18.02 Review

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1 Review of multivariable calculus (18.02) constructs

1.1 Introduction

These notes are a terse summary of what we'll need from multivariable calculus. If, after reading these, some parts are still unclear, you should consult your notes or book from your multivariable calculus or ask about it at office hours. We've also posted a more detailed review of line integrals and Green's theorem. You should consult that if needed.

We've seen that complex exponentials make trigonometric functions easier to work with and give insight into many of the properties of trig functions. Similarly, we'll eventually reformulate some material from 18.02 in complex form. We'll see that it's easier to present and the main properties are more transparent in complex form.

1.2 Terminology and notation

Vectors. We'll denote vectors in the plane by (x, y)

Note. In physics and in 18.02 we usually write vectors in the plane as $x\mathbf{i} + y\mathbf{j}$. This use of $\mathbf{i}$ and $\mathbf{j}$ would be confusing in 18.04, so we will write this vector as (x, y).

In 18.02 you might have used angled brackets $\langle x, y\rangle$ for vectors and round brackets (x, y) for points. In 18.04 we will adopt the more standard mathematical convention and use round brackets for both vectors and points. It shouldn't lead to any confusion.

Orthogonal. Orthogonal is a synonym for perpendicular. Two vectors are orthogonal if their dot product is zero, i.e. $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ are orthogonal if

$$\mathbf{v} \cdot \mathbf{w} = (v_1, v_2) \cdot (w_1, w_2) = v_1 w_1 + v_2 w_2 = 0.$$

Composition. Composition of functions will be denoted $f(g(z))$ or $f \circ g(z)$, which is read as 'f composed with g'

1.3 Parametrized curves

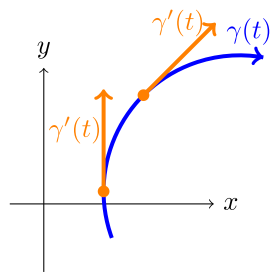
We often use the greek letter gamma for a paramtrized curve, i.e.

$$\gamma(t) = (x(t), y(t)).$$

We think of this as a moving point tracing out a curve in the plane. The tangent vector

$$\gamma'(t) = (x'(t), y'(t))$$

is tangent to the curve at the point $(x(t), y(t))$. It's length $|\gamma'(t)|$ is the instantaneous speed of the moving point.



Parametrized curve $\gamma(t)$ with some tangent vectors $\gamma'(t)$.

Example Rev.1. Parametrize the straight line from the point (x_0, y_0) to (x_1, y_1).

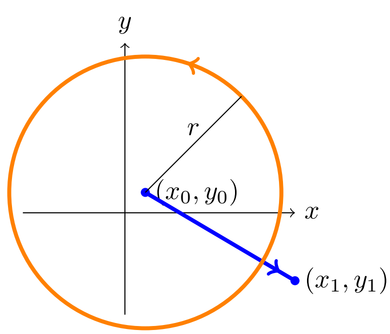
answer: There are always many parametrizations of a given curve. A standard one for straight lines is

$$\gamma(t) = (x, y) = (x_0, y_0) + t(x_1 - x_0, y_1 - y_0), \text{ with } 0 \le t \le 1.$$

Example Rev.2. Parametrize the circle of radius r around the point (x_0, y_0).

answer: Again there are many parametrizations. Here is the standard one with the circle traversed in the counterclockwise direction:

$$\gamma(t) = (x, y) = (x_0, y_0) + r(\cos(t), \sin(t)), \text{ with } 0 \le t \le 2\pi.$$



Line from (x_0, y_0) to (x_1, y_1) and circle around (x_0, y_0).

1.4 Chain rule

For a function $f(x, y)$ and a curve $\gamma(t) = (x(t), y(t))$ the chain rule gives

$$\frac{df(\gamma(t))}{dt} = \left.\frac{\partial f}{\partial x}\right|_{\gamma(t)} x'(t) + \left.\frac{\partial f}{\partial y}\right|_{\gamma(t)} y'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) \text{ dot product of vectors.}$$

Here ∇f is the gradient of f defined in the next section.

1.5 Grad, curl and div

Gradient. For a function $f(x, y)$, the gradient is defined as $\textbf{grad} f = \nabla f = (f_x, f_y)$. A vector field $\mathbf{F}$ which is the gradient of some function is called a gradient vector field.

Curl. For a vector in the plane $\mathbf{F}(x,y) = (M(x,y), N(x,y))$ we define

$$\text{curl}\mathbf{F} = N_x - M_y.$$

Note. The curl is a scalar. In 18.02 and in general, the curl of a vector field is another vector field. However, for vectors fields in the plane the curl is always in the $\widehat{k}$ direction, so we have simply dropped the $\widehat{k}$ and made curl a scalar.

Divergence. The divergence of the vector field $\mathbf{F} = (M, N)$ is

$$\text{div}\mathbf{F} = M_x + N_y.$$

1.6 Level curves

Recall that the level curves of a function $f(x,y)$ are the curves given by $f(x,y) = \text{constant}$.

Recall also that the gradient ∇f is orthogonal to the level curves of f

1.7 Line integrals

The ingredients for line (also called path or contour) integrals are the following:

- A vector field $\mathbf{F} = (M, N)$
- A curve $\gamma(t) = (x(t), y(t))$ defined for $a \leq t \leq b$

Then the line integral of $\mathbf{F}$ along γ is defined by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt = \int_\gamma M dx + N dy.$$

Example Rev.3. Let $\mathbf{F} = (-y/r^2, x/r^2)$ and let γ be the unit circle. Compute line integral of $\mathbf{F}$ along γ.

answer: You should be able to supply the answer to this example

1.7.1 Properties of line integrals

1. Independent of parametrization.
2. Reverse direction on curve $\Rightarrow$ change sign. That is,

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

   (Here, $-C$ means the same curve traversed in the opposite direction.)

3. If C is closed then we sometimes indicate this with the notation $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M\, dx + N\, dy$.

1.7.2 Fundamental theorem for gradient fields

Theorem Rev.4. (Fundamental theorem for gradient fields)
If $\mathbf{F} = \nabla f$ then $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = f(P) - f(Q)$, where Q, P are the beginning and endpoints respectively of γ.

Proof. By the chain rule we have

$$\frac{df(\gamma(t))}{dt} = \nabla f(\gamma(t)) \cdot \gamma'(t) = \mathbf{F}(\gamma(t)) \cdot \gamma'(t).$$

The last equality follows from our assumption that $\mathbf{F} = \nabla f$. Now we can this when we compute the line integral:

$$\begin{aligned} \int_\gamma \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_a^b \frac{df(\gamma(t))}{dt}\, dt \\ &= f(\gamma(b)) - f(\gamma(a)) \\ &= f(P) - f(Q) \end{aligned}$$

Notice that the third equality follows from the fundamental theorem of calculus.

Definition. If a vector field $\mathbf{F}$ is a gradient field, with $\mathbf{F} = \nabla f$, then we call f a a potential function for $\mathbf{F}$.

Note: the usual physics terminology would be to call $-f$ the potential function for $\mathbf{F}$.

1.7.3 Path independence and conservative functions

Definition. For a vector field $\mathbf{F}$, the line integral $\displaystyle\int \mathbf{F} \cdot d\mathbf{r}$ is called path independent if, for any two points P and Q, the line integral has the same value for *every* path between P and Q.

Theorem. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent is equivalent to $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Sketch of proof. Draw two paths from Q to P. Following one from Q to P and the reverse of the other back to P is a closed path. The equivalence follows easily. We refer you to the more detailed review of line integrals and Green's theorem for more details.

Definition. A vector field with path independent line integrals, equivalently a field whose line integrals around any closed loop is 0 is called a conservative vector field.

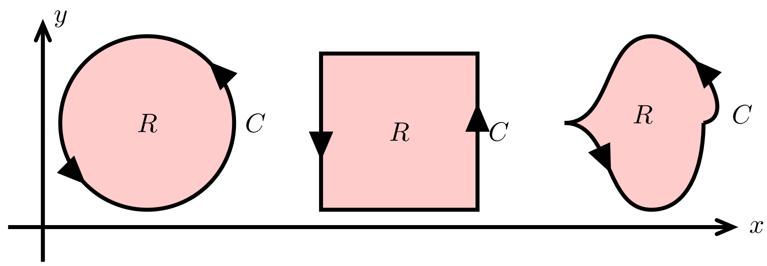
Theorem Rev.5. We have the following equivalence: On a connected region, a gradient field is conservative and a conservative field is a gradient field.

Proof. Again we refer you to the more detailed review for details. Essentially, if $\mathbf{F}$ is conservative then we can define a potential function $f(x, y)$ as the line integral of $\mathbf{F}$ from some base point to (x, y).

1.8 Green's Theorem

Ingredients: C a *simple* closed curve (i.e. no self-intersection), and R the interior of C.

C must be *positively oriented* (traversed so interior region R is on the left) and *piecewise smooth* (a few corners are okay).



Theorem Rev.6. Green's Theorem: If the vector field $\mathbf{F} = (M, N)$ is defined and differentiable on R then

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

In vector form this is written

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\,dA.$$

where the curl is defined as $\text{curl}\mathbf{F} = (N_x - M_y)$.

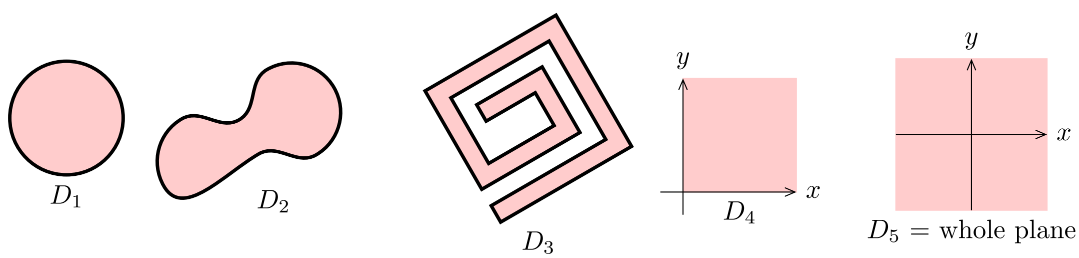
Proof of Green's Theorem. See the more detailed notes on Green's theorem and line integrals for the proof.

1.9 Extensions and applications of Green's theorem

1.9.1 Simply connected regions

Definition: A region D in the plane is simply connected if it has "no holes". Said differently, it is simply connected for every simple closed curve C in D, the interior of C is fully contained in D.

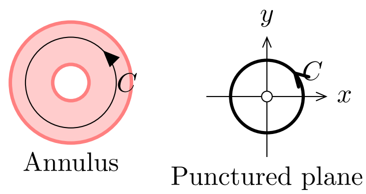
Examples:



D1-D5 are simply connected. For any simple closed curve C inside any of these regions the interior of C is entirely inside the region.

Note. Sometimes we say any curve can be shrunk to a point without leaving the region.

The regions below are not simply connected. For each, the interior of the curve C is not entirely in the region.



1.9.2 Potential Theorem

Here is an application of Green's theorem which tells us how to spot a conservative field on a simply connected region. The theorem does not have a standard name, so we choose to call it the Potential Theorem.

Theorem Rev.7. (Potential Theorem) Take $\mathbf{F} = (M, N)$ defined and differentiable on a region D.

(a) If $\mathbf{F} = \nabla f$ then curl$\mathbf{F} = N_x - M_y = 0$.

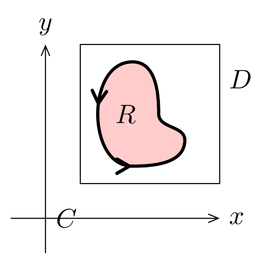
(b) If D is simply connected and curl$\mathbf{F} = 0$ on D, then $\mathbf{F} = \nabla f$ for some f.

We know that on a connected region, being a gradient field is equivalent to being conservative. So we can restate the Potential Theorem as: on a simply connected region, $\mathbf{F}$ is conservative is equivalent to curl$\mathbf{F} = 0$.

Proof of (a): $\mathbf{F} = (f_x, f_y)$, so curl$\mathbf{F} = f_{yx} - f_{xy} = 0$.

Proof of (b): Suppose C is a simple closed curve in D. Since D is simply connected the interior of C is also in D. Therefore, using Green's theorem we have,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA = 0.$$



This shows that $\mathbf{F}$ is conservative in D. Therefore, by Theorem Rev.5 $\mathbf{F}$ is a gradient field.

Summary: Suppose the vector field $\mathbf{F} = (M, N)$ is defined on a simply connected region D. Then, the following statements are equivalent.

(1) $\displaystyle\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is path independent.

(2) $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C.

(3) $\mathbf{F} = \nabla f$ for some f in D

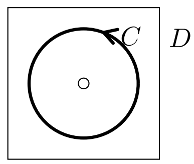
(4) $\mathbf{F}$ is conservative in D.

If $\mathbf{F}$ is continuously differentiable then 1,2,3,4 all imply 5:

(5) $\text{curl}\mathbf{F} = N_x - M_y = 0$ in D

1.9.3 Why we need simply connected in the Potential Theorem

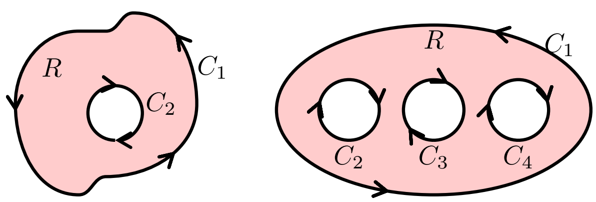
If there is a hole then $\mathbf{F}$ might not be defined on the interior of C. (See the example on the tangential field below.)



1.9.4 Extended Green's Theorem

We can extend Green's theorem to a region R which has multiple boundary curves.

Suppose R is the region between the two simple closed curves C_1 and C_2.



(Note R is always to the left as you traverse either curve in the direction indicated.)

Then we can extend Green's theorem to this setting by

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA.$$

Likewise for more than two curves:

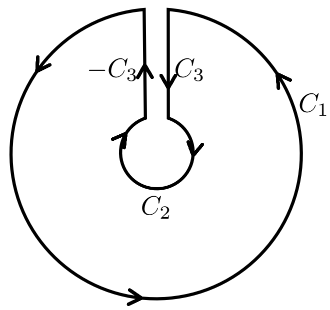
$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_4} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA.$$

Proof. The proof is based on the following figure. We 'cut' both C_1 and C_2 and connect them by two copies of C_3, one in each direction. (In the figure we have drawn the two copies of C_3 as separate curves, in reality they are the same curve traversed in opposite directions.)

Now the curve $C = C_1 + C_3 + C_2 - C_3$ is a simple closed curve and Green's theorem holds on it. But the region inside C is exactly R and the contributions of the two copies of C_3 cancel. That is, we have shown that

$$\iint_R \text{curl}\mathbf{F}\, dA = \int_{C_1+C_3+C_2-C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r}.$$

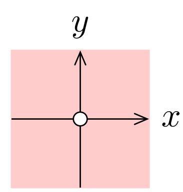
This is exactly Green's theorem, which we wanted to prove.



The punctured plane.

Example Rev.8. Let $\mathbf{F} = \dfrac{(-y, x)}{r^2}$ ("tangential field")

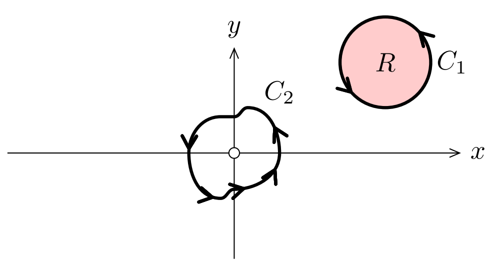
$\mathbf{F}$ is defined on D = plane - (0,0) = the punctured plane. (Shown below.)



It's easy to compute (we've done it before) that curl$\mathbf{F} = 0$ in D.

Question: For the tangential field $\mathbf{F}$ what values can $\oint_C \mathbf{F} \cdot d\mathbf{r}$ take for C a simple closed curve (positively oriented)?

<u>**answer:**</u> We have two cases (i) C_1 not around 0 (ii) C_2 around 0



In case (i) Green's theorem applies because the interior does not contain the problem point at the origin. Thus,

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA = 0.$$

For case (ii) we will show that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi$.

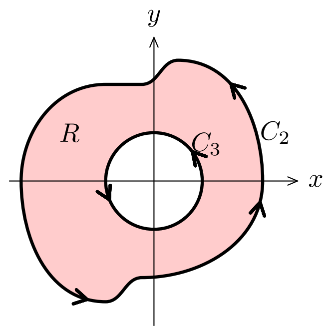
Let C_3 be a small circle of radius a, entirely inside C_2. By the extended Green's theorem we have

$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA = 0.$$

Thus, $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$.

Using the usual parametrization of a circle we can easily compute that the line integral is

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1\, dt = 2\pi. \quad QED.$$



Answer to the question: The only possible values are 0 and 2π.

We can extend this answer in the following way:

If C is not simple, then the possible values of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ are $2\pi n$, where n is the number of times C goes (counterclockwise) around (0,0).

Not for class: n is called the *winding number* of C around 0. n also equals the number of times C crosses the positive x-axis, counting +1 from below and −1 from above.