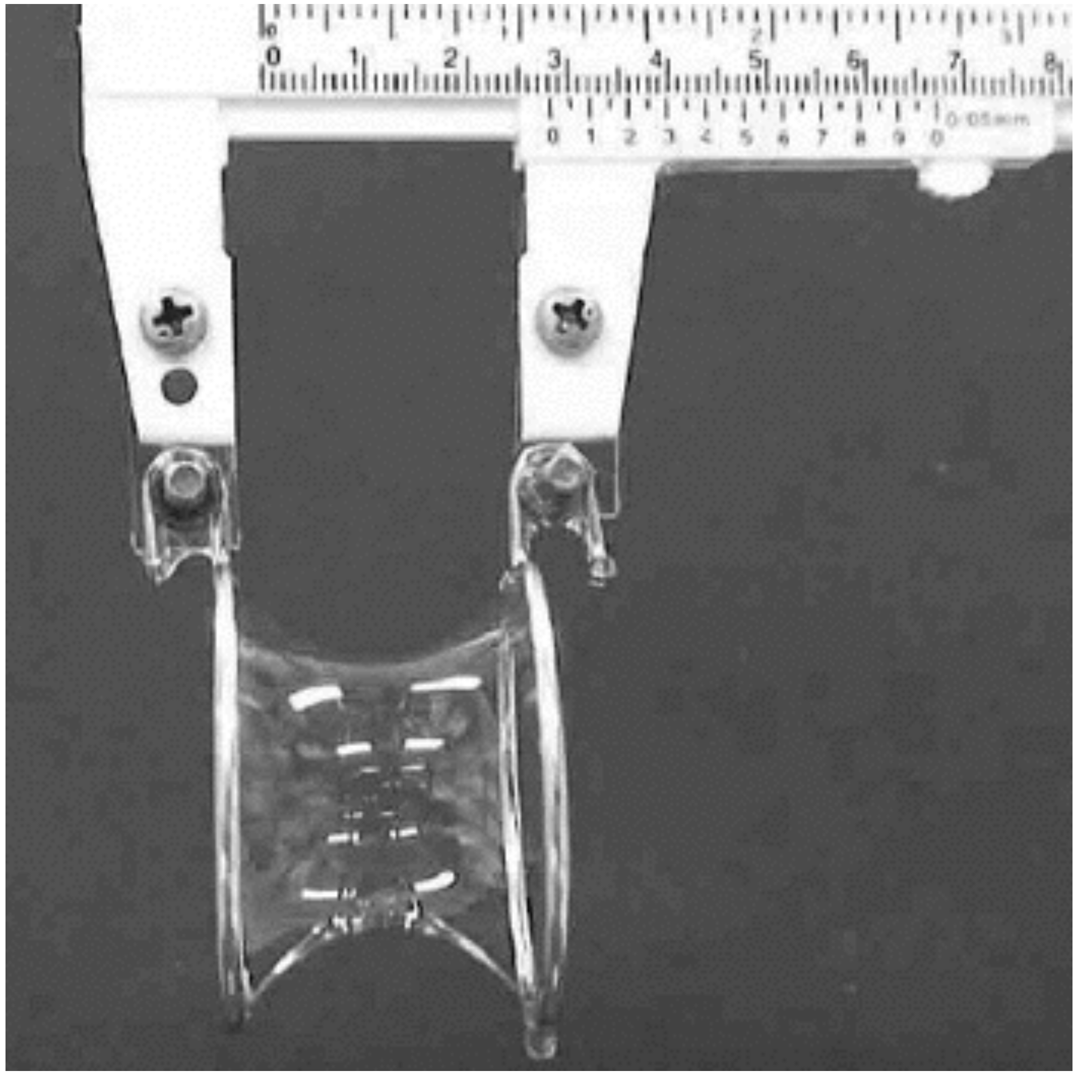
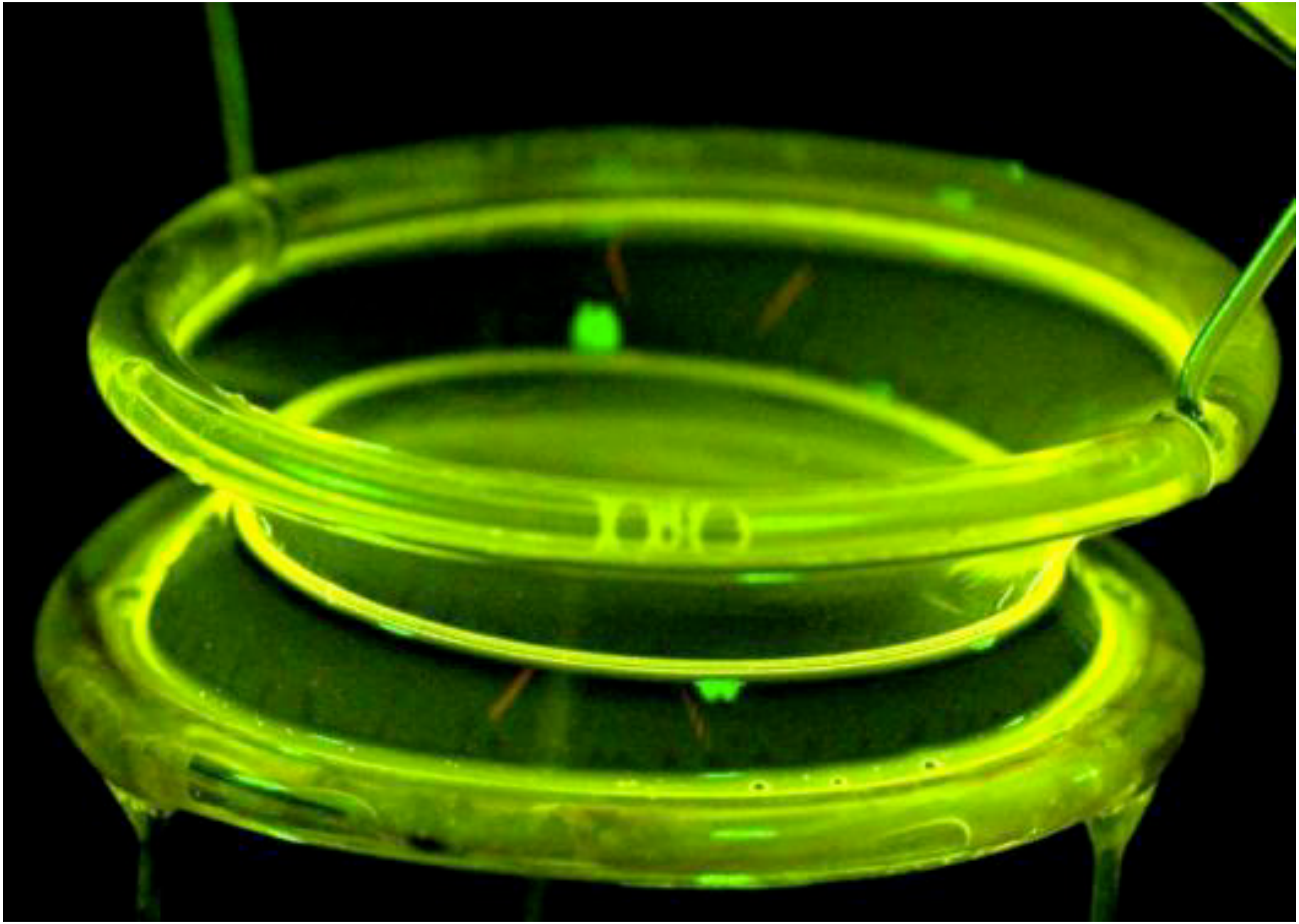


photo: Andrej Kosmrlj

Surface tension





Goldstein lab, Cambridge

Large drop in microgravity

Hydrodynamics

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \rho \mathbf{u} \cdot \mathbf{n} dS = -\int_V \nabla \cdot (\rho \mathbf{u}) dV.$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

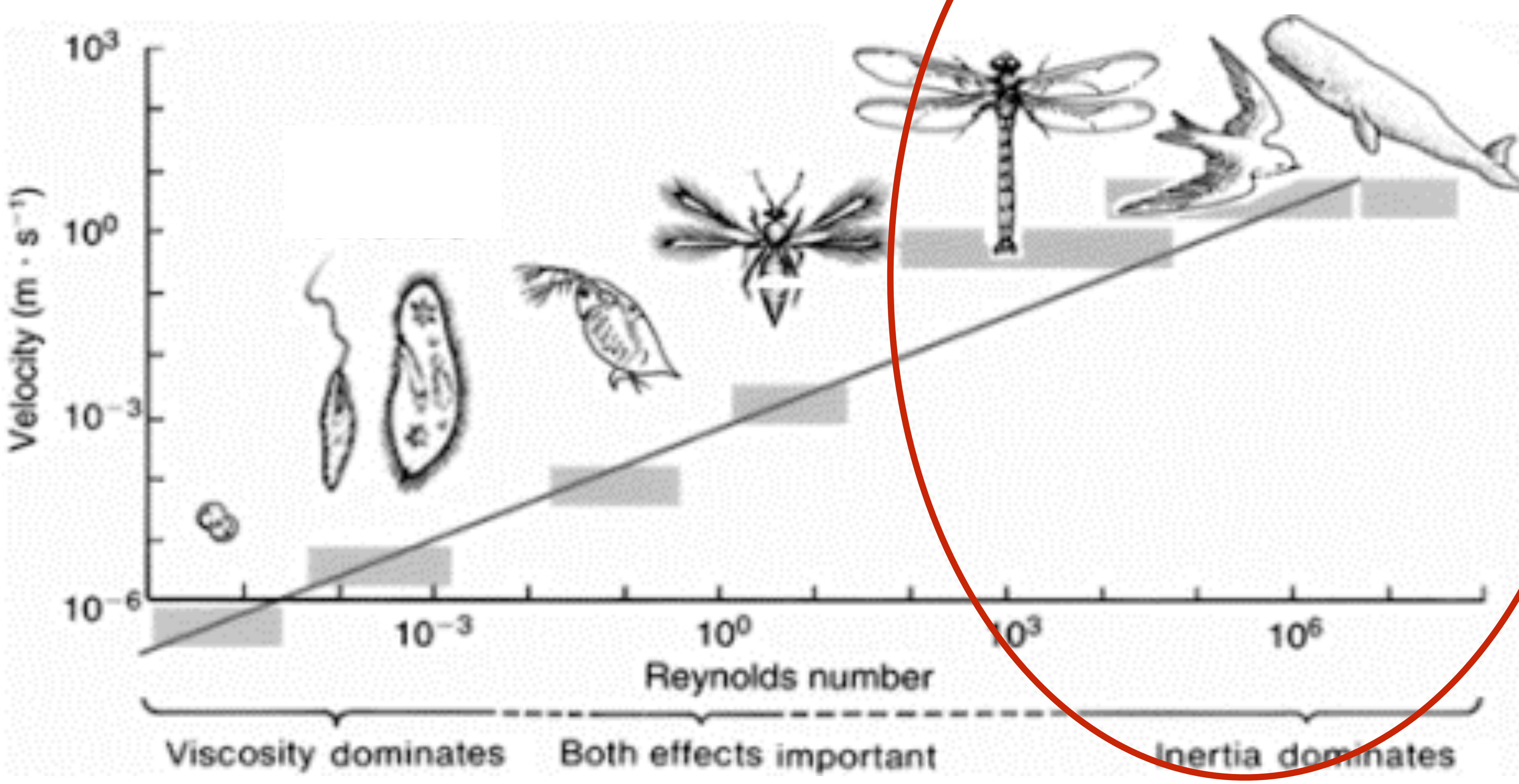
$$\int_{V(t)} \rho \frac{D\mathbf{u}}{Dt} dV = \int_{V(t)} (-\nabla p + \rho \mathbf{g}) dV$$

$$\frac{D\mathbf{u}}{Dt} = \frac{-\nabla p}{\rho} + \mathbf{g}.$$

Typical Reynolds numbers

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} - \tfrac{2}{3}\mu\nabla(\nabla\cdot\mathbf{u}) + \rho\mathbf{g}$$

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

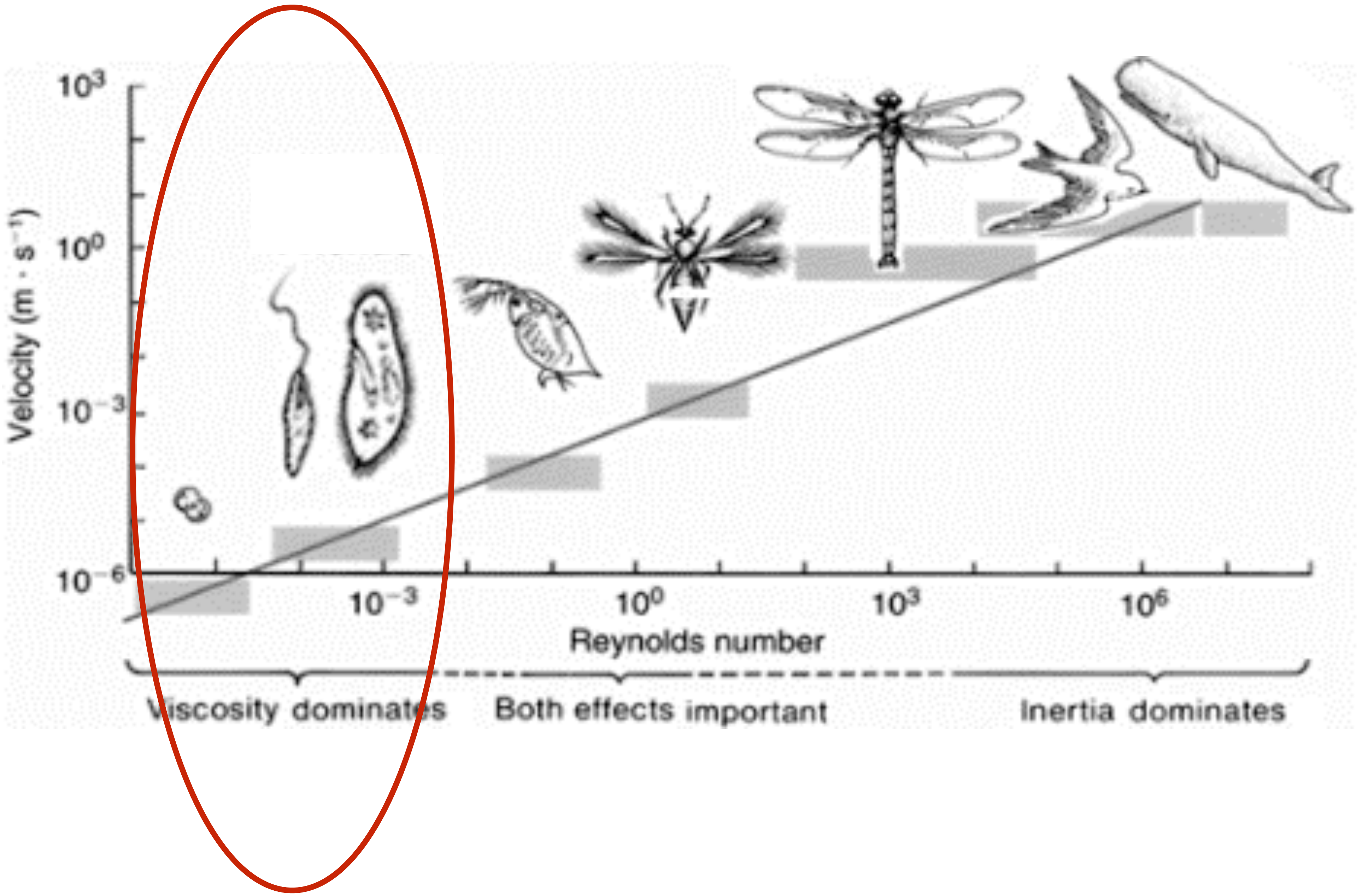


Macro-
world

~ meters

What happens at low Reynolds numbers ?

$$Re = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$



Low-Re (laminar) flow

Swimming at low Reynolds number

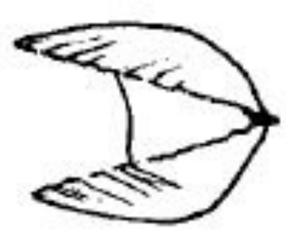
Navier - Stokes :

$$-\nabla p + \eta \nabla^2 \vec{v} = \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v}$$

If $\mathcal{R} \sim UL\rho/\eta \ll 1$

Time doesn't matter. The pattern of motion is the same, whether slow or fast, whether forward or backward in time.

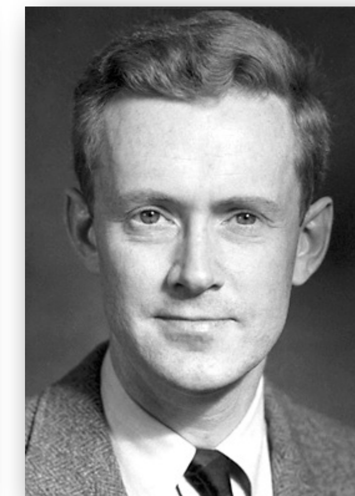
The Scallop Theorem



American Journal of Physics, Vol. 45, No. 1, January 1977

Geoffrey Ingram Taylor

James Lighthill



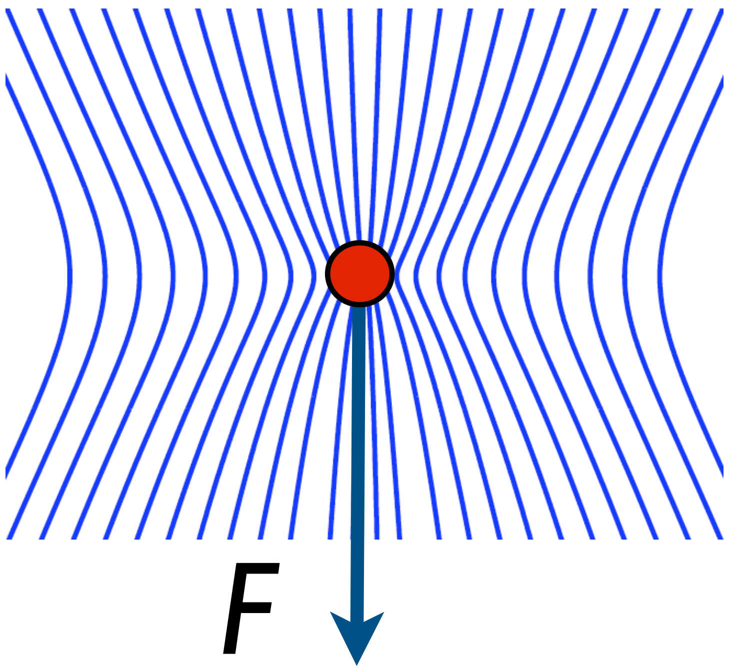
Edward Purcell

$$\begin{aligned} 0 &= \mu \nabla^2 \boldsymbol{u} - \nabla p + \boldsymbol{f}, \\ 0 &= \nabla \cdot \boldsymbol{u}. \end{aligned}$$

+ time-dependent BCs

Superposition of singularities

stokeslet

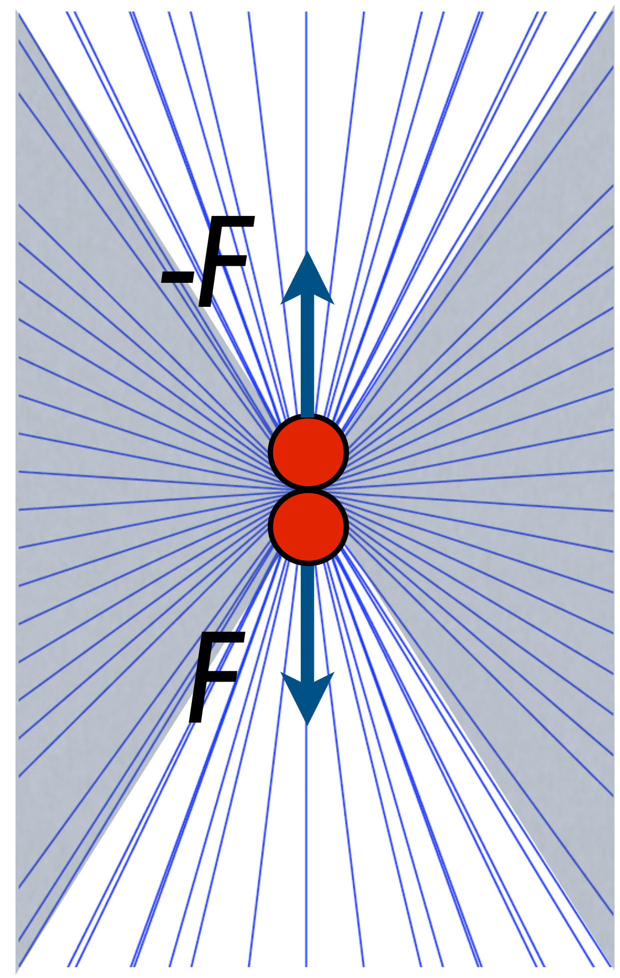


$$p(\boldsymbol{r}) = \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{F}}{4\pi r^2} + p_0$$

$$v_i(\boldsymbol{r}) = \frac{(8\pi\mu)^{-1}}{r}[\delta_{ij} + \hat{r}_i \hat{r}_j]F_j$$

flow ~ r^{-1}

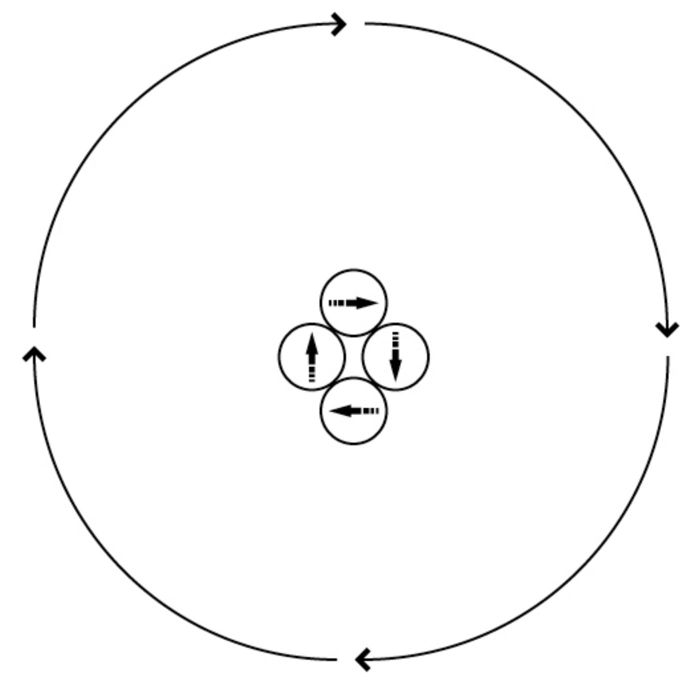
2x stokeslet = symmetric dipole



r^{-2}

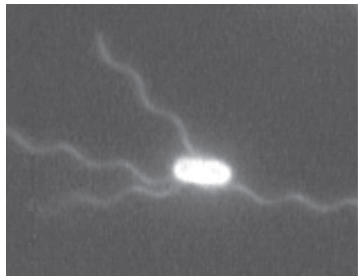
'pusher'

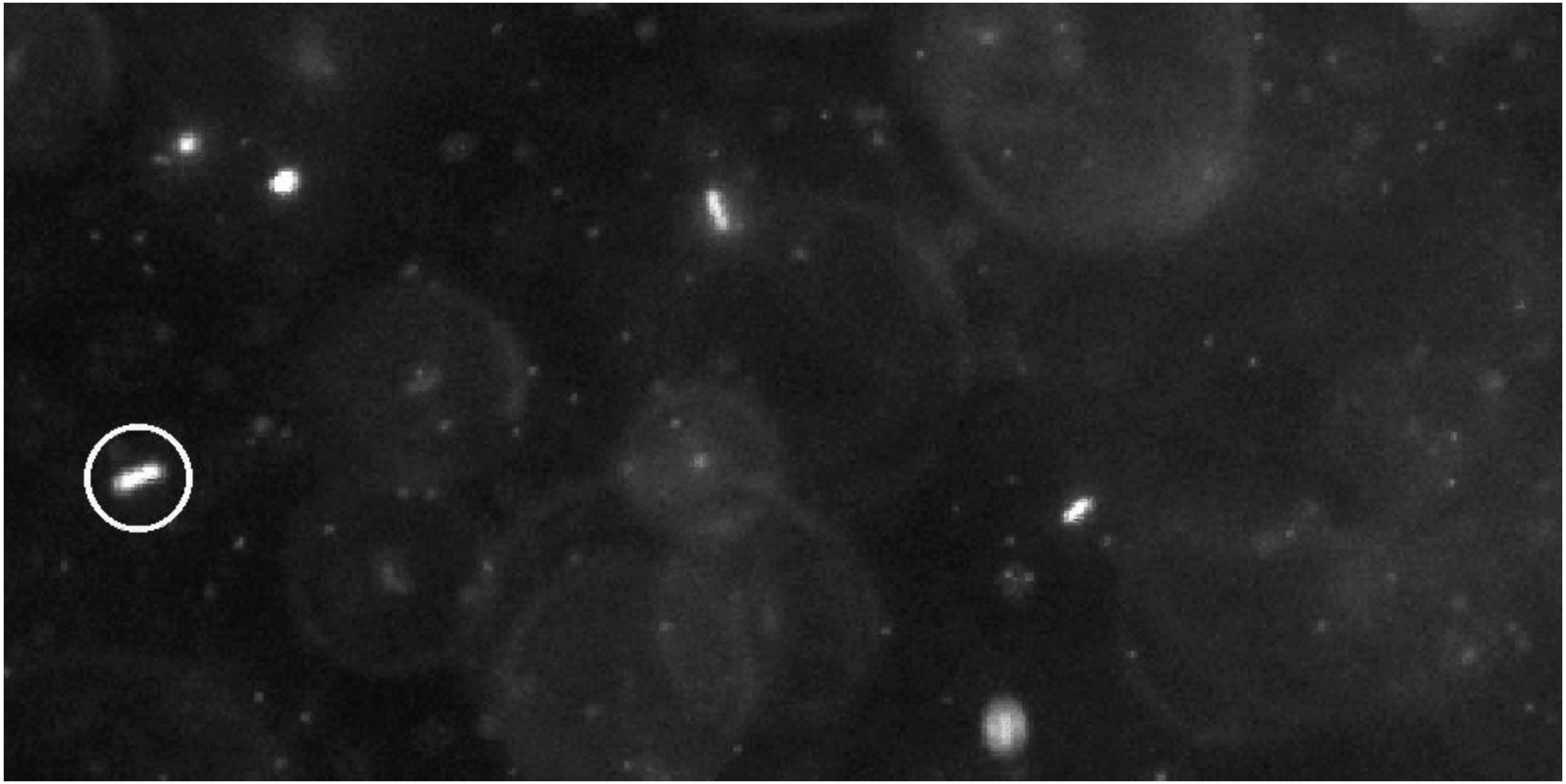
rotlet



r^{-2}

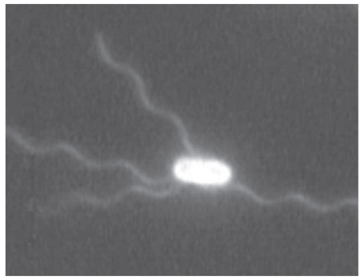
E coli (non-tumbling HCB 437)

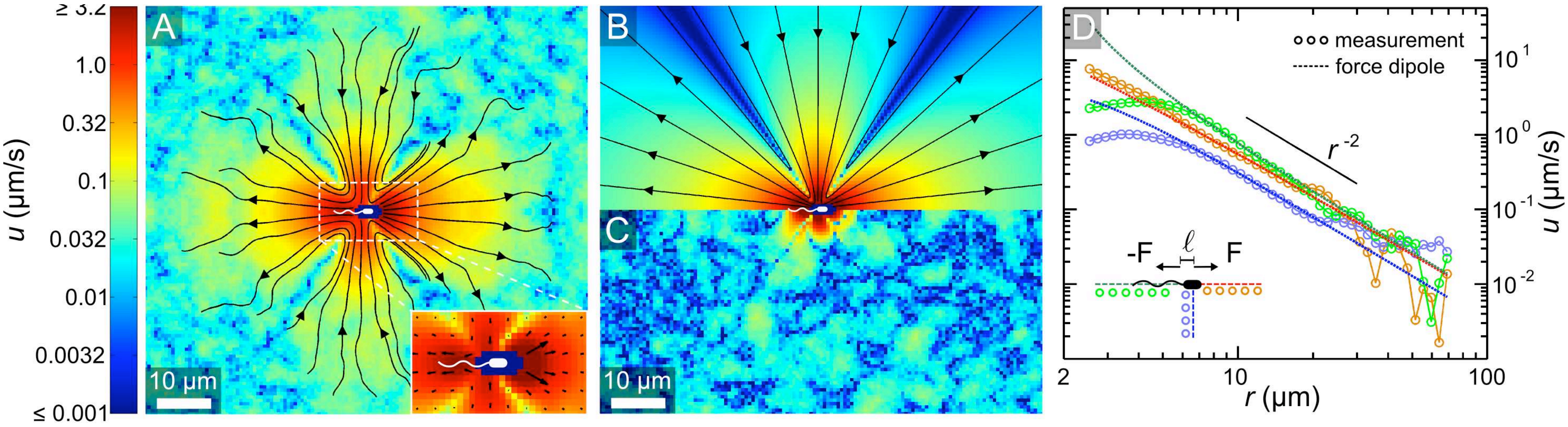




Drescher et al (2011) PNAS

E coli (non-tumbling HCB 437)





$$\boldsymbol{u}(\boldsymbol{r}) = \frac{A}{|\boldsymbol{r}|^2}\left[3(\hat{\boldsymbol{r}}.\hat{\boldsymbol{d}})^2 - 1\right]\hat{\boldsymbol{r}}, \quad A = \frac{\ell F}{8\pi\eta}, \quad \hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$$

$$V_0 = 22 \pm 5 \;\; \mu\text{m/s}$$

$$\ell = 1.9 \;\; \mu\text{m}$$

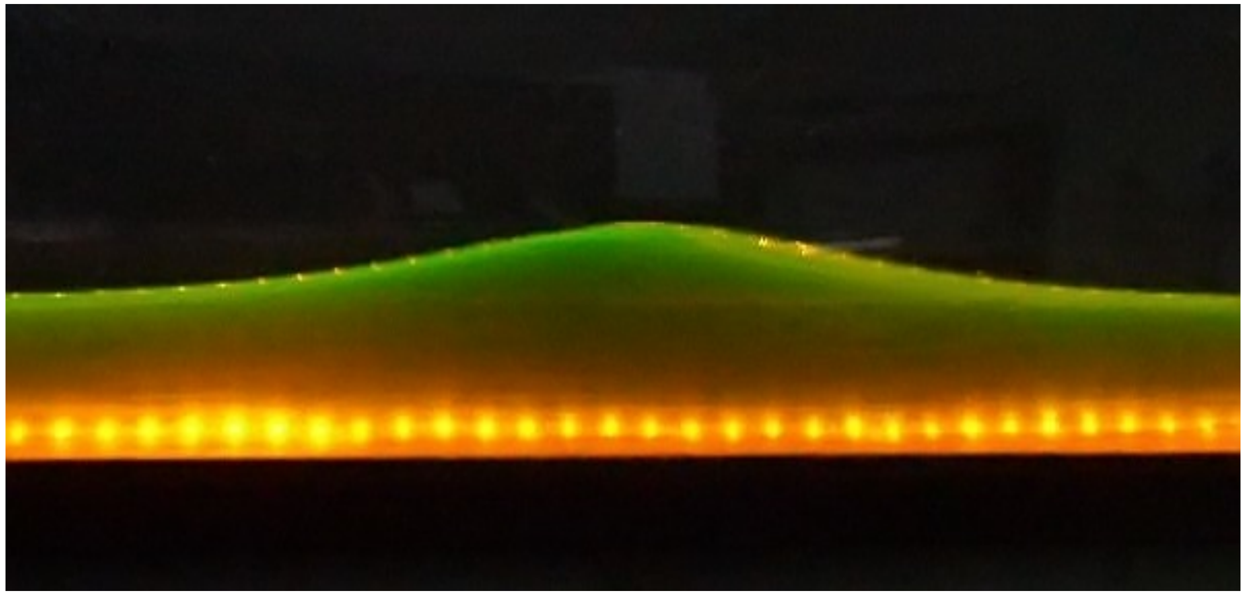
$$F = 0.42 \;\; \text{pN}$$

Drescher et al (2011) PNAS

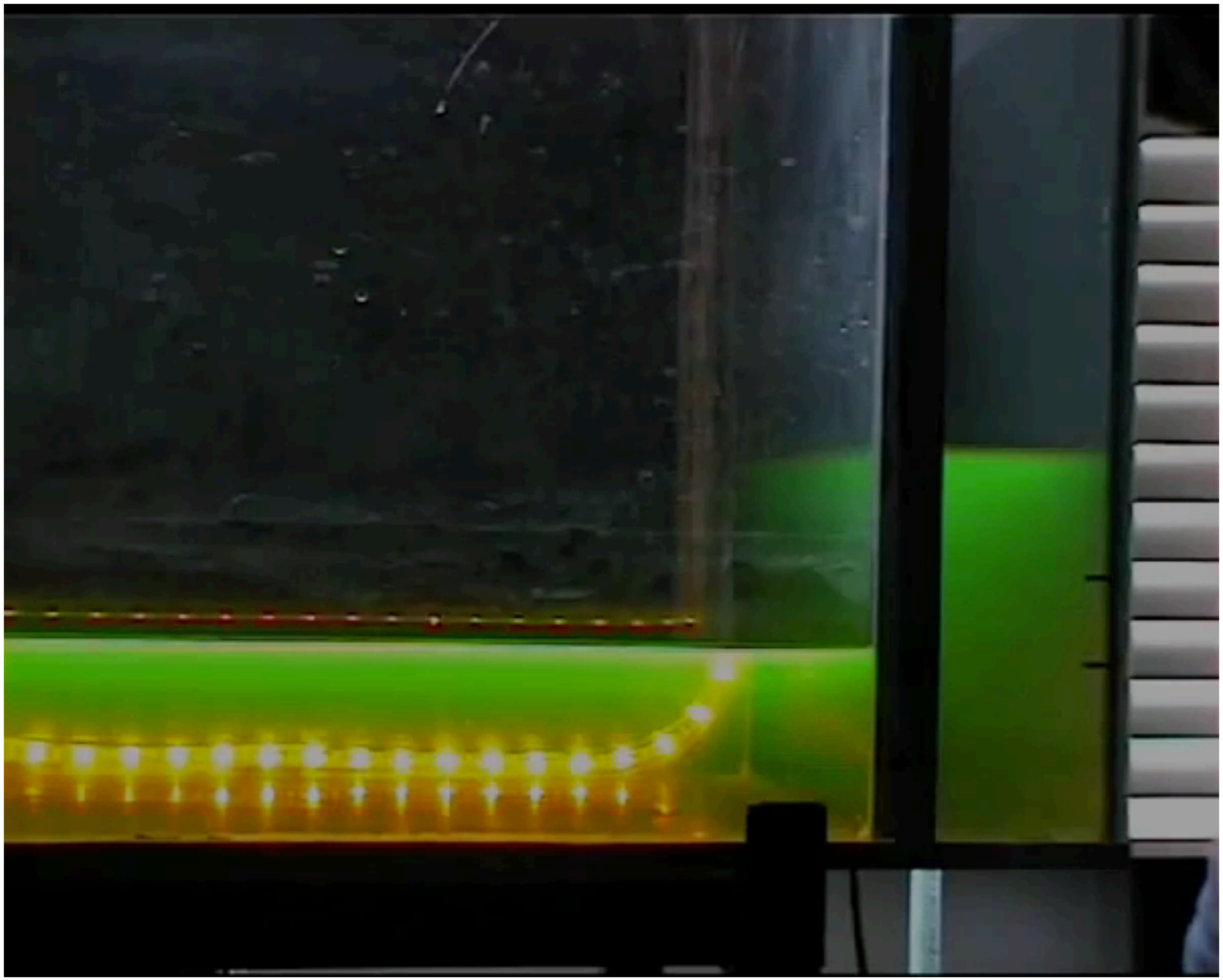
Nonlinear HD: Solitons

KdV equation

$$\partial_t \phi + \partial_x^2 \phi + 6\phi\partial_x \phi = 0$$



Solitons

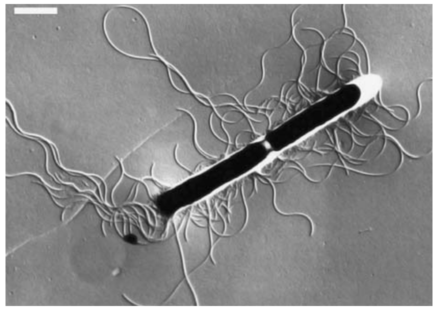


credit: Christophe Finot

Active fluids

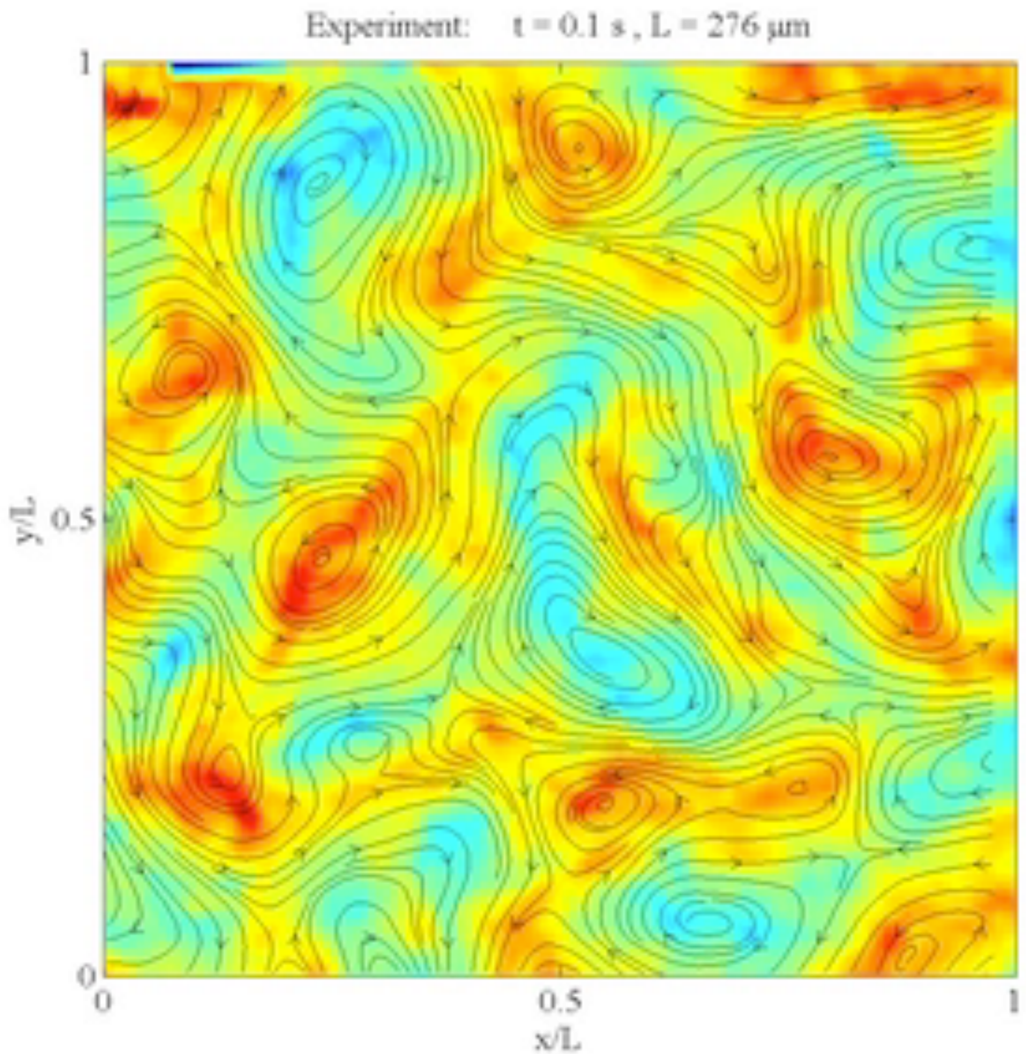
Collective non-equilibrium dynamics in a multicellular system

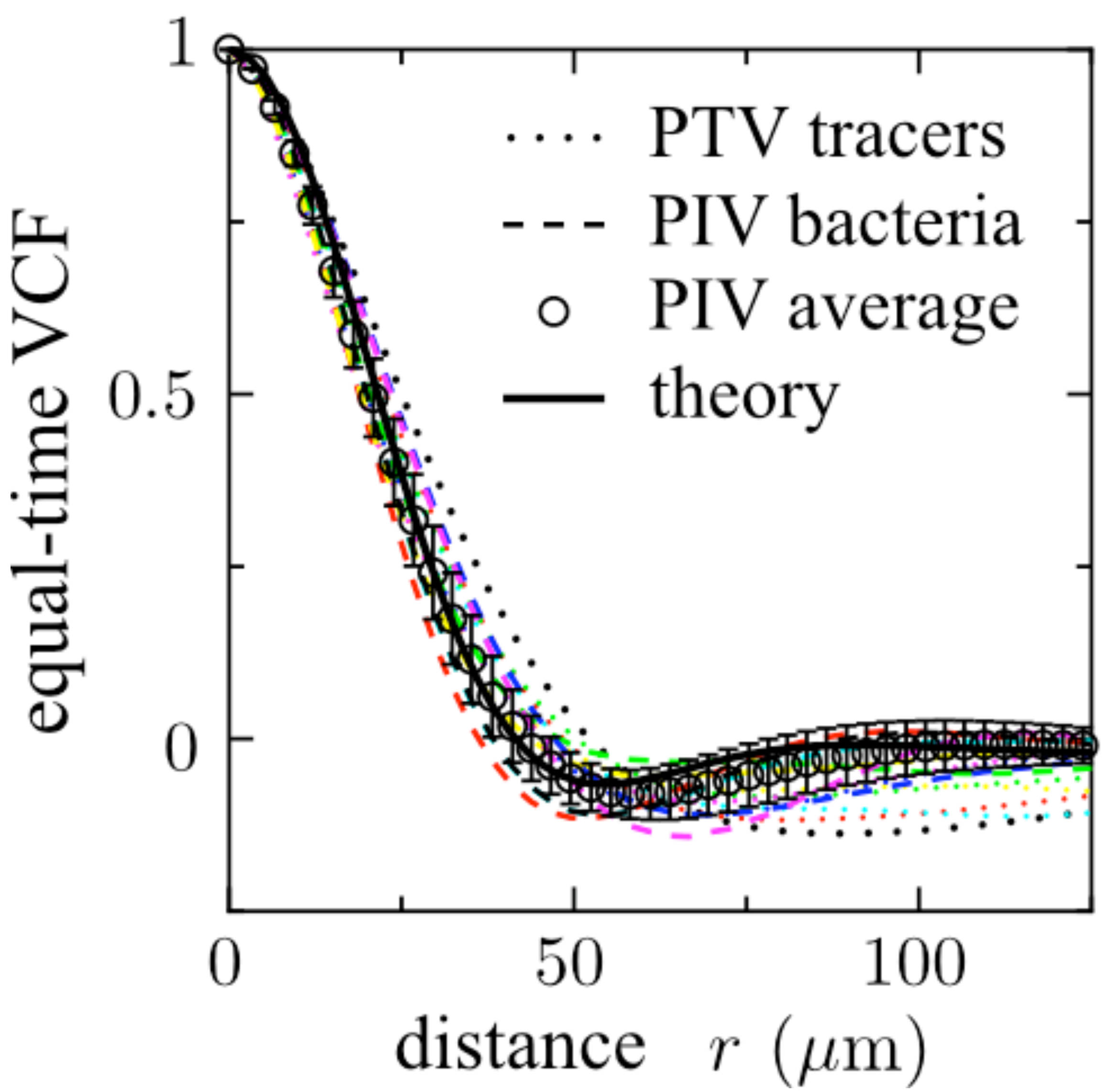
Cisneros et al, Exp Fluids 2017



~ 5μm long

Bacillus subtilis



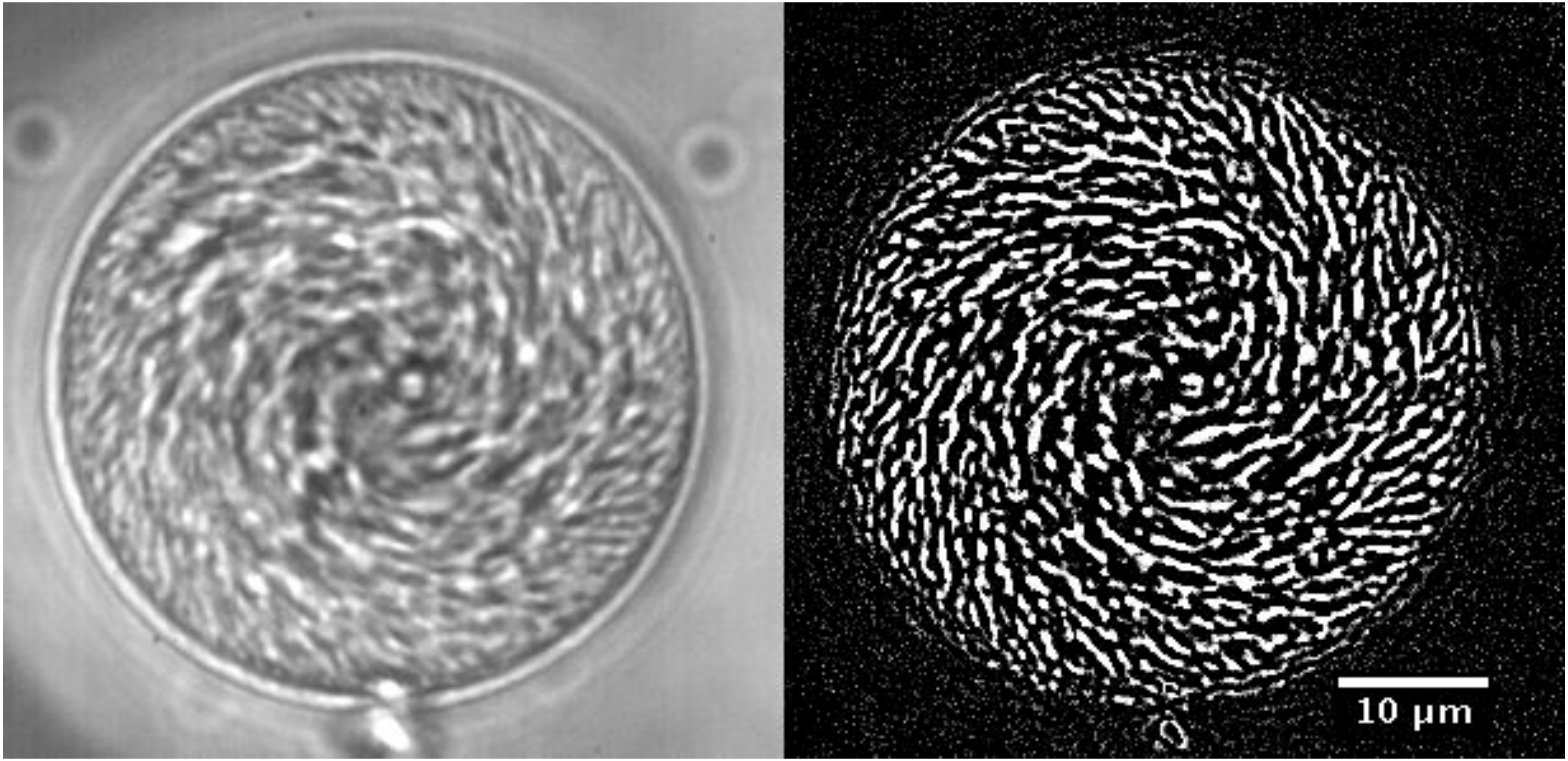


Vortex life time ~ seconds

Vortex diameter ~ 60 μm

PNAS 2012 & PRL 2013

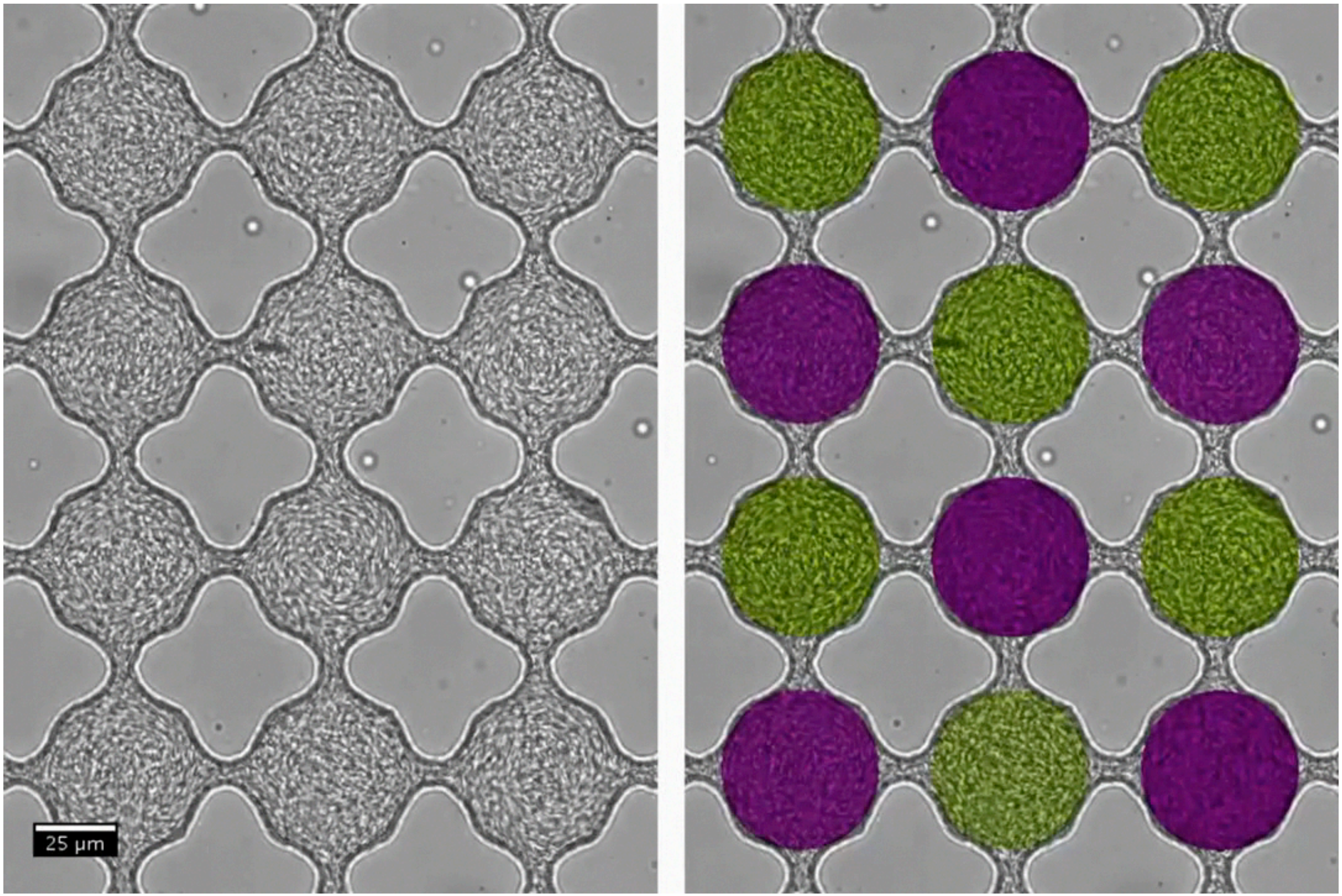
Confined bacterial suspensions



Spontaneously broken mirror-symmetry & edge currents

Wioland et al (2013) PRL

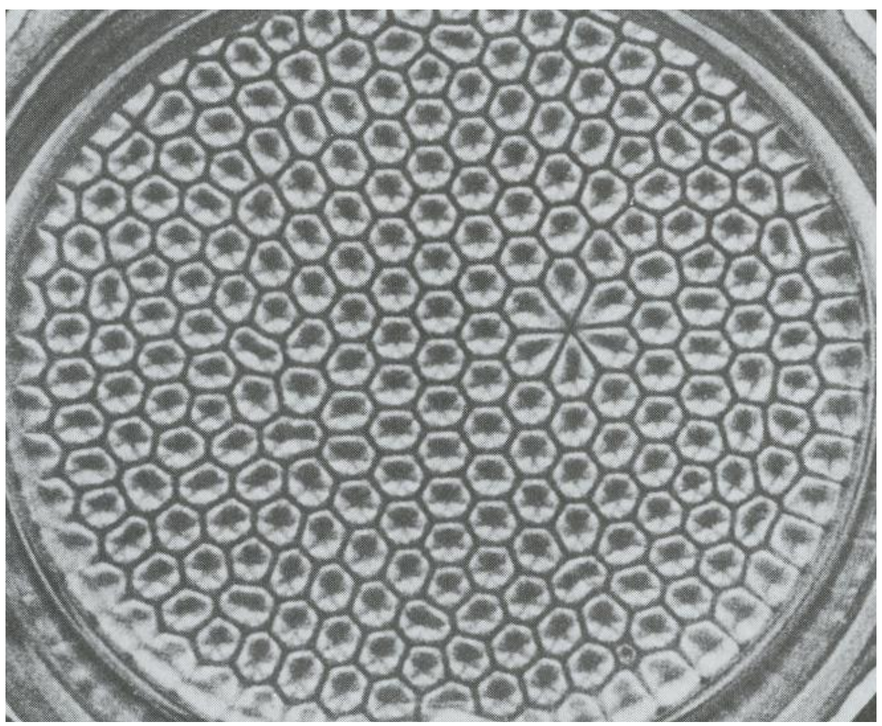
Bacterial spin lattices: Anti-ferromagnetic order



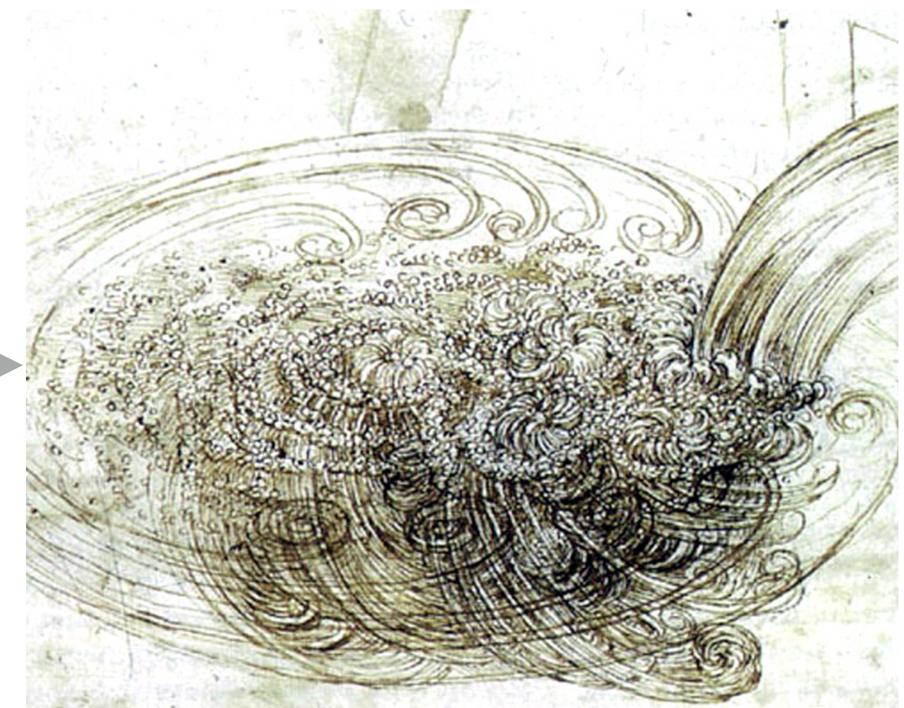
Wioland et al, Nature Physics 2016

Generalized Navier-Stokes equations

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0, \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} &= -\nabla p + \Gamma_0 \nabla^2 \boldsymbol{v} - \Gamma_2 \nabla^4 \boldsymbol{v} + \Gamma_4 \nabla^6 \boldsymbol{v} \end{aligned}$$

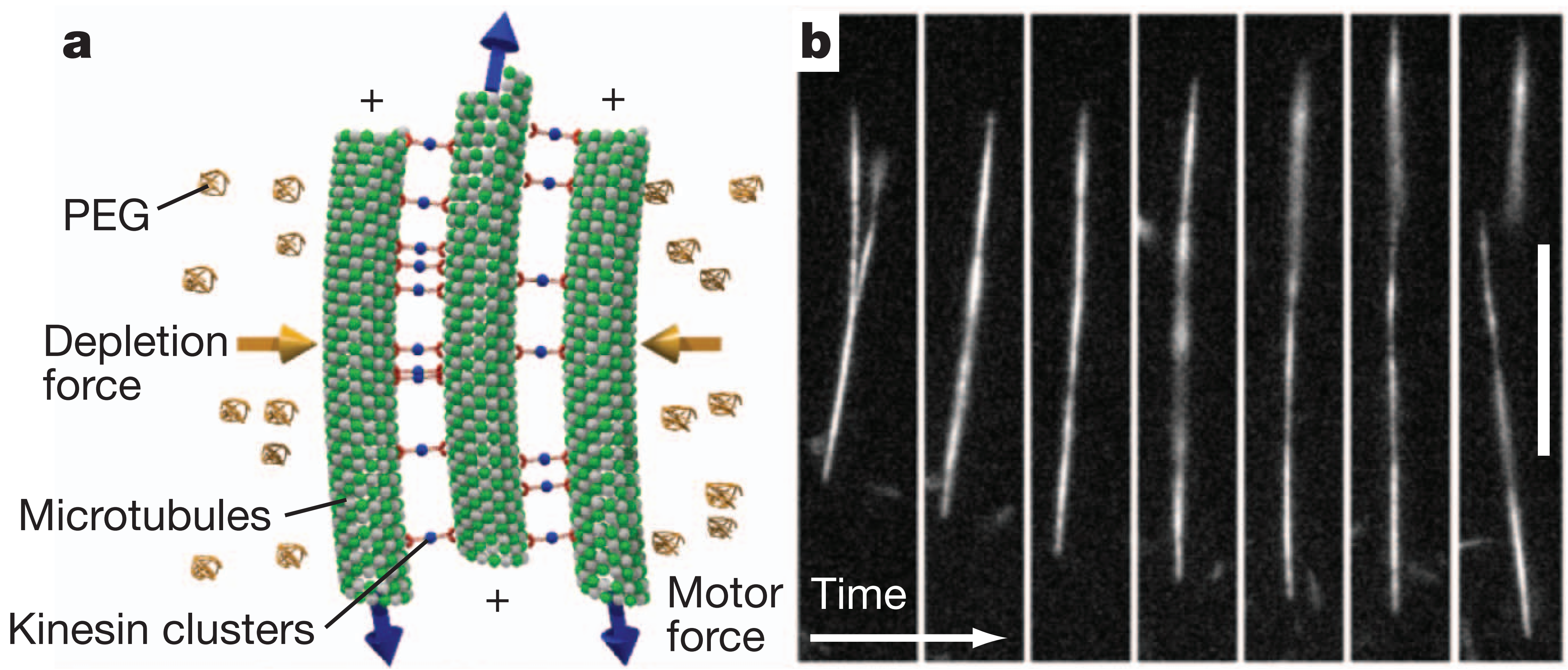


Classical
pattern formation theory



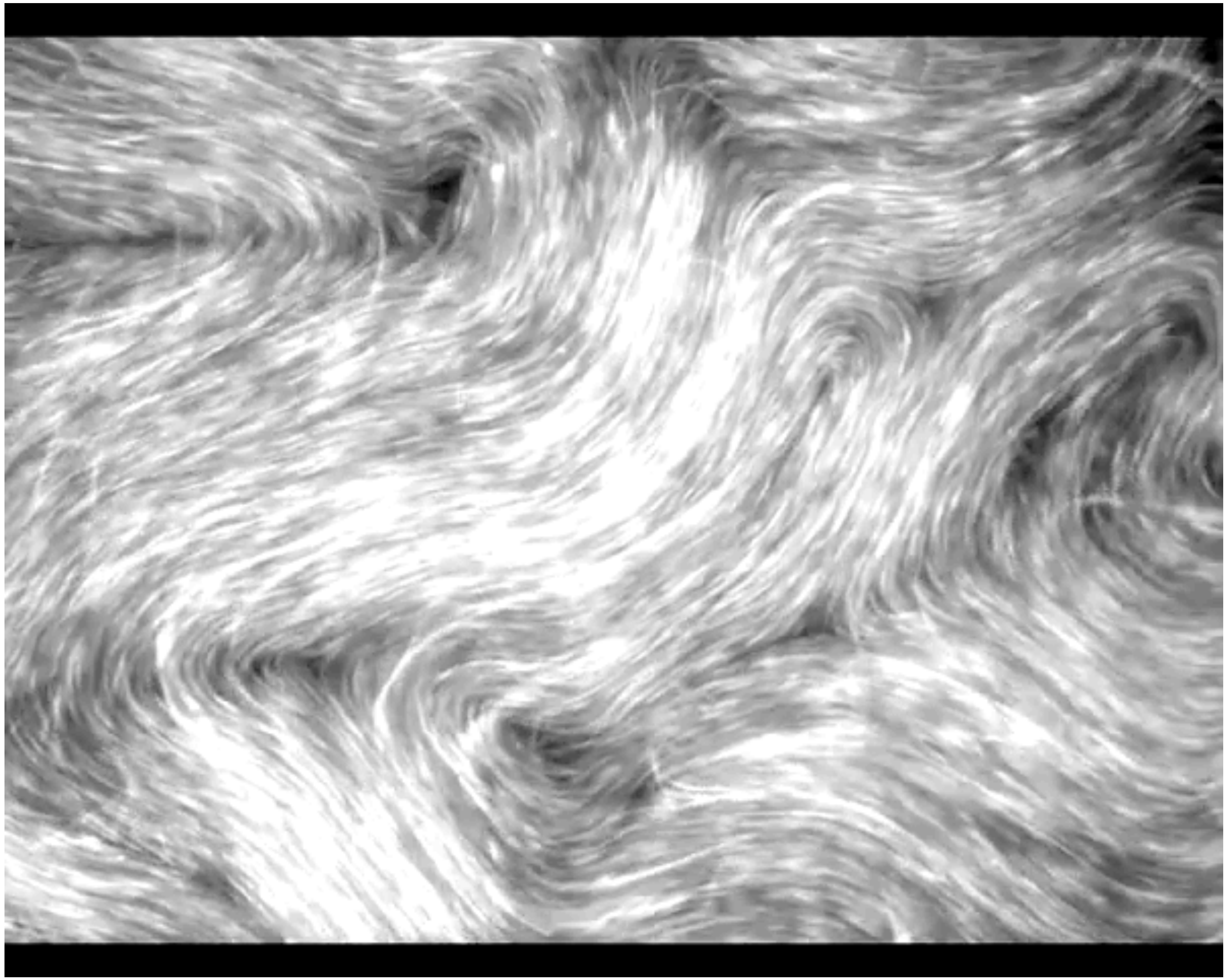
Classical
turbulence theory

Active liquid crystals



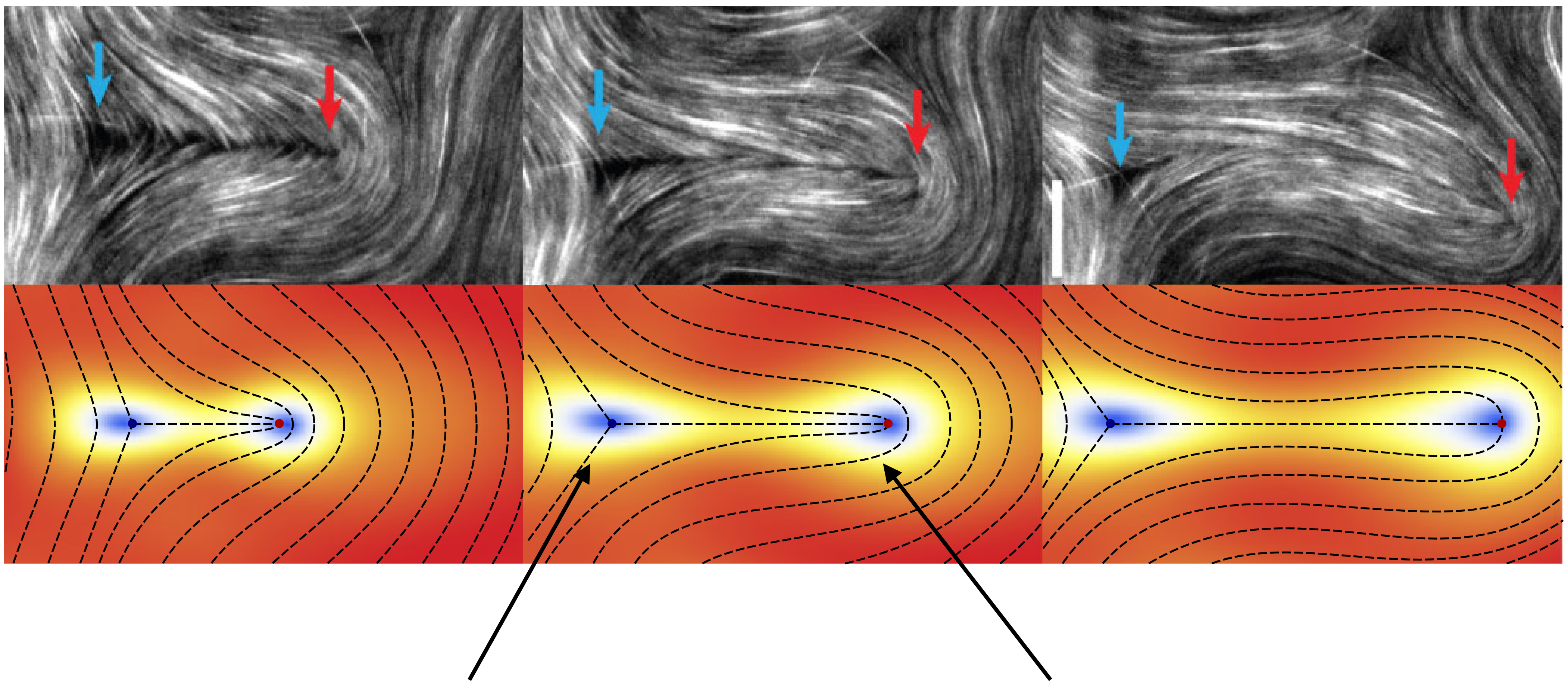
Dogic lab (Brandeis) Nature 2012

Active liquid crystals



Dogic lab, Nature 2012

Active liquid crystal defects

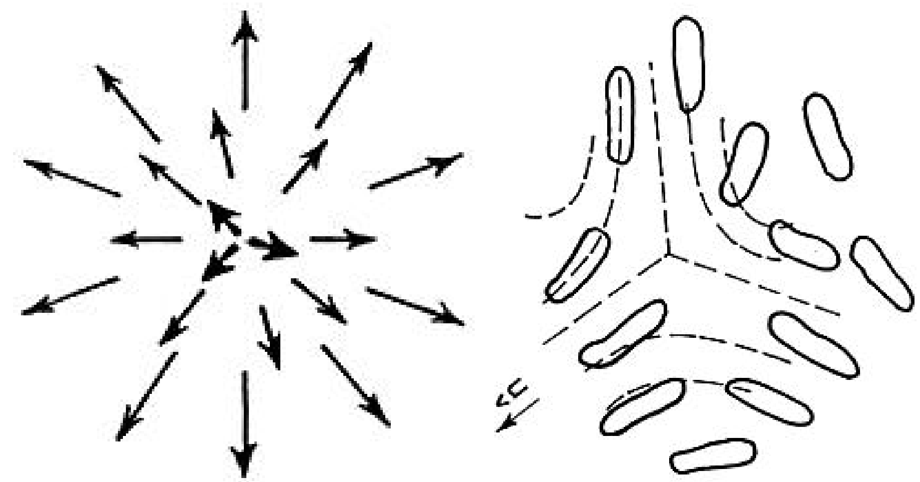


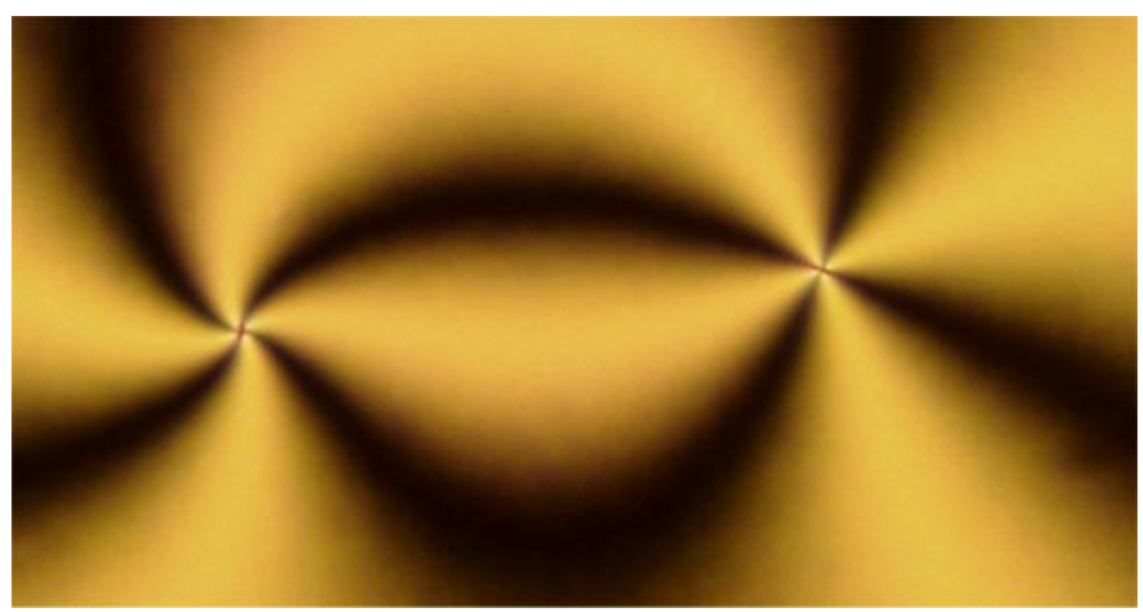
-1/2 +1/2

Giomi et al PRL 2012

Topological defects in order-parameter fields

- optical effects
- work hardening, etc





‘umbilic defects’ in a nematic liquid crystal