18.04 Complex analysis with applications

Spring 2019 lecture notes

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This PDF is an adaption and extension of the original by Andre Nachbin and Jeremy Orloff. Credit for course design and content should go to them; responsibility for typos and errors lies with me.

Contents

1 Brief course description

Complex analysis is a beautiful, tightly integrated subject. It revolves around complex analytic functions. These are functions that have a complex derivative. Unlike calculus using real variables, the mere existence of a complex derivative has strong implications for the properties of the function.

Complex analysis is a basic tool in many mathematical theories. By itself and through some of these theories it also has a great many practical applications.

There are a small number of far-reaching theorems that we'll explore in the first part of the class. Along the way, we'll touch on some mathematical and engineering applications of these theorems. The last third of the class will be devoted to a deeper look at applications.

The main theorems are Cauchy's Theorem, Cauchy's integral formula, and the existence of Taylor and Laurent series. Among the applications will be harmonic functions, two dimensional fluid flow, easy methods for computing (seemingly) hard integrals, Laplace transforms, and Fourier transforms with applications to engineering and physics.

1.1 Topics needed from prerequisite math classes

We will review these topics as we need them:

- Limits
- Power series
- Vector fields
- Line integrals
- Green's theorem

1.2 Level of mathematical rigor

We will make careful arguments to justify our results. Though, in many places we will allow ourselves to skip some technical details if they get in the way of understanding the main point, but we will note what was left out.

1.3 Speed of the class

(Borrowed from R. Rosales 18.04 OCW 1999)

Do not be fooled by the fact things start slow. This is the kind of course where things keep on building up continuously, with new things appearing rather often. Nothing is really very hard, but the total integration can be staggering - and it will sneak up on you if you do not watch it. Or, to express it in mathematically sounding lingo, this course is 'locally easy' but 'globally hard'. That means that if you keep up-to-date with the homework and lectures, and read the class notes regularly, you should not have any problems.

2 Complex algebra and the complex plane

We will start with a review of the basic algebra and geometry of complex numbers. Most likely you have encountered this previously in 18.03 or elsewhere.

2.1 Motivation

The equation $x^2 = -1$ has no real solutions, yet we know that this equation arises naturally and we want to use its roots. So we make up a new symbol for the roots and call it a complex number.

Definition. The symbols $\pm i$ will stand for the solutions to the equation $x^2 = -1$. We will call these new numbers complex numbers. We will also write

$$\sqrt{-1} = \pm i$$

where i is also called an **imaginary number**.[1] This is a historical term. These are perfectly valid numbers that don't happen to lie on the real number line.[2] We're going to look at the algebra, geometry and, most important for us, the exponentiation of complex numbers.

Before starting a systematic exposition of complex numbers, we'll work a simple example.

Example. Solve the equation $z^2 + z + 1 = 0$.

Solution: We can apply the quadratic formula to get

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}.$$

Q: Do you know how to solve quadratic equations by completing the square? This is how the quadratic formula is derived and is well worth knowing!

2.2 Fundamental theorem of algebra

One of the reasons for using complex numbers is because allowing complex roots means every polynomial has exactly the expected number of roots. This is the **fundamental theorem of algebra:**

A polynomial of degree n has exactly n complex roots (repeated roots are counted with multiplicity).

In a few weeks, we will be able to prove this theorem as a remarkably simple consequence of one of our main theorems.

[1] Engineers typically use j instead of i. We'll follow mathematical custom in 18.04.

[2] Our motivation for using complex numbers is *not* the same as the historical motivation. Historically, mathematicians were willing to say $x^2 = -1$ had no solutions. The issue that pushed them to accept complex numbers had to do with the formula for the roots of cubics. Cubics always have at least one real root, and when square roots of negative numbers appeared in this formula, even for the real roots, mathematicians were forced to take a closer look at these (seemingly) exotic objects.

2.3 Terminology and basic arithmetic

Definitions

- **Complex numbers** are defined as the set of all numbers

$$z = x + yi,$$

  where x and y are real numbers.

- We denote the set of all complex numbers by $\mathbf{C}$. (On the blackboard we will usually write $\mathbb{C}$ –this font is called *blackboard bold.*)

- We call x the **real part** of z. This is denoted by $x = \text{Re}(z)$.

- We call y the **imaginary part** of z. This is denoted by $y = \text{Im}(z)$.

Important: The imaginary part of z is a **real number**. It **does not** include the i.

The basic arithmetic operations follow the standard rules. All you have to remember is that $i^2 = -1$. We will go through these quickly using some simple examples. It almost goes without saying that in 18.04 it is essential that you become fluent with these manipulations.

- **Addition:** $(3 + 4i) + (7 + 11i) = 10 + 15i$

- **Subtraction:** $(3 + 4i) - (7 + 11i) = -4 - 7i$

- **Multiplication:**

$$(3 + 4i)(7 + 11i) = 21 + 28i + 33i + 44i^2 = -23 + 61i$$

  Here we have used the fact that $44i^2 = -44$.

Before talking about division and absolute value we introduce a new operation called conjugation. It will prove useful to have a name and symbol for this, since we will use it frequently.

Complex conjugation is denoted with a bar and defined by

$$\overline{x + iy} = x - iy.$$

If $z = x + iy$ then its conjugate is $\overline{z} = x - iy$ and we read this as "z-bar $= x - iy$".

Example.

$$\overline{3 + 5i} = 3 - 5i$$

The following is a very **useful property of conjugation**: If $z = x + iy$ then

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

is real. We will use it in the next example to help with division.

Example. (Division.) Write $\dfrac{3 + 4i}{1 + 2i}$ in the standard form $x + iy$.

Solution: We use the useful property of conjugation to clear the denominator:

$$\frac{3+4i}{1+2i} = \frac{3+4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{11-2i}{5} = \frac{11}{5} - \frac{2}{5}i.$$

In the next section we will discuss the geometry of complex numbers, which give some insight into the meaning of the magnitude of a complex number. For now we just give the definition.

Definition. The **magnitude** of the complex number $x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

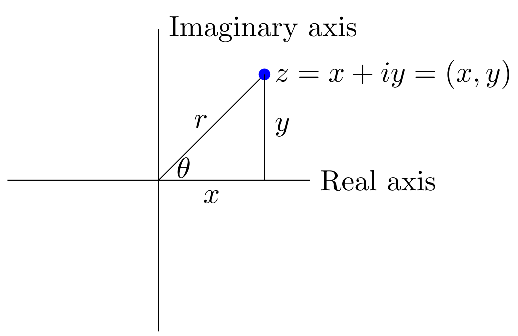
The magnitude is also called the **absolute value**, **norm** or **modulus**.

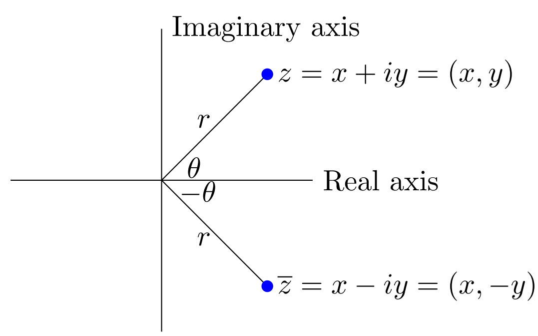
Example. The norm of $3 + 5i = \sqrt{9 + 25} = \sqrt{34}$.
Important. The norm is the sum of x^2 and y^2. **It does not include the i and is therefore always positive**.

2.4 The complex plane

2.4.1 The geometry of complex numbers

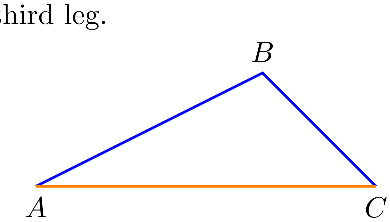
Because it takes two numbers x and y to describe the complex number $z = x + iy$ we can visualize complex numbers as points in the xy-plane. When we do this we call it the complex plane. Since x is the real part of z we call the x-axis the real axis. Likewise, the y-axis is the imaginary axis.





2.4.2 The triangle inequality

The triangle inequality says that for a triangle the sum of the lengths of any two legs is greater than the length of the third leg.

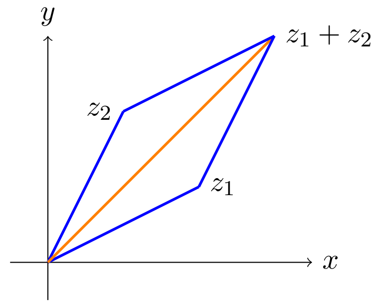


Triangle inequality: $|AB| + |BC| > |AC|$

For complex numbers the triangle inequality translates to a statement about complex magnitudes. Precisely: for complex numbers z_1, z_2

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

with equality only if one of them is 0 or $\arg(z_1) = \arg(z_2)$. This is illustrated in the following figure.



Triangle inequality: $|z_1| + |z_2| \geq |z_1 + z_2|$

We get equality only if z_1 and z_2 are on the same ray from the origin.

2.5 Polar coordinates

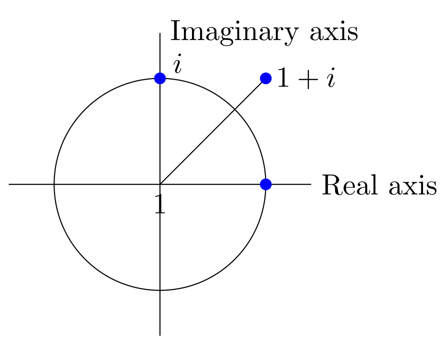
In the figures above we have marked the length r and polar angle θ of the vector from the origin to the point $z = x + iy$. These are the same polar coordinates you saw in 18.02 and 18.03. There are a number of synonyms for both r and θ

$$r = |z| = \text{magnitude} = \text{length} = \text{norm} = \text{absolute value} = \text{modulus}$$
$$\theta = \arg(z) = \text{ argument of } z = \text{polar angle of } z$$

As in 18.02 you should be able to visualize polar coordinates by thinking about the distance r from the origin and the angle θ with the x-axis.

Example. Let's make a table of z, r and θ for some complex numbers. Notice that θ is not uniquely defined since we can always add a multiple of 2π to θ and still be at the same point in the plane.

$z = a + bi$	r	θ	
1	1	$0, 2\pi, 4\pi, \ldots$	Argument $= 0$, means z is along the x-axis
i	1	$\pi/2, \pi/2 + 2\pi \ldots$	Argument $= \pi/2$, means z is along the y-axis
$1 + i$	$\sqrt{2}$	$\pi/4, \pi/4 + 2\pi \ldots$	Argument $= \pi/4$, means z is along the ray at $45°$ to the x-axis



When we want to be clear which value of θ is meant, we will specify a **branch** of arg. For example, $0 \le \theta < 2\pi$ or $-\pi < \theta \le \pi$. This will be discussed in much more detail in the coming weeks. Keeping careful track of the branches of arg will turn out to be one of the key requirements of complex analysis.

2.6 Euler's Formula

Euler's (pronounced 'oilers') formula connects complex exponentials, polar coordinates, and sines and cosines. It turns messy trig identities into tidy rules for exponentials. We will use it a lot. The formula is the following:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{1}$$

There are many ways to approach Euler's formula. Our approach is to simply take Equation 1 as the definition of complex exponentials. This is legal, but does not show that it's a good definition. To do that we need to show the $e^{i\theta}$ obeys all the rules we expect of an exponential. To do that we go systematically through the properties of exponentials and check that they hold for complex exponentials.

2.6.1 $e^{i\theta}$ behaves like a true exponential

P1. e^{it} differentiates as expected:

$$\frac{de^{it}}{dt} = ie^{it}$$

Proof. This follows directly from the definition:

$$\frac{de^{it}}{dt} = \frac{d}{dt}(\cos(t) + i\sin(t)) = -\sin(t) + i\cos(t) = i(\cos(t) + i\sin(t)) = ie^{it}$$

P2. $e^{i\cdot 0} = 1$.

Proof.

$$e^{i\cdot 0} = \cos(0) + i\sin(0) = 1$$

P3. The usual rules of exponents hold:

$$e^{ia}e^{ib} = e^{i(a+b)}$$

Proof. This relies on the cosine and sine addition formulas.

$$\begin{aligned} e^{ia} \cdot e^{ib} &= (\cos(a) + i\sin(a)) \cdot (\cos(b) + i\sin(b)) \\ &= \cos(a)\cos(b) - \sin(a)\sin(b) + i\,(\cos(a)\sin(b) + \sin(a)\cos(b)) \\ &= \cos(a+b) + i\sin(a+b) = e^{i(a+b)}. \end{aligned}$$

P4. The definition of $e^{i\theta}$ is consistent with the power series for e^x.

Proof. To see this we have to recall the power series for e^x, $\cos(x)$ and $\sin(x)$. They are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots && \text{(2a)} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots && \text{(2b)} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots && \text{(2c)} \end{aligned}$$

Now we can write the power series for $e^{i\theta}$ and then split it into the power series for sine and cosine:

$$\begin{aligned} e^{i\theta} &= \sum_0^\infty \frac{(i\theta)^n}{n!} \\ &= \sum_0^\infty (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_0^\infty (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

So the Euler formula definition is consistent with the usual power series for e^x.

Properties **P1-P4** should convince you that $e^{i\theta}$ behaves like an exponential.

2.6.2 Complex exponentials and polar form

Now let's turn to the relation between polar coordinates and complex exponentials.

Suppose $z = x + iy$ has polar coordinates r and θ. That is, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus, we get the important relationship

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

This is so important you shouldn't proceed without understanding. We also record it without the intermediate equation.

$$z = x + iy = re^{i\theta}. \tag{3}$$

Because r and θ are the polar coordinates of (x, y) we call $z = re^{i\theta}$ the polar form of z. We next show that

Magnitude, argument, conjugate, multiplication and division are easy in polar form.

Magnitude. $|e^{i\theta}| = 1$.
Proof.

$$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

In words, this says that $e^{i\theta}$ is always on the unit circle – this is useful to remember! Likewise, if $z = re^{i\theta}$ then $|z| = r$. You can calculate this, but it should be clear from the definitions: $|z|$ is the distance from z to the origin, which is exactly the same definition as for r.

Argument. If $z = re^{i\theta}$ then $\arg(z) = \theta$.
Proof. This is again the definition: the argument is the polar angle θ.

Conjugate. $\overline{(re^{i\theta})} = re^{-i\theta}$.
Proof.

$$\overline{(re^{i\theta})} = \overline{r(\cos(\theta) + i\sin(\theta))} = r(\cos(\theta) - i\sin(\theta)) = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}.$$

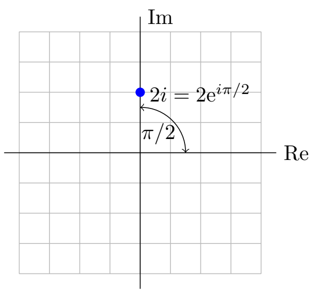
Thus, complex conjugation changes the sign of the argument.
Multiplication. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

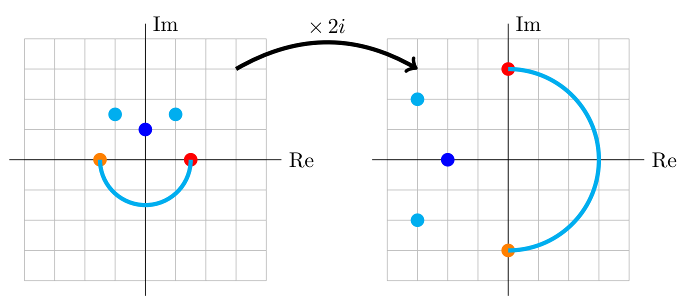
This is what mathematicians call trivial to see, just write the multiplication down. In words, the formula says the for $z_1 z_2$ the magnitudes multiply and the arguments add.
Division. Again it's trivial that

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Example. (Multiplication by $2i$) Here's a simple but important example. By looking at the graph we see that the number $2i$ has magnitude 2 and argument $\pi/2$. So in polar coordinates it equals $2e^{i\pi/2}$. This means that multiplication by $2i$ multiplies lengths by 2 and adds $\pi/2$ to arguments, i.e. rotates by 90°. The effect is shown in the figures below



$|2i| = 2, \quad \arg(2i) = \pi/2$



Multiplication by $2i$ rotates by $\pi/2$ and scales by 2

Example. (Raising to a power) Let's compute $(1+i)^6$ and $\left(\frac{1+i\sqrt{3}}{2}\right)^3$.

Solution: $1 + i$ has magnitude $= \sqrt{2}$ and arg $= \pi/4$, so $1 + i = \sqrt{2}e^{i\pi/4}$. Raising to a power is now easy:

$$(1+i)^6 = \left(\sqrt{2}e^{i\pi/4}\right)^6 = 8e^{6i\pi/4} = 8e^{3i\pi/2} = -8i$$

Similarly, $\dfrac{1+i\sqrt{3}}{2} = e^{i\pi/3}$, so $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3 = (1 \cdot e^{i\pi/3})^3 = e^{i\pi} = -1$

2.6.3 Complexification or complex replacement

In the next example we will illustrate the technique of complexification or complex replacement. This can be used to simplify a trigonometric integral. It will come in handy when we need to compute certain integrals.

Example. Use complex replacement to compute

$$I = \int dx\ \mathrm{e}^x \cos(2x)$$

Solution: We have Euler's formula

$$\mathrm{e}^{2ix} = \cos(2x) + i\sin(2x)$$

so $\cos(2x) = \mathrm{Re}(\mathrm{e}^{2ix})$. The complex replacement trick is to replace $\cos(2x)$ by e^{2ix}. We get (justification below)

$$I_c = \int dx\ (\mathrm{e}^x \cos 2x + i\mathrm{e}^x \sin 2x)\ , \qquad I = \mathrm{Re}(I_c).$$

Computing I_c is straightforward:

$$I_c = \int dx\ \mathrm{e}^x \mathrm{e}^{i2x} = \int dx\ \mathrm{e}^{x(1+2i)} = \frac{\mathrm{e}^{x(1+2i)}}{1+2i}.$$

Here we will do the computation first in rectangular coordinates[3]

$$\begin{aligned} I_c &= \frac{\mathrm{e}^{x(1+2i)}}{1+2i} \cdot \frac{1-2i}{1-2i} \\ &= \frac{\mathrm{e}^x(\cos(2x) + i\ \sin(2x))(1-2i)}{5} \\ &= \frac{1}{5}\mathrm{e}^x(\cos(2x) + 2\sin(2x) + i(-2\cos(2x) + \sin(2x))) \end{aligned}$$

So,

$$I = \mathrm{Re}(I_c) = \frac{1}{5}\mathrm{e}^x(\cos(2x) + 2\sin(2x)).$$

Justification of complex replacement. The trick comes by cleverly adding a new integral to I as follows. Let $J = \displaystyle\int \mathrm{e}^x \sin(2x)\, dx$. Then we let

$$I_c = I + iJ = \int dx\ \mathrm{e}^x(\cos(2x) + i\sin(2x)) = \int dx\ \mathrm{e}^x \mathrm{e}^{2ix}$$

Clearly, by construction, $\mathrm{Re}(I_c) = I$ as claimed above.

[3]In applications, for example throughout 18.03, polar form is often preferred because it is easier and gives the answer in a more useable form.

We can use polar coordinates to simplify the expression for I_c: In polar form, we have $1+2i = r\mathrm{e}^{i\phi}$, where $r=\sqrt{5}$ and $\phi = \arg(1+2i) = \tan^{-1}(2)$ in the first quadrant. Then:

$$I_c = \frac{\mathrm{e}^{x(1+2i)}}{\sqrt{5}\mathrm{e}^{i\phi}} = \frac{\mathrm{e}^{x}}{\sqrt{5}}\mathrm{e}^{i(2x-\phi)} = \frac{\mathrm{e}^{x}}{\sqrt{5}}(\cos(2x-\phi) + i\,\sin(2x-\phi)).$$

Thus,

$$I = \mathrm{Re}(I_c) = \frac{\mathrm{e}^{x}}{\sqrt{5}}\cos(2x-\phi)$$

2.6.4 Nth roots

We are going to need to be able to find the nth roots of complex numbers, i.e., solve equations of the form

$$z^N = c$$

where c is a given complex number. This can be done most conveniently by expressing c and z in polar form, $c = Re^{i\phi}$ and $z = re^{i\theta}$. Then, upon substituting, we have to solve

$$r^N e^{iN\theta} = Re^{i\phi}$$

For the complex numbers on the left and right to be equal, their absolute values must be same and the arguments can only differ by an integer-multiple of 2π, which gives

$$r = R^{1/N}\ , \qquad N\theta = \phi + 2\pi n\ , \qquad n = 0, \pm 1, \pm 2, \ldots \tag{4}$$

Solving for θ, we have

$$\theta = \frac{\phi}{N} + 2\pi\frac{n}{N}. \tag{5}$$

Example. Find all 5 fifth roots of 2.

Solution: For $c=2$, we have $R=2$ and $\phi=0$, so the fifth roots of 2 are

$$z_n = 2^{1/5}\mathrm{e}^{2n\pi i/5},\ \text{where } n = 0, \pm 1, \pm 2, \ldots$$

Looking at the right hand side we see that for $n=5$ we have $2^{1/5}\mathrm{e}^{2\pi i}$ which is exactly the same as the root when $n=0$, i.e. $2^{1/5}\mathrm{e}^{0i}$. Likewise $n=6$ gives exactly the same root as $n=1$, and so on. This means, we have 5 different roots corresponding to $n = 0, 1, 2, 3, 4$.

$$z_n = 2^{1/5},\ 2^{1/5}\mathrm{e}^{2\pi i/5},\ 2^{1/5}\mathrm{e}^{4\pi i/5},\ 2^{1/5}\mathrm{e}^{6\pi i/5},\ 2^{1/5}\mathrm{e}^{8\pi i/5}$$

Similarly we can say that in general $c = R\mathrm{e}^{i\phi}$ has N distinct Nth roots:

$$z_n = r^{1/N}\mathrm{e}^{i\phi/N + i\,2\pi(n/N)}\ \text{ for }\ n = 0, 1, 2, \ldots N-1.$$

Example. Find the 4 fourth roots of 1.

Solution: Need to solve $z^4 = 1$, so $\phi = 0$. So the 4 distinct fourth roots are in polar form

$$z_n = 1,\ \mathrm{e}^{i\pi/2},\ \mathrm{e}^{i\pi},\ \mathrm{e}^{i3\pi/2}$$

and in Cartesian representation

$$z_n = 1,\ i,\ -1,\ -i$$

Example. Find the 3 cube roots of -1.

Solution: $z^2 = -1 = \mathrm{e}^{i\,\pi + i\,2\pi n}$. So, $z_n = \mathrm{e}^{i\,\pi/3 + i\,2\pi(n/3)}$ and the 3 cube roots are $\mathrm{e}^{i\pi/3}$, $\mathrm{e}^{i\pi}$, $\mathrm{e}^{i5\pi/3}$. Since $\pi/3$ radians is 60° we can simpify:

$$\mathrm{e}^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\,\frac{\sqrt{3}}{2} \qquad \Rightarrow \qquad z_n = -1,\ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Example. Find the 5 fifth roots of $1+i$.

Solution: $z^5 = 1 + i = \sqrt{2}\mathrm{e}^{i(\pi/4+2n\pi)}$, for $n = 0, 1, 2, \ldots$. So, the 5 fifth roots are

$$2^{1/10}\mathrm{e}^{i\pi/20},\ 2^{1/10}\mathrm{e}^{i9\pi/20},\ 2^{1/10}\mathrm{e}^{i17\pi/20},\ 2^{1/10}\mathrm{e}^{i25\pi/20},\ 2^{1/10}\mathrm{e}^{i33\pi/20}.$$

Using a calculator we could write these numerically as $a + bi$, but there is no easy simplification.

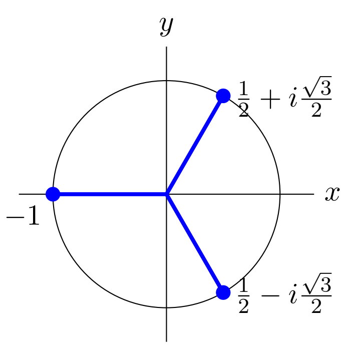
Example. We should check that our technique works as expected for a simple problem. Find the 2 square roots of 4.

Solution: $z^2 = 4\mathrm{e}^{i\,2\pi n}$. So, $z_n = 2\mathrm{e}^{i\,\pi n}$ with $n = 0, 1$. So the two roots are $2\mathrm{e}^0 = +2$ and $2\mathrm{e}^{i\pi} = -2$ as expected.

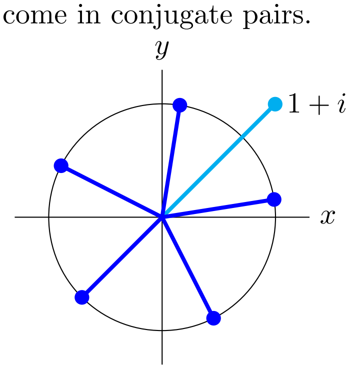
2.6.5 The geometry of Nth roots

Looking at the examples above we see that roots are always spaced evenly around a circle centered at the origin. For example, the fifth roots of $1+i$ are spaced at increments of $2\pi/5$ radians around the circle of radius $2^{1/5}$.

Note also that the roots of real numbers always come in conjugate pairs.



Cube roots of -1



Fifth roots of $1+i$

2.7 Inverse Euler formula

Euler's formula gives a complex exponential in terms of sines and cosines. We can turn this around to get the inverse Euler formulas. Euler's formula says:

$$\mathrm{e}^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad \mathrm{e}^{-it} = \cos(t) - i\sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{\mathrm{e}^{it} + \mathrm{e}^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{\mathrm{e}^{it} - \mathrm{e}^{-it}}{2i}.$$

Please take note of these formulas we will use them frequently!

2.8 de Moivre's formula

For positive integers n we have de Moivre's formula:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. This is a simple consequence of Euler's formula:

$$(\cos(\theta) + i\sin(\theta))^n = (\mathrm{e}^{i\theta})^n = \mathrm{e}^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

The reason this simple fact has a name is that historically de Moivre stated it before Euler's formula was known. Without Euler's formula there is not such a simple proof.

2.9 Representing complex multiplication as matrix multiplication

Consider two complex numbers $z_1 = a + bi$ and $z_2 = x + yi$ and their product

$$z_1 z_2 = (a + bi)(x + iy) = (ax - by) + i(bx + ay) =: w \tag{6}$$

Now let's define two matrices

$$Z_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \tag{7}$$

Note that these matrices store the same information as z_1 and z_2, respectively. Let's compute their matrix product

$$Z_1 Z_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{bmatrix} = W \tag{8}$$

Comparing with Eq. (6), we see that W is indeed the matrix corresponding to the complex number $w = z_1 z_2$. Thus, we can represent any complex number z equivalently by the matrix

$$Z = \begin{bmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{bmatrix} \tag{9}$$

and complex multiplication then simply becomes matrix multiplication. Further note that we can write

$$Z = \operatorname{Re} z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \operatorname{Im} z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{10}$$

i.e., the imaginary unit i corresponds to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $i^2 = -1$ becomes

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

Polar form (decomposition). Writing $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we find

$$Z = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \tag{12}$$

corresponding to a 2D rotation matrix multiplied by the stretch factor r. In particular, multiplication by i corresponds to the rotation with angle $\theta = \pi/2$ and $r = 1$.

3 Complex functions

3.1 The exponential function

We have Euler's formula: $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$. We can extend this to the complex exponential function e^z.

Definition. For $z = x + iy$ the complex exponential function is defined as

$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x \mathrm{e}^{iy} = \mathrm{e}^x(\cos(y) + i\sin(y)).$$

In this definition e^x is the usual exponential function for a real variable x. It is easy to see that all the usual rules of exponents hold:

1. $\mathrm{e}^0 = 1$
2. $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$
3. $(\mathrm{e}^z)^n = \mathrm{e}^{nz}$ for positive integers n.
4. $(\mathrm{e}^z)^{-1} = \mathrm{e}^{-z}$
5. $\mathrm{e}^z \neq 0$

   It will turn out that the property $\dfrac{d\mathrm{e}^z}{dz} = \mathrm{e}^z$ also holds, but we can't prove this yet because we haven't defined what we mean by the complex derivative $\dfrac{d}{dz}$.

   Here are some more simple, but extremely important properties of e^z. You should become fluent in their use and know how to prove them.

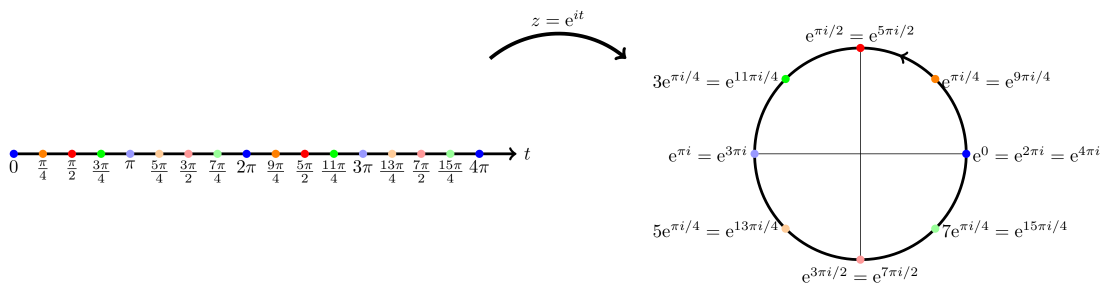
6. $|e^{i\theta}| = 1$

   Proof.
   $$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

7. $|e^{x+iy}| = e^x$ (as usual $z = x + iy$ and x, y are real).

   Proof. You should be able to supply this. If not: ask a teacher or TA.

8. The path e^{it} for $0 < t < \infty$ wraps counterclockwise around the unit circle. It does so infinitely many times. This is illustrated in the following picture.



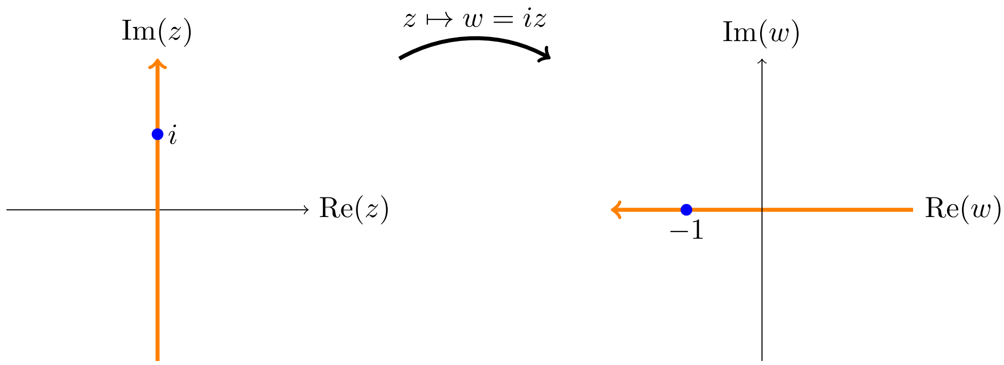
The map $t \to e^{it}$ wraps the real axis around the unit circle.

3.2 Complex functions as mappings

A complex function $w = f(z)$ is hard to graph because it takes 4 dimensions: 2 for z and 2 for w. So, to visualize them we will think of complex functions as mappings. That is we will think of $w = f(z)$ as taking a point in the complex z-plane and sending it to a point in the complex w-plane.

We will use the following terms and symbols to discuss mappings.

- A function $w = f(z)$ will also be called a mapping of z to w.
- Alternatively we will write $z \mapsto w$ or $z \mapsto f(z)$. This is read as "z maps to w".
- We will say that "w is the image of z under the mapping" or more simply "w is the image of z".
- If we have a set of points in the z-plane we will talk of the image of that set under the mapping. For example, under the mapping $z \mapsto iz$ the image of the imaginary z-axis is the real w-axis.



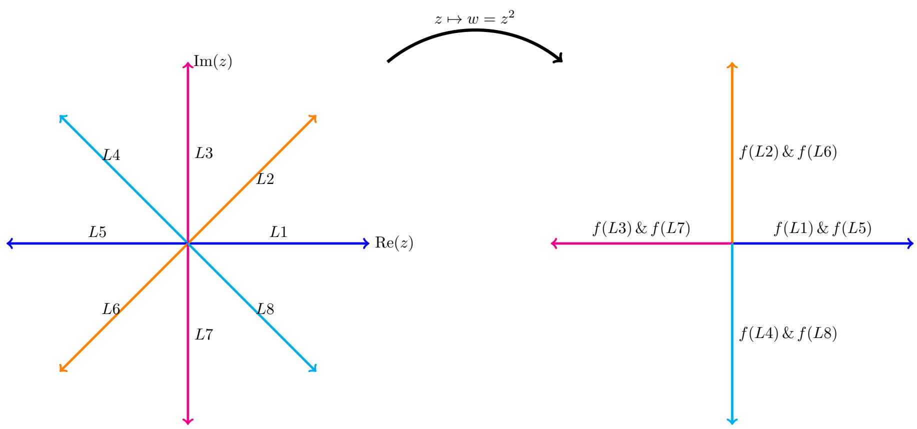
The image of the imaginary axis under $z \mapsto iz$.

Next, we'll illustrate visualizing mappings with some examples:

Example. The mapping $w = z^2$. We visualize this by putting the z-plane on the left and the w-plane on the right. We then draw various curves and regions in the z-plane and the corresponding image under z^2 in the w-plane.

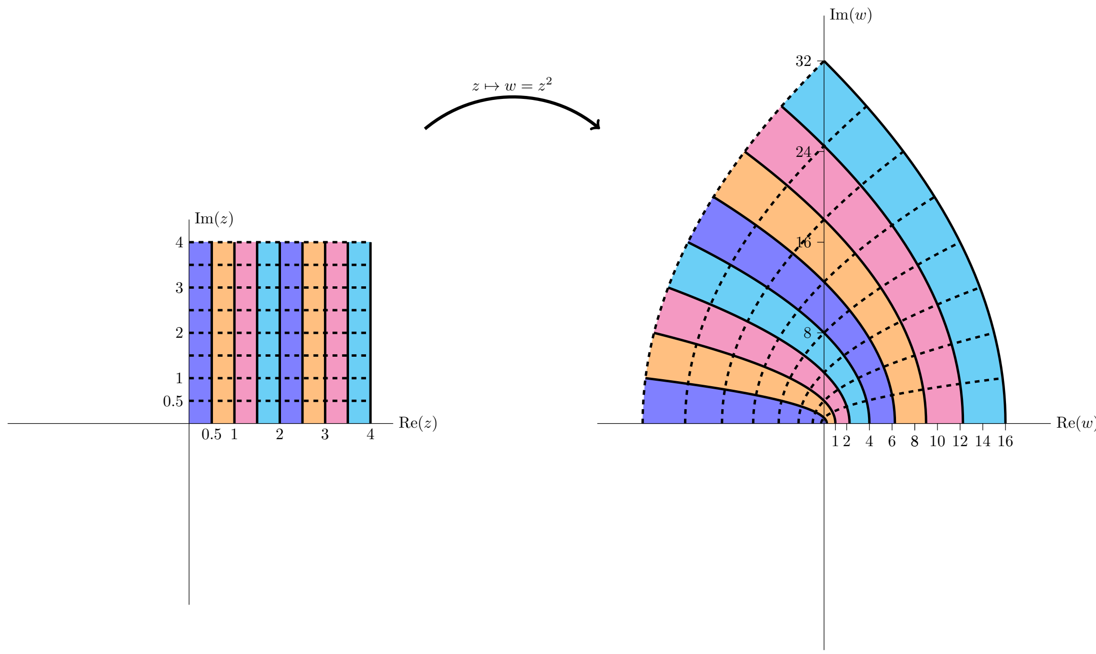
In the first figure we show that rays from the origin are mapped by z^2 to rays from the origin. We see that

1. The ray L_2 at $\pi/4$ radians is mapped to the ray $f(L_2)$ at $\pi/2$ radians.
2. The rays L_2 and L_6 are both mapped to the same ray. This is true for each pair of diametrically opposed rays.
3. A ray at angle θ is mapped to the ray at angle 2θ.



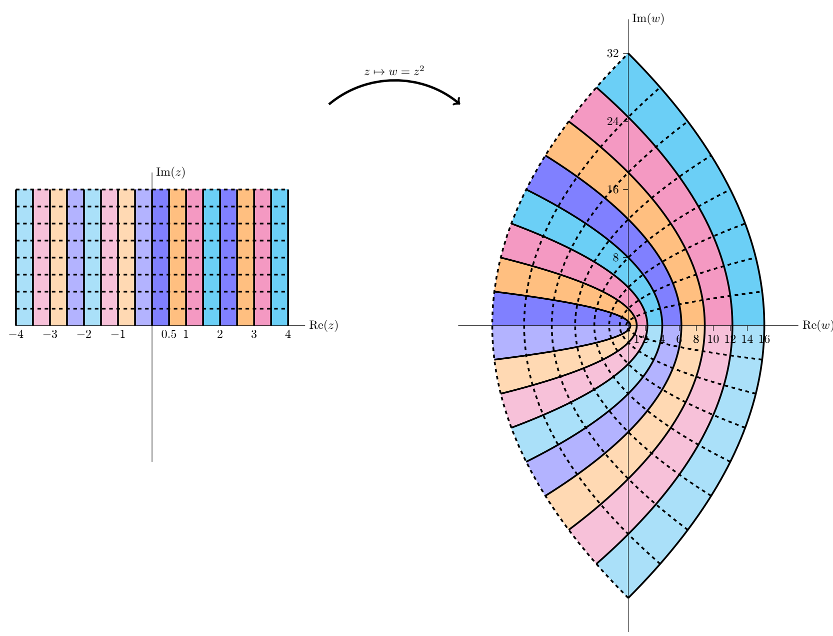
$f(z) = z^2$ maps rays from the origin to rays from the origin.

The next figure gives another view of the mapping. Here we see vertical stripes in the first quadrant are mapped to parabolic stripes that live in the first and second quadrants.



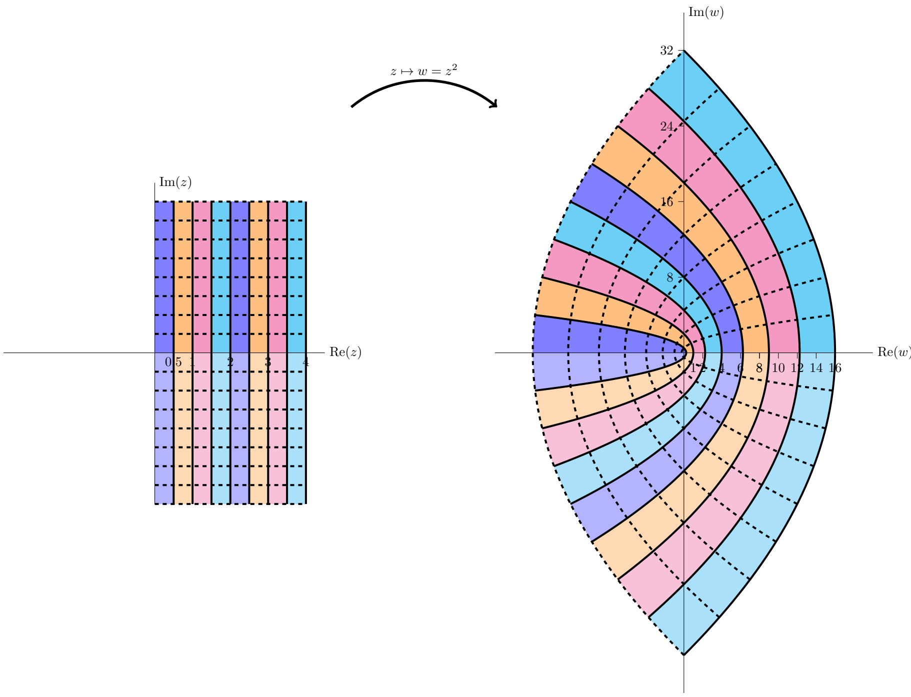
$z^2 = (x^2 - y^2) + i2xy$ maps vertical lines to left facing parabolas.

The next figure is similar to the previous one, except in this figure we look at vertical stripes in both the first and second quadrants. We see that they map to parabolic stripes that live in all four quadrants.

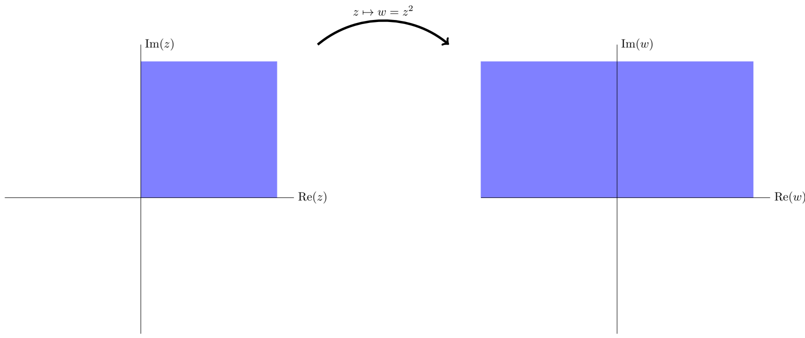


$f(z) = z^2$ maps the first two quadrants to the entire plane.

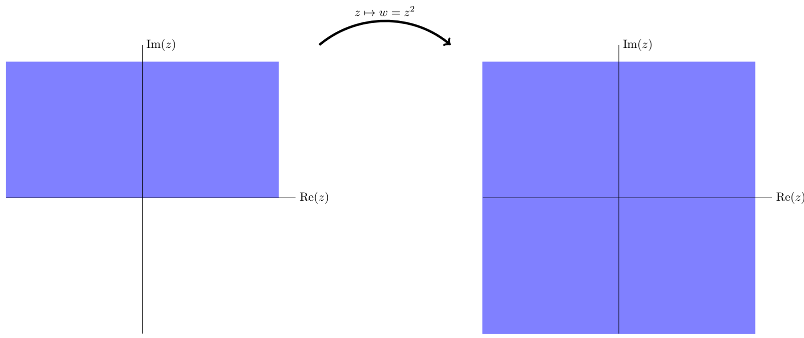
The next figure shows the mapping of stripes in the first and fourth quadrants. The image map is identical to the previous figure. This is because the fourth quadrant is minus the second quadrant, but $z^2 = (-z)^2$.



Vertical stripes in quadrant 4 are mapped identically to vertical stripes in quadrant 2.

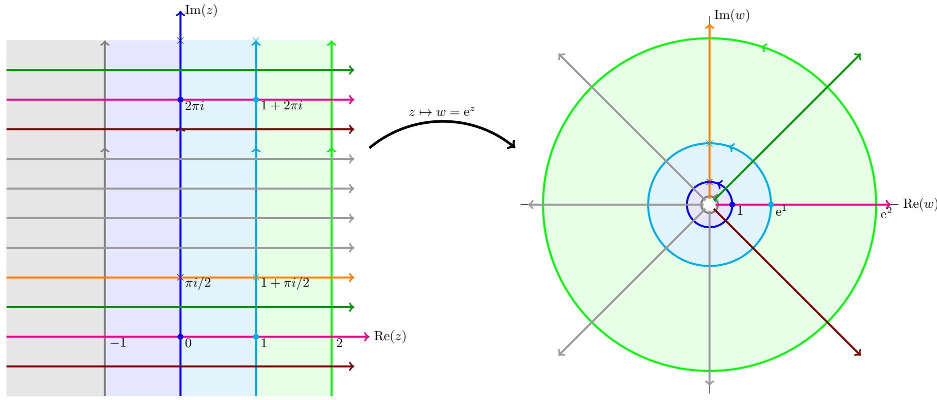


Simplified view of the first quadrant being mapped to the first two quadrants.



Simplified view of the first two quadrants being mapped to the entire plane.

Example. The mapping $w = \mathrm{e}^z$. Here we present a series of plots showing how the exponential function maps z to w.

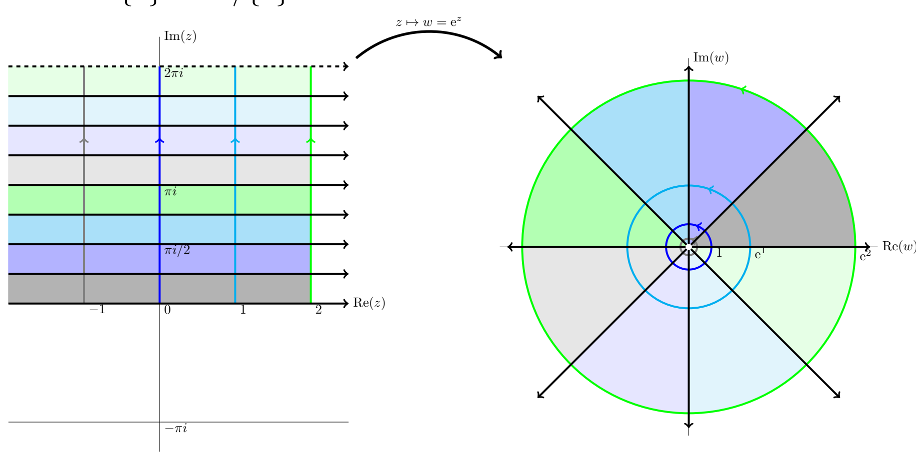


Notice that vertical lines are mapped to circles and horizontal lines to rays from the origin.

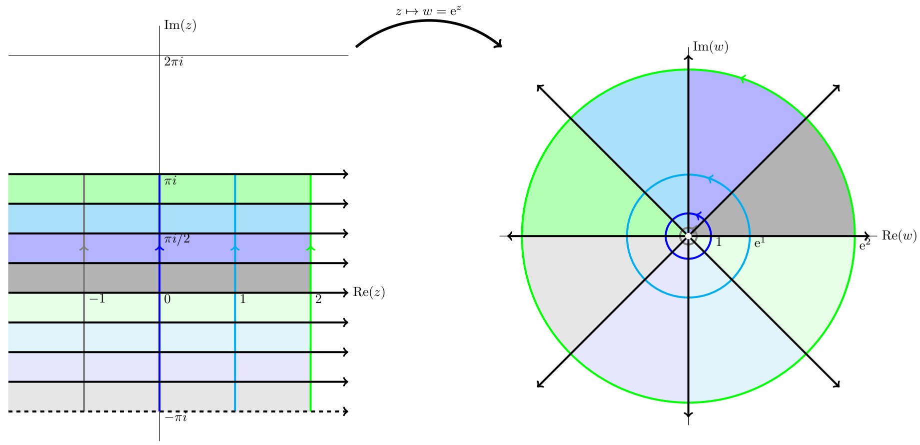
The next four figures all show essentially the same thing: the exponential function maps horizontal stripes to circular sectors. Any horizontal stripe of width 2π gets mapped to the entire plane minus the origin,

Because the plane minus the origin comes up frequently we give it a name:

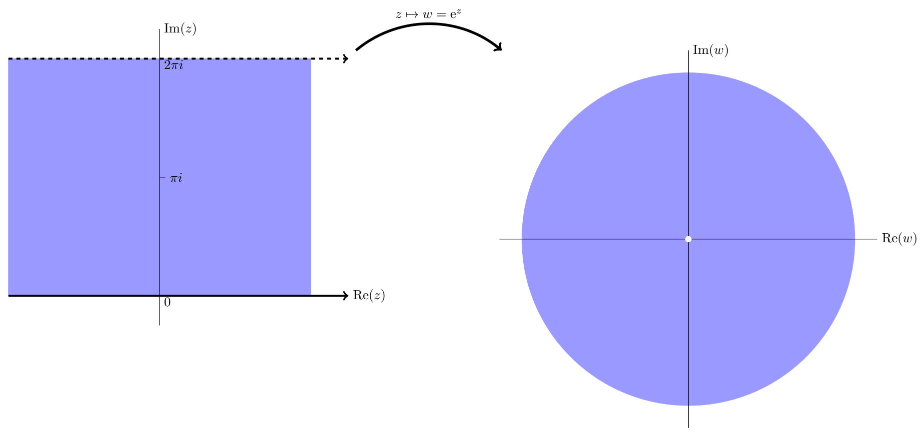
Definition. The punctured plane is the complex plane minus the origin. In symbols we can write it as $\mathbf{C} - \{0\}$ or $\mathbf{C}/\{0\}$.



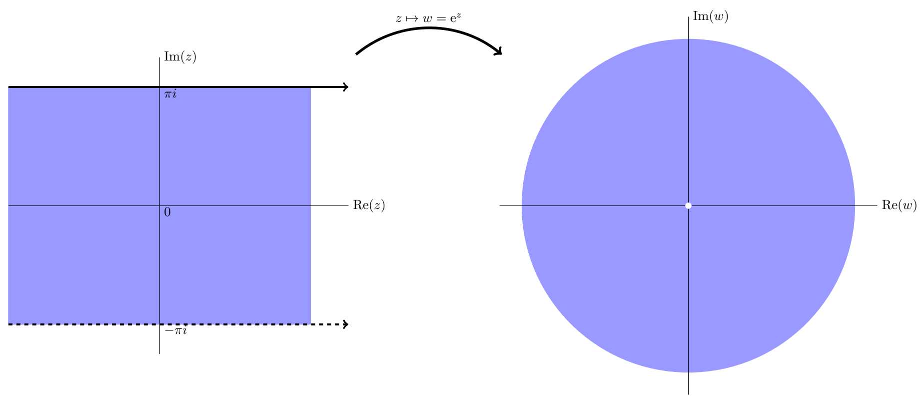
The horizontal strip $0 \leq y < 2\pi$ is mapped to the punctured plane



The horizontal strip $-\pi < y \leq \pi$ is mapped to the punctured plane



Simplified view showing e^z maps the horizontal stripe $0 \leq y < 2\pi$ to the punctured plane.



Simplified view showing e^z maps the horizontal stripe $-\pi < y \leq \pi$ to the punctured plane.

3.3 The function $\arg(z)$

3.3.1 Many-to-one functions

The function $f(z) = z^2$ maps $\pm z$ to the same value, e.g. $f(2) = f(-2) = 4$. We say that $f(z)$ is a 2-to-1 function. That is, it maps 2 different points to each value. (Technically, it only maps one point to 0, but we will gloss over that for now.) Here are some other examples of many-to-one functions.

Example 1. $w = z^3$ is a 3-to-1 function. For example, 3 different z values get mapped to $w = 1$:

$$1^3 = \left(\frac{-1+\sqrt{3}\,i}{2}\right)^3 = \left(\frac{-1-\sqrt{3}\,i}{2}\right)^3 = 1$$

Example 2. The function $w = \mathrm{e}^z$ maps infinitely many points to each value. For example

$$\mathrm{e}^0 = \mathrm{e}^{2\pi i} = \mathrm{e}^{4\pi i} = \ldots = \mathrm{e}^{n2\pi i} = \ldots = 1$$
$$\mathrm{e}^{i\pi/2} = \mathrm{e}^{i\pi/2+2\pi i} = \mathrm{e}^{i\pi/2+4\pi i} = \ldots = \mathrm{e}^{i\pi/2+n2\pi i} = \ldots = i$$

In general, $\mathrm{e}^{z+n\,2\pi i}$ has the same value for every integer n.

3.3.2 Branches of $\arg(z)$

Important note. You should master this section. Branches of $\arg(z)$ are the key that really underlies all our other examples. Fortunately it is reasonably straightforward.

The key point is that the argument is only defined up to multiples of $2\pi i$ so every z produces infinitely many values for $\arg(z)$. Because of this we will say that $\arg(z)$ is a multiple-valued function.

Note. In general a function should take just one value. What that means in practice is that whenever we use such a function will have to be careful to specify which of the possible values we mean. This is known as specifying a branch of the function.

Definition. By a branch of the argument function we mean a choice of range so that it becomes single-valued. By specifying a branch we are saying that we will take the single value of $\arg(z)$ that lies in the branch. Let's look at several different branches to understand how they work:

(i) If we specify the branch as $0 \le \arg(z) < 2\pi$ then we have the following arguments.

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = 3\pi/2$$

This branch and these points are shown graphically in Figure (i) below.

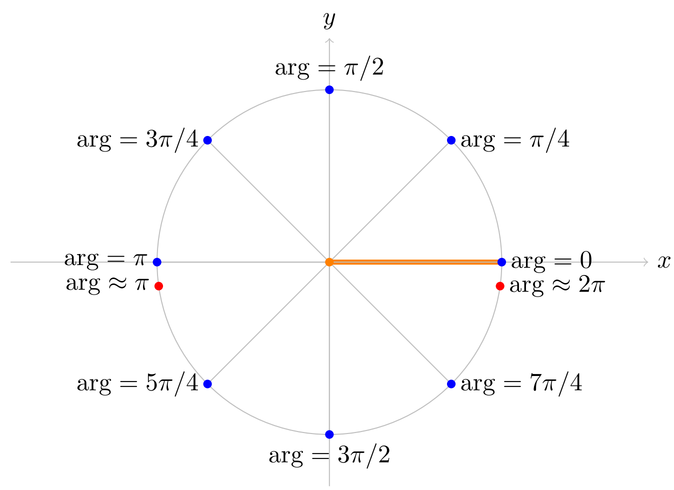


Figure (i): The branch $0 \le \arg(z) < 2\pi$ of $\arg(z)$.

Notice that if we start at $z = 1$ on the positive real axis we have $\arg(z) = 0$. Then $\arg(z)$ increases as we move counterclockwise around the circle. The argument is continuous until we get back to the positive real axis. There it jumps from almost 2π back to 0.

There is no getting around (no pun intended) this discontinuity. If we need $\arg(z)$ to be continuous we will need to remove (cut) the points of discontinuity out of the domain. The branch cut for this branch of $\arg(z)$ is shown as a thick orange line in the figure. If we make the branch cut then the domain for $\arg(z)$ is the plane minus the cut, i.e. we will only consider $\arg(z)$ for z not on the cut.

For future reference you should note that, on this branch, $\arg(z)$ is continuous near the negative real axis, i.e. the arguments of nearby points are close to each other.

(ii) If we specify the branch as $-\pi < \arg(z) \le \pi$ then we have the following arguments:

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = -\pi/2$$

This branch and these points are shown graphically in Figure (ii) below.

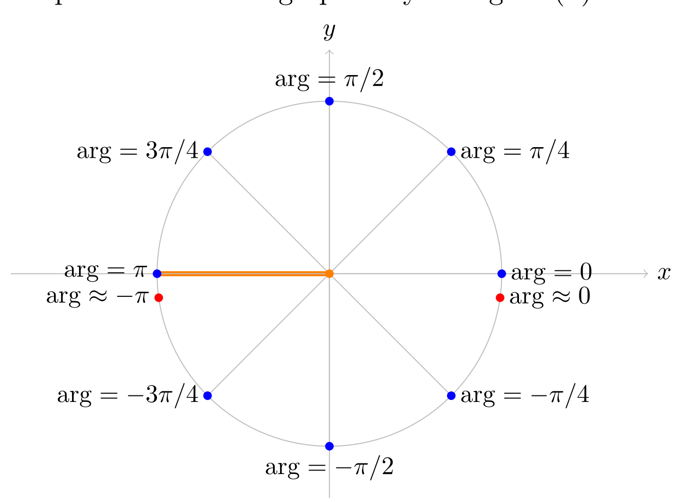


Figure (ii): The branch $-\pi < \arg(z) \le \pi$ of $\arg(z)$.

Compare Figure (ii) with Figure (i). The values of $\arg(z)$ are the same in the upper half plane, but in the lower half plane they differ by 2π.

For this branch the branch cut is along the negative real axis. As we cross the branch cut the value of $\arg(z)$ jumps from π to something close to $-\pi$.

(iii) Figure (iii) shows the branch of $\arg(z)$ with $\pi/4 \le \arg(z) < 9\pi/4$.

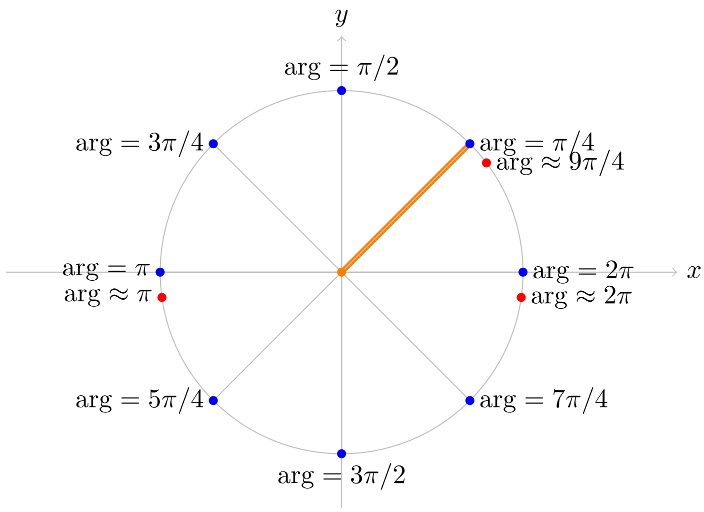


Figure (iii): The branch $\pi/4 \le \arg(z) < 9\pi/4$ of $\arg(z)$.

Notice that on this branch $\arg(z)$ is continuous at both the positive and negative real axes. The jump of 2π occurs along the ray at angle $\pi/4$.

(iv) Obviously, there are many many possible branches. For example,

$$42 < \arg(z) \le 42 + 2\pi.$$

(v) We won't make use of this in 18.04, but, in fact, the branch cut doesn't have to be a straight line. Any curve that goes from the origin to infinity will do. The argument will be continuous except for a jump by 2π when z crosses the branch cut.

3.3.3 The principal branch of $\arg(z)$

Branch (ii) in the previous section is singled out and given a name:

Definition. The branch $-\pi < \arg(z) \le \pi$ is called the principal branch of $\arg(z)$. We will use the notation $\mathrm{Arg}(z)$ (capital A) to indicate that we are using the principal branch. (Of course, in cases where we don't want there to be any doubt we will say explicitly that we are using the principal branch.)

Note. The examples above show that there is no getting around the jump of 2π as we cross the branch cut. This means that when we need $\arg(z)$ to be continuous we will have to restrict its domain to the plane minus a branch cut.

3.4 Concise summary of branches and branch cuts

We discussed branches and branch cuts for $\arg(z)$. Before talking about $\log(z)$ and its branches and branch cuts we will give a short review of what these terms mean. You should probably scan this section now and then come back to it after reading about $\log(z)$.

Consider the function $w = f(z)$. Suppose that $z = x + iy$ and $w = u + iv$.

Domain. The domain of f is the set of z where we are allowed to compute $f(z)$.

Range. The range (image) of f is the set of all $f(z)$ for z in the domain, i.e. the set of all w reached by f.

Branch. For a multiple-valued function, a branch is a choice of range for the function. We choose the range to exclude all but one possible value for each element of the domain.

Branch cut. A branch cut removes (cuts) points out of the domain. This is done to remove points where the function is discontinuous.

3.5 The function $\log(z)$

Our goal in this section is to define the log function. We want $\log(z)$ to be the inverse of e^z. That is, we want $e^{\log(z)} = z$. We will see that $\log(z)$ is multiple-valued, so when we use it we will have to specify a branch.

We start by looking at the simplest example which illustrates that $\log(z)$ is multiple-valued.

Example. Find $\log(1)$.

Solution: We know that $e^0 = 1$, so $\log(1) = 0$ is one answer. We also know that $e^{2\pi i} = 1$, so $\log(1) = 2\pi i$ is another possible answer. In fact, we can choose any multiple of $2\pi i$:

$$\log(1) = n\, 2\pi i, \ \text{ where } n \text{ is any integer}$$

This example leads us to consider the polar form for z as we try to define $\log(z)$. If $z = re^{i\theta}$ then one possible value for $\log(z)$ is

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta,$$

here $\log(r)$ is the usual logarithm of a real positive number. For completeness we show explicitly that with this definition $e^{\log(z)} = z$:

$$e^{\log(z)} = e^{\log(r)+i\theta} = e^{\log(r)}e^{i\theta} = re^{i\theta} = z.$$

Since $r = |z|$ and $\theta = \arg(z)$ we have arrived at our definition.

Definition. The function $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z),$$

where $\log(|z|)$ is the usual natural logarithm of a positive real number.

Remarks.

1. Since $\arg(z)$ has infinitely many possible values, so does $\log(z)$.
2. $\log(0)$ is not defined. (Both because $\arg(0)$ is not defined and $\log(|0|)$ is not defined.)
3. Choosing a branch for $\arg(z)$ makes $\log(z)$ single valued. The usual terminology is to say we have chosen a branch of the log function.
4. The principal branch of log comes from the principal branch of arg. That is,

$$\log(z) = \log(|z|) + i\arg(z), \text{ where } -\pi < \arg(z) \leq \pi \quad \text{(principal branch).}$$

Example. Compute all the values of $\log(i)$. Specify which one comes from the principal branch.

Solution: We have that $|i| = 1$ and $\arg(i) = \dfrac{\pi}{2} + 2\pi n$, so

$$\log(i) = \log(1) + i\frac{\pi}{2} + i2\pi n = i\frac{\pi}{2} + i2\pi n, \text{ where } n \text{ is any integer.}$$

The principal branch of $\arg(z)$ is between $-\pi$ and π, so $\text{Arg}(i) = \pi/2$. The value of $\log(i)$ from the principal branch is therefore $i\pi/2$.

Example. Compute all the values of $\log(-1 - \sqrt{3}\,i)$. Specify which one comes from the principal branch.

Solution: Let $z = -1 - \sqrt{3}\,i$. Then $|z| = 2$ and in the principal branch $\text{Arg}(z) = -2\pi/3$. So all the values of $\log(z)$ are

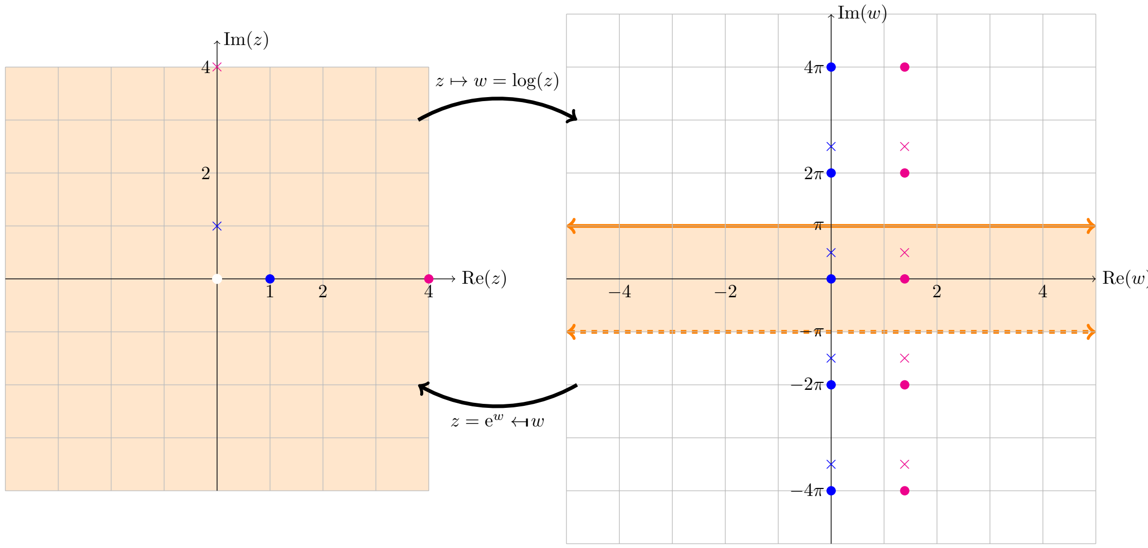
$$\log(z) = \log(2) - i\frac{2\pi}{3} + i2\pi n.$$

The value from the principal branch is $\log(z) = \log(2) - i2\pi/3$.

3.5.1 Figures showing $w = \log(z)$ as a mapping

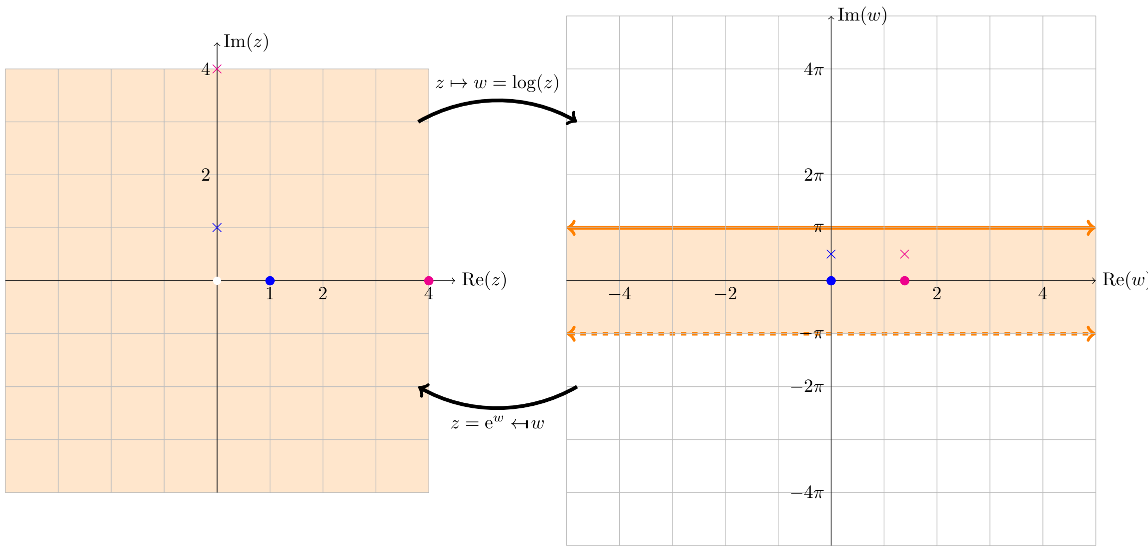
The figures below show different aspects of the mapping given by $\log(z)$.

In the first figure we see that a point z is mapped to (infinitely) many values of w. In this case we show $\log(1)$ (blue dots), $\log(4)$ (red dots), $\log(i)$ (blue cross), and $\log(4i)$ (red cross). The values in the principal branch are inside the shaded region in the w-plane. Note that the values of $\log(z)$ for a given z are placed at intervals of $2\pi i$ in the w-plane.



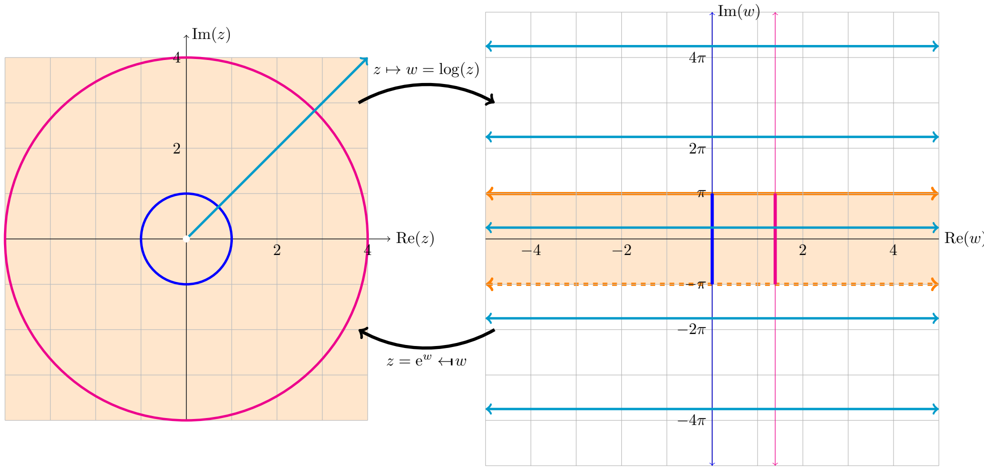
Mapping $\log(z)$: $\log(1)$, $\log(4)$, $\log(i)$, $\log(4i)$

The next figure illustrates that the principal branch of log maps the punctured plane to the horizontal strip $-\pi < \text{Im}(w) \leq \pi$. We again show the values of $\log(1)$, $\log(4)$, $\log(i)$ and $\log(4i)$. Since we've chosen a branch, there is only one value shown for each log.



Mapping $\log(z)$: the principal branch and the punctured plane

The third figure shows how circles centered on 0 are mapped to vertical lines, and rays from the origin are mapped to horizontal lines. If we restrict ourselves to the principal branch the circles are mapped to vertical line segments and rays to a single horizontal line in the principal (shaded) region of the w-plane.



Mapping $\log(z)$: mapping circles and rays

3.5.2 Complex powers

We can use the log function to define complex powers.

Definition. Let z and a be complex numbers then the power z^a is defined as

$$z^a = \mathrm{e}^{a\log(z)}.$$

This is generally multiple-valued, so to specify a single value requires choosing a branch of $\log(z)$.

Example. Compute all the values of $\sqrt{2i}$. Give the value associated to the principal branch of $\log(z)$.

Solution: We have

$$\log(2i) = \log(2\mathrm{e}^{\frac{i\pi}{2}}) = \log(2) + i\frac{\pi}{2} + i2n\pi$$

So,

$$\sqrt{2i} = (2i)^{1/2} = \mathrm{e}^{\frac{\log(2i)}{2}} = \mathrm{e}^{\frac{\log(2)}{2}+\frac{i\pi}{4}+in\pi} = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}+in\pi}.$$

(As usual n is an integer.) As we saw earlier, this only gives two distinct values. The principal branch has $\mathrm{Arg}(2i) = \pi/2$, so

$$\sqrt{2i} = \sqrt{2}\mathrm{e}^{(\frac{i\pi}{4})} = \sqrt{2}\frac{(1+i)}{\sqrt{2}} = 1+i.$$

The other distinct value is when $n = 1$ and gives minus the value just above.

Example. Cube roots: Compute all the cube roots of i. Give the value which comes from the principal branch of $\log(z)$.

Solution: We have $\log(i) = i\dfrac{\pi}{2} + i2\pi n$, where n is any integer. So,

$$i^{1/3} = \mathrm{e}^{\frac{\log(i)}{3}} = \mathrm{e}^{i\frac{\pi}{6}+i\frac{2n\pi}{3}}$$

This gives only three distinct values

$$\mathrm{e}^{i\pi/6}, \quad \mathrm{e}^{i5\pi/6}, \quad \mathrm{e}^{i9\pi/6}$$

On the principal branch $\log(i) = i\dfrac{\pi}{2}$, so the value of $i^{1/3}$ which comes from this is

$$\mathrm{e}^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Example. Compute all the values of 1^i. What is the value from the principal branch?

Solution: This is similar to the problems above. $\log(1) = 2n\pi i$, so

$$1^i = \mathrm{e}^{i\log(1)} = \mathrm{e}^{i2n\pi i} = \mathrm{e}^{-2n\pi}, \text{ where } n \text{ is an integer.}$$

The principal branch has $\log(1) = 0$ so $1^i = 1$.

4 Analytic functions

The main goal of this section is to define and give some of the important properties of complex analytic functions. A function $f(z)$ is analytic if it has a complex derivative $f'(z)$. In general, the rules for computing derivatives will be familiar to you from single variable calculus. However, a much richer set of conclusions can be drawn about a complex analytic function than is generally true about real differentiable functions.

4.1 The derivative: preliminaries

In calculus we defined the derivative as a limit. In complex analysis we will do the same.

$$f'(z) = \lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

Before giving the derivative our full attention we are going to have to spend some time exploring and understanding limits. To motivate this we'll first look at two simple examples – one positive and one negative.

Example 4.1. Find the derivative of $f(z) = z^2$.

Solution: We compute using the definition of the derivative as a limit.

$$\lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} 2z + \Delta z = 2z.$$

That was a positive example. Here's a negative one which shows that we need a careful understanding of limits.

Example 4.2. Let $f(z) = \overline{z}$. Show that the limit for $f'(0)$ does not converge.

Solution: Let's try to compute $f'(0)$ using a limit:

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Here we used $\Delta z = \Delta x + i\Delta y$.

Now, $\Delta z \to 0$ means both Δx and Δy have to go to 0. There are lots of ways to do this. For example, if we let Δz go to 0 along the x-axis then, $\Delta y = 0$ while Δx goes to 0. In this case, we would have

$$f'(0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand, if we let Δz go to 0 along the positive y-axis then

$$f'(0) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

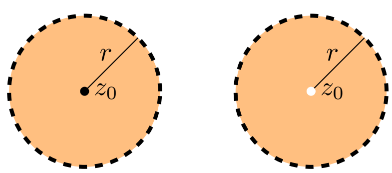
The limits **don't** agree! The problem is that the limit depends on how Δz approaches 0. If we came from other directions we'd get other values. This means that the limit does not exist.

We next explore limits in more detail.

4.2 Open disks, open deleted disks, open regions

Definition. The open disk of radius r around z_0 is the set of points z with $|z - z_0| < r$, i.e. all points within distance r of z_0.

The open deleted disk of radius r around z_0 is the set of points z with $0 < |z - z_0| < r$. That is, we remove the center z_0 from the open disk. A deleted disk is also called a punctured disk.



Left: an open disk around z_0; right: a deleted open disk around z_0

Definition. An open region in the complex plane is a set A with the property that every point in A can be be surrounded by an open disk that lies entirely in A. We will often drop the word open and simply call A a region.

In the figure below, the set A on the left is an open region because for every point in A we can draw a little circle around the point that is completely in A. (The dashed boundary line indicates that the boundary of A is not part of A.) In contrast, the set B is not an open region. Notice the point z shown is on the boundary, so every disk around z contains points outside B.

Left: an open region A; right: B is not an open region

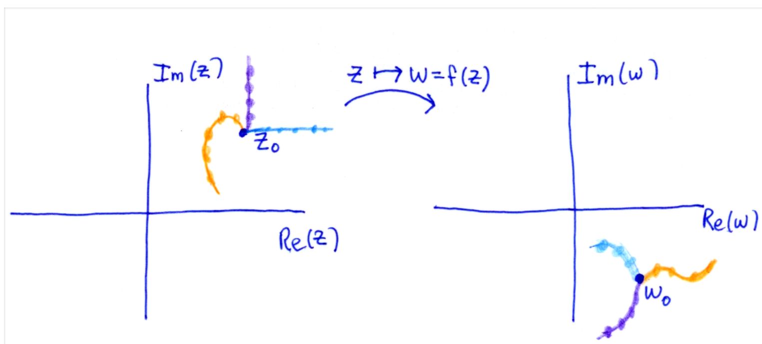
4.3 Limits and continuous functions

Definition. If $f(z)$ is defined on a punctured disk around z_0 then we say

$$\lim_{z \to z_0} f(z) = w_0$$

if $f(z)$ goes to w_0 no matter what direction z approaches z_0.

The figure below shows several sequences of points that approach z_0. If $\lim_{z \to z_0} f(z) = w_0$ then $f(z)$ must go to w_0 along each of these sequences.



Sequences going to z_0 are mapped to sequences going to w_0.

Example 4.3. Many functions have obvious limits. For example:

$$\lim_{z \to 2} z^2 = 4$$

and

$$\lim_{z \to 2} (z^2 + 2)/(z^3 + 1) = 6/9$$

Here is an example where the limit doesn't exist because different sequences give different limits.

Example 4.4. (No limit) Show that

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{z \to 0} \frac{x + iy}{x - iy}$$

does not exist.

Solution: On the real axis we have

$$\frac{z}{\bar{z}} = \frac{x}{x} = 1,$$

so the limit as $z \to 0$ along the real axis is 1. By contrast, on the imaginary axis we have

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1,$$

so the limit as $z \to 0$ along the imaginary axis is -1. Since the two limits do not agree the limit as $z \to 0$ does not exist!

4.3.1 Properties of limits

We have the usual properties of limits. Suppose

$$\lim_{z \to z_0} f(z) = w_1 \text{ and } \lim_{z \to z_0} g(z) = w_2$$

then

- $\lim\limits_{z \to z_0} f(z) + g(z) = w_1 + w_2$.
- $\lim\limits_{z \to z_0} f(z)g(z) = w_1 \cdot w_2$.
- If $w_2 \neq 0$ then $\lim\limits_{z \to z_0} f(z)/g(z) = w_1/w_2$
- If $h(z)$ is continuous and defined on a neighborhood of w_1 then $\lim\limits_{z \to z_0} h(f(z)) = h(w_1)$ (Note: we will give the official definition of continuity in the next section.)

We won't give a proof of these properties. As a challenge, you can try to supply it using the formal definition of limits given in the appendix.

We can restate the definition of limit in terms of functions of (x, y). To this end, let's write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

and abbreviate

$$P = (x, y)\ , \qquad P_0 = (x_0, y_0)\ , \qquad w_0 = u_0 + iv_0$$

Then

$$\lim_{z \to z_0} f(z) = w_0 \qquad \text{iff} \qquad \begin{cases} \lim_{P \to P_0} u(x, y) = u_0 \\ \lim_{P \to P_0} v(x, y) = v_0. \end{cases}$$

Note. The term 'iff' stands for 'if and only if' which is another way of saying 'is equivalent to'.

4.3.2 Continuous functions

A function is continuous if it doesn't have any sudden jumps. This is the gist of the following definition.

Definition. If the function $f(z)$ is defined on an open disk around z_0 and $\lim\limits_{z \to z_0} f(z) = f(z_0)$ then we say f is continuous at z_0. If f is defined on an open region A then the phrase 'f is continuous on A' means that f is continuous at every point in A.

As usual, we can rephrase this in terms of functions of (x, y):

Fact. $f(z) = u(x, y) + iv(x, y)$ is continuous iff $u(x, y)$ and $v(x, y)$ are continuous as functions of two variables.

Example 4.5. (Some continuous functions)
(i) A polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

is continuous on the entire plane. Reason: it is clear that each power $(x+iy)^k$ is continuous as a function of (x, y).
(ii) The exponential function is continuous on the entire plane. Reason:

$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x \cos(y) + i\mathrm{e}^x \sin(y),$$

so the both the real and imaginary parts are clearly continuous as a function of (x, y).
(iii) The principal branch $\mathrm{Arg}(z)$ is continuous on the plane minus the non-positive real axis. Reason: this is clear and is the reason we defined branch cuts for arg. We have to remove the negative real axis because $\mathrm{Arg}(z)$ jumps by 2π when you cross it. We also have to remove $z = 0$ because $\mathrm{Arg}(z)$ is not even defined at 0.
(iv) The principal branch of the function $\log(z)$ is continuous on the plane minus the non-positive real axis. Reason: the principal branch of log has

$$\log(z) = \log(r) + i\,\mathrm{Arg}(z),$$

so the continuity of $\log(z)$ follows from the continuity of $\mathrm{Arg}(z)$.

4.3.3 Properties of continuous functions

Since continuity is defined in terms of limits, we have the following properties of continuous functions.

Suppose $f(z)$ and $g(z)$ are continuous on a region A. Then

- $f(z) + g(z)$ is continuous on A.
- $f(z)g(z)$ is continuous on A.
- $f(z)/g(z)$ is continuous on A except (possibly) at points where $g(z) = 0$.
- If h is continuous on $f(A)$ then $h(f(z))$ is continuous on A.

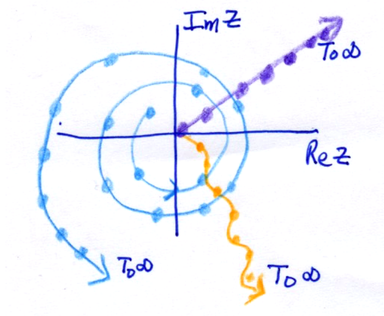
Using these properties we can claim continuity for each of the following functions:

- e^{z^2}
- $\cos(z) = (\mathrm{e}^{iz} + \mathrm{e}^{-iz})/2$
- If $P(z)$ and $Q(z)$ are polynomials then $P(z)/Q(z)$ is continuous except at roots of $Q(z)$.

4.4 The point at infinity

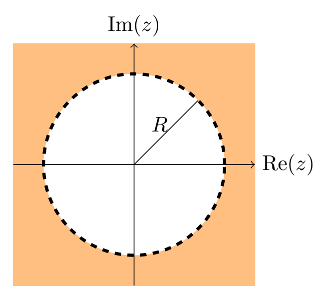
By definition the extended complex plane $= \mathbf{C} \cup \{\infty\}$. That is, we have **one** point at infinity to be thought of in a limiting sense described as follows.

A sequence of points $\{z_n\}$ goes to infinity if $|z_n|$ goes to infinity. This "point at infinity" is approached in any direction we go. All of the sequences shown in the figure below are growing, so they all go to the (same) "point at infinity".



Various sequences all going to infinity.

If we draw a large circle around 0 in the plane, then we call the region **outside** this circle a neighborhood of infinity.



The shaded region outside the circle of radius R is a neighborhood of infinity.

4.4.1 Limits involving infinity

The key idea is $1/\infty = 0$. By this we mean

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

We then have the following facts:

- $\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} 1/f(z) = 0$
- $\lim_{z \to \infty} f(z) = w_0 \Leftrightarrow \lim_{z \to 0} f(1/z) = w_0$
- $\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \frac{1}{f(1/z)} = 0$

Example 4.6. $\lim_{z\to\infty} e^z$ is not defined because it has different values if we go to infinity in different directions, e.g. we have $e^z = e^x e^{iy}$ and

$$\lim_{x\to-\infty} e^x e^{iy} = 0$$

$$\lim_{x\to+\infty} e^x e^{iy} = \infty$$

$\lim_{y\to+\infty} e^x e^{iy}$ is not defined, since x is constant, so $e^x e^{iy}$ loops in a circle indefinitely.

Example 4.7. Show $\lim_{z\to\infty} z^n = \infty$ (for n a positive integer).

Solution: We need to show that $|z^n|$ gets large as $|z|$ gets large. Write $z = Re^{i\theta}$, then

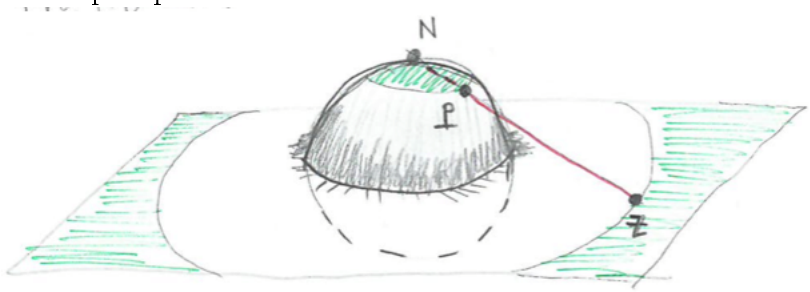
$$|z^n| = |R^n e^{in\theta}| = R^n = |z|^n$$

Clearly, as $|z| = R \to \infty$ also $|z|^n = R^n \to \infty$.

4.4.2 Stereographic projection from the Riemann sphere

This is a lovely section and we suggest you read it. However it will be a while before we use it in 18.04.

One way to visualize the point at ∞ is by using a (unit) Riemann sphere and the associated stereographic projection. The figure below shows a sphere whose equator is the unit circle in the complex plane.



Stereographic projection from the sphere to the plane.

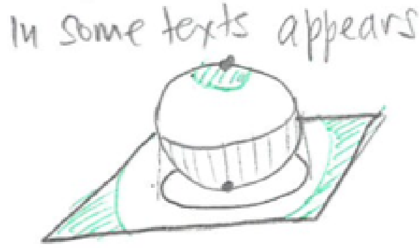
Stereographic projection from the sphere to the plane is accomplished by drawing the secant line from the north pole N through a point on the sphere and seeing where it intersects the plane. This gives a 1-1 correspondence between a point on the sphere P and a point in the complex plane z. It is easy to see show that the formula for stereographic projection is

$$P = (a, b, c) \mapsto z = \frac{a}{1-c} + i\frac{b}{1-c}.$$

The point $N = (0, 0, 1)$ is special, the secant lines from N through P become tangent lines to the sphere at N which never intersect the plane. We consider N the point at infinity.

In the figure above, the region outside the large circle through the point z is a neighborhood of infinity. It corresponds to the small circular cap around N on the sphere. That is, the small cap around N is a neighborhood of the point at infinity on the sphere!

The figure below shows another common version of stereographic projection. In this figure the sphere sits with its south pole at the origin. We still project using secant lines from the north pole.



4.5 Derivatives

The definition of the complex derivative of a complex function is similar to that of a real derivative of a real function: For a function $f(z)$ the derivative f at z_0 is defined as

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

Provided, of course, that the limit exists. If the limit exists we say f is analytic at z_0 or f is differentiable at z_0.

Remember: The limit has to exist and be the same no matter how you approach z_0!

If f is analytic at all the points in an open region A then we say f is analytic on A. As usual with derivatives there are several alternative notations. For example, if $w = f(z)$ we can write

$$f'(z_0) = \left.\frac{dw}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Example 4.8. Find the derivative of $f(z) = z^2$.

Solution: We did this above in Example 4.1. Take a look at that now. Of course, $f'(z) = 2z$.

Example 4.9. Show $f(z) = \overline{z}$ is not differentiable at any point z.

Solution: We did this above in Example 4.2. Take a look at that now.

Challenge. Use polar coordinates to show the limit in the previous example can be any value with modulus 1 depending on the angle at which z approaches z_0.

4.5.1 Derivative rules

It wouldn't be much fun to compute every derivative using limits. Fortunately, we have the same differentiation formulas as for real-valued functions. That is, assuming f and g are differentiable we have:

- Sum rule: $\dfrac{d}{dz}(f(z) + g(z)) = f' + g'$

- Product rule: $\dfrac{d}{dz}(f(z)g(z)) = f'g + fg'$
- Quotient rule: $\dfrac{d}{dz}(f(z)/g(z)) = \dfrac{f'g - fg'}{g^2}$
- Chain rule: $\dfrac{d}{dz}g(f(z)) = g'(f(z))f'(z)$
- Inverse rule: $\dfrac{df^{-1}(z)}{dz} = \dfrac{1}{f'(f^{-1}(z))}$

To give you the flavor of these arguments we'll prove the product rule.

$$\begin{aligned}
\frac{d}{dz}(f(z)g(z)) &= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \\
&= \lim_{z\to z_0} \frac{(f(z) - f(z_0))g(z) + f(z_0)(g(z) - g(z_0))}{z - z_0} \\
&= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} g(z) + f(z_0)\frac{(g(z) - g(z_0))}{z - z_0} \\
&= f'(z_0)g(z_0) + f(z_0)g'(z_0)
\end{aligned}$$

Here is an important fact that you would have guessed. We will prove it in the next section.

Theorem. If $f(z)$ is defined and differentiable on an open disk and $f'(z) = 0$ on the disk then $f(z)$ is constant.

4.6 Cauchy-Riemann equations

The Cauchy-Riemann equations are our first consequence of the fact that the limit defining $f(z)$ must be the same no matter which direction you approach z from. The Cauchy-Riemann equations will be one of the most important tools in our toolbox.

4.6.1 Partial derivatives as limits

Before getting to the Cauchy-Riemann equations we remind you about partial derivatives. If $u(x, y)$ is a function of two variables then the partial derivatives of u are defined as

$$\frac{\partial u}{\partial x}(x, y) = \lim_{\Delta x\to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

i.e. the derivative of u holding y constant.

$$\frac{\partial u}{\partial y}(x, y) = \lim_{\Delta y\to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

i.e. the derivative of u holding x constant.

4.6.2 The Cauchy-Riemann equations

The Cauchy-Riemann equations use the partial derivatives of u and v to do two things: first, to check if f has a complex derivative and second, how to compute that derivative. We start by stating the equations as a theorem.

Theorem. (Cauchy-Riemann equations) If $f(z) = u(x, y) + iv(x, y)$ is analytic (complex differentiable) then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

In particular,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This last set of partial differential equations is what is usually meant by the Cauchy-Riemann equations. Here is the short form of the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Proof. Let's suppose that $f(z)$ is differentiable in some region A and

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We'll compute $f'(z)$ by approaching z first from the horizontal direction and then from the vertical direction. We'll use the formula

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

where $\Delta z = \Delta x + i\Delta y$.

Horizontal direction: $\Delta y = 0$, $\Delta z = \Delta x$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(u(x + \Delta x, y) + iv(x + \Delta x, y)) - (u(x, y) + iv(x, y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) \end{aligned}$$

Vertical direction: $\Delta x = 0$, $\Delta z = i\Delta y$ (We'll do this one a little faster.)

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta y \to 0} \frac{(u(x, y+\Delta y) + iv(x, y+\Delta y)) - (u(x,y) + iv(x,y))}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \\
&= \frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \\
&= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y)
\end{aligned}$$

We have found two different representations of $f'(z)$ in terms of the partials of u and v. If put them together we have the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

It turns out that the converse is true and will be very useful to us.

Theorem. Consider the function $f(z) = u(x,y) + iv(x,y)$ defined on a region A. If u and v satisfy the Cauchy-Riemann equations and have continuous partials then $f(z)$ is differentiable on A.

The proof of this is a tricky exercise in analysis. It is somewhat beyond the scope of this class, so we will skip it.

4.6.3 Using the Cauchy-Riemann equations

The Cauchy-Riemann equations provide us with a direct way of checking that a function is differentiable and computing its derivative.

Example 4.10. Use the Cauchy-Riemann equations to show that e^z is differentiable and its derivative is e^z.

Solution: We write $e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$. So

$$u(x,y) = e^x \cos(y) \ \text{ and } \ v(x,y) = e^x \sin(y).$$

Computing partial derivatives we have

$$\begin{aligned}
u_x &= e^x \cos(y)\ , \qquad u_y = -e^x \sin(y) \\
v_x &= e^x \sin(y)\ , \qquad v_y = e^x \cos(y)
\end{aligned}$$

We see that $u_x = v_y$ and $u_y = -v_x$, so the Cauchy-Riemann equations are satisfied. Thus, e^z is differentiable and

$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos(y) + ie^x \sin(y) = e^z.$$

Example 4.11. Use the Cauchy-Riemann equations to show that $f(z) = \overline{z}$ is not differentiable.

Solution: $f(x+iy) = x - iy$, so $u(x,y) = x$, $v(x,y) = -y$. Taking partial derivatives

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1$$

Since $u_x \neq v_y$ the Cauchy-Riemann equations are not satisfied and therefore f is not differentiable.

Theorem. If $f(z)$ is differentiable on a disk and $f'(z) = 0$ on the disk then $f(z)$ is constant.
Proof. Since f is differentiable and $f'(z) \equiv 0$, the Cauchy-Riemann equations show that

$$u_x(x,y) = u_y(x,y) = v_x(x,y) = v_y(x,y) = 0$$

We know from multivariable calculus that a function of (x,y) with both partials identically zero is constant. Thus u and v are constant, and therefore so is f.

4.6.4 $f'(z)$ as a 2×2 matrix

Recall that we could represent a complex number $a + ib$ as a 2×2 matrix

$$a + ib \quad \leftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{13}$$

Now if we write $f(z)$ in terms of (x,y) we have

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \quad \leftrightarrow \quad f(x,y) = (u(x,y),\, v(x,y))$$

We have

$$f'(z) = u_x + iv_x$$

so we can represent $f'(z)$ as

$$\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$$

Using the Cauchy-Riemann equations we can replace $-v_x$ by u_y and u_x by v_y which gives us the representation

$$f'(z) \quad \leftrightarrow \quad \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

i.e. $f'(z)$ is just the Jacobian of $f(x,y)$.

For me, it is easier to remember the Jacobian than the Cauchy-Riemann equations. Since $f'(z)$ is a complex number I can use the matrix representation in Equation (13) to remember the Cauchy-Riemann equations!

4.7 Geometric interpretation & linear elasticity theory

Consider a mapping $\mathbb{R}^2 \to \mathbb{R}^2$

$$z = (x, y) \quad \to \quad w = f(z) = (u(x, y), v(x, y)) \tag{14}$$

We can interpret $w = f(z)$ as the (in general nonlinear) deformation map of a two-dimensional planar continuum body – that is, an infinite flat elastic sheet. Let's consider small deformations, corresponding to the assumption that the displacement field

$$d(z) := f(z) - z \tag{15}$$

is small,

$$|d(z)| = |f(z) - z| \ll 1 \tag{16}$$

In terms of the components d_1 and d_2 of d, this means that

$$d_1(x, y) = u(x, y) - x \qquad \text{and} \qquad d_2(x, y) = v(x, y) - y \tag{17}$$

are small everywhere. Picking some point z, we can Taylor-expand $d(z)$ at $z + \epsilon$, where

$$\epsilon = (\delta x, \delta y)$$

is a small shift vector. Defining

$$x_1 = x \ , \qquad x_2 = y \ , \qquad d_{i,j} = \frac{\partial}{\partial x_j} d_i(x_1, x_2)$$

and adopting Einstein's summation convention, $a_i b_i = \sum_{i=1}^{2} a_i b_i$, the Taylor expansion reads

$$d_i(z + \epsilon) = d_i(z) + d_{i,j}(z)\epsilon_j + \mathcal{O}(\epsilon^2). \tag{18}$$

The coefficients $d_{i,j}(z)$ of the linear term are the elements of the Jacobian matrix of the displacement map $d(z)$,

$$(d_{i,j}) = \begin{pmatrix} u_x - 1 & u_y \\ v_x & v_y - 1 \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} - \boldsymbol{I} \tag{19}$$

where the first matrix on the right-hand side is the Jacobian of the deformation $f(z)$ and $\boldsymbol{I} = (\delta_{ij})$ the 2×2 identity matrix. By splitting into symmetric and antisymmetric parts, we can decompose this matrix into the form

$$\begin{aligned} d_{i,j} &= \frac{1}{2}(d_{i,j} + d_{j,i}) + \frac{1}{2}(d_{i,j} - d_{j,i}) - \delta_{ij} \\ &= \frac{1}{2} d_{i,i}\delta_{ij} + \frac{1}{2}\left[(d_{i,j} + d_{j,i}) - d_{i,i}\delta_{ij}\right] + \frac{1}{2}(d_{i,j} - d_{j,i}) - \delta_{ij} \end{aligned} \tag{20}$$

In the second line, we have still split off the trace from the symmetric part. Substituting back $u(x, y)$ and $v(x, y)$, we have

$$(d_{i,j}) \quad = \quad \frac{1}{2}(u_x + v_y)\boldsymbol{I} + \frac{1}{2}\begin{pmatrix} u_x - v_y & u_y + v_x \\ v_x + u_y & v_y - u_x \end{pmatrix} - \left\{\boldsymbol{I} + \frac{1}{2}(u_y - v_x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} \tag{21}$$

Now recall that an infinitesimal rotation is represented by the matrix

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \approx \boldsymbol{I} + \phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{22}$$

we can interpret the contributions in terms of intuitive fundamental deformations:

- The first term

  $$\frac{1}{2}(u_x + v_y)\boldsymbol{I} = \frac{1}{2}(\nabla \cdot d)\boldsymbol{I} \tag{23a}$$

  represents stretching or compression.

- The last term term

  $$\left\{ \boldsymbol{I} + \frac{1}{2}(u_y - v_x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \tag{23b}$$

  represents an infinitesimal rotation by an angle $\phi = \frac{1}{2}(u_y - v_x) = \frac{1}{2}\nabla \wedge d$.

- The middle term

  $$\frac{1}{2}\begin{pmatrix} u_x - v_y & v_x + u_y \\ v_x + u_y & -(u_x - v_y) \end{pmatrix} = \frac{1}{2}(u_x - v_y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}(v_x + u_y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{23c}$$

  is the sum of a scaled *reflection* (via the diagonal components) and *shear* strain (via the off-diagonal components).

Thus, deformations that preserve orientation <u>and</u> angles locally must satisfy

$$u_x = v_y\ , \qquad u_y = -v_x \tag{24}$$

But these are just the Cauchy-Riemann conditions !

4.8 Cauchy-Riemann all the way down

We've defined an analytic function as one having a complex derivative. The following theorem shows that if f is analytic then so is f'. Thus, there are derivatives all the way down!

Theorem 4.12. Assume the second order partials of u and v exist. If $f(z) = u + iv$ is analytic, then so is $f'(z)$.

Proof. To show this we have to prove that $f'(z)$ satisfies the Cauchy-Riemann equations. If $f = u + iv$ we know

$$u_x = v_y\ , \qquad u_y = -v_x\ , \qquad f' = u_x + iv_x$$

Let's write

$$f' = U + iV$$

so

$$U = u_x = v_y \ , \qquad V = v_x = -u_y$$

We want to show that $U_x = V_y$ and $U_y = -V_x$. We do them one at a time.

To prove $U_x = V_y$, we note that

$$U_x = v_{yx} \ , \qquad V_y = v_{xy}$$

Since $v_{xy} = v_{yx}$, we have $U_x = V_y$.

Similarly, to show $U_y = -V_x$, we compute

$$U_y = u_{xy} \ , \qquad V_x = -u_{yx}.$$

Thus, $U_y = -V_x$.

Technical point. We've assumed as many partials as we need. So far we can't guarantee that all the partials exist. Soon we will have a theorem which says that an analytic function has derivatives of all order. We'll just assume that for now. In any case, in most examples this will be obvious.

4.9 Gallery of functions

In this section we'll look at many of the functions you know and love as functions of z. For each one we'll have to do three things.

1. Define how to compute it.
2. Specify a branch (if necessary) giving its range.
3. Specify a domain (with branch cut if necessary) where it is analytic.
4. Compute its derivative.

Most often, we can compute the derivatives of a function using the algebraic rules like the quotient rule. If necessary we can use the Cauchy-Riemann equations or, as a last resort, even the definition of the derivative as a limit.

Before we start on the gallery we define the term "entire function".

Definition. A function that is analytic at every point in the complex plane is called an entire function. We will see that e^z, z^n, $\sin(z)$ are all entire functions.

4.9.1 Gallery of functions, derivatives and properties

The following is a concise list of a number of functions and their complex derivatives. None of the derivatives will surprise you. We also give important properties for some of the functions. The proofs for each follow below.

1. $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$.

   Domain = all of $\mathbf{C}$ (f is entire).

   $f'(z) = e^z$.

2. $f(z) \equiv c$ (constant)

   Domain = all of $\mathbf{C}$ (f is entire).

   $f'(z) = 0$.

3. $f(z) = z^n$ (n an integer ≥ 0)

   Domain = all of $\mathbf{C}$ (f is entire).

   $f'(z) = nz^{n-1}$.

4. $P(z)$ (polynomial)

   A polynomial has the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$.

   Domain = all of $\mathbf{C}$ ($P(z)$ is entire).

   $P'(z) = na_n z^{n-1} + (n-1)a_{n-1} z^{n-1} + \ldots + 2a_2 z + a_1$.

5. $f(z) = 1/z$

   Domain = $\mathbf{C} - \{0\}$ (the punctured plane).

   $f'(z) = -1/z^2$.

6. $f(z) = P(z)/Q(z)$ (rational function).

   When P and Q are polynomials $P(z)/Q(z)$ is called a rational function.

   If we assume that P and Q have no common roots, then:

   Domain = $\mathbf{C} - \{\text{roots of } Q\}$

   $f'(z) = \dfrac{P'Q - PQ'}{Q^2}$.

7. $\sin(z)$, $\cos(z)$

   Definition. $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

   (By Euler's formula we know this is consistent with $\cos(x)$ and $\sin(x)$ when $z = x$ is real.)

   Domain: these functions are entire.

   $$\frac{d\cos(z)}{dz} = -\sin(z), \quad \frac{d\sin(z)}{dz} = \cos(z).$$

   Other key properties of sin and cos:

   - $\cos^2(z) + \sin^2(z) = 1$
   - $e^z = \cos(z) + i\sin(z)$
   - Periodic in x with period 2π, e.g. $\sin(x + 2\pi + iy) = \sin(x + iy)$.
   - They are not bounded!

- In the form $f(z) = u(x, y) + iv(x, y)$ we have

$$\begin{aligned} \cos(z) &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \\ \sin(z) &= \sin(x)\cosh(y) + i\cos(x)\sinh(y) \end{aligned}$$

(cosh and sinh are defined below.)

- The zeros of $\sin(z)$ are $z = n\pi$ for n any integer.
The zeros of $\cos(z)$ are $z = \pi/2 + n\pi$ for n any integer.
(That is, they have only real zeros that you learned about in your trig. class.)

8. Other trig functions $\cot(z)$, $\mathrm{s}(z)$ etc.

Definition. The same as for the real versions of these function, e.g. $\cot(z) = \cos(z)/\sin(z)$, $\mathrm{s}(z) = 1/\cos(z)$.

Domain: The entire plane minus the zeros of the denominator.

Derivative: Compute using the quotient rule, e.g.

$$\frac{d\tan(z)}{dz} = \frac{d}{dz}\left(\frac{\sin(z)}{\cos(z)}\right) = \frac{\cos(z)\cos(z) - \sin(z)(-\sin(z))}{\cos^2(z)} = \frac{1}{\cos^2(z)} = \mathrm{s}^2 z$$

(No surprises there!)

9. $\sinh(z)$, $\cosh(z)$ (hyperbolic sine and cosine)

Definition.

$$\cosh(z) = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}, \quad \sinh(z) = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}$$

Domain: these functions are entire.

$$\frac{d\cosh(z)}{dz} = \sinh(z), \quad \frac{d\sinh(z)}{dz} = \cosh(z)$$

Other key properties of cosh and sinh:

- $\cosh^2(z) - \sinh^2(z) = 1$
- For real x, $\cosh(x)$ is real and positive, $\sinh(x)$ is real.
- $\cosh(iz) = \cos(z)$, $\sinh(z) = -i\sin(iz)$.

10. $\log(z)$ (See Topic 1.)

Definition. $\log(z) = \log(|z|) + i\arg(z)$.

Branch: Any branch of $\arg(z)$.

Domain: $\mathbf{C}$ minus a branch cut where the chosen branch of $\arg(z)$ is discontinuous.

$$\frac{d}{dz}\log(z) = \frac{1}{z}$$

11. z^a (any complex a) (See Topic 1.)

    Definition. $z^a = \mathrm{e}^{a\log(z)}$.

    Branch: Any branch of $\log(z)$.

    Domain: Generally the domain is $\mathbf{C}$ minus a branch cut of log. If a is an integer ≥ 0 then z^a is entire. If a is a negative integer then z^a is defined and analytic on $\mathbf{C} - \{0\}$.

$$\frac{dz^a}{dz} = az^{a-1}.$$

12. $\sin^{-1}(z)$

    Definition. $\sin^{-1}(z) = -i\log(iz + \sqrt{1-z^2})$.

    The definition is chosen so that $\sin(\sin^{-1}(z)) = z$. The derivation of the formula is as follows. Let $w = \sin^{-1}(z)$, so $z = \sin(w)$. Then,

$$z = \frac{\mathrm{e}^{iw} - \mathrm{e}^{-iw}}{2i} \;\Rightarrow\; \mathrm{e}^{2iw} - 2iz\mathrm{e}^{iw} - 1 = 0$$

    Solving the quadratic in e^{iw} gives

$$\mathrm{e}^{iw} = \frac{2iz + \sqrt{-4z^2+4}}{2} = iz + \sqrt{1-z^2}.$$

    Taking the log gives

$$iw = \log(iz + \sqrt{1-z^2}) \;\Leftrightarrow\; w = -i\log(iz + \sqrt{1-z^2}). \quad \blacksquare$$

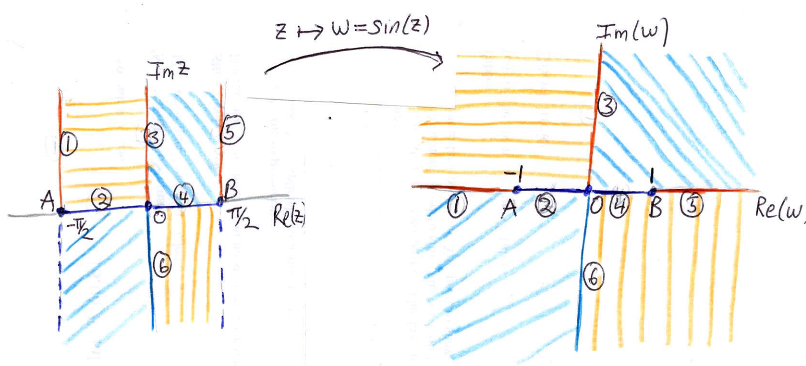
    From the definition we can compute the derivative:

$$\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}.$$

    Choosing a branch is tricky because both the square root and the log require choices. We will look at this more carefully in the future.

    For now, the following discussion and figure are for your amusement.

    Sine (likewise cosine) is not a 1-1 function, so if we want $\sin^{-1}(z)$ to be single-valued then we have to choose a region where $\sin(z)$ is 1-1. (This will be a branch of $\sin^{-1}(z)$, i.e. a range for the image,) The figure below shows a domain where $\sin(z)$ is 1-1. The domain consists of the vertical strip $z = x + iy$ with $-\pi/2 < x < \pi/2$ together with the two rays on boundary where $y \geq 0$ (show as red lines). The figure indicates how the regions making up the domain in the z-plane are mapped to the quadrants in the w-plane.



A domain where $z \mapsto w = \sin(z)$ is one-to-one

4.9.2 A few proofs

Here we prove at least some of the facts stated in the list just above.

1. $f(z) = e^z$. This was done in Example 4.10 using the Cauchy-Riemann equations.
2. $f(z) \equiv c$ (constant). This case is trivial.
3. $f(z) = z^n$ (n an integer ≥ 0): show $f'(z) = nz^{n-1}$

   It's probably easiest to use the definition of derivative directly. Before doing that we note the factorization

   $$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \ldots + z^2 z_0^{n-3} + z z_0^{n-2} + z_0^{n-1})$$

   Now

   $$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} \\ &= \lim_{z \to z_0} (z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \ldots + z^2 z_0^{n-3} + z z_0^{n-2} + z_0^{n-1}) \\ &= n z_0^{n-1}. \end{aligned}$$

   Since we showed directly that the derivative exists for all z, the function must be *entire*[4].
4. $P(z)$ (polynomial). Since a polynomial is a sum of monomials, the formula for the derivative follows from the derivative rule for sums and the case $f(z) = z^n$. Likewise the fact the $P(z)$ is entire.
5. $f(z) = 1/z$. This follows from the quotient rule.
6. $f(z) = P(z)/Q(z)$. This also follows from the quotient rule.
7. $\sin(z)$, $\cos(z)$. All the facts about $\sin(z)$ and $\cos(z)$ follow from their definition in terms of exponentials.

[4] An entire function is a complex-valued function that is complex differentiable at all finite points over the whole complex plane.

8. Other trig functions $\cot(z)$, $\mathrm{s}(z)$ etc. Since these are all defined in terms of cos and sin, all the facts about these functions follow from the derivative rules.

9. $\sinh(z)$, $\cosh(z)$. All the facts about $\sinh(z)$ and $\cosh(z)$ follow from their definition in terms of exponentials.

10. $\log(z)$. The derivative of $\log(z)$ can be found by differentiating the relation $\mathrm{e}^{\log(z)} = z$ using the chain rule. Let $w = \log(z)$, so $\mathrm{e}^w = z$ and

$$\frac{d}{dz}\mathrm{e}^w = \frac{dz}{dz} = 1 \quad \Rightarrow \quad \frac{d\mathrm{e}^w}{dw}\frac{dw}{dz} = 1 \quad \Rightarrow \quad \mathrm{e}^w \frac{dw}{dz} = 1 \quad \Rightarrow \quad \frac{dw}{dz} = \frac{1}{\mathrm{e}^w}$$

Using $w = \log(z)$ we get

$$\frac{d\log(z)}{dz} = \frac{1}{z}$$

11. z^a (any complex a). The derivative for this follows from the formula

$$z^a = \mathrm{e}^{a\log(z)} \qquad \Rightarrow \qquad \frac{dz^a}{dz} = \mathrm{e}^{a\log(z)} \cdot \frac{a}{z} = \frac{az^a}{z} = az^{a-1}$$

4.10 Branch cuts and function composition

We often compose functions, i.e. $f(g(z))$. In general in this case we have the chain rule to compute the derivative. However we need to specify the domain for z where the function is analytic. And when branches and branch cuts are involved we need to take care.

Example 4.13. Let $f(z) = \mathrm{e}^{z^2}$. Since e^z and z^2 are both entire functions, so is $f(z) = \mathrm{e}^{z^2}$. The chain rule gives us

$$f'(z) = \mathrm{e}^{z^2}(2z).$$

Example 4.14. Let $f(z) = \mathrm{e}^z$ and $g(z) = 1/z$. $f(z)$ is entire and $g(z)$ is analytic everywhere but 0. So $f(g(z))$ is analytic except at 0 and

$$\frac{df(g(z))}{dz} = f'(g(z))g'(z) = \mathrm{e}^{1/z} \cdot \frac{-1}{z^2}.$$

Example 4.15. Let $h(z) = 1/(\mathrm{e}^z - 1)$. Clearly h is entire except where the denominator is 0. The denominator is 0 when $\mathrm{e}^z - 1 = 0$. That is, when $z = 2\pi ni$ for any integer n. Thus, $h(z)$ is analytic on the set

$$\mathbf{C} - \{2\pi ni, \text{ where } n \text{ is any integer}\}$$

The quotient rule gives $h'(z) = -\mathrm{e}^z/(\mathrm{e}^z - 1)^2$. A little more formally: $h(z) = f(g(z))$. where $f(w) = 1/w$ and $w = g(z) = \mathrm{e}^z - 1$. We know that $g(z)$ is entire and $f(w)$ is analytic everywhere except $w = 0$. Therefore, $f(g(z))$ is analytic everywhere except where $g(z) = 0$.

Example 4.16. It can happen that the derivative has a larger domain where it is analytic

than the original function. The main example is $f(z) = \log(z)$. This is analytic on C minus a branch cut. However
$$\frac{d}{dz}\log(z) = \frac{1}{z}$$
is analytic on $\mathbf{C} - \{0\}$. The converse can't happen.

Example 4.17. Define a region where $\sqrt{1-z}$ is analytic.

Solution: Choosing the principal branch of argument, we have $\sqrt{w}$ is analytic on

$$\mathbf{C} - \{x \le 0,\, y = 0\}, \quad \text{(see figure below.)}.$$

So $\sqrt{1-z}$ is analytic except where $w = 1 - z$ is on the branch cut, i.e. where $w = 1 - z$ is real and ≤ 0. It's easy to see that

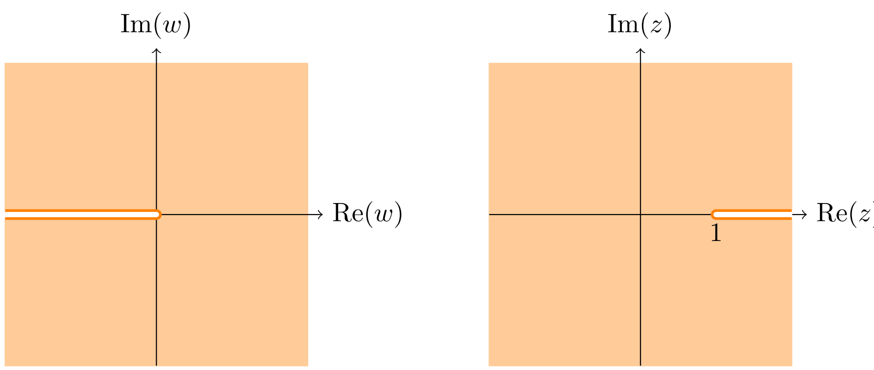
$$w = 1 - z \text{ is real and } \le 0 \Leftrightarrow z \text{ is real and } \ge 1.$$

So $\sqrt{1-z}$ is analytic on the region (see figure below)

$$\mathbf{C} - \{x \ge 1,\, y = 0\}$$

Note. A different branch choice for $\sqrt{w}$ would lead to a different region where $\sqrt{1-z}$ is analytic.

The figure below shows the domains with branch cuts for this example.



domain for $\sqrt{w}$ domain for $\sqrt{1-z}$

Example 4.18. Define a region where $f(z) = \sqrt{1 + e^z}$ is analytic.

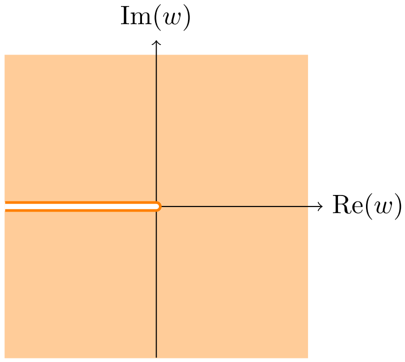
Solution: Again, let's take $\sqrt{w}$ to be analytic on the region

$$\mathbf{C} - \{x \le 0,\, y = 0\}$$

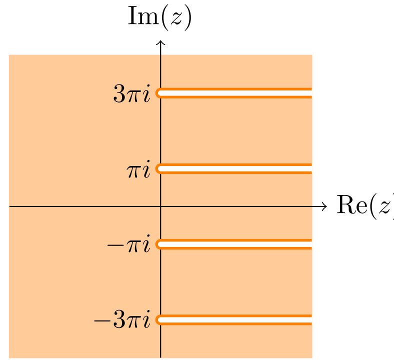
So, $f(z)$ is analytic except where $1 + e^z$ is real and ≤ 0. That is, except where e^z is real and ≤ -1. Now, $e^z = e^x e^{iy}$ is real only when y is a multiple of π. It is negative only when y is an odd mutltiple of π. It has magnitude greater than 1 only when $x > 0$. Therefore $f(z)$ is analytic on the region

$$\mathbf{C} - \{x \ge 0, \;\; y = \text{odd multiple of } \pi\}$$

The figure below shows the domains with branch cuts for this example.



domain for $\sqrt{w}$



domain for $\sqrt{e^z + 1}$

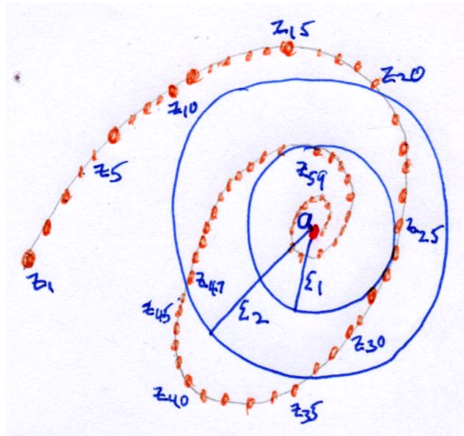
4.11 Appendix: Limits

The intuitive idea behind limits is relatively simple. Still, in the 19th century mathematicians were troubled by the lack of rigor, so they set about putting limits and analysis on a firm footing with careful definitions and proofs. In this appendix we give you the formal definition and connect it to the intuitive idea. In 18.04 we will not need this level of formality. Still, it's nice to know the foundations are solid, and some of you may find this interesting.

4.11.1 Limits of sequences

Intuitively, we say a sequence of complex numbers $z_1, z_2, \ldots$ converges to a if for large n, z_n is really close to a. To be a little more precise, if we put a small circle of radius ϵ around a then eventually the sequence should stay inside the circle. Let's refer to this as the sequence being captured by the circle. This has to be true for any circle no matter how small, though it may take longer for the sequence to be 'captured' by a smaller circle.

This is illustrated in the figure below. The sequence is strung along the curve shown heading towards a. The bigger circle of radius ϵ_2 captures the sequence by the time $n = 47$, the smaller circle doesn't capture it till $n = 59$. Note that z_{25} is inside the larger circle, but since later points are outside the circle we don't say the sequence is captured at $n = 25$



A sequence of points converging to a

Definition. The sequence $z_1, z_2, z_3, \ldots$ converges to the value a if for every $\epsilon > 0$ there is a number N_ϵ such that $|z_n - a| < \epsilon$ for all $n > N_\epsilon$. We write this as

$$\lim_{n\to\infty} z_n = a.$$

Again, the definition just says that eventually the sequence is within ϵ of a, no matter how small you choose ϵ.

Example 4.19. Show that the sequence $z_n = (1/n + i)^2$ has limit -1.

Solution: This is clear because $1/n \to 0$. For practice, let's phrase it in terms of epsilons: given $\epsilon > 0$ we have to choose N_ϵ such that

$$|z_n - (-1)| < \epsilon \text{ for all } n > N_\epsilon$$

One strategy is to look at $|z_n + 1|$ and see what N_ϵ should be. We have

$$|z_n - (-1)| = \left| \left(\frac{1}{n} + i\right)^2 + 1 \right| = \left| \frac{1}{n^2} + \frac{2i}{n} \right| < \frac{1}{n^2} + \frac{2}{n}$$

So all we have to do is pick N_ϵ large enough that

$$\frac{1}{N_\epsilon^2} + \frac{2}{N_\epsilon} < \epsilon$$

Since this can clearly be done we have proved that $z_n \to i$.

This was clearly more work than we want to do for every limit. Fortunately, most of the time we can apply general rules to determine a limit without resorting to epsilons!

Remarks.

1. In 18.04 we will be able to spot the limit of most concrete examples of sequences. The formal definition is needed when dealing abstractly with sequences.

2. To mathematicians ϵ is one of the go-to symbols for a small number. The prominent and rather eccentric mathematician Paul Erdos used to refer to children as epsilons, as in 'How are the epsilons doing?'

3. The term 'captured by the circle' is not in common usage, but it does capture what is happening.

4.11.2 $\lim_{z \to z_0} f(z)$

Sometimes we need limits of the form $\lim_{z \to z_0} f(z) = a$. Again, the intuitive meaning is clear: as z gets close to z_0 we should see $f(z)$ get close to a. Here is the technical definition

Definition. Suppose $f(z)$ is defined on a punctured disk $0 < |z - z_0| < r$ around z_0. We say $\lim_{z \to z_0} f(z) = a$ if for every $\epsilon > 0$ there is a δ such that

$$|f(z) - a| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

This says exactly that as z gets closer (within δ) to z_0 we have $f(z)$ is close (within ϵ) to a. Since ϵ can be made as small as we want, $f(z)$ must go to a.

Remarks.

1. Using the punctured disk (also called a deleted neighborhood) means that $f(z)$ does not have to be defined at z_0 and, if it is then $f(z_0)$ does not necessarily equal a. If $f(z_0) = a$ then we say the f is continuous at z_0.

2. Ask any mathematician to complete the phrase "For every ϵ" and the odds are that they will respond "there is a δ ..."

4.11.3 Connection between limits of sequences and limits of functions

Here's an equivalent way to define limits of functions: the limit $\lim_{z \to z_0} f(z) = a$ if, for every sequence of points $\{z_n\}$ with limit z_0 the sequence $\{f(z_n)\}$ has limit a.

5 Line integrals and Cauchy's theorem

The basic theme here is that complex line integrals will mirror much of what we've seen for multivariable calculus line integrals. But, just like working with $e^{i\theta}$ is easier than working with sine and cosine, complex line integrals are easier to work with than their multivariable analogs. At the same time they will give deep insight into the workings of these integrals.

To define complex line integrals, we will need the following ingredients

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \le t \le b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

5.1 Complex line integrals

Line integrals are also called path or contour integrals. Given the above ingredients, we define the complex line integral by

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{25a}$$

You should note that this notation looks just like integrals of a real variable. We don't need the vectors and dot products of line integrals in $\mathbf{R}^2$. Also, make sure you understand that the product $f(\gamma(t))\gamma'(t)$ is just a product of complex numbers.

An alternative notation uses $dz = dx + idy$ to write

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) \tag{25b}$$

Let's check that Equations (25a) and (25b) are the same. Equation (25b) is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_\gamma f(z)\,dz = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t)]\,\big(x'(t) + iy'(t)\big)\,dt$$

But,

$$u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$$

and

$$x'(t) + iy'(t) = \gamma'(t)$$

so the right hand side of this equation is

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

That is, it is exactly the same as the expression in Equation (25a).

Example 5.1. Compute $\int_\gamma z^2 dz$ along the straight line from 0 to $1 + i$.

Solution: We parametrize the curve as $\gamma(t) = t(1 + i)$ with $0 \le t \le 1$. So $\gamma'(t) = 1 + i$. The line integral is

$$\int z^2 dz = \int_0^1 t^2(1 + i)^2(1 + i)\,dt = \frac{2i(1 + i)}{3}.$$

Example 5.2. Compute $\int_\gamma \overline{z}\,dz$ along the straight line from 0 to $1 + i$.

Solution: We can use the same parametrization as in the previous example. So,

$$\int_\gamma \overline{z}\,dz = \int_0^1 t(1 - i)(1 + i)\,dt = 1$$

Example 5.3. Compute $\int_\gamma z^2\,dz$ along the unit circle.

Solution: We parametrize the unit circle by $\gamma(\theta) = \mathrm{e}^{i\theta}$, where $0 \le \theta \le 2\pi$. We have $\gamma'(\theta) = i\mathrm{e}^{i\theta}$. So, the integral becomes

$$\int_\gamma z^2\,dz = \int_0^{2\pi} \mathrm{e}^{2i\theta} i\mathrm{e}^{i\theta}\,d\theta = \int_0^{2\pi} i\mathrm{e}^{3i\theta}\,d\theta = \left.\frac{\mathrm{e}^{i3\theta}}{3}\right|_0^{2\pi} = 0.$$

Example 5.4. Compute $\displaystyle\int \overline{z}\,dz$ along the unit circle.

Solution: Parametrize C: $\gamma(t) = \mathrm{e}^{it}$, with $0 \le t \le 2\pi$. So, $\gamma'(t) = i\mathrm{e}^{it}$. Putting this into the integral gives

$$\int_C \overline{z}\,dz = \int_0^{2\pi} \overline{\mathrm{e}^{it}}\,i\,\mathrm{e}^{it}\,dt = \int_0^{2\pi} i\,dt = \boxed{2\pi i.}$$

5.2 Fundamental theorem for complex line integrals

This is exactly analogous to the fundamental theorem of calculus.

Theorem 5.5. (Fundamental theorem of complex line integrals) If $f(z)$ is a complex analytic function and γ is a curve from z_0 to z_1 then

$$\int_\gamma f'(z)\,dz = f(z_1) - f(z_0).$$

Proof. This is an application of the chain rule. We have

$$\frac{df(\gamma(t))}{dt} = f'(\gamma(t))\,\gamma'(t).$$

So

$$\int_\gamma f'(z)\,dz = \int_{z_0}^{z_1} f'(\gamma(t))\,\gamma'(t)\,dt = \int_{z_0}^{z_1} \frac{df(\gamma(t))}{dt}\,dt = f(\gamma(t))\Big|_{z_0}^{z_1} = f(z_1) - f(z_0).$$

Another equivalent way to state the fundamental theorem is: if f has an antiderivative F, i.e. $F' = f$ then

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$

Example 5.6. Redo $\int_\gamma z^2\,dz$, with γ the straight line from 0 to $1+i$.

Solution: We can check by inspection that z^2 has an antiderivative $F(z) = z^3/3$. Therefore the fundamental theorem implies

$$\int_\gamma z^2\,dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3} = \frac{2i(1+i)}{3}.$$

Example 5.7. Redo $\int_\gamma z^2\,dz$, with γ the unit circle.

Solution: Again, since z^2 had antiderivative $z^3/3$ we can evaluate the integral by plugging the endpoints of γ into the $z^3/3$. Since the endpoints are the same the resulting difference will be 0!

5.3 Path independence

We say the integral $\int_\gamma f(z)\,dz$ is path independent if it has the same value for any two paths with the same endpoints. More precisely, if $f(z)$ is defined on a region A then $\int_\gamma f(z)\,dz$ is path independent in A, if it has the same value for any two paths in A with the same endpoints. This statement follows directly from the fundamental theorem.

Theorem 5.8. (Path independence) If $f(z)$ has an antiderivative in a simply connected open region A, then the path integral $\int_\gamma f(z)\,dz$ is path independent for all paths in A.

Proof. Since $f(z)$ has an antiderivative of $f(z)$, the fundamental theorem tells us that the integral only depends on the endpoint of γ, i.e.

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$$

where z_0 and z_1 are the beginning and end point of γ.

An alternative way to express path independence uses closed paths.

Theorem 5.9. The following two things are equivalent.

1. The integral $\displaystyle\int_\gamma f(z)\,dz$ is path independent.

2. The integral $\displaystyle\int_\gamma f(z)\,dz$ around any closed path is 0.

Proof. This is essentially identical to the corresponding multivariable proof. We have to show two things:

(i) Path independence implies the line integral around any closed path is 0.

(ii) If the line integral around all closed paths is 0 then we have path independence.

To see *(i)*, assume path independence and consider the closed path C shown in figure (i) below. Since the starting point z_0 is the same as the endpoint z_1 the line integral $\int_C f(z)\,dz$ must have the same value as the line integral over the curve consisting of the single point z_0. Since that is clearly 0 we must have the integral over C is 0.

To see *(ii)*, assume $\int_C f(z)\,dz = 0$ for any closed curve. Consider the two curves C_1 and C_2 shown in figure (ii). Both start at z_0 and end at z_1. By the assumption that integrals over closed paths are 0 we have $\int_{C_1 - C_2} f(z)\, dz = 0$. So,

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

That is, any two paths from z_0 to z_1 have the same line integral. This shows that the line integrals are path independent.

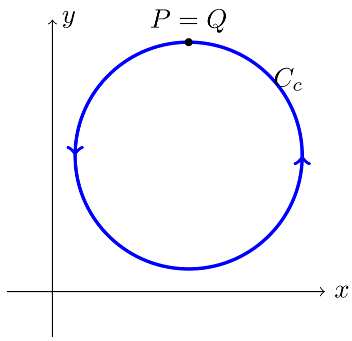


Figure (i)

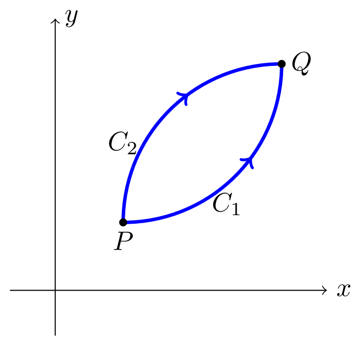


Figure (ii)

5.4 Examples

Example 5.10. Why can't we compute $\int_\gamma \overline{z}\, dz$ using the fundamental theorem.

Solution: Because $\overline{z}$ doesn't have an antiderivative. We can also see this by noting that if $\overline{z}$ had an antiderivative, then its integral around the unit circle would have to be 0. But, we saw in Example 5.4 that this is not the case.

Example 5.11. Compute $\int_\gamma z^{-1}\, dz$ over several contours

(i) The line from 1 to $1 + i$.
(ii) The circle of radius 1 around $z = 3$.
(iii) The unit circle.

Solution: For parts (i) and (ii) there is no problem using the antiderivative $\log(z)$ because these curves are contained in a simply connected region that doesn't contain the origin.

(i)

$$\int_\gamma \frac{1}{z}\, dz = \log(1+i) - \log(1) = \log(\sqrt{2}) + i\frac{\pi}{4}$$

(ii) Since the beginning and end points are the same

$$\int_\gamma \frac{1}{z}\, dz = 0$$

(iii) We parametrize the unit circle by $\gamma(\theta) = \mathrm{e}^{i\theta}$ with $0 \le \theta \le 2\pi$. We compute $\gamma'(\theta) = i\mathrm{e}^{i\theta}$. So the integral becomes

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\mathrm{e}^{i\theta}}\, i\mathrm{e}^{i\theta}\, dt = \int_0^{2\pi} i\, dt = 2\pi i$$

Notice that we could use $\log(z)$ if we were careful to let the argument increase by 2π as it went around the origin once.

Example 5.12. Compute $\int_\gamma z^{-2}\, dz$, where γ is the unit circle in two ways.

(i) Using the fundamental theorem.

(ii) Directly from the definition.

Solution: (i) Let $f(z) = -1/z$. Since $f'(z) = 1/z^2$, the fundamental theorem says

$$\int_\gamma \frac{1}{z^2}\, dz = \int_\gamma f'(z)\, dz = f(\text{endpoint}) - f(\text{start point}) = 0.$$

It equals 0 because the start and endpoints are the same.

(ii) As usual, we parametrize the unit circle as $\gamma(\theta) = \mathrm{e}^{i\theta}$ with $0 \le \theta \le 2\pi$. So, $\gamma'(\theta) = i\mathrm{e}^{i\theta}$ and the integral becomes

$$\int_\gamma \frac{1}{z^2}\, dz = \int_0^{2\pi} \frac{1}{\mathrm{e}^{2i\theta}}\, i\mathrm{e}^{i\theta}\, d\theta = \int_0^{2\pi} i\mathrm{e}^{-i\theta}\, d\theta = -\mathrm{e}^{-i\theta}\Big|_0^{2\pi} = 0.$$

5.5 Cauchy's theorem

Cauchy's theorem is analogous to Green's theorem for curl free vector fields.

Theorem 5.13. (Cauchy's theorem) Suppose A is a simply connected region, $f(z)$ is analytic on A and C is a simple closed curve in A. Then the following three things hold:

(i)

$$\int_C f(z)\, dz = 0$$

(i′) We can drop the requirement that C is simple in part (i).

(ii) Integrals of f on paths within A are path independent. That is, two paths with same endpoints integrate to the same value.

(iii) f has an antiderivative in A.

Proof. We will prove (i) using Green's theorem – we could give a proof that didn't rely on Green's, but it would be quite similar in flavor to the proof of Green's theorem.

Let R be the region inside the curve. And write $f = u + iv$. Now we write out the integral as follows

$$\int_C f(z)\, dz = \int_C (u + iv)\,(dx + idy) = \int_C (u\, dx - v\, dy) + i(v\, dx + u\, dy).$$

Let's apply Green's theorem to the real and imaginary pieces separately. First the real piece:

$$\int_C u\,dx - v\,dy = \int_R (-v_x - u_y)\,dx\,dy = 0.$$

We get 0 because the Cauchy-Riemann equations say $u_y = -v_x$, so $-v_x - u_y = 0$.

Likewise for the imaginary piece:

$$\int_C v\,dx + u\,dy = \int_R (u_x - v_y)\,dx\,dy = 0.$$

We get 0 because the Cauchy-Riemann equations say $u_x = v_y$, so $u_x - v_y = 0$.

To see part (i′) you should draw a few curves that intersect themselves and convince yourself that they can be broken into a sum of simple closed curves. Thus, (i′) follows from (i).[5]

Part (ii) follows from (i) and Theorem 5.9.

To see (iii), pick a base point $z_0 \in A$ and let

$$F(z) = \int_{z_0}^{z} f(w)\,dw.$$

Here the integral is over any path in A connecting z_0 to z. By part (ii), $F(z)$ is well defined. If we can show that $F'(z) = f(z)$ then we'll be done. Doing this amounts to managing the notation to apply the fundamental theorem of calculus and the Cauchy-Riemann equations. Let's write

$$f(z) = u(x,y) + iv(x,y)\ , \qquad F(z) = U(x,y) + iV(x,y)$$

Then we can write

$$\frac{\partial f}{\partial x} = u_x + iv_x, \text{ etc.}$$

We can formulate the Cauchy-Riemann equations for $F(z)$ as

$$F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y} \tag{26a}$$

i.e.

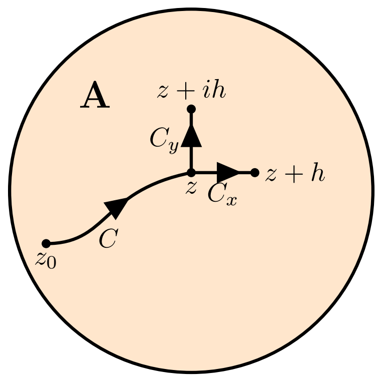
$$F'(z) = U_x + iV_x = \frac{1}{i}(U_y + iV_y) = V_y - iU_y. \tag{26b}$$

For reference, we note that using the path $\gamma(t) = x(t) + iy(t)$, with $\gamma(0) = z_0$ and $\gamma(b) = z$ we have

$$\begin{aligned} F(z) = \int_{z_0}^{z} f(w)\,dw &= \int_{z_0}^{z} (u(x,y) + iv(x,y))(dx + idy) \\ &= \int_0^b (u(x(t),y(t)) + iv(x(t),y(t))(x'(t) + iy'(t))\,dt. \end{aligned} \tag{27}$$

[5] In order to truly prove part (i′) we would need a more technically precise definition of simply connected so we could say that all closed curves within A can be continuously deformed to each other.

Our goal now is to prove that the Cauchy-Riemann equations given in Equation (27) hold for $F(z)$. The figure below shows an arbitrary path from z_0 to z, which can be used to compute $F(z)$. To compute the partials of F we'll need the straight lines that continue C to $z+h$ or $z+ih$.



Paths for proof of Cauchy's theorem

To prepare the rest of the argument we remind you that the fundamental theorem of calculus implies

$$\lim_{h\to 0} \frac{\int_0^h g(t)\,dt}{h} = g(0). \tag{28}$$

(That is, the derivative of the integral is the original function.)

First we'll look at $\dfrac{\partial F}{\partial x}$. So, fix $z = x + iy$. Looking at the paths in the figure above we have

$$F(z+h) - F(z) = \int_{C+C_x} f(w)\,dw - \int_C f(w)\,dw = \int_{C_x} f(w)\,dw.$$

The curve C_x is parametrized by $\gamma(t) = x + t + iy$, with $0 \le t \le h$. So,

$$\begin{aligned}
\frac{\partial F}{\partial x} = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} &= \lim_{h\to 0} \frac{\int_{C_x} f(w)\,dw}{h} \\
&= \lim_{h\to 0} \frac{\int_0^h u(x+t,y) + iv(x+t,y)\,dt}{h} \\
&= u(x,y) + iv(x,y) \\
&= f(z)
\end{aligned} \tag{29}$$

The second to last equality follows from Equation (28).

Similarly, we get (remember: $w = z + it$, so $dw = i\,dt$)

$$\begin{aligned}
\frac{1}{i}\frac{\partial F}{\partial y} = \lim_{h\to 0} \frac{F(z+ih)-F(z)}{ih} &= \lim_{h\to 0} \frac{\int_{C_y} f(w)\,dw}{ih} \\
&= \lim_{h\to 0} \frac{\int_0^h u(x,y+t) + iv(x,y+t)\,i\,dt}{ih} \\
&= u(x,y) + iv(x,y) \\
&= f(z).
\end{aligned} \tag{30}$$

Together Equations (29) and (30) show

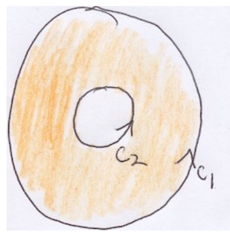
$$f(z) = \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$$

By Equation (26) we have shown that F is analytic and $F' = f$. $\square$

5.6 Extensions of Cauchy's theorem

Cauchy's theorem requires that the function $f(z)$ be analytic on a simply connected region. In cases where it is not, we can extend it in a useful way.

Suppose R is the region between the two simple closed curves C_1 and C_2. Note, both C_1 and C_2 are oriented in a counterclockwise direction.



Theorem 5.14. (Extended Cauchy's theorem) If $f(z)$ is analytic on R then

$$\int_{C_1 - C_2} f(z)\, dz = 0.$$

Proof. The proof is based on the following figure. We 'cut' both C_1 and C_2 and connect them by two copies of C_3, one in each direction. (In the figure we have drawn the two copies of C_3 as separate curves, in reality they are the same curve traversed in opposite directions.)

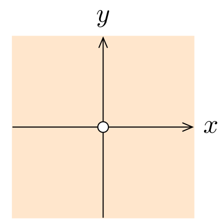
With C_3 acting as a cut, the region enclosed by $C_1 + C_3 - C_2 - C_3$ is simply connected, so Cauchy's Theorem 5.13 applies. We get

$$\int_{C_1 + C_3 - C_2 - C_3} f(z)\, dz = 0$$

The contributions of C_3 and $-C_3$ cancel, which leaves $\int_{C_1 - C_2} f(z)\, dz = 0$. $\square$

Note. This clearly implies $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

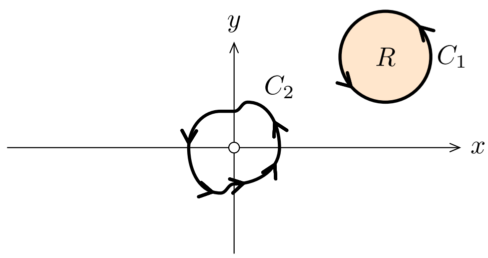
Example 5.15. Let $f(z) = 1/z$. $f(z)$ is defined and analytic on the punctured plane.



Punctured plane: $\mathbf{C} - \{0\}$

What values can $\displaystyle\int_C f(z)\,dz$ take for C a simple closed curve (positively oriented) in the plane?

Solution: We have two cases (i) C_1 not around 0, and (ii), C_2 around 0



Case (i): Cauchy's theorem applies directly because the interior does not contain the problem point at the origin. Thus,

$$\int_{C_1} f(z)\,dz = 0$$

Case (ii): we will show that

$$\int_{C_2} f(z)\,dz = 2\pi i$$

Let C_3 be a small circle of radius a centered at 0 and entirely inside C_2.

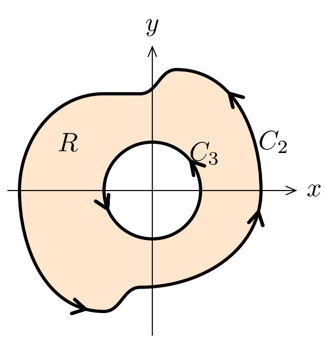


Figure for part (ii)

By the extended Cauchy theorem we have

$$\int_{C_2} f(z)\,dz = \int_{C_3} f(z)\,dz.$$

Using the usual parametrization of a circle we can easily compute that the line integral is

$$\int_{C_2} f(z)\,dz = \int_{C_3} f(z)\,dz = \int_0^{2\pi} i\,dt = 2\pi i$$

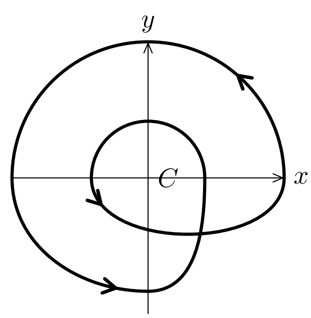
We can extend this answer in the following way:

If C is *not* simple, then the possible values of

$$\int_C f(z)\,dz$$

are $2\pi n i$, where n is the number of times C goes (counterclockwise) around the origin 0.

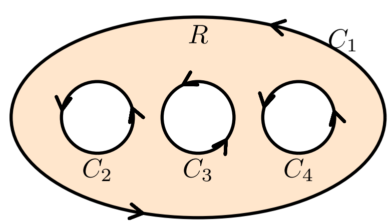
Definition. n is called the *winding number* of C around 0. n also equals the number of times C crosses the positive x-axis, counting $+1$ for crossing from below and -1 for crossing from above.



A curve with winding number 2 around the origin.

Example 5.16. A further extension: using the same trick of cutting the region by curves to make it simply connected we can show that if f is analytic in the region R shown below then

$$\int_{C_1 - C_2 - C_3 - C_4} f(z)\,dz = 0.$$



That is, $C_1 - C_2 - C_3 - C_4$ is the boundary of the region R.

Orientation. It is important to get the orientation of the curves correct. One way to do this is to make sure that the region R is always to the left as you traverse the curve. In the above example. The region is to the right as you traverse C_2, C_3 or C_4 in the direction indicated. This is why we put a minus sign on each when describing the boundary.

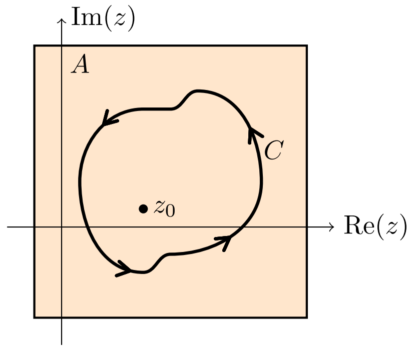
6 Cauchy's integral formula

Cauchy's theorem is a big theorem which we will use almost daily from here on out. Right away it will reveal a number of interesting and useful properties of analytic functions. More will follow as the course progresses. If you learn just one theorem this week it should be Cauchy's integral formula! We start with a statement of the theorem for functions. After some examples, we'll give a generalization to all derivatives of a function. After some more examples we will prove the theorems. After that we will see some remarkable consequences that follow fairly directly from the Cauchy's formula.

6.1 Cauchy's integral for functions

Theorem 6.1. (Cauchy's integral formula) Suppose C is a simple closed curve and the function $f(z)$ is analytic on a region containing C and its interior. We assume C is oriented counterclockwise. Then for any z_0 inside C:

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz} \tag{31}$$



Cauchy's integral formula: simple closed curve C, $f(z)$ analytic on and inside C.

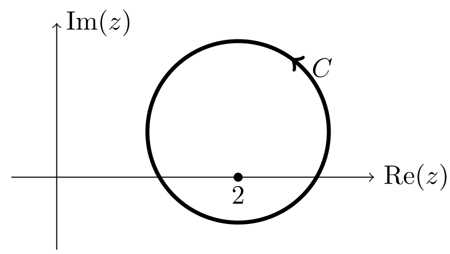
This is remarkable: it says that knowing the values of f on the boundary curve C means we know everything about f inside C!! This is probably unlike anything you've encountered with functions of real variables.

Aside 1. With a slight change of notation (z becomes w and z_0 becomes z) we often write the formula as

$$\boxed{f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\, dw} \tag{32}$$

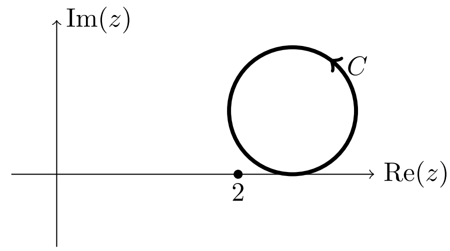
Aside 2. We're not being entirely fair to functions of real variables. We will see that for $f = u + iv$ the real and imaginary parts u and v have many similar remarkable properties. u and v are called conjugate harmonic functions.

Example 6.2. Compute $\displaystyle\int_c \frac{e^{z^2}}{z-2}\, dz$, where C is the curve shown.



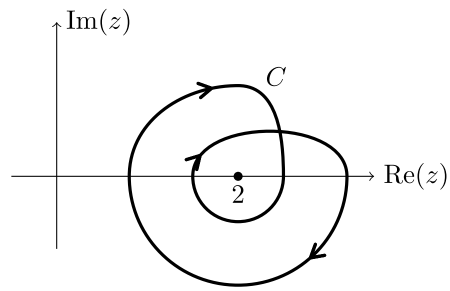
Solution: Let $f(z) = \mathrm{e}^{z^2}$. $f(z)$ is entire. Since C is a simple closed curve (counterclockwise) and $z = 2$ is inside C, Cauchy's integral formula says that the integral is $2\pi i f(2) = 2\pi i \mathrm{e}^4$.

Example 6.3. Do the same integral as the previous example with C the curve shown.

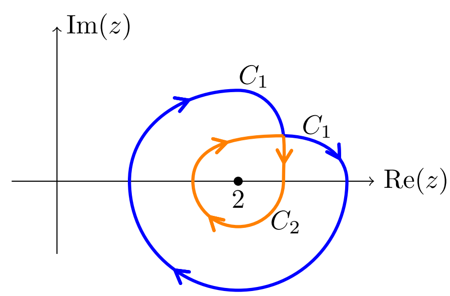


Solution: Since $f(z) = \mathrm{e}^{z^2}/(z-2)$ is analytic on and inside C, Cauchy's theorem says that the integral is 0.

Example 6.4. Do the same integral as the previous examples with C the curve shown.



Solution: This one is trickier. Let $f(z) = \mathrm{e}^{z^2}$. The curve C goes around 2 twice in the *clockwise* direction, so we break C into $C_1 + C_2$ as shown in the next figure.



These are both simple closed curves, so we can apply the Cauchy integral formula to each separately. (The negative signs are because they go clockwise around $z = 2$.)

$$\int_C \frac{f(z)}{z-2}\,dz = \int_{C_1} \frac{f(z)}{z-2}\,dz + \int_{C_2} \frac{f(z)}{z-2}\,dz = -2\pi i f(2) - 2\pi i f(2) = -4\pi i f(2)$$

6.2 Cauchy's integral formula for derivatives

Cauchy's integral formula is worth repeating several times. So, now we give it for all derivatives $f^{(n)}(z)$ of f. This will include the formula for functions as a special case.

Theorem 6.5. Cauchy's integral formula for derivatives. If $f(z)$ and C satisfy the same hypotheses as for Cauchy's integral formula then, for all z inside C we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\,dw, \quad n = 0, 1, 2, \ldots \tag{33}$$

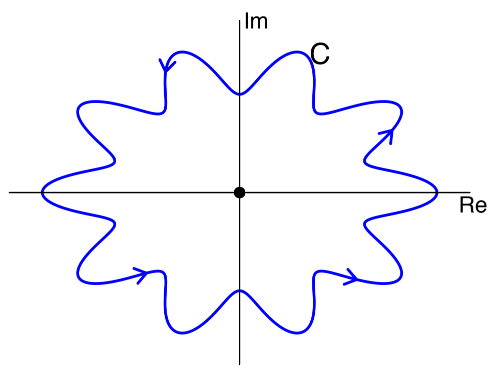
where, C is a simple closed curve, oriented counterclockwise, z is inside C and $f(w)$ is analytic on and inside C.

Example 6.6. Evaluate $\quad I = \displaystyle\int_C \frac{e^{2z}}{z^4}\,dz \quad$ where $C : |z| = 1$.

Solution: With Cauchy's formula for derivatives this is easy. Let $f(z) = e^{2z}$. Then,

$$I = \int_C \frac{f(z)}{z^4}\,dz = \frac{2\pi i}{3!} f'''(0) = \frac{8}{3}\pi i.$$

Example 6.7. Now Let C be the contour shown below and evaluate the same integral as in the previous example.



Solution: Again this is easy: the integral is the same as the previous example, i.e. $I = \dfrac{8}{3}\pi i$.

6.2.1 Another approach to some basic examples

Suppose C is a simple closed curve around 0. We have seen that

$$\int_C \frac{1}{z}\,dz = 2\pi i$$

The Cauchy integral formula gives the same result. That is, let $f(z) = 1$, then the formula says

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}\,dz = f(0) = 1.$$

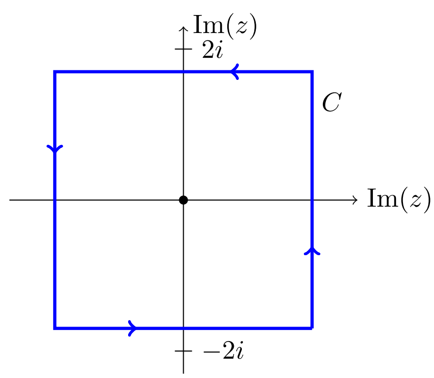
Likewise Cauchy's formula for derivatives shows

$$\int_C \frac{1}{(z)^n}\,dz = \int_C \frac{f(z)}{z^{n+1}}\,dz = f^{(n)}(0) = 0, \tag{34}$$

for integers $n > 1$

6.2.2 More examples

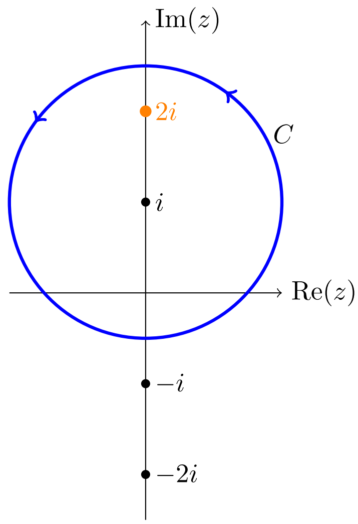
Example 6.8. Compute $\displaystyle\int_C \frac{\cos(z)}{z(z^2+8)}\,dz$ over the contour shown.



Solution: Let $f(z) = \cos(z)/(z^2+8)$. $f(z)$ is analytic on and inside the curve C. That is, the roots of z^2+8 are outside the curve. So, we rewrite the integral as

$$\int_C \frac{\cos(z)/(z^2+8)}{z}\,dz = \int_C \frac{f(z)}{z}\,dz = 2\pi i f(0) = 2\pi i\,\frac{1}{8} = \frac{\pi i}{4}$$

Example 6.9. Compute $\displaystyle\int_C \frac{1}{(z^2+4)^2}\,dz$ over the contour shown.



Solution: We factor the denominator as

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$$

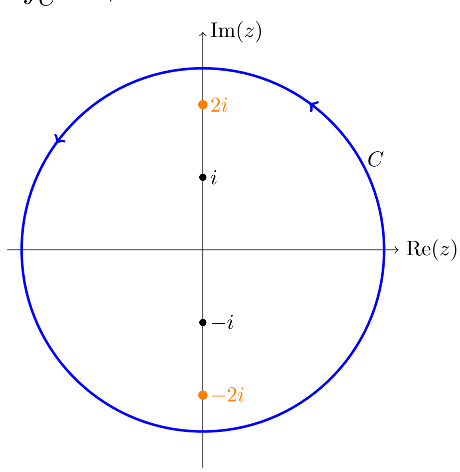
Let

$$f(z) = \frac{1}{(z+2i)^2}$$

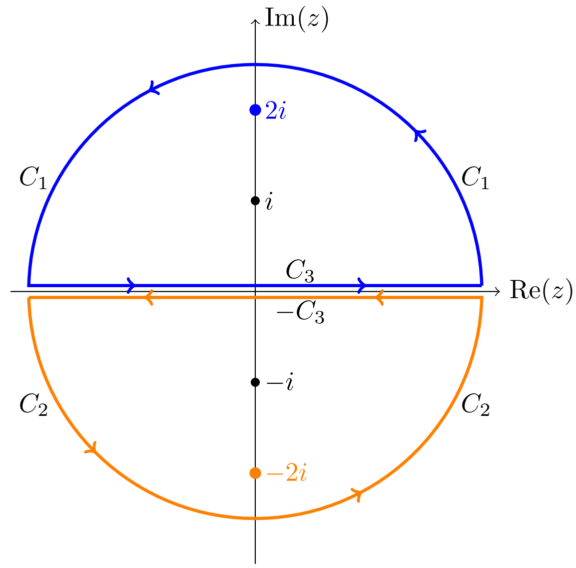
Clearly $f(z)$ is analytic inside C. So, by Cauchy's formula for derivatives:

$$\int_C \frac{1}{(z^2+4)^2}\,dz = \int_C \frac{f(z)}{(z-2i)^2} = 2\pi i f'(2i) = 2\pi i \left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{4\pi i}{64i} = \frac{\pi}{16}$$

Example 6.10. Compute $\displaystyle\int_C \frac{z}{z^2+4}\,dz$ over the curve C shown below.



Solution: The integrand has singularities at $\pm 2i$ and the curve C encloses them both. The solution to the previous solution won't work because we can't find an appropriate $f(z)$ that is analytic on the whole interior of C. Our solution is to split the curve into two pieces. Notice that C_3 is traversed both forward and backward.



Split the original curve C into 2 pieces that each surround just one singularity.

We have

$$\frac{z}{z^2+4} = \frac{z}{(z-2i)(z+2i)}$$

We let

$$f_1(z) = \frac{z}{(z+2i)} \qquad\qquad f_2(z) = \frac{z}{(z-2i)}$$

So

$$\frac{z}{z^2+4} = \frac{f_1(z)}{z-2i} = \frac{f_2(z)}{z+2i}$$

The integral, can be written out as

$$\int_C \frac{z}{z^2+4}\,dz = \int_{C_1+C_3-C_3+C_2} \frac{z}{z^2+4}\,dz = \int_{C_1+C_3} \frac{f_1(z)}{z-2i}\,dz + \int_{C_2-C_3} \frac{f_2(z)}{z+2i}\,dz$$

Since f_1 is analytic inside the simple closed curve $C_1 + C_3$ and f_2 is analytic inside the simple closed curve $C_2 - C_3$, Cauchy's formula applies to both integrals. The total integral equals

$$2\pi i(f_1(2i) + f_2(-2i)) = 2\pi i(1/2 + 1/2) = 2\pi i.$$

Remarks. 1. We could also have done this problem using partial fractions

$$\frac{z}{(z-2i)(z+2i)} = \frac{A}{z-2i} + \frac{B}{z+2i}.$$

It will turn out that $A = f_1(2i)$ and $B = f_2(-2i)$. It is easy to apply the Cauchy integral formula to both terms.

2. **Important note.** In an upcoming topic we will formulate the Cauchy residue theorem. This will allow us to compute the integrals in Examples 6.8-6.10 in an easier and less ad hoc manner.

6.2.3 The triangle inequality for integrals

We discussed the triangle inequality in the Topic 1 notes. It says that

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$

with equality if and only if z_1 and z_2 lie on the same ray from the origin.

Since an integral is basically a sum, this translates to the triangle inequality for integrals. We'll state it in two ways that will be useful to us.

Theorem 6.11. (Triangle inequality for integrals I) Suppose $g(t)$ is a complex valued function of a real variable, defined on $a \leq t \leq b$. Then

$$\left| \int_a^b g(t)\, dt \right| \leq \int_a^b |g(t))|\, dt,$$

with equality if and only if the values of $g(t)$ all lie on the same ray from the origin.

Proof. This follows by approximating the integral as a Riemann sum.

$$\left| \int_a^b g(t)\, dt \right| \approx \left| \sum g(t_k) \Delta t \right| \leq \sum |g(t_k)| \Delta t \approx \int_a^b |g(t)|\, dt.$$

The middle inequality is just the standard triangle inequality for sums of complex numbers. □

Theorem 6.12. (Triangle inequality for integrals II) For any function $f(z)$ and any curve γ, we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz|.$$

Here $dz = \gamma'(t)\, dt$ and $|dz| = |\gamma'(t)|\, dt$.

Proof. This follows immediately from the previous theorem:

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, dt \right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt = \int_\gamma |f(z)|\, |dz|.$$

Corollary. If $|f(z)| < M$ on C then

$$\left| \int_C f(z)\, dz \right| \leq M \cdot (\text{length of } C)$$

Proof. Let $\gamma(t)$, with $a \le t \le b$, be a parametrization of C. Using the triangle inequality

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz| = \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \le \int_a^b M|\gamma'(t)|\,dt = M \cdot (\text{length of } C).$$

Here we have used that

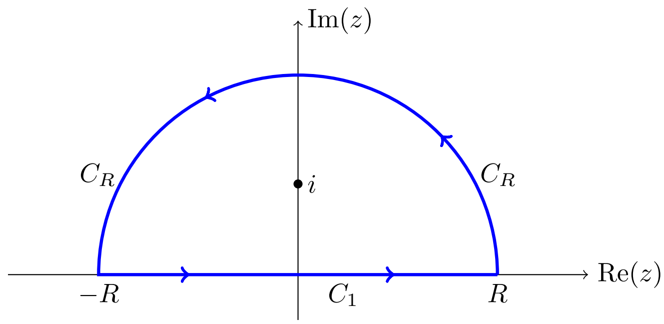
$$|\gamma'(t)|\,dt = \sqrt{(x')^2 + (y')^2}\,dt = ds,$$

the arclength element. □

Example 6.13. Compute the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$$

Solution: The trick is to integrate $f(z) = 1/(z^2+1)^2$ over the closed contour $C_1 + C_R$ shown, and then show that the contribution of C_R to this integral vanishes as R goes to ∞.



The only singularity of

$$f(z) = \frac{1}{(z+i)^2(z-i)^2}$$

inside the contour is at $z = i$. Let

$$g(z) = \frac{1}{(z+i)^2}$$

Since g is analytic on and inside the contour, Cauchy's formula gives

$$\int_{C_1+C_R} f(z)\,dz = \int_{C_1+C_R} \frac{g(z)}{(z-i)^2}\,dz = 2\pi i g'(i) = 2\pi i \frac{-2}{(2i)^3} = \frac{\pi}{2}.$$

We parametrize C_1 by

$$\gamma(x) = x\ , \qquad\qquad -R \le x \le R$$

So

$$\int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)^2}\,dx.$$

This goes to I (the value we want to compute) as $R \to \infty$. Next, we parametrize C_R by

$$\gamma(\theta) = R\mathrm{e}^{i\theta}\ , \qquad\qquad 0 \le \theta \le \pi$$

So

$$\int_{C_R} f(z)\,dz = \int_0^\pi \frac{1}{(R^2 e^{2i\theta}+1)^2}\, iRe^{i\theta}\,d\theta$$

By the triangle inequality for integrals, if $R > 1$

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_0^\pi \left|\frac{1}{(R^2 e^{2i\theta}+1)^2}\, iRe^{i\theta}\right|\,d\theta. \tag{35}$$

From the triangle equality for complex numbers

$$R^2 = |R^2 e^{2i\theta}| = |R^2 e^{2i\theta} + 1 + (-1)| \le |R^2 e^{2i\theta} + 1| + |-1| = |R^2 e^{2i\theta} + 1| + 1$$

we get

$$|R^2 e^{2i\theta} + 1| \ge R^2 - 1$$

Thus

$$\frac{1}{|R^2 e^{2i\theta}+1|^2} \le \frac{1}{(R^2-1)^2}$$

Using Equation (35), we then have

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_0^\pi \left|\frac{1}{(R^2 e^{2i\theta}+1)^2}\, iRe^{i\theta}\right|\,d\theta \le \int_0^\pi \frac{R}{(R^2-1)^2}\,d\theta = \frac{\pi R}{(R^2-1)^2}$$

Clearly this goes to 0 as $R \to \infty$. Thus, the integral over the contour $C_1 + C_R$ goes to I as R gets large. But

$$\int_{C_1+C_R} f(z)\,dz = \pi/2$$

for all $R > 1$. We can therefore conclude that $I = \pi/2$.

As a sanity check, we note that our answer is real and positive as it needs to be.

6.3 Proof of Cauchy's integral formula

6.3.1 A useful theorem

Before proving the theorem we'll need a theorem that will be useful in its own right.

Theorem 6.14. (A second extension of Cauchy's theorem) Suppose that A is a simply connected region containing the point z_0. Suppose g is a function which is

1. Analytic on $A - \{z_0\}$
2. Continuous on A. (In particular, g does not blow up at z_0.)

Then,

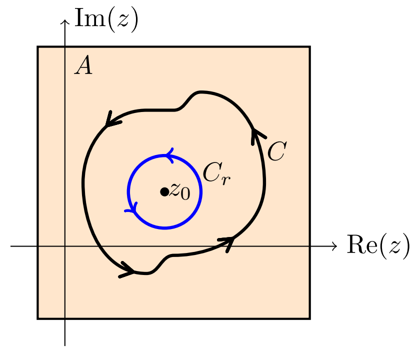
$$\int_C g(z)\,dz = 0$$

for all closed curves C in A.

Proof. The extended version of Cauchy's theorem tells us that

$$\int_C g(z)\,dz = \int_{C_r} g(z)\,dz,$$

where C_r is a circle of radius r around z_0.



Since $g(z)$ is continuous we know that $|g(z)|$ is bounded inside C_r. Say, $|g(z)| < M$. The corollary to the triangle inequality says that

$$\left|\int_{C_r} g(z)\,dz\right| \le M\,(\text{length of } C_r) = M\,2\pi r.$$

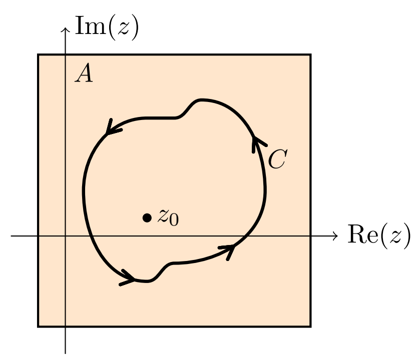
Since r can be as small as we want, this implies that

$$\int_{C_r} g(z)\,dz = 0$$

Note. Using this, we can show that $g(z)$ is, in fact, analytic at z_0. The proof will be the same as in our proof of Cauchy's theorem that $g(z)$ has an antiderivative.

6.3.2 Proof of Cauchy's integral formula

We reiterate Cauchy's integral formula from Equation (31): $\qquad f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z - z_0}\,dz$



Proof. (of Cauchy's integral formula) We use a trick that is useful enough to be worth remembering. Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

Since $f(z)$ is analytic on A, we know that $g(z)$ is analytic on $A-\{z_0\}$. Since the derivative of f exists at z_0, we know that

$$\lim_{z\to z_0} g(z) = f'(z_0)$$

That is, if we define $g(z_0) = f'(z_0)$ then g is continuous at z_0. From the extension of Cauchy's theorem just above, we have

$$\int_C g(z)\,dz = 0, \qquad \text{i.e.} \qquad \int_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$$

Thus

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_C \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\int_C \frac{1}{z-z_0}\,dz = 2\pi i f(z_0).$$

The last equality follows from our, by now, well known integral of $1/(z-z_0)$ on a loop around z_0.

6.4 Proof of Cauchy's integral formula for derivatives

Recall that Cauchy's integral formula in Equation (33) says

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw, \quad n = 0, 1, 2, \ldots$$

First we'll offer a quick proof which captures the reason behind the formula, and then a formal proof.

Quick proof: We have an integral representation for $f(z)$, $z \in A$, we use that to find an integral representation for $f'(z)$, $z \in A$.

$$f'(z) = \frac{d}{dz}\left[\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw\right] = \frac{1}{2\pi i}\int_C \frac{d}{dz}\left(\frac{f(w)}{w-z}\right)dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$$

(Note, since $z \in A$ and $w \in C$, we know that $w - z \neq 0$) Thus,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$$

Now, by iterating this process, i.e. by mathematical induction, we can show the formula for higher order derivatives.

Formal proof: We do this by taking the limit of

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

using the integral representation of both terms:

$$f(z+\Delta z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z-\Delta z}\,dw\,, \qquad\qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$

Now, using a little algebraic manipulation we get

$$\begin{aligned}
\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{2\pi i\,\Delta z}\int_C \frac{f(w)}{w-z-\Delta z}-\frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi i\,\Delta z}\int_C \frac{f(w)\Delta z}{(w-z-\Delta z)(w-z)}\,dw \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2-\Delta z(w-z)}\,dw
\end{aligned}$$

Letting Δz go to 0, we get Cauchy's formula for $f'(z)$

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$$

There is no problem taking the limit under the integral sign because everything is continuous and the denominator is never 0. □

6.5 Consequences of Cauchy's integral formula

6.5.1 Existence of derivatives

Theorem. Suppose $f(z)$ is analytic on a region A. Then, f has derivatives of all order.

Proof. This follows from Cauchy's integral formula for derivatives. That is, we have a formula for all the derivatives, so in particular the derivatives all exist. A little more precisely: for any point z in A we can put a small disk around z_0 that is entirely contained in A. Let C be the boundary of the disk, then Cauchy's formula gives a formula for all the derivatives $f^{(n)}(z_0)$ in terms of integrals over C. In particular, those derivatives exist. □

Remark. If you look at the proof of Cauchy's formula for derivatives you'll see that f having derivatives of all orders boils down to $1/(w-z)$ having derivatives of all orders for w on a curve not containing z.

Important remark. We have at times assumed that for $f=u+iv$ analytic, u and v have continuous higher order partial derivatives. This theorem confirms that fact. In particular, $u_{xy}=u_{yx}$, etc.

6.5.2 Cauchy's inequality

Theorem 6.15. (Cauchy's inequality) Let C_R be the circle $|z-z_0|=R$. Assume that $f(z)$ is analytic on C_R and its interior, i.e. on the disk $|z-z_0|\le R$. Finally let $M_R=\max|f(z)|$ over z on C_R. Then

$$|f^{(n)}(z_0)|\le\frac{n!M_R}{R^n}\,,\qquad n=1,2,3,\ldots \tag{36}$$

Proof. Using Cauchy's integral formula for derivatives (Equation 33) we have

$$|f^{(n)}(z_0)|\le\frac{n!}{2\pi}\int_{C_R}\frac{|f(w)|}{|w-z_0|^{n+1}}\,|dw|\le\frac{n!}{2\pi}\frac{M_R}{R^{n+1}}\int_{C_R}|dw|=\frac{n!}{2\pi}\frac{M_R}{R^{n+1}}\cdot 2\pi R$$

6.5.3 Liouville's theorem

Theorem 6.16. (Liouville's theorem) Assume $f(z)$ is entire and suppose it is bounded in the complex plane, namely $|f(z)| < M$ for all $z \in \mathbf{C}$ then $f(z)$ is constant.

Proof. For any circle of radius R around z_0 the Cauchy inequality says $|f'(z_0)| \leq \dfrac{M}{R}$. But, R can be as large as we like so we conclude that $|f'(z_0)| = 0$ for every $z_0 \in \mathbf{C}$. Since the derivative is 0, the function itself is constant.

In short:

$$\boxed{\text{If } f \text{ is entire and bounded then } f \text{ is constant.}}$$

Note. $P(z) = a_n z^n + \ldots + a_0$, $\sin(z)$, e^z are all entire but **not bounded**.

Now, practically for free, we get the fundamental theorem of algebra.

Corollary. (Fundamental theorem of algebra) Any polynomial P of degree $n \geq 1$, i.e.

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n,\ a_n \neq 0,$$

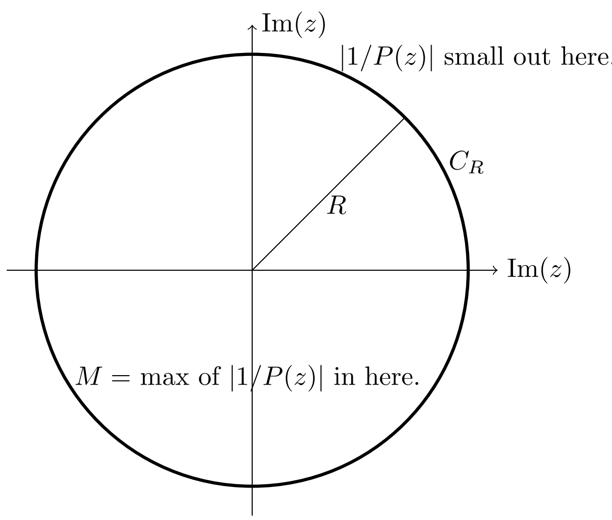
has exactly n roots.

Proof. There are two parts to the proof.

Hard part: Show that P has at least one root.

This is done by contradiction, together with Liouville's theorem. Suppose $P(z)$ **does not** have a zero. Then

1. $f(z) = 1/P(z)$ is entire. This is obvious because (by assumption) $P(z)$ has no zeros.
2. $f(z)$ is bounded. This follows because $1/P(z)$ goes to 0 as $|z|$ goes to ∞.



(It is clear that $|1/P(z)|$ goes to 0 as z goes to infinity, i.e. $|1/P(z)|$ is small outside a large circle. So $|1/P(z)|$ is bounded by M.)

So, by Liouville's theorem $f(z)$ is constant, and therefore $P(z)$ must be constant as well. But this a contradiction, so P must have a zero.

Easy part: P has exactly n zeros. Let z_0 be one zero. We can factor $P(z) = (z-z_0)Q(z)$. $Q(z)$ has degree $n-1$. If $n-1>0$, then we can apply the result to $Q(z)$. We can continue this process until the degree of Q is 0.

6.5.4 Maximum modulus principle

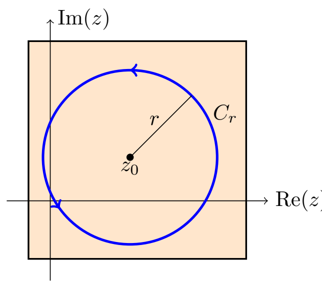
Briefly, the maximum modulus principle states that if f is analytic and not constant in a domain A then $|f(z)|$ has no relative maximum in A and the absolute maximum of $|f|$ occurs on the boundary of A.

In order to prove the maximum modulus principle we will first prove the mean value property. This will give you a good feel for the maximum modulus principle. It is also important and interesting in its own right.

Theorem 6.17. (Mean value property) Suppose $f(z)$ is analytic on the closed disk of radius r centered at z_0, i.e. the set $|z-z_0| \le r$. Then,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \tag{37}$$

Proof. This is an application of Cauchy's integral formula on the disk $D_r = |z-z_0| \le r$.



We can parametrize C_r, the boundary of D_r, as

$$\gamma(t) = z_0 + re^{i\theta}, \text{ with } 0 \le \theta \le 2\pi, \quad \text{so } \gamma'(\theta) = ire^{i\theta}.$$

By Cauchy's formula we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

This proves the property. □

In words, the mean value property says $f(z_0)$ is the arithmetic mean of the values on the circle.

We can now prove maximum modulus principle.

Theorem 6.18. (Maximum modulus principle) Suppose $f(z)$ is analytic in a connected region A and z_0 is a point in A.

1. If $|f|$ has a relative maximum at z_0 then $f(z)$ is constant in a neighborhood of z_0.

2. If A is bounded and connected, and f is continuous on A and its boundary, then either f is constant or the absolute maximum of $|f|$ occurs only on the boundary of A.

Proof. Part (1): The argument for is a little fussy. We will use the mean value property and the triangle inequality from Theorem 6.11.

Since z_0 is a relative maximum of $|f|$, for every small enough circle $C : |z - z_0| = r$ around z_0 we have $|f(z)| \leq |f(z_0)|$ for z on C. Therefore, by the mean value property and the triangle inequality

$$\begin{aligned}
|f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})\,d\theta\right| && \text{(mean value property)} \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + r\mathrm{e}^{i\theta})|\,d\theta && \text{(triangle inequality)} \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta && (|f(z_0 + r\mathrm{e}^{i\theta})| \leq |f(z_0)|) \\
&= |f(z_0)|
\end{aligned}$$

Since the beginning and end of the above are both $|f(z_0)|$ all the inequalities in the chain must be equalities.

The first inequality can only be an equality if for all θ, $f(z_0 + r\mathrm{e}^{i\theta})$ lie on the same ray from the origin, i.e. have the same argument or are 0.

The second inequality can only be an equality if all $|f(z_0 + r\mathrm{e}^{i\theta})| = |f(z_0)|$. So we have all $f(z_0 + r\mathrm{e}^{i\theta})$ have the same magnitude and the same argumeny. This implies they are all the same.

Finally, if $f(z)$ is constant along the circle and $f(z_0)$ is the average of $f(z)$ over the circle then $f(z) = f(z_0)$, i.e. f is constant on a small disk around z_0.

Part (2): The assumptions that A is bounded and f is continuous on A and its boundary serve to guarantee that $|f|$ has an absolute maximum (on A combined with its boundary). Part (1) guarantees that the absolute maximum can not lie in the interior of the region A unless f is constant. (This requires a bit more argument. Do you see why?) If the absolute maximum is not in the interior it must be on the boundary.□

Example 6.19. Find the maximum modulus of e^z on the unit square with $0 \leq x, y \leq 1$.
Solution:

$$|\mathrm{e}^{x+iy}| = \mathrm{e}^x$$

so the maximum is when $x = 1$, $0 \leq y \leq 1$ is arbitrary. This is indeed on the boundary of the unit square

Example 6.20. Find the maximum modulus for $\sin(z)$ on the square $[0, 2\pi] \times [0, 2\pi]$.
Solution: We use the formula

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y$$

So

$$\begin{aligned}
|\sin(z)|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}$$

We know the maximum over x of $\sin^2(x)$ is at $x = \pi/2$ and $x = 3\pi/2$. The maximum of $\sinh^2 y$ is at $y = 2\pi$. So maximum modulus is

$$\sqrt{1 + \sinh^2(2\pi)} = \sqrt{\cosh^2(2\pi)} = \cosh(2\pi)$$

This occurs at the points

$$z = x + iy = \frac{\pi}{2} + 2\pi i\ , \qquad z = \frac{3\pi}{2} + 2\pi i$$

Both these points are on the boundary of the region.

Example 6.21. Suppose $f(z)$ is entire. Show that if $\lim_{z\to\infty} f(z) = 0$ then $f(z) \equiv 0$.

Solution: This is a standard use of the maximum modulus principle. The strategy is to show that the maximum of $|f(z)|$ is not on the boundary (of the appropriately chosen region), so $f(z)$ must be constant.

Fix z_0. For $R > |z_0|$ let M_R be the maximum of $|f(z)|$ on the circle $|z| = R$. The maximum modulus theorem says that $|f(z_0)| < M_R$. Since $f(z)$ goes to 0, as R goes to infinity, we must have M_R also goes to 0. This means $|f(z_0)| = 0$. Since this is true for any z_0, we have $f(z) \equiv 0$.

Example 6.22. Here is an example of why you need A to be bounded in the maximum modulus theorem. Let A be the upper half-plane

$$\mathrm{Im}(z) > 0$$

So the boundary of A is the real axis.

Let $f(z) = \mathrm{e}^{-iz}$. We have

$$|f(x)| = |\mathrm{e}^{-ix}| = 1$$

for x along the real axis. Since $|f(2i)| = |\mathrm{e}^2| > 1$, we see $|f|$ cannot take its maximum along the boundary of A.

Of course, it can't take its maximum in the interior of A either. What happens here is that $f(z)$ doesn't have a maximum modulus. Indeed $|f(z)|$ goes to infinity along the positive imaginary axis.

7 Introduction to harmonic functions

Harmonic functions appear regularly and play a fundamental role in math, physics and engineering. In this topic we'll learn the definition, some key properties and their tight connection to complex analysis. The key connection to 18.04 is that both the real and imaginary parts of analytic functions are harmonic. We will see that this is a simple consequence of the Cauchy-Riemann equations. In the next topic we will look at some applications to hydrodynamics.

7.1 Harmonic functions

We start by defining harmonic functions and looking at some of their properties.

Definition 7.1. A function $u(x, y)$ is called harmonic if it is twice continuously differentiable and satisfies the following partial differential equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0. \tag{38}$$

Equation (38) is called Laplace's equation. So a function is harmonic if it satisfies Laplace's equation. The operator ∇^2 is called the Laplacian and $\nabla^2 u$ is called the Laplacian of u.

7.2 Del notation

Here's a quick reminder on the use of the notation $\boldsymbol{\nabla}$. For a function $u(x, y)$ and a vector field $\mathbf{F}(x, y) = (u, v)$, we have

(i) $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ in Cartesian coordinates

(ii) $\operatorname{grad} u = \boldsymbol{\nabla} u = (u_x, u_y)$

(iii) $\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = (v_x - u_y)$

(iv) $\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = u_x + v_y$

(v) $\operatorname{div} \operatorname{grad} u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \nabla^2 u = u_{xx} + u_{yy}$

(vi) $\operatorname{curl} \operatorname{grad} u = \boldsymbol{\nabla} \times \boldsymbol{\nabla}\, u = 0$

7.2.1 Analytic functions have harmonic pieces

The connection between analytic and harmonic functions is very strong. In many respects it mirrors the connection between e^z and sine and cosine.

Let $z = x + iy$ and write $f(z) = u(x, y) + iv(x, y)$.

Theorem 7.2. If $f(z) = u(x, y) + iv(x, y)$ is analytic on a region A then both u and v are harmonic functions on A.

Proof. This is a simple consequence of the Cauchy-Riemann equations. Since $u_x = v_y$ we have

$$u_{xx} = v_{yx}$$

Likewise, $u_y = -v_x$ implies

$$u_{yy} = -v_{xy}$$

Since $v_{xy} = v_{yx}$ we have

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

Therefore u is harmonic. We can handle v similarly. □

Note. Since we know an analytic function is infinitely differentiable we know u and v have the required two continuous partial derivatives. This also ensures that the mixed partials agree, i.e. $v_{xy} = v_{yx}$.

To complete the tight connection between analytic and harmonic functions we show that any harmonic function is the real part of an analytic function.

Theorem 7.3. If $u(x, y)$ is harmonic on a simply connected region A, then u is the real part of an analytic function $f(z) = u(x, y) + iv(x, y)$.

Proof. This is similar to our proof that an analytic function has an antiderivative. First we come up with a candidate for $f(z)$ and then show it has the properties we need. Here are the details broken down into steps 1-4.

1. Find a candidate, call it $g(z)$, for $f'(z)$:

   If we had an analytic f with $f = u+iv$, then Cauchy-Riemann says that $f' = u_x - iu_y$. So, let's define

   $$g = u_x - iu_y$$

   This is our candidate for f'.

2. Show that $g(z)$ is analytic:

   Write $g = \phi + i\psi$, where $\phi = u_x$ and $\psi = -u_y$. Checking the Cauchy-Riemann equations we have

   $$\begin{bmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{bmatrix} = \begin{bmatrix} u_{xx} & u_{xy} \\ -u_{yx} & -u_{yy} \end{bmatrix}$$

   Since u is harmonic we know $u_{xx} = -u_{yy}$, so $\phi_x = \psi_y$. It is clear that $\phi_y = -\psi_x$. Thus g satisfies the Cauchy-Riemann equations, so it is analytic.

3. Let f be an antiderivative of g:

   Since A is simply connected our statement of Cauchy's theorem guarantees that $g(z)$ has an antiderivative in A. We'll need to fuss a little to get the constant of integration exactly right. So, pick a base point z_0 in A. Define the antiderivative of $g(z)$ by

   $$f(z) = \int_{z_0}^{z} g(z)\, dz + u(x_0, y_0).$$

   (Again, by Cauchy's theorem this integral can be along any path in A from z_0 to z.)

4. Show that the real part of f is u.

   Let's write $f = U + iV$. So, $f'(z) = U_x - iU_y$. By construction

   $$f'(z) = g(z) = u_x - iu_y$$

   This means the first partials of U and u are the same, so U and u differ by at most a constant. However, also by construction,

   $$f(z_0) = u(x_0, y_0) = U(x_0, y_0) + iV(x_0, y_0)$$

   So, $U(x_0, y_0) = u(x_0, y_0)$ (and $V(x_0, y_0) = 0$). Since they agree at one point we must have $U = u$, i.e. the real part of f is u as we wanted to prove.

Important corollary. u is infinitely differentiable.

Proof. By definition we only require a harmonic function u to have continuous second partials. Since the analytic f is infinitely differentiable, we have shown that so is u!

7.2.2 Harmonic conjugates

Definition. If u and v are the real and imaginary parts of an analytic function, then we say u and v are harmonic conjugates.

Note. If $f(z) = u + iv$ is analytic then so is $if(z) = -v + iu$. So, u and v are harmonic conjugates and so are u and $-v$.

7.3 A second proof that u and v are harmonic

This fact is important enough that we will give a second proof using Cauchy's integral formula. One benefit of this proof is that it reminds us that Cauchy's integral formula can transfer a general question on analytic functions to a question about the function $1/z$. We start with an easy to derive fact.

Fact. The real and imaginary parts of $f(z) = 1/z$ are harmonic away from the origin. Likewise for

$$g(z) = f(z-a) = \frac{1}{z-a}$$

away from the point $z = a$.

Proof. We have

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

It is a simple matter to apply the Laplacian and see that you get 0. We'll leave the algebra to you! The statement about $g(z)$ follows in either exactly the same way, or by noting that the Laplacian is translation invariant.

Second proof that f analytic implies u and v are harmonic. We are proving that if $f = u + iv$ is analytic then u and v are harmonic. So, suppose f is analytic at the point z_0. This means there is a disk of some radius, say r, around z_0 where f is analytic. Cauchy's formula says

$$f(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\,dw,$$

where C_r is the circle $|w - z_0| = r$ and z is in the disk $|z - z_0| < r$.

Now, since the real and imaginary parts of $1/(w - z)$ are harmonic, the same must be true of the integral, which is limit of linear combinations of such functions. Since the circle is finite and f is continuous, interchanging the order of integration and differentiation is not a problem.

7.4 Maximum principle and mean value property

These are similar to the corresponding properties of analytic functions. Indeed, we deduce them from those corresponding properties.

Theorem. (Mean value property) If u is a harmonic function then u satisfies the mean value property. That is, suppose u is harmonic on and inside a circle of radius r centered at $z_0 = x_0 + iy_0$ then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

Proof. Let $f = u + iv$ be an analytic function with u as its real part. The mean value property for f says

$$\begin{aligned} u(x_0, y_0) + iv(x_0, y_0) = f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta})\, d\theta \end{aligned}$$

Looking at the real parts of this equation proves the theorem.

Theorem. (Maximum principle) Suppose $u(x, y)$ is harmonic on a open region A.

(i) Suppose z_0 is in A. If u has a relative maximum or minimum at z_0 then u is constant on a disk centered at z_0.

(ii) If A is bounded and connected and u is continuous on the boundary of A then the absolute maximum and absolute minimum of u occur on the boundary.

Proof. The proof for maxima is identical to the one for the maximum modulus principle. The proof for minima comes by looking at the maxima of $-u$.

Note. For analytic functions we only talked about maxima because we had to use the modulus in order to have real values. Since $|-f| = |f|$ we couldn't use the trick of turning minima into maxima by using a minus sign.

7.5 Orthogonality of curves

An important property of harmonic conjugates u and v is that their level curves are orthogonal. We start by showing their gradients are orthogonal.

Lemma 7.4. Let $z = x + iy$ and suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic. Then the dot product of their gradients is 0, i.e.

$$\nabla u \cdot \nabla v = 0.$$

Proof. The proof is an easy application of the Cauchy-Riemann equations.

$$\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = v_y v_x - v_x v_y = 0$$

In the last step we used the Cauchy-Riemann equations to substitute v_y for u_x and $-v_x$ for u_y.$\square$

The lemma holds whether or not the gradients are 0. To guarantee that the level curves are smooth the next theorem requires that $f'(z) \neq 0$.

Theorem. Let $z = x + iy$ and suppose that

$$f(z) = u(x, y) + iv(x, y)$$

is analytic. If $f'(z) \neq 0$ then the level curve of u through (x, y) is orthogonal to the level curve v through (x, y).

Proof. The technical requirement that $f'(z) \neq 0$ is needed to be sure that the level curves are smooth. We need smoothness so that it even makes sense to ask if the curves are orthogonal. We'll discuss this below. Assuming the curves are smooth the proof of the theorem is trivial: We know from 18.02 that the gradient ∇u is orthogonal to the level curves of u and the same is true for ∇v and the level curves of v. Since, by Lemma 7.4, the gradients are orthogonal this implies the curves are orthogonal.

Finally, we show that $f'(z) \neq 0$ means the curves are smooth. First note that

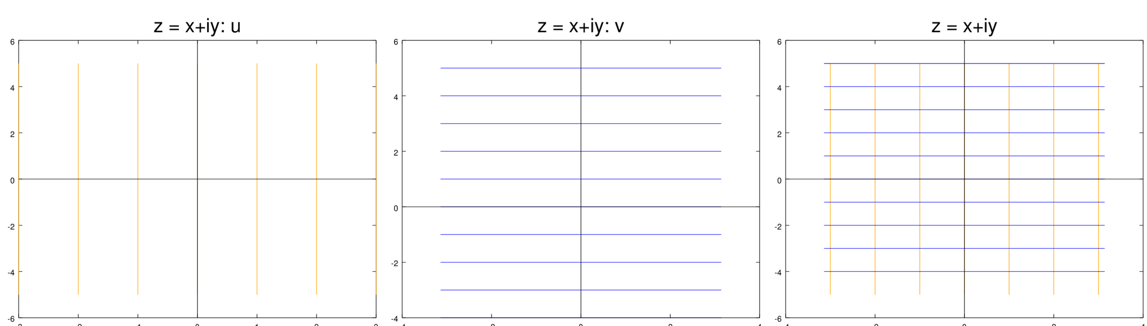
$$f'(z) = u_x(x, y) - iu_y(x, y) = v_y(x, y) + iv_x(x, y).$$

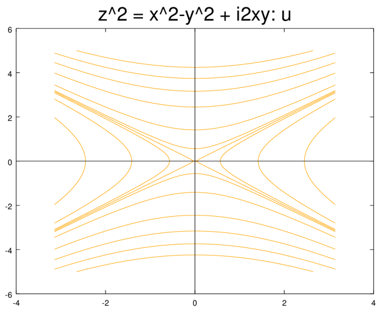
Now, since $f'(z) \neq 0$ we know that

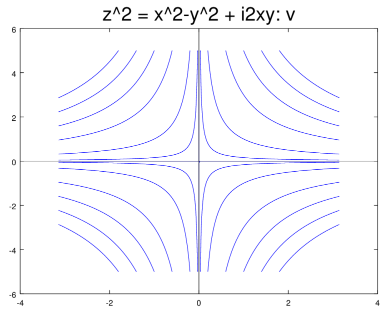
$$\nabla u = (u_x, u_y) \neq 0$$

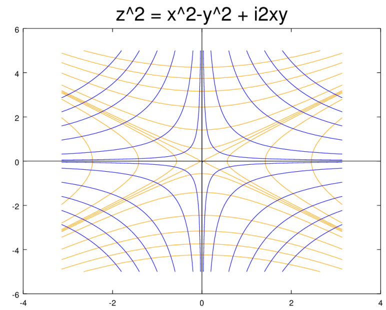
Likewise, $\nabla v \neq 0$. Thus, the gradients are not zero and the level curves must be smooth. $\square$

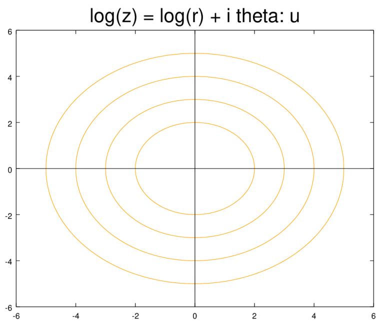
Example 7.5. The figures below show level curves of u and v for a number of functions. In all cases, the level curves of u are in orange and those of v are in blue. For each case we show the level curves separately and then overlayed on each other.

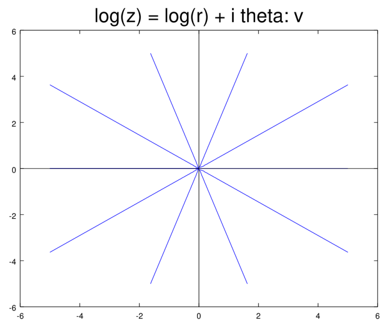


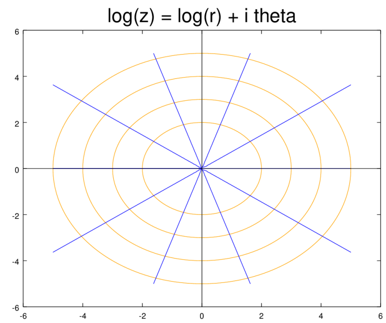

z^2 = x^2-y^2 + i2xy: u

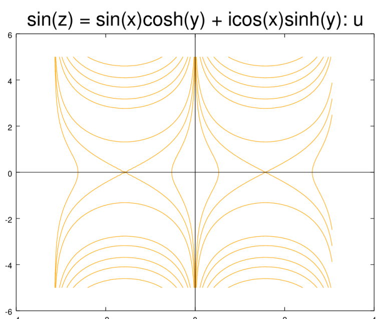

z^2 = x^2-y^2 + i2xy: v

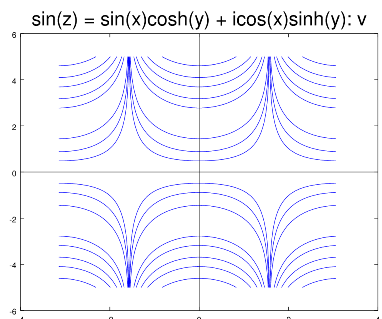

z^2 = x^2-y^2 + i2xy

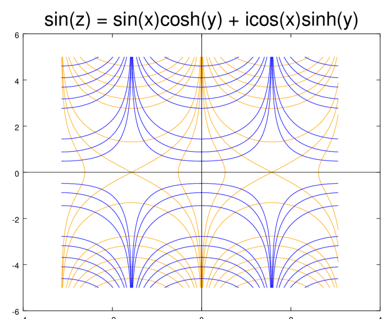

log(z) = log(r) + i theta: u

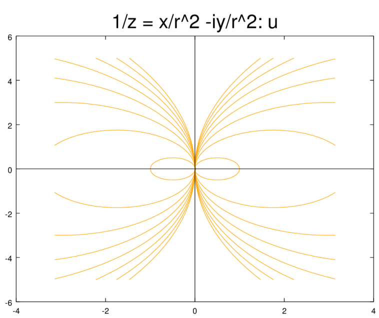

log(z) = log(r) + i theta: v

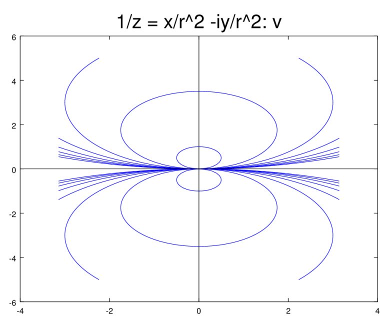

log(z) = log(r) + i theta

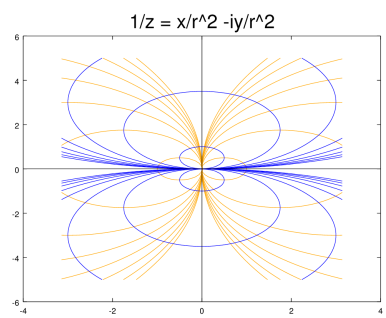

sin(z) = sin(x)cosh(y) + icos(x)sinh(y): u


sin(z) = sin(x)cosh(y) + icos(x)sinh(y): v

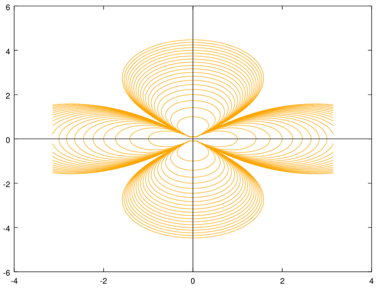

sin(z) = sin(x)cosh(y) + icos(x)sinh(y)


1/z = x/r^2 -iy/r^2: u

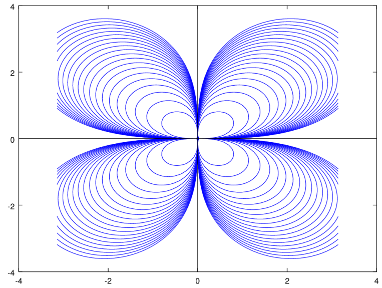

1/z = x/r^2 -iy/r^2: v


1/z = x/r^2 -iy/r^2

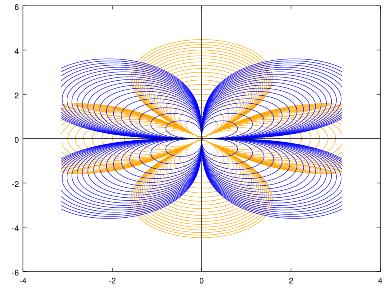
1/z^2: u



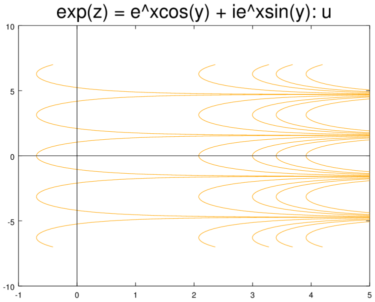
1/z^2: v



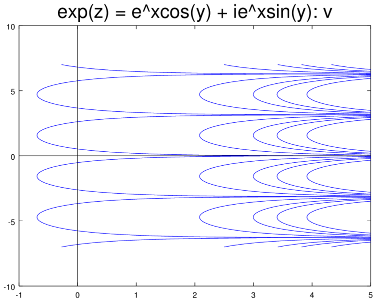
1/z^2



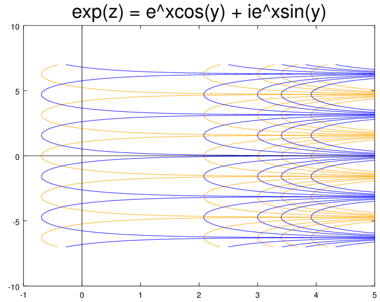
exp(z) = e^xcos(y) + ie^xsin(y): u



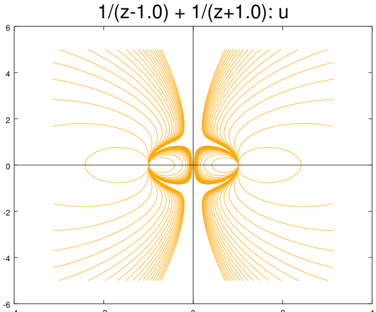
exp(z) = e^xcos(y) + ie^xsin(y): v



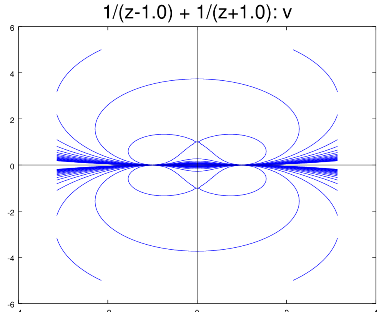
exp(z) = e^xcos(y) + ie^xsin(y)



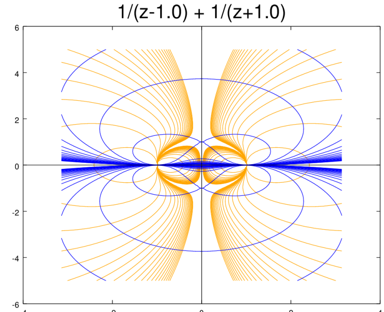
1/(z-1.0) + 1/(z+1.0): u



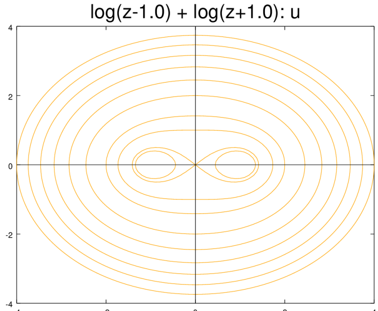
1/(z-1.0) + 1/(z+1.0): v



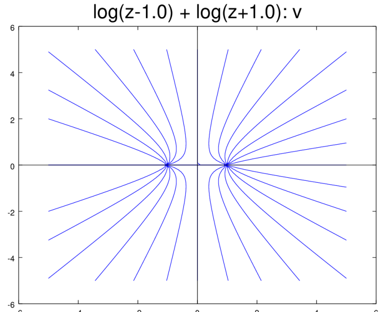
1/(z-1.0) + 1/(z+1.0)



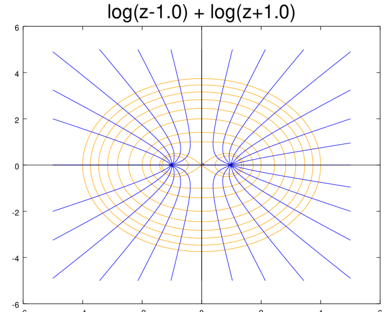
log(z-1.0) + log(z+1.0): u



log(z-1.0) + log(z+1.0): v



log(z-1.0) + log(z+1.0)



Example 7.6. Let's work out the gradients in a few simple examples.

(i) Let

$$f(z) = z^2 = (x^2 - y^2) + i2xy$$

So

$$\nabla u = (2x, -2y) \qquad \text{and} \qquad \nabla v = (2y, 2x).$$

It's trivial to check that $\nabla u \cdot \nabla v = 0$, so they are orthogonal.

(ii) Let

$$f(z) = \frac{1}{z} = \frac{x}{r^2} - i\frac{y}{r^2}$$

So, it's easy to compute

$$\nabla u = \left(\frac{y^2 - x^2}{r^4}, \frac{-2xy}{r^4}\right) \qquad \text{and} \qquad \nabla v = \left(\frac{2xy}{r^4}, \frac{y^2 - x^2}{r^4}\right).$$

Again it's trivial to check that $\nabla u \cdot \nabla v = 0$, so they are orthogonal.

Example 7.7. (Degenerate points: $f'(z) = 0$.) Consider

$$f(z) = z^2$$

From the previous example we have

$$u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy, \qquad \nabla u = (2x, -2y), \qquad \nabla v = (2y, 2x).$$

At $z = 0$, the gradients are both 0 so the theorem on orthogonality doesn't apply.

Let's look at the level curves through the origin. The level curve (really the 'level set') for

$$u = x^2 - y^2 = 0$$

is the pair of lines $y = \pm x$. At the origin this is not a smooth curve.

Look at the figures for z^2 above. It does appear that away from the origin the level curves of u intersect the lines where $v = 0$ at right angles. The same is true for the level curves of v and the lines where $u = 0$. You can see the degeneracy forming at the origin: as the level curves head towards 0 they get pointier and more right angled. So the level curve $u = 0$ is more properly thought of as four right angles. The level curve of $u = 0$, not knowing which leg of $v = 0$ to intersect orthogonally takes the average and comes into the origin at 45°.

8 Two dimensional hydrodynamics and complex potentials

Laplace's equation and harmonic functions show up in many physical models. As we have just seen, harmonic functions in two dimensions are closely linked with complex analytic functions. In this section we will exploit this connection to look at two dimensional hydrodynamics, i.e. fluid flow.

Since static electric fields and steady state temperature distributions are also harmonic, the ideas and pictures we use can be repurposed to cover these topics as well.

8.1 Velocity fields

Suppose we have water flowing in a region A of the plane. Then at every point (x, y) in A the water has a velocity. In general, this velocity will change with time. We'll let $\mathbf{F}$ stand for the velocity vector field and we can write

$$\mathbf{F}(x, y, t) = (u(x, y, t),\ v(x, y, t)).$$

The arguments (x, y, t) indicate that the velocity depends on these three variables. In general, we will shorten the name to velocity field.

A beautiful and mesmerizing example of a velocity field is at `http://hint.fm/wind/index.html`. This shows the current velocity of the wind at all points in the continental U.S.

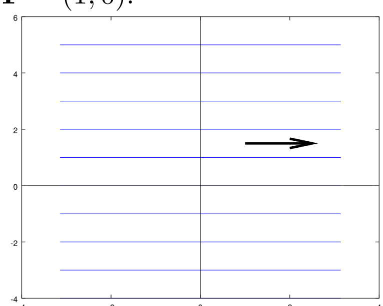
8.2 Stationary flows

If the velocity field is unchanging in time we call the flow a stationary flow. In this case, we can drop t as an argument and write:

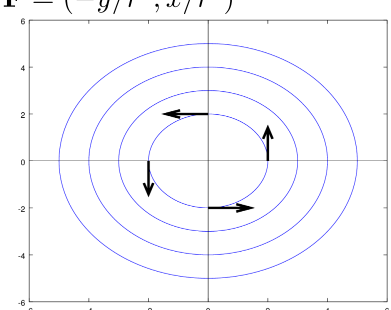
$$\mathbf{F}(x, y) = (u(x, y), v(x, y))$$

Here are a few examples. These pictures show the streamlines from similar figures in topic 5. We've added arrows to indicate the direction of flow.

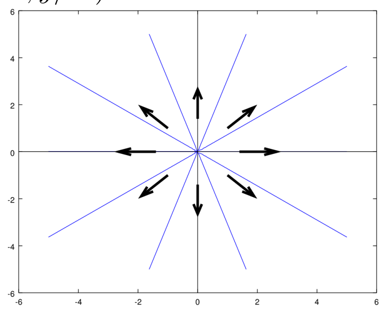
Example 8.1. Uniform flow. $\mathbf{F} = (1, 0)$.



Example 8.2. Eddy (vortex) $\mathbf{F} = (-y/r^2, x/r^2)$



Example 8.3. Source $\mathbf{F} = (x/r^2, y/r^2)$



8.3 Physical assumptions, mathematical consequences

This is a wordy section, so we'll start by listing the mathematical properties that will follow from our assumptions about the velocity field $\mathbf{F} = u + iv$.

(A) $\mathbf{F} = \mathbf{F}(x, y)$ is a function of x,y, but not time t (stationary).

(B) $\text{div}\,\mathbf{F} = 0$ (divergence free).

(C) $\text{curl}\,\mathbf{F} = 0$ (curl free).

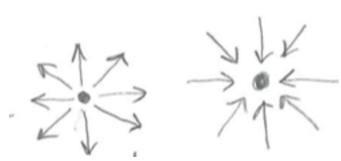
8.3.1 Physical assumptions

We will make some standard physical assumptions. These don't apply to all flows, but they do apply to a good number of them and they are a good starting point for understanding fluid flow more generally. More important to 18.04, these are the flows that are readily susceptible to complex analysis. Our three assumptions are that the flow is stationary , incompressible and irrotational. We have already discussed stationarity in Sec. 8.2, so let's still discuss the other two properties.

Incompressibility. We will assume throughout that the fluid is incompressible. This means that the density of the fluid is constant across the domain. Mathematically this says that the velocity field $\mathbf{F}$ must be divergence free, i.e. for $\mathbf{F} = (u, v)$:

$$\text{div}\mathbf{F} \equiv \boldsymbol{\nabla} \cdot \mathbf{F} = u_x + v_y = 0.$$

To understand this, recall that the divergence measures the infinitesimal flux of the field. If the flux is not zero at a point (x_0, y_0) then near that point the field looks like



Left: Divergent field: $\text{div}\mathbf{F} > 0$, right: Convergent field: $\text{div}\mathbf{F} < 0$

If the field is diverging or converging then the density must be changing! That is, the flow is not incompressible.

As a fluid flow the left hand picture represents a source and the right represents a sink. In electrostatics where $\mathbf{F}$ expresses the electric field, the left hand picture is the field of a positive charge density and the right is that of a negative charge density.

If you prefer a non-infinitesimal explanation, we can recall Green's theorem in flux form. It says that for a simple closed curve C and a field $\mathbf{F} = (u, v)$, differentiable on and inside C, the flux of $\mathbf{F}$ through C satisfies

$$\text{Flux of } \mathbf{F} \text{ across } C \;= \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_R \text{div}\mathbf{F}\, dx\, dy,$$

where R is the region inside C. Now, suppose that $\text{div}\mathbf{F}(x_0, y_0) > 0$, then $\text{div}\mathbf{F}(x, y) > 0$ for all (x, y) close to (x_0, y_0). So, choose a small curve C around (x_0, y_0) such that $\text{div}\mathbf{F} > 0$ on and inside C. By Green's theorem

$$\text{Flux of } \mathbf{F} \text{ through } C \;= \int\int_R \text{div}\mathbf{F}\, dx\, dy > 0.$$

Clearly, if there is a net flux out of the region the density is decreasing and the flow is not incompressible. The same argument would hold if $\text{div}\mathbf{F}(x_0, y_0) < 0$. We conclude that incompressible is equivalent to divergence free.

Irrotational flow. We will assume that the fluid is irrotational. This means that the there are no infinitesimal vortices in A. Mathematically this says that the velocity field $\mathbf{F}$ must be curl free, i.e. for $\mathbf{F} = (u, v)$:

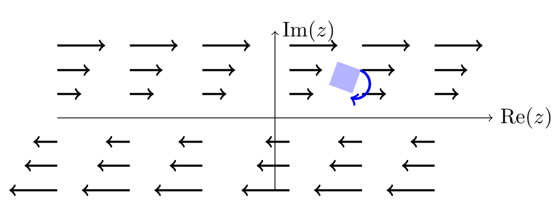
$$\text{curl}\mathbf{F} \equiv \boldsymbol{\nabla} \times \mathbf{F} = v_x - u_y = 0.$$

To understand this, recall that the curl measures the infinitesimal rotation of the field. Physically this means that a small paddle placed in the flow will not spin as it moves with the flow.

8.3.2 Examples

Example 8.4. The eddy is irrotational! The eddy from Example 8.2 is irrotational. The vortex at the origin is not in $A = \mathbf{C} - \{0\}$ and you can easily check that $\text{curl}\mathbf{F} = 0$ everywhere in A. This is not physically impossible: if you placed a small paddle wheel in the flow it would travel around the origin without spinning!

Example 8.5. Shearing flows are rotational. Here's an example of a vector field that has rotation, though not necessarily swirling.



Shearing flow: box turns because current is faster at the top.

The field $\mathbf{F} = (ay, 0)$ is horizontal, but $\text{curl}\mathbf{F} = -a \neq 0$. Because the top moves faster than the bottom it will rotate a square parcel of fluid. The minus sign tells you the parcel will rotate clockwise! This is called a shearing flow. The water at one level will be sheared away from the level above it.

8.3.3 Summary

(A) Stationary: $\mathbf{F}$ depends on x, y, but not t, i.e.,

$$\mathbf{F}(x, y) = (u(x, y), v(x, y))$$

(B) Incompressible: divergence free

$$\text{div}\mathbf{F} = u_x + v_y = 0$$

(C) Irrotational: curl free

$$\text{curl}\mathbf{F} = v_x - u_y = 0$$

For future reference we put the last two equalities in a numbered equation:

$$u_x = -v_y \qquad \text{and} \qquad u_y = v_x \tag{39}$$

These look almost like the Cauchy-Riemann equations (with sign differences)!

8.4 Complex potentials

There are different ways to do this. We'll start by seeing that every complex analytic function leads to an irrotational, incompressible flow. Then we'll go backwards and see that all such flows lead to an analytic function. We will learn to call the analytic function the complex potential of the flow.

Annoyingly, we are going to have to switch notation. Because u and v are already taken by the vector field $\mathbf{F}$, we will call our complex potential

$$\Phi = \phi + i\psi$$

8.4.1 Analytic functions give us incompressible, irrotational flows

Let $\Phi(z)$ be an analytic function on a region A. For $z = x + iy$ we write

$$\Phi(z) = \phi(x, y) + i\psi(x, y).$$

From this we can define a vector field

$$\mathbf{F} = \boldsymbol{\nabla}\phi = (\phi_x, \phi_y) =: (u, v),$$

here we mean that u and v are defined by ϕ_x and ϕ_y.

From our work on analytic and harmonic functions we can make a list of properties of these functions.

1. ϕ and ψ are both harmonic.
2. The level curves of ϕ and ψ are orthogonal.
3. $\Phi' = \phi_x - i\phi_y$.
4. $\mathbf{F}$ is divergence and curl free (proof just below). That is, the analytic function Φ has given us an incompressible, irrotational vector field $\mathbf{F}$.

It is standard terminology to call ϕ a potential function for the vector field $\mathbf{F}$. We will also call Φ a complex potential function for $\mathbf{F}$. The function ψ will be called the stream function of $\mathbf{F}$ (the name will be explained soon). The function Φ' will be called the complex velocity.

Proof. ($\mathbf{F}$ is curl and divergence free.) This is an easy consequence of the definition. We find

$$\text{curl}\mathbf{F} = v_x - u_y = \phi_{yx} - \phi_{xy} = 0$$

$$\text{div}\mathbf{F} = u_x + v_y = \phi_{xx} + \phi_{yy} = 0 \ \text{ (since } \phi \text{ is harmonic).}$$

We'll postpone examples until after deriving the complex potential from the flow.

8.4.2 Incompressible, irrotational flows always have complex potential functions

For technical reasons we need to add the assumption that A is simply connected. This is not usually a problem because we often work locally in a disk around a point (x_0, y_0).

Theorem. Assume $\mathbf{F} = (u, v)$ is an incompressible, irrotational field on a simply connected region A. Then there is an analytic function Φ which is a complex potential function for $\mathbf{F}$.

Proof. We have done all the heavy lifting for this in previous topics. The key is to use the property $\Phi' = u - iv$ to guess Φ'. Working carefully we define

$$g(z) = u - iv$$

Step 1: Show that g is analytic. Keeping the signs straight, the Cauchy Riemann equations are

$$u_x = (-v)_y \ \text{ and } u_y = -(-v)_x = v_x.$$

But, these are exactly the equations in Equation (39). Thus $g(z)$ is analytic.

Step 2: Since A is simply connected, Cauchy's theorem says that $g(z)$ has an antiderivative on A. We call the antiderivative $\Phi(z)$.

Step 3: Show that $\Phi(z)$ is a complex potential function for $\mathbf{F}$. This means we have to show that if we write $\Phi = \phi + i\psi$, then $\mathbf{F} = \nabla\phi$. To do this we just unwind the definitions.

$$\begin{aligned} \Phi' &= \phi_x - i\phi_y && \text{(standard formula for } \Phi'\text{)} \\ \Phi' &= g = u - iv && \text{(definition of } \Phi \text{ and } g\text{)} \end{aligned}$$

Comparing these equations we get

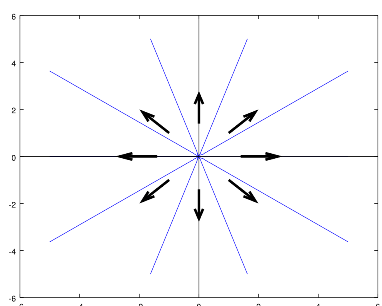
$$\phi_x = u, \qquad \phi_y = v.$$

But this says precisely that $\nabla\phi = \mathbf{F}$. $\square$

Example 8.6. Source fields. The vector field

$$\mathbf{F} = a\left(\frac{x}{r^2}, \frac{y}{r^2}\right)$$

models a source pushing out water or the 2D electric field of a positive charge at the origin. If you prefer a 3D model, it is the field of an infinite wire with uniform charge density along the z-axis. Show that $\mathbf{F}$ is curl-free and divergence-free and find its complex potential.



We could compute directly that this is curl-free and divergence-free away from 0. An alternative method is to look for a complex potential Φ. If we can find one then this will show $\mathbf{F}$ is curl and divergence free and find ϕ and ψ all at once. If there is no such Φ then we'll know that $\mathbf{F}$ is not both curl and divergence free.

One standard method is to use the formula for Φ':

$$\Phi' = u - iv = a\frac{(x - iy)}{r^2} = a\frac{\overline{z}}{(\overline{z}z)} = \frac{a}{z}.$$

This is analytic and we have

$$\Phi(z) = a\log(z)$$

8.5 Stream functions

In everything we did above poor old ψ just tagged along as the harmonic conjugate of the potential function ϕ. Let's turn our attention to it and see why it's called the stream function.

Theorem. Suppose that

$$\Phi = \phi + i\psi$$

is the complex potential for a velocity field $\mathbf{F}$. Then the fluid flows along the level curves of ψ. That is, the $\mathbf{F}$ is everywhere tangent to the level curves of ψ. The level curves of ψ are called streamlines and ψ is called the stream function.

Proof. Again we have already done most of the heavy lifting to prove this. Since $\mathbf{F}$ is the velocity of the flow at each point, the flow is always tangent to $\mathbf{F}$. You also need to remember that $\nabla\phi$ is perpendicular to the level curves of ϕ. So we have:

1. The flow is parallel to $\mathbf{F}$.

2. $\mathbf{F} = \nabla\phi$, so the flow is orthogonal to the level curves of ϕ.

3. Since ϕ and ψ are harmonic conjugates, the level curves of ψ are orthogonal to the level curves of ϕ.

Combining 2 and 3 we see that the flow must be along the level curves of ψ.

8.5.1 Examples

We'll illustrate the streamlines in a series of examples that start by defining the complex potential for a vector field.

Example 8.7. Uniform flow. Let

$$\Phi(z) = z.$$

Find $\mathbf{F}$ and draw a plot of the streamlines. Indicate the direction of the flow.

Solution: Write

$$\Phi = x + iy.$$

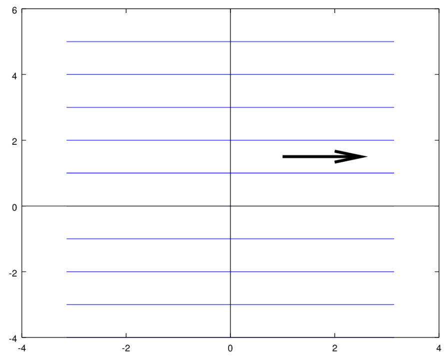
So

$$\phi = x\ , \qquad \mathbf{F} = \nabla\phi = (1, 0)$$

which says the flow has uniform velocity and points to the right. We also have

$$\psi = y,$$

so the streamlines are the horizontal lines $y =$ constant.



Uniform flow to the right.

Note that another way to see that the flow is to the right is to check the direction that the potential ϕ increases. The previous section showed pictures of this complex potential which show both the streamlines and the equipotential lines.

Example 8.8. Linear source. Let

$$\Phi(z) = \log(z).$$

Find $\mathbf{F}$ and draw a plot of the streamlines. Indicate the direction of the flow.

Solution: Write

$$\Phi = \log(r) + i\theta$$

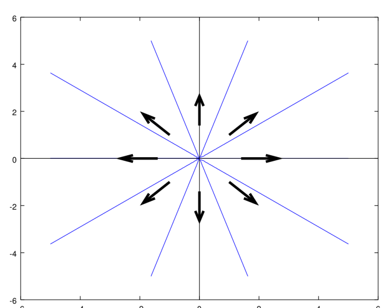
So

$$\phi = \log(r)\ , \qquad \mathbf{F} = \boldsymbol{\nabla}\phi = (x/r^2, y/r^2)$$

which says the flow is radial and decreases in speed as it gets farther from the origin. The field is not defined at $z = 0$. We also have

$$\psi = \theta$$

so the streamlines are rays from the origin.



Linear source: radial flow from the origin.

8.5.2 Stagnation points

A stagnation point is one where the velocity field is 0.

Stagnation points. If Φ is the complex potential for a field $\mathbf{F}$ then the stagnation points $\mathbf{F} = 0$ are exactly the points z where $\Phi'(z) = 0$.

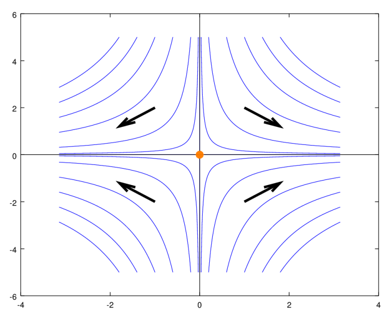
Proof. This is clear since $\mathbf{F} = (\phi_x, \phi_y)$ and $\Phi' = \phi_x - i\phi_y$.

Example 8.9. Stagnation points. Draw the streamlines and identify the stagnation points for the potential $\Phi(z) = z^2$.

Solution: (We drew the level curves for this in the previous section.) We have

$$\Phi = (x^2 - y^2) + i2xy$$

So the streamlines are the hyperbolas: $2xy =$ constant. Since $\phi = x^2 - y^2$ increases as $|x|$ increases and decreases as $|y|$ increases, the arrows, which point in the direction of increasing ϕ, are as shown on the figure below.



Stagnation flow: stagnation point at $z = 0$.

The stagnation points are the zeros of

$$\Phi'(z) = 2z$$

i.e. the only stagnation point is at the $z = 0$.

Note. The stagnation points are what we called the critical points of a vector field in 18.03.

8.6 More examples

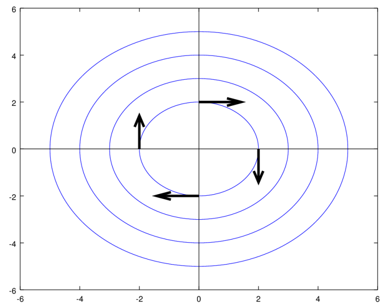
Example 8.10. Linear vortex. Analyze the flow with complex potential function

$$\Phi(z) = i\log(z).$$

Solution: Multiplying by i switches the real and imaginary parts of $\log(z)$ (with a sign change). We have

$$\Phi = -\theta + i\log(r)$$

The stream lines are the curves $\log(r) =$ constant, i.e. circles with center at $z = 0$. The flow is clockwise because the potential $\phi = -\theta$ increases in the clockwise direction.



Linear vortex.

This flow is called a linear vortex. We can find $\mathbf{F}$ using Φ'.

$$\Phi' = \frac{i}{z} = \frac{y}{r^2} + i\frac{x}{r^2} = \phi_x - i\phi_y.$$

So

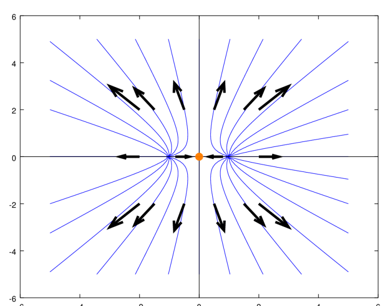
$$\mathbf{F} = (\phi_x, \phi_y) = (y/r^2, -x/r^2)$$

By now this should be a familiar vector field. There are no stagnation points, but there is a singularity at the origin.

Example 8.11. Double source. Analyze the flow with complex potential function

$$\Phi(z) = \log(z-1) + \log(z+1)$$

Solution: This is a flow with linear sources at ± 1. We used Octave to plot the level curves of $\psi = \mathrm{Im}(\Phi)$.



Two sources.

We can analyze this flow further as follows.

- Near each source the flow looks like a linear source.
- On the y-axis the flow is along the axis. That is, the y-axis is a streamline. It's worth seeing three different ways of arriving at this conclusion.

Reason 1: By symmetry of vector fields associated with each linear source, the x components cancel and the combined field points along the y-axis.

Reason 2: We can write

$$\Phi(z) = \log(z-1) + \log(z+1) = \log((z-1)(z+1)) = \log(z^2-1).$$

So

$$\Phi'(z) = \frac{2z}{z^2-1}$$

On the imaginary axis

$$\Phi'(iy) = \frac{2iy}{-y^2-1}$$

Thus,

$$\mathbf{F} = \left(0, \frac{2y}{y^2+1}\right)$$

which is along the axis.
Reason 3: On the imaginary axis $\Phi(iy) = \log(-y^2-1)$. Since this has constant imaginary part, the axis is a streamline.

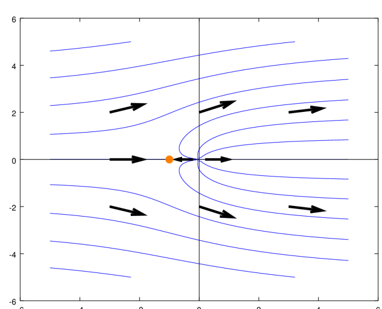
Because of the branch cut for $\log(z)$ we should probably be a little more careful here. First note that the vector field $\mathbf{F}$ comes from $\Phi' = 2z/(z^2-1)$, which doesn't have a branch cut. So we shouldn't really have a problem. Now, as z approaches the y-axis from one side or the other, the argument of $\log(z^2-1)$ approaches either π or $-\pi$. That is, as such limits, the imaginary part is constant. So the streamline on the y-axis is the limit case of streamlines near the axis.

Since $\Phi'(z) = 0$ when $z = 0$, the origin is a stagnation point. This is where the fields from the two sources exactly cancel each other.

Example 8.12. A source in uniform flow. Consider the flow with complex potential

$$\Phi(z) = z + \frac{Q}{2\pi}\log(z).$$

This is a combination of uniform flow to the right and a source at the origin. The figure below was drawn using Octave. It shows that the flow looks like a source near the origin. Farther away from the origin the flow stops being radial and is pushed to the right by the uniform flow.



A source in uniform flow.

Since the components of Φ' and $\mathbf{F}$ are the same except for signs, we can understand the flow by considering

$$\Phi'(z) = 1 + \frac{Q}{2\pi z}.$$

Near $z = 0$ the singularity of $1/z$ is most important and

$$\Phi' \approx Q/(2\pi z)$$

So, the vector field looks a linear source. Far away from the origin the $1/z$ term is small and $\Phi'(z) \approx 1$, so the field looks like uniform flow.

Setting $\Phi'(z) = 0$ we find one stagnation point

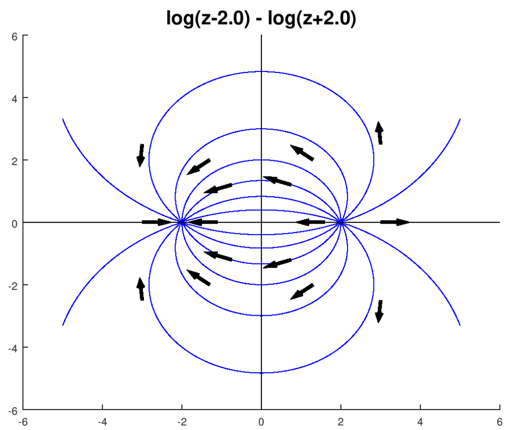
$$z = -Q/(2\pi)$$

It is the point on the x-axis where the flow from the source exactly balances that from the uniform flow. For bigger values of Q the source pushes fluid farther out before being overwhelmed by the uniform flow. That is why Q is called the source strength.

Example 8.13. Source + sink. Consider the flow with complex potential

$$\Phi(z) = \log(z - 2) - \log(z + 2).$$

This is a combination of source $(\log(z - 2))$ at $z = 2$ and a sink $(-\log(z + 2))$ at $z = -2$.



A source plus a sink.

9 Taylor and Laurent series

We originally defined an analytic function as one where the derivative, defined as a limit of ratios, existed. We went on to prove Cauchy's theorem and Cauchy's integral formula. These revealed some deep properties of analytic functions, e.g. the existence of derivatives of all orders.

Our goal in this topic is to express analytic functions as infinite power series. This will lead us to Taylor series. When a complex function has an isolated singularity at a point we will replace Taylor series by Laurent series. Not surprisingly we will derive these series from Cauchy's integral formula.

Although we come to power series representations after exploring other properties of analytic functions, they will be one of our main tools in understanding and computing with analytic functions.

9.1 Geometric series

Having a detailed understanding of geometric series will enable us to use Cauchy's integral formula to understand power series representations of analytic functions. We start with the definition:

Definition. A finite geometric series has one of the following (all equivalent) forms.

$$\begin{aligned} S_n &= a(1 + r + r^2 + r^3 + \ldots + r^n) \\ &= a + ar + ar^2 + ar^3 + \ldots + ar^n \\ &= \sum_{j=0}^{n} ar^j \\ &= a \sum_{j=0}^{n} r^j \end{aligned}$$

The number r is called the ratio of the geometric series because it is the ratio of consecutive terms of the series.

Theorem. The sum of a finite geometric series is given by

$$S_n = a(1 + r + r^2 + r^3 + \ldots + r^n) = \frac{a(1 - r^{n+1})}{1 - r}. \tag{40}$$

Proof. This is a standard trick that you've probably seen before.

$$\begin{array}{lllll} S_n & = a+ & ar + ar^2 + \ldots + ar^n & \\ rS_n & = & ar + ar^2 + \ldots + ar^n & +ar^{n+1} \end{array}$$

When we subtract these two equations most terms cancel and we get

$$S_n - rS_n = a - ar^{n+1}$$

Some simple algebra now gives us the formula in Equation (40).

Definition. An infinite geometric series has the same form as the finite geometric series except there is no last term:

$$S = a + ar + ar^2 + \ldots = a \sum_{j=0}^{\infty} r^j.$$

Note. We will usually simply say 'geometric series' instead of 'infinite geometric series'.

Theorem. If $|r| < 1$ then the infinite geometric series converges to

$$S = a \sum_{j=0}^{\infty} r^j = \frac{a}{1 - r} \tag{41}$$

If $|r| \geq 1$ then the series does not converge.

Proof. This is an easy consequence of the formula for the sum of a finite geometric series. Simply let $n \to \infty$ in Equation (40).

9.1.1 Connection to Cauchy's integral formula

Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw$$

Inside the integral we have the expression

$$\frac{1}{w - z}$$

which looks a lot like the sum of a geometric series. We will make frequent use of the following manipulations of this expression.

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1 - z/w} = \frac{1}{w}\left(1 + (z/w) + (z/w)^2 + \ldots\right) \tag{42}$$

The geometric series in this equation has ratio z/w. Therefore, the series converges, i.e. the formula is valid, whenever $|z/w| < 1$, or equivalently when

$$|z| < |w|$$

Similarly

$$\frac{1}{w-z} = -\frac{1}{z} \cdot \frac{1}{1 - w/z} = -\frac{1}{z}\left(1 + (w/z) + (w/z)^2 + \ldots\right) \tag{43}$$

The series converges, i.e. the formula is valid, whenever $|w/z| < 1$, or equivalently when

$$|z| > |w|$$

9.2 Convergence of power series

When we include powers of the variable z in the series we will call it a power series. In this section we'll state the main theorem we need about the convergence of power series.

Theorem 9.1. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

There is a number $R \geq 0$ such that:

1. If $R > 0$ then the series converges absolutely to an analytic function for $|z - z_0| < R$.

2. The series diverges for $|z - z_0| > R$. R is called the radius of convergence. The disk $|z - z_0| < R$ is called the disk of convergence.

3. The derivative is given by term-by-term differentiation

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

The series for f' also has radius of convergence R.

4. If γ is a bounded curve inside the disk of convergence then the integral is given by term-by-term integration

$$\int_\gamma f(z)\,dz = \sum_{n=0}^{\infty} \int_\gamma a_n(z-z_0)^n$$

Notes.

- The theorem doesn't say what happens when $|z - z_0| = R$.
- If $R = \infty$ the function $f(z)$ is entire.
- If $R = 0$ the series only converges at the point $z = z_0$. In this case, the series does not represent an analytic function on any disk around z_0.
- Often (not always) we can find R using the ratio test.

9.2.1 Ratio test and root test

Here are two standard tests from calculus on the convergence of infinite series.

Ratio test. Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_{n+1}/c_n|$ exists, then:

1. If $L < 1$ then the series converges absolutely.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ then the test gives no information.

Note. In words, L is the limit of the absolute ratios of consecutive terms.

Example 9.2. Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the absolute ratios of consecutive terms is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|}{|z^n|} = |z|$$

Thus, the ratio test agrees that the geometric series converges when $|z| < 1$. We know this converges to $1/(1-z)$. Note, the disk of convergence ends exactly at the singularity $z = 1$.

Example 9.3. Consider the series $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. The limit from the ratio test is

$$L = \lim_{n\to\infty} \frac{|z^{n+1}|/(n+1)!}{|z^n|/n!} = \lim \frac{|z|}{n+1} = 0.$$

Since $L < 1$ this series converges for every z. Thus, by Theorem 9.1, the radius of convergence for this series is ∞. That is, $f(z)$ is entire. Of course we know that $f(z) = \mathrm{e}^z$.

Root test. Consider the series $\sum_0^\infty c_n$. If $L = \lim_{n\to\infty} |c_n|^{1/n}$ exists, then:

1. If $L < 1$ then the series converges absolutely.

2. If $L > 1$ then the series diverges.

3. If $L = 1$ then the test gives no information .

Note. In words, L is the limit of the nth roots of the (absolute value) of the terms.

The geometric series is so fundamental that we should check the root test on it.

Example 9.4. Consider the geometric series $1 + z + z^2 + z^3 + \ldots$. The limit of the nth roots of the terms is

$$L = \lim_{n\to\infty} |z^n|^{1/n} = \lim |z| = |z|$$

Happily, the root test agrees that the geometric series converges when $|z| < 1$.

9.3 Taylor series

The previous section showed that a power series converges to an analytic function inside its disk of convergence. Taylor's theorem completes the story by giving the converse: around each point of analyticity an analytic function equals a convergent power series.

Theorem 9.5. (Taylor's theorem) Suppose $f(z)$ is an analytic function in a region A. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in A. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \tag{44}$$

(Where γ is any simple closed curve in A around z_0, with its interior entirely in A.)

We call the series the power series representing f around z_0.

The proof will be given below. First we look at some consequences of Taylor's theorem.

Corollary. The power series representing an analytic function around a point z_0 is unique. That is, the coefficients are uniquely determined by the function $f(z)$.

Proof. Taylor's theorem gives a formula for the coefficients.

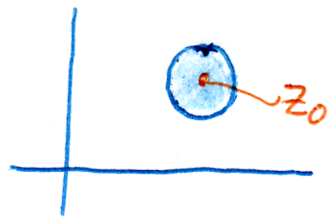
9.3.1 Order of a zero

Theorem. Suppose $f(z)$ is analytic on the disk $|z - z_0| < r$ and f is not identically 0. Then there is an integer $k \geq 0$ such that $a_k \neq 0$ and f has Taylor series around z_0 given by

$$f(z) = (z - z_0)^k (a_k + a_{k+1}(z - z_0) + \ldots) = (z - z_0)^k \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}. \tag{45}$$

Proof. Since $f(z)$ is not identically 0, not all the Taylor coefficients are zero. So, we take k to be the index of the first nonzero coefficient.

Theorem 9.6. Zeros are isolated. If $f(z)$ is analytic and not identically zero then the zeros of f are isolated (by isolated we mean that we can draw a small disk around any zeros that doesn't contain any other zeros).



Isolated zero at z_0: $f(z_0) = 0$, $f(z) \neq 0$ elsewhere in the disk.

Proof. Suppose $f(z_0) = 0$. Write f as in Equation (45). There are two factors:

$$(z - z_0)^k$$

and

$$g(z) = a_k + a_{k+1}(z - z_0) + \ldots$$

Clearly $(z - z_0)^k \neq 0$ if $z \neq z_0$. We have $g(z_0) = a_k \neq 0$, so $g(z)$ is not 0 on some small neighborhood of z_0. We conclude that on this neighborhood the product is only zero when $z = z_0$, i.e. z_0 is an isolated 0.

Definition. The integer k in Theorem 9.6 is called the order of the zero of f at z_0.

Note, if $f(z_0) \neq 0$ then z_0 is a zero of order 0.

9.3.2 Taylor series examples

The uniqueness of Taylor series along with the fact that they converge on any disk around z_0 where the function is analytic allows us to use lots of computational tricks to find the series and be sure that it converges.

Example 9.7. Use the formula for the coefficients in terms of derivatives to give the Taylor series of $f(z) = e^z$ around $z = 0$.

Solution: Since $f'(z) = e^z$, we have $f^{(n)}(0) = e^0 = 1$. So,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example 9.8. Expand $f(z) = z^8 e^{3z}$ in a Taylor series around $z = 0$.

Solution: Let $w = 3z$. So,

$$e^{3z} = e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{k=0}^{\infty} \frac{3^n}{n!} z^n$$

Thus,

$$f(z) = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+8}.$$

Example 9.9. Find the Taylor series of $\sin(z)$ around $z = 0$ (Sometimes the Taylor series around 0 is called the Maclaurin series.)

Solution: We give two methods for doing this.

Method 1.

$$f^{(n)}(0) = \frac{d^n \sin(z)}{dz^n} = \begin{cases} (-1)^m & \text{for } n = 2m+1 = \text{odd}, \ m = 0, 1, 2, \ldots \\ 0 & \text{for } n \text{ even} \end{cases}$$

Method 2. Using

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

we have

$$\sin(z) = \frac{1}{2i} \left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right] = \frac{1}{2i} \sum_{n=0}^{\infty} [(1 - (-1)^n)] \frac{i^n z^n}{n!}$$

(We need absolute convergence to add series like this.)

Conclusion:

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

which converges for $|z| < \infty$.

Example 9.10. Expand the rational function

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5}$$

around $z = 0$.

Solution: Note that f has a singularity at 0, so we can't expect a convergent Taylor series expansion. We'll aim for the next best thing using the following shortcut.

$$f(z) = \frac{1}{z^3} \frac{2(1 + z^2) - 1}{1 + z^2} = \frac{1}{z^3} \left[2 - \frac{1}{1 + z^2} \right].$$

Using the geometric series we have

$$\frac{1}{1 + z^2} = \frac{1}{1 - (-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = 1 - z^2 + z^4 - z^6 + \ldots$$

Putting it all together

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + \ldots\right) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$$

Note. The first terms are called the singular part, i.e. those with negative powers of z. The summation is called the regular or analytic part. Since the geometric series for $1/(1+z^2)$ converges for $|z| < 1$, the entire series is valid in $0 < |z| < 1$

Example 9.11. Find the Taylor series for

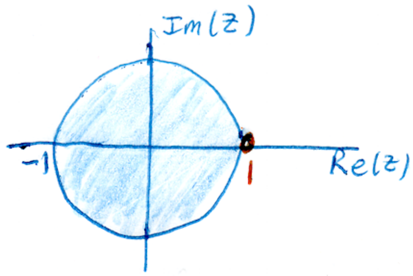
$$f(z) = \frac{e^z}{1-z}$$

around $z = 0$. Give the radius of convergence.

Solution: We start by writing the Taylor series for each of the factors and then multiply them out.

$$\begin{aligned} f(z) &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)\left(1 + z + z^2 + z^3 + \ldots\right) \\ &= 1 + (1+1)z + \left(1 + 1 + \frac{1}{2!}\right)z^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)z^3 + \ldots \end{aligned}$$

The biggest disk around $z = 0$ where f is analytic is $|z| < 1$. Therefore, by Taylor's theorem, the radius of convergence is $R = 1$.



$f(z)$ is analytic on $|z| < 1$ and has a singularity at $z = 1$.

Example 9.12. Find the Taylor series for

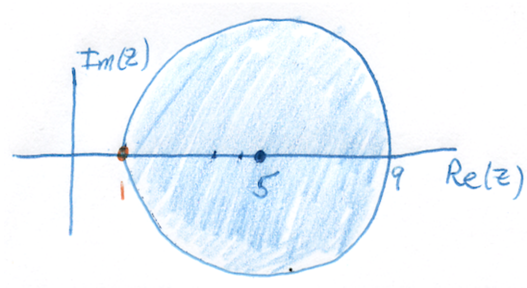
$$f(z) = \frac{1}{1-z}$$

around $z = 5$. Give the radius of convergence.

Solution: We have to manipulate this into standard geometric series form.

$$f(z) = \frac{1}{-4(1 + (z-5)/4)} = -\frac{1}{4}\left(1 - \left(\frac{z-5}{4}\right) + \left(\frac{z-5}{4}\right)^2 - \left(\frac{z-5}{4}\right)^3 + \ldots\right)$$

Since $f(z)$ has a singularity at $z = 1$ the radius of convergence is $R = 4$. We can also see this by considering the geometric series. The geometric series ratio is $(z-5)/4$. So the series converges when $|z-5|/4 < 1$, i.e. when $|z-5| < 4$, i.e. $R = 4$.



Disk of convergence stops at the singularity at $z = 1$.

Example 9.13. Find the Taylor series for

$$f(z) = \log(1 + z)$$

around $z = 0$. Give the radius of convergence.

Solution: We know that f is analytic for $|z| < 1$ and not analytic at $z = -1$. So, the radius of convergence is $R = 1$. To find the series representation we take the derivative and use the geometric series.

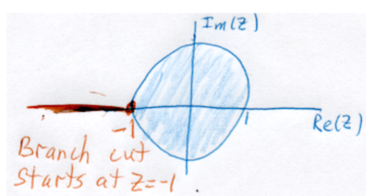
$$f'(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \ldots$$

Integrating term by term (allowed by Theorem 9.1) we have

$$f(z) = a_0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots = a_0 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

Here a_0 is the constant of integration. We find it by evalating at $z = 0$.

$$f(0) = a_0 = \log(1) = 0.$$



Disk of convergence for $\log(1 + z)$ around $z = 0$.

Example 9.14. Can the series

$$\sum a_n (z - 2)^n$$

converge at $z = 0$ and diverge at $z = 3$.

Solution: No! We have $z_0 = 2$. We know the series diverges everywhere outside its radius of convergence. So, if the series converges at $z = 0$, then the radius of convergence is at least 2. Since $|3 - z_0| < 2$ we would also have that $z = 3$ is inside the disk of convergence.

9.3.3 Proof of Taylor's theorem

For convenience we restate Taylor's Theorem 9.5.

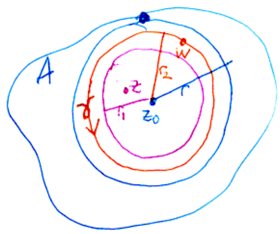
Taylor's theorem. (Taylor series) Suppose $f(z)$ is an analytic function in a region A. Let $z_0 \in A$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on any disk $|z - z_0| < r$ contained in A. Furthermore, we have formulas for the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (46)$$

Proof. In order to handle convergence issues we fix $0 < r_1 < r_2 < r$. We let γ be the circle $|w - z_0| = r_2$ (traversed counterclockise).



Disk of convergence extends to the boundary of A
$r_1 < r_2 < r$, but r_1 and r_2 can be arbitrarily close to r.

Take z inside the disk $|z - z_0| < r_1$. We want to express $f(z)$ as a power series around z_0. To do this we start with the Cauchy integral formula and then use the geometric series.

As preparation we note that for w on γ and $|z - z_0| < r_1$ we have

$$|z - z_0| < r_1 < r_2 = |w - z_0|$$

so

$$\frac{|z - z_0|}{|w - z_0|} < 1.$$

Therefore,

$$\frac{1}{w - z} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

Using this and the Cauchy formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}}\,(z-z_0)^n\,dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n \end{aligned}$$

The last equality follows from Cauchy's formula for derivatives. Taken together the last two equalities give Taylor's formula. □

9.4 Singularities

Definition. A function $f(z)$ is singular at a point z_0 if it is not analytic at z_0

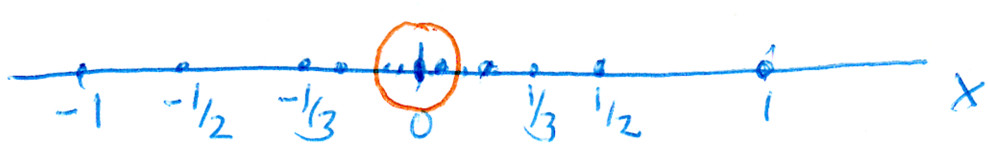
Definition. For a function $f(z)$, the singularity z_0 is an isolated singularity if f is analytic on the deleted disk $0 < |z-z_0| < r$ for some $r > 0$.

Example 9.15. $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has isolated singularities at $z = 0, \pm i$.

Example 9.16. $f(z) = \mathrm{e}^{1/z}$ has an isolated singularity at $z = 0$.

Example 9.17. $f(z) = \log(z)$ has a singularity at $z = 0$, but it is not isolated because a branch cut, starting at $z = 0$, is needed to have a region where f is analytic.

Example 9.18. $f(z) = \dfrac{1}{\sin(\pi/z)}$ has singularities at $z = 0$ and $z = 1/n$ for $n = \pm 1, \pm 2, \ldots$ The singularities at $\pm 1/n$ are isolated, but the one at $z = 0$ is not isolated.



Every neighborhood of 0 contains zeros at $1/n$ for large n.

9.5 Laurent series

Theorem 9.19. (Laurent series). Suppose that $f(z)$ is analytic on the annulus

$$A: \ r_1 < |z-z_0| < r_2$$

Then $f(z)$ can be expressed as a series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The coefficients have the formulus

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}} \, dw \\ b_n &= \frac{1}{2\pi i} \int_{\gamma} f(w)(w - z_0)^{n-1} \, dw, \end{aligned}$$

where γ is any circle $|w - z_0| = r$ inside the annulus, i.e. $r_1 < r < r_2$.

Furthermore

- The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to an analytic function for $|z - z_0| < r_2$.
- The series $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to an analytic function for $|z - z_0| > r_1$.
- Together, the series both converge on the annulus A where f is analytic.

The proof is given below. First we define a few terms.

Definition. The entire series is called the Laurent series for f around z_0. The series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called the analytic or regular part of the Laurent series. The series

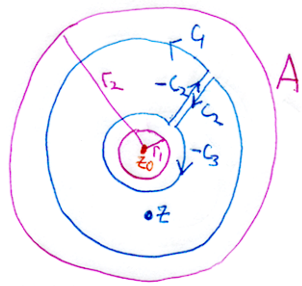
$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is called the singular or principal part of the Laurent series.

Note. Since $f(z)$ may not be analytic (or even defined) at z_0 we don't have any formulas for the coefficients using derivatives.

Proof. (Laurent series). Choose a point z in A. Now set circles C_1 and C_3 close enough to the boundary that z is inside $C_1 + C_2 - C_3 - C_2$ as shown. Since this curve and its interior are contained in A, Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i} \int_{C_1 + C_2 - C_3 - C_2} \frac{f(w)}{w - z} \, dw$$



The contour used for proving the formulas for Laurent series.

The integrals over C_2 cancel, so we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1 - C_3} \frac{f(w)}{w - z}\, dw.$$

Next, we divide this into two pieces and use our trick of converting to a geometric series. The calculuations are just like the proof of Taylor's theorem. On C_1 we have

$$\frac{|z - z_0|}{|w - z_0|} < 1$$

so

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw &= \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z_0} \cdot \frac{1}{\left(1 - \frac{z - z_0}{w - z_0}\right)}\, dw \\
&= \frac{1}{2\pi i} \int_{C_1} \sum_{n=0}^{\infty} \frac{f(w)}{(w - z_0)^{n+1}} (z - z_0)^n\, dw \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right) (z - z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z - z_0)^n
\end{aligned}$$

Here a_n is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on z is that $|z - z_0| < r_2$. So, this series converges for all such z.

Similarly on C_3 we have

$$\frac{|w - z_0|}{|z - z_0|} < 1$$

so

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_3}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\int_{C_3} -\frac{f(w)}{z-z_0}\cdot\frac{1}{\left(1-\frac{w-z_0}{z-z_0}\right)}\,dw \\
&= -\frac{1}{2\pi i}\int_{C_3}\sum_{n=0}^{\infty} f(w)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\,dw \\
&= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{C_1} f(w)(w-z_0)^n\,dw\right)(z-z_0)^{-n-1} \\
&= -\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}
\end{aligned}$$

In the last equality we changed the indexing to match the indexing in the statement of the theorem. Here b_n is defined by the integral formula given in the statement of the theorem. Examining the above argument we see that the only requirement on z is that $|z-z_0| > r_1$. So, this series converges for all such z.

Combining these two formulas we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1-C_3}\frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

The last thing to note is that the integrals defining a_n and b_n do not depend on the exact radius of the circle of integration. Any circle inside A will produce the same values. We have proved all the statements in the theorem on Laurent series. □

9.5.1 Examples of Laurent series

In general, the integral formulas are not a practical way of computing the Laurent coefficients. Instead we use various algebraic tricks. Even better, as we shall see, is the fact that often we don't really need all the coefficients and we will develop more techniques to compute those that we do need.

Example 9.20. Find the Laurent series for

$$f(z) = \frac{z+1}{z}$$

around $z_0 = 0$. Give the region where it is valid.

Solution: The answer is simply

$$f(z) = 1 + \frac{1}{z}$$

This is a Laurent series, valid on the infinite region $0 < |z| < \infty$.

Example 9.21. Find the Laurent series for

$$f(z) = \frac{z}{z^2+1}$$

around $z_0 = i$. Give the region where your answer is valid. Identify the singular (principal) part.

Solution: Using partial fractions we have

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{2} \cdot \frac{1}{z+i}$$

Since $\dfrac{1}{z+i}$ is analytic at $z = i$ it has a Taylor series expansion. We find it using geometric series:

$$\frac{1}{z+i} = \frac{1}{2i} \cdot \frac{1}{1 + (z-i)/(2i)} = \frac{1}{2i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

So the Laurent series is

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-i} + \frac{1}{4i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{2i}\right)^n$$

The singular (principal) part is given by the first term. The region of convergence is $0 < |z - i| < 2$.

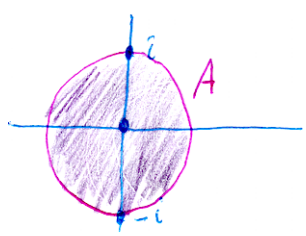
Note. We could have looked at $f(z)$ on the region $2 < |z - i| < \infty$. This would have produced a different Laurent series. We discuss this further in an upcoming example.

Example 9.22. Compute the Laurent series for

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

on the region $A:\ 0 < |z| < 1$ centered at $z = 0$.

Solution: This function has isolated singularities at $z = 0, \pm i$. Therefore it is analytic on the region A.



$f(z)$ has singularities at $z = 0, \pm i$.

At $z = 0$ we have

$$f(z) = \frac{1}{z^3}(1+z)(1 - z^2 + z^4 - z^6 + \ldots)$$

Multiplying this out we get

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} - 1 + z + z^2 - z^3 - \ldots.$$

The following example shows that the Laurent series depends on the region under consideration.

Example 9.23. Find the Laurent series around $z = 0$ for $f(z) = \dfrac{1}{z(z-1)}$ in each of the following regions:

- (i) the region $A_1 : 0 < |z| < 1$
- (ii) the region $A_2 : 1 < |z| < \infty$.

Solution: For (i)

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + \ldots) = -\frac{1}{z} - 1 - z - z^2 - \ldots$$

For (ii): Since the usual geometric series for $1/(1-z)$ does not converge on A_2 we need a different form,

$$f(z) = \frac{1}{z} \cdot \frac{1}{z(1 - 1/z)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Since $|1/z| < 1$ on A_2 our use of the geometric series is justified.

One lesson from this example is that the Laurent series depends on the region as well as the formula for the function.

9.6 Digression to differential equations

Here is a standard use of series for solving differential equations.

Example 9.24. Find a power series solution to the equation

$$f'(x) = f(x) + 2\ , \qquad f(0) = 0$$

Solution: We look for a solution of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the initial condition we find $f(0) = 0 = a_0$. Substituting the series into the differential equation we get

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^3 + \ldots = f(x) + 2 = a_0 + 2 + a_1 x + a_2 x^2 + \ldots$$

Equating coefficients and using $a_0 = 0$ we have

$$\begin{aligned}
a_1 &= a_0 + 2 \qquad &\Rightarrow \qquad & a_1 = 2 \\
2a_2 &= a_1 &\Rightarrow \qquad & a_2 = a_1/2 = 1 \\
3a_3 &= a_2 &\Rightarrow \qquad & a_3 = 1/3 \\
4a_4 &= a_3 &\Rightarrow \qquad & a_4 = 1/(3 \cdot 4)
\end{aligned}$$

In general

$$(n+1)a_{n+1} = a_n \qquad \Rightarrow \qquad a_{n+1} = \frac{a_n}{(n+1)} = \frac{1}{3 \cdot 4 \cdot 5 \cdots (n+1)}.$$

You can check using the ratio test that this function is entire.

9.7 Poles

Poles refer to isolated singularities. So, we suppose $f(z)$ is analytic on $0 < |z - z_0| < r$ and has Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

Definition of poles. If only a finite number of the coefficients b_n are nonzero we say z_0 is a finite pole of f. In this case, if $b_k \neq 0$ and $b_n = 0$ for all $n > k$ then we say z_0 is a pole of order k.

- If z_0 is a pole of order 1 we say it is a simple pole of f.
- If an infinite number of the b_n are nonzero we say that z_0 is an essential singularity or a pole of infinite order of f.
- If all the b_n are 0, then z_0 is called a removable singularity. That is, if we define $f(z_0) = a_0$ then f is analytic on the disk $|z - z_0| < r$.

The terminology can be a bit confusing. So, imagine that I tell you that f is defined and analytic on the punctured disk $0 < |z - z_0| < r$. Then, a priori we assume f has a singularity at z_0. But, if after computing the Laurent series we see there is no singular part we can extend the definition of f to the full disk, thereby 'removing the singularity'.

We can explain the term essential singularity as follows. If $f(z)$ has a pole of order k at z_0 then $(z - z_0)^k f(z)$ is analytic (has a removable singularity) at z_0. So, $f(z)$ itself is not much harder to work with than an analytic function. On the other hand, if z_0 is an essential singularity then no algebraic trick will change $f(z)$ into an analytic function at z_0.

9.7.1 Examples of poles

We'll go back through many of the examples from the previous sections.

Example 9.25. The rational function

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5}$$

expanded to

$$f(z) = \left(\frac{1}{z^3} + \frac{1}{z}\right) - \sum_{n=0}^{\infty} (-1)^n z^{2n+1}.$$

Thus, $z = 0$ is a pole of order 3.

Example 9.26. Consider

$$f(z) = \frac{z+1}{z} = 1 + \frac{1}{z}$$

Thus, $z = 0$ is a pole of order 1, i.e. a simple pole.

Example 9.27. Consider

$$f(z) = \frac{z}{z^2 + 1} = \frac{1}{2} \cdot \frac{1}{z - i} + g(z),$$

where $g(z)$ is analytic at $z = i$. So, $z = i$ is a simple pole.

Example 9.28. The function

$$f(z) = \frac{1}{z(z-1)}$$

has isolated singularities at $z = 0$ and $z = 1$. Show that both are simple poles.

Solution: In a neighborhood of $z = 0$ we can write

$$f(z) = \frac{g(z)}{z}\,, \qquad g(z) = \frac{1}{z-1}$$

Since $g(z)$ is analytic at 0, $z = 0$ is a finite pole. Since $g(0) \neq 0$, the pole has order 1, i.e. it is simple.

Likewise, in a neighborhood of $z = 1$,

$$f(z) = \frac{h(z)}{z-1}\,, \qquad h(z) = \frac{1}{z}$$

Since h is analytic at $z = 1$, f has a finite pole there. Since $h(1) \neq 0$ it is simple.

Example 9.29. Consider

$$\mathrm{e}^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

So, $z = 0$ is an essential singularity.

Example 9.30. $\log(z)$ has a singularity at $z = 0$. Since the singularity is not isolated, it can't be classified as either a pole or an essential singularity.

9.7.2 Residues

In preparation for discussing the residue theorem in the next topic we give the definition and an example here.

Note well, residues have to do with isolated singularites.

Definition 9.31. Consider the function $f(z)$ with an isolated singularity at z_0, i.e. defined on $0 < |z - z_0| < r$ and with Laurent series

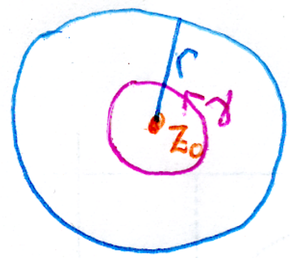
$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

The residue of f at z_0 is b_1. This is denoted

$$\mathrm{Res}(f, z_0) \quad \text{or} \quad \operatorname*{Res}_{z=z_0} f = b_1.$$

What is the importance of the residue? If γ is a small, simple closed curve that goes counterclockwise around z_0 then

$$\int_\gamma f(z) = 2\pi i b_1.$$



γ is small enough to be inside $|z - z_0| < r$, and surround z_0.

This is easy to see by integrating the Laurent series term by term. The only nonzero integral comes from the term b_1/z.

Example 9.32. The function

$$f(z) = e^{1/(2z)} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \ldots$$

has an isolated singularity at 0. From the Laurent series we see that

$$\text{Res}(f, 0) = \frac{1}{2}$$

10 Residue Theorem

10.1 Poles and zeros

We remind you of the following terminology: Suppose $f(z)$ is analytic at z_0 and

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \ldots$$

with $a_n \neq 0$. Then we say f has a zero of order n at z_0. If $n = 1$ we say z_0 is a simple zero.

Suppose f has an *isolated* singularity at z_0 and Laurent series

$$f(z) = \frac{b_n}{(z - z_0)^n} + \frac{b_{n-1}}{(z - z_0)^{n-1}} + \ldots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

which converges on $0 < |z - z_0| < R$ and with $b_n \neq 0$. Then we say f has a pole of order n at z_0. If $n = 1$ we say z_0 is a simple pole. There were several examples in the previous section. Here is one more

Example 10.1.

$$f(z) = \frac{z + 1}{z^3(z^2 + 1)}$$

has isolated singularities at $z = 0, \pm i$ and a zero at $z = -1$. We will show that $z = 0$ is a pole of order 3, $z = \pm i$ are poles of order 1 and $z = -1$ is a zero of order 1. The style of argument is the same in each case.

At $z = 0$:

$$f(z) = \frac{1}{z^3} \cdot \frac{z+1}{z^2+1}$$

Call the second factor $g(z)$. Since $g(z)$ is analytic at $z = 0$ and $g(0) = 1$, it has a Taylor series

$$g(z) = \frac{z+1}{z^2+1} = 1 + a_1 z + a_2 z^2 + \ldots$$

Therefore

$$f(z) = \frac{1}{z^3} + \frac{a_1}{z^2} + \frac{a_2}{z} + \ldots$$

This shows $z = 0$ is a pole of order 3.

At $z = i$:

$$f(z) = \frac{1}{z-i} \cdot \frac{z+1}{z^3(z+i)}$$

Call the second factor $g(z)$. Since $g(z)$ is analytic at $z = i$, it has a Taylor series

$$g(z) = \frac{z+1}{z^3(z+i)} = a_0 + a_1(z-i) + a_2(z-i)^2 + \ldots$$

where $a_0 = g(i) \neq 0$. Therefore

$$f(z) = \frac{a_0}{z-i} + a_1 + a_2(z-i) + \ldots$$

This shows $z = i$ is a pole of order 1.

The arguments for $z = -i$ and $z = -1$ are similar.

10.2 Words: Holomorphic and meromorphic

Definition. A function that is analytic on a region A is called holomorphic on A.
A function that is analytic on A except for a set of poles of finite order is called meromorphic on A.

Example 10.2. Let

$$f(z) = \frac{z + z^2 + z^3}{(z-2)(z-3)(z-4)(z-5)}.$$

This is meromorphic on $\mathbf{C}$ with (simple) poles at $z = 2, 3, 4, 5$.

10.3 Behavior of functions near zeros and poles

The basic idea is that near a zero of order n, a function behaves like $(z - z_0)^n$ and near a pole of order n, a function behaves like $1/(z - z_0)^n$. The following make this a little more precise.

Behavior near a zero. If f has a zero of order n at z_0 then near z_0,

$$f(z) \approx a_n(z - z_0)^n$$

for some constant a_n.

Proof. By definition f has a Taylor series around z_0 of the form

$$\begin{aligned} f(z) &= a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \ldots \\ &= a_n(z-z_0)^n \left(1 + \frac{a_{n+1}}{a_n}(z-z_0) + \frac{a_{n+2}}{a_n}(z-z_0)^2 + \ldots\right) \end{aligned}$$

Since the second factor equals 1 at z_0, the claim follows.

Behavior near a finite pole. If f has a pole of order n at z_0 then near z_0,

$$f(z) \approx \frac{b_n}{(z-z_0)^n},$$

for some constant b_n.

Proof. This is nearly identical to the previous argument. By definition f has a Laurent series around z_0 of the form

$$\begin{aligned} f(z) &= \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \ldots + \frac{b_1}{z-z_0} + a_0 + \ldots \\ &= \frac{b_n}{(z-z_0)^n}\left(1 + \frac{b_{n-1}}{b_n}(z-z_0) + \frac{b_{n-2}}{b_n}(z-z_0)^2 + \ldots\right) \end{aligned}$$

Since the second factor equals 1 at z_0, the claim follows.

10.3.1 Picard's theorem and essential singularities

Near an essential singularity we have Picard's theorem. We won't prove or make use of this theorem in 18.04. Still, we feel it is pretty enough to warrant showing to you.

Picard's theorem. If $f(z)$ has an essential singularity at z_0 then in every neighborhood of z_0, $f(z)$ takes on all possible values infinitely many times, with the possible exception of one value.

Example 10.3. It is easy to see that in any neighborhood of $z = 0$ the function $w = \mathrm{e}^{1/z}$ takes every value except $w = 0$.

10.3.2 Quotients of functions

We have the following statement about quotients of functions. We could make similar statements if one or both functions has a pole instead of a zero.

Theorem. Suppose f has a zero of order m at z_0 and g has a zero of order n at z_0. Let

$$h(z) = \frac{f(z)}{g(z)}.$$

Then

- If $n > m$ then $h(z)$ has a pole of order $n - m$ at z_0.
- If $n < m$ then $h(z)$ has a zero of order $m - n$ at z_0.
- If $n = m$ then $h(z)$ is analytic and nonzero at z_0.

We can paraphrase this as $h(z)$ has 'pole' of order $n - m$ at z_0. If $n - m$ is negative then the 'pole' is actually a zero.

Proof. You should be able to supply the proof. It is nearly identical to the proofs above: express f and g as Taylor series and take the quotient.

Example 10.4. Let

$$h(z) = \frac{\sin(z)}{z^2}$$

We know $\sin(z)$ has a zero of order 1 at $z = 0$ and z^2 has a zero of order 2. So, $h(z)$ has a pole of order 1 at $z = 0$. Of course, we can see this easily using Taylor series

$$h(z) = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \ldots\right)$$

10.4 Residues

In this section we'll explore calculating residues. We've seen enough already to know that this will be useful. We will see that even more clearly when we look at the residue theorem in the next section.

We introduced residues in the previous topic. We repeat the definition here for completeness.

Definition. Consider the function $f(z)$ with an isolated singularity at z_0, i.e. defined on the region $0 < |z - z_0| < r$ and with Laurent series (on that region)

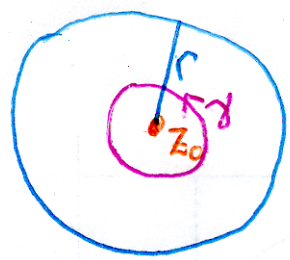
$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The residue of f at z_0 is b_1. This is denoted

$$\operatorname{Res}(f, z_0) = b_1 \quad \text{or} \quad \operatorname*{Res}_{z=z_0} f = b_1.$$

What is the importance of the residue? If γ is a small, simple closed curve that goes counterclockwise around b_1 then

$$\int_\gamma f(z) = 2\pi i b_1.$$



γ small enough to be inside $|z - z_0| < r$, surround z_0 and contain no other singularity of f.

This is easy to see by integrating the Laurent series term by term. The only nonzero integral comes from the term b_1/z.

Example 10.5.

$$f(z) = e^{1/2z} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \ldots$$

has an isolated singularity at 0. From the Laurent series we see that $\text{Res}(f, 0) = 1/2$.

Example 10.6.

(i) Let

$$f(z) = \frac{1}{z^3} + \frac{2}{z^2} + \frac{4}{z} + 5 + 6z$$

f has a pole of order 3 at $z = 0$ and $\text{Res}(f, 0) = 4$.

(ii) Suppose

$$f(z) = \frac{2}{z} + g(z)$$

where g is analytic at $z = 0$. Then, f has a simple pole at 0 and $\text{Res}(f, 0) = 2$.

(iii) Let

$$f(z) = \cos(z) = 1 - z^2/2! + \ldots$$

Then f is analytic at $z = 0$ and $\text{Res}(f, 0) = 0$.

(iv) Let

$$f(z) = \frac{\sin(z)}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \ldots\right) = 1 - \frac{z^2}{3!} + \ldots$$

So, f has a removable singularity at $z = 0$ and $\text{Res}(f, 0) = 0$.

Example 10.7. Using partial fractions. Let

$$f(z) = \frac{z}{z^2 + 1}$$

Find the poles and residues of f.

Solution: Using partial fractions we write

$$f(z) = \frac{z}{(z - i)(z + i)} = \frac{1}{2} \cdot \frac{1}{z - i} + \frac{1}{2} \cdot \frac{1}{z + i}$$

The poles are at $z = \pm i$. We compute the residues at each pole:

At $z = i$:

$$f(z) = \frac{1}{2} \cdot \frac{1}{z - i} + \text{something analytic at } i$$

Therefore the pole is simple and $\text{Res}(f, i) = 1/2$.

At $z = -i$:

$$f(z) = \frac{1}{2} \cdot \frac{1}{z + i} + \text{something analytic at } -i$$

Therefore the pole is simple and $\text{Res}(f, -i) = 1/2$.

Example 10.8. Mild warning! Let

$$f(z) = -\frac{1}{z(1-z)}$$

then we have the following Laurent expansions for f around $z = 0$.

On $0 < |z| < 1$:

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z} = -\frac{1}{z}(1 + z + z^2 + \ldots)$$

Therefore the pole at $z = 0$ is simple and $\text{Res}(f, 0) = -1$.

On $1 < |z| < \infty$:

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1 - 1/z} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Even though this is a valid Laurent expansion you must not use it to compute the residue at 0. This is because the definition of residue requires that we use the Laurent series on the region $0 < |z - z_0| < r$.

Example 10.9. Let

$$f(z) = \log(1 + z)$$

This has a singularity at $z = -1$, but it is not isolated, so not a pole and therefore there is no residue at $z = -1$.

10.4.1 Residues at simple poles

Simple poles occur frequently enough that we'll study computing their residues in some detail. Here are a number of ways to spot a simple pole and compute its residue. The justification for all of them goes back to Laurent series.

Suppose $f(z)$ has an isolated singularity at $z = z_0$. Then we have the following properties.

Property 1. If the Laurent series for $f(z)$ has the form

$$\frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

then f has a simple pole at z_0 and $\text{Res}(f, z_0) = b_1$.

Property 2 If

$$g(z) = (z - z_0)f(z)$$

is analytic at z_0 then z_0 is either a simple pole or a removable singularity. In either case $\mathrm{Res}(f, z_0) = g(z_0)$. (In the removable singularity case the residue is 0.)

Proof. Directly from the Laurent series for f around z_0.

Property 3. If f has a simple pole at z_0 then

$$\lim_{z \to z_0} (z - z_0)f(z) = \mathrm{Res}(f, z_0)$$

This says that the limit exists and equals the residue. Conversely, if the limit exists then either the pole is simple, or f is analytic at z_0. In both cases the limit equals the residue.

Proof. Directly from the Laurent series for f around z_0.

Property 4. If f has a simple pole at z_0 and $g(z)$ is analytic at z_0 then

$$\mathrm{Res}(fg, z_0) = g(z_0)\,\mathrm{Res}(f, z_0)$$

If $g(z_0) \neq 0$ then

$$\mathrm{Res}(f/g, z_0) = \frac{1}{g(z_0)}\,\mathrm{Res}(f, z_0)$$

Proof. Since z_0 is a simple pole,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0)$$

Since g is analytic,

$$g(z) = c_0 + c_1(z - z_0) + \ldots,$$

where $c_0 = g(z_0)$. Multiplying these series together it is clear that

$$\mathrm{Res}(fg, z_0) = c_0 b_1 = g(z_0)\,\mathrm{Res}(f, z_0). \qquad \square$$

The statement about quotients f/g follows from the proof for products because $1/g$ is analytic at z_0.

Property 5. If $g(z)$ has a simple zero at z_0 then $1/g(z)$ has a simple pole at z_0 and

$$\mathrm{Res}(1/g, z_0) = \frac{1}{g'(z_0)}$$

Proof. The algebra for this is similar to what we've done several times above. The Taylor expansion for g is

$$g(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots,$$

where $a_1 = g'(z_0)$. So

$$\frac{1}{g(z)} = \frac{1}{a_1(z-z_0)} \left(\frac{1}{1 + \frac{a_2}{a_1}(z-z_0) + \ldots} \right)$$

The second factor on the right is analytic at z_0 and equals 1 at z_0. Therefore we know the Laurent expansion of $1/g$ is

$$\frac{1}{g(z)} = \frac{1}{a_1(z-z_0)} \left(1 + c_1(z-z_0) + \ldots\right)$$

Clearly the residue is $1/a_1 = 1/g'(z_0)$. $\square$

Example 10.10. Let

$$f(z) = \frac{2+z+z^2}{(z-2)(z-3)(z-4)(z-5)}.$$

Show all the poles are simple and compute their residues.

Solution: The poles are at $z = 2, 3, 4, 5$. They are all isolated. We'll look at $z = 2$ the others are similar. Multiplying by $z - 2$ we get

$$g(z) = (z-2)f(z) = \frac{2+z+z^2}{(z-3)(z-4)(z-5)}.$$

This is analytic at $z = 2$ and $g(2) = \dfrac{8}{-6} = -\dfrac{4}{3}$. So the pole is simple and $\text{Res}(f, 2) = -4/3$.

Example 10.11. Let

$$f(z) = \frac{1}{\sin(z)}$$

Find all the poles and their residues.

Solution: The poles of $f(z)$ are the zeros of $\sin(z)$, i.e. $n\pi$ for n an integer. Since the derivative

$$\sin'(n\pi) = \cos(n\pi) \neq 0$$

the zeros are simple and by Property 5 above

$$\text{Res}(f, n\pi) = \frac{1}{\cos(n\pi)} = (-1)^n$$

Example 10.12. Let

$$f(z) = \frac{1}{z(z^2+1)(z-2)^2}$$

Identify all the poles and say which ones are simple.

Solution: Clearly the poles are at $z = 0, \pm i, 2$.

At $z = 0$:

$$g(z) = zf(z)$$

is analytic at 0 and $g(0) = 1/4$. So the pole is simple and the residue is $g(0) = 1/4$.

At $z = i$:

$$g(z) = (z - i)f(z) = \frac{1}{z(z + i)(z - 2)^2}$$

is analytic at i, the pole is simple and the residue is $g(i)$.

At $z = -i$: This is similar to the case $z = i$. The pole is simple.

At $z = 2$:

$$g(z) = (z - 2)f(z) = \frac{1}{z(z^2 + 1)(z - 2)}$$

is not analytic at 2, so the pole is not simple. (It should be obvious that it's a pole of order 2.)

Example 10.13. Let $p(z)$, $q(z)$ be analytic at $z = z_0$. Assume $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$. Find

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)}$$

Solution: Since $q'(z_0) \neq 0$, q has a simple zero at z_0. So $1/q$ has a simple pole at z_0 and

$$\text{Res}(1/q, z_0) = \frac{1}{q'(z_0)}$$

Since $p(z_0) \neq 0$ we know

$$\text{Res}(p/q, z_0) = p(z_0)\,\text{Res}(1/q, z_0) = \frac{p(z_0)}{q'(z_0)}$$

10.4.2 Residues at finite poles

For higher-order poles we can make statements similar to those for simple poles, but the formulas and computations are more involved. The general principle is the following

Higher order poles. If $f(z)$ has a pole of order k at z_0 then

$$g(z) = (z - z_0)^k f(z)$$

is analytic at z_0 and if

$$g(z) = a_0 + a_1(z - z_0) + \ldots$$

then

$$\text{Res}(f, z_0) = a_{k-1} = \frac{g^{(k-1)}(z_0)}{(k - 1)!}$$

Proof. This is clear using Taylor and Laurent series for g and f.

Example 10.14. Let

$$f(z) = \frac{\sinh(z)}{z^5}$$

find the residue at $z = 0$.

Solution: We know the Taylor series for

$$\sinh(z) = z + z^3/3! + z^5/5! + \ldots$$

(You can find this using $\sinh(z) = (e^z - e^{-z})/2$ and the Taylor series for e^z.) Therefore,

$$f(z) = \frac{1}{z^4} + \frac{1}{3!z^2} + \frac{1}{5!} + \ldots$$

We see $\text{Res}(f, 0) = 0$.

Note, we could have seen this by realizing that $f(z)$ is an even function.

Example 10.15. Let

$$f(z) = \frac{\sinh(z)e^z}{z^5}$$

Find the residue at $z = 0$.

Solution: It is clear that $\text{Res}(f, 0)$ equals the coefficient of z^4 in the Taylor expansion of $\sinh(z)e^z$. We compute this directly as

$$\sinh(z)e^z = \left(z + \frac{z^3}{3!} + \ldots\right)\left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots\right) = \ldots + \left(\frac{1}{4!} + \frac{1}{3!}\right)z^4 + \ldots$$

So

$$\text{Res}(f, 0) = \frac{1}{4!} + \frac{1}{3!} = \frac{5}{24}$$

Example 10.16. Find the residue of

$$f(z) = \frac{1}{z(z^2 + 1)(z - 2)^2}$$

at $z = 2$.

Solution:

$$g(z) = (z - 2)^2 f(z) = \frac{1}{z(z^2 + 1)}$$

is analytic at $z = 2$. So, the residue we want is the a_1 term in its Taylor series, i.e. $g'(2)$. This is easy, if dull, to compute

$$\text{Res}(f, 2) = g'(2) = -\frac{13}{100}$$

10.4.3 $\cot(z)$

The function $\cot(z)$ turns out to be very useful in applications. This stems largely from the fact that it has simple poles at all multiples of π and the residue is 1 at each pole. We show that first.

Fact. $f(z) = \cot(z)$ has simple poles at $n\pi$ for n an integer and $\text{Res}(f, n\pi) = 1$.

Proof.

$$f(z) = \frac{\cos(z)}{\sin(z)}$$

This has poles at the zeros of sin, i.e. at $z = n\pi$. At poles f is of the form p/q where q has a simple zero at z_0 and $p(z_0) \neq 0$. Thus we can use the formula

$$\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

In our case, we have

$$\text{Res}(f, n\pi) = \frac{\cos(n\pi)}{\cos(n\pi)} = 1$$

as claimed.

Sometimes we need more terms in the Laurent expansion of $\cot(z)$. There is no known easy formula for the terms, but we can easily compute as many as we need using the following technique.

Example 10.17. Compute the first several terms of the Laurent expansion of $\cot(z)$ around $z = 0$.

Solution: Since $\cot(z)$ has a simple pole at 0 we know

$$\cot(z) = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

We also know

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{1 - z^2/2 + z^4/4! - \ldots}{z - z^3/3! + z^5/5! - \ldots}$$

Cross multiplying the two expressions we get

$$\left(\frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \ldots$$

We can do the multiplication and equate the coefficients of like powers of z.

$$b_1 + a_0 z + \left(-\frac{b_1}{3!} + a_1\right) z^2 + \left(-\frac{a_0}{3!} + a_2\right) z^3 + \left(\frac{b_1}{5!} - \frac{a_1}{3!} + a_3\right) z^4 = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$$

So,

$$\begin{array}{rcl} b_1 = 1\ , & & a_0 = 0 \\ -b_1/3! + a_1 = -1/2! & \Rightarrow & a_1 = -1/3 \\ -a_0/3! + a_2 = 0 & \Rightarrow & a_2 = 0 \\ b_1/5! - a_1/3! + a_3 = 1/4! & \Rightarrow & a_3 = -1/45 \end{array}$$

As noted above, all the even terms are 0 as they should be. We have

$$\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \ldots$$

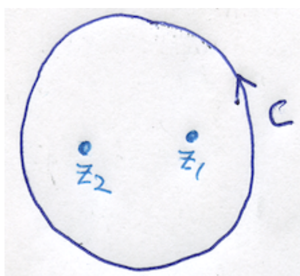
10.5 Cauchy Residue Theorem

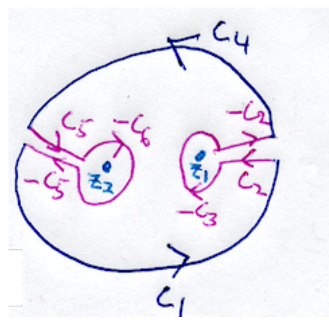
This is one of the major theorems in 18.04. It will allow us to make systematic our previous somewhat ad hoc approach to computing integrals on contours that surround singularities.

Theorem. (**Cauchy's residue theorem**) Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then

$$\int_C f(z)\,dz = 2\pi i \sum \text{residues of } f \text{ inside } C$$

Proof. The proof is based of the following figures. They only show a curve with two singularities inside it, but the generalization to any number of signularities is straightforward. In what follows we are going to abuse language and say pole when we mean isolated singularity, i.e. a finite order pole or an essential singularity ('infinite order pole').





The left figure shows the curve C surrounding two poles z_1 and z_2 of f. The right figure shows the same curve with some cuts and small circles added. It is chosen so that there are no poles of f inside it and so that the little circles around each of the poles are so small that there are no other poles inside them. The right hand curve is

$$\tilde{C} = C_1 + C_2 - C_3 - C_2 + C_4 + C_5 - C_6 - C_5$$

The left hand curve is $C = C_1 + C_4$. Since there are no poles inside $\tilde{C}$ we have, by Cauchy's theorem,

$$\int_{\tilde{C}} f(z)\,dz = \int_{C_1+C_2-C_3-C_2+C_4+C_5-C_6-C_5} f(z)\,dz = 0$$

Dropping C_2 and C_5, which are both added and subtracted, this becomes

$$\int_{C_1+C_4} f(z)\,dz = \int_{C_3+C_6} f(z)\,dz \qquad (47)$$

If

$$f(z) = \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ldots$$

is the Laurent expansion of f around z_1 then

$$\int_{C_3} f(z)\,dz = \int_{C_3} \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ldots\, dz = 2\pi i b_1$$
$$= 2\pi i\,\mathrm{Res}(f, z_1)$$

Likewise

$$\int_{C_6} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, z_2)$$

Using these residues and the fact that $C = C_1 + C_4$, Equation (47) becomes

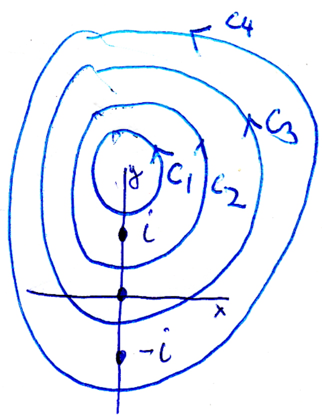
$$\int_{C} f(z)\,dz = 2\pi i\,[\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2)]$$

That proves the residue theorem for the case of two poles. As we said, generalizing to any number of poles is straightforward.

Example 10.18. Let

$$f(z) = \frac{1}{z(z^2+1)}$$

Compute $\int f(z)\,dz$ over each of the contours C_1, C_2, C_3, C_4 shown.



Solution: The poles of $f(z)$ are at $z = 0, \pm i$. Using the residue theorem we just need to compute the residues of each of these poles.
At $z = 0$:

$$g(z) = zf(z) = \frac{1}{z^2+1}$$

is analytic at 0 so the pole is simple and

$$\mathrm{Res}(f, 0) = g(0) = 1$$

At $z = i$:

$$g(z) = (z-i)f(z) = \frac{1}{z(z+i)}$$

is analytic at i so the pole is simple and

$$\operatorname{Res}(f, i) = g(i) = -1/2$$

At $z = -i$:

$$g(z) = (z+i)f(z) = \frac{1}{z(z-i)}$$

is analytic at $-i$ so the pole is simple and

$$\operatorname{Res}(f, -i) = g(-i) = -1/2$$

Using the residue theorem we have

$$\begin{aligned}
\int_{C_1} f(z)\, dz &= 0 \text{ (since } f \text{ is analytic inside } C_1) \\
\int_{C_2} f(z)\, dz &= 2\pi i \operatorname{Res}(f, i) = -\pi i \\
\int_{C_3} f(z)\, dz &= 2\pi i \left[\operatorname{Res}(f, i) + \operatorname{Res}(f, 0)\right] = \pi i \\
\int_{C_4} f(z)\, dz &= 2\pi i \left[\operatorname{Res}(f, i) + \operatorname{Res}(f, 0) + \operatorname{Res}(f, -i)\right] = 0
\end{aligned}$$

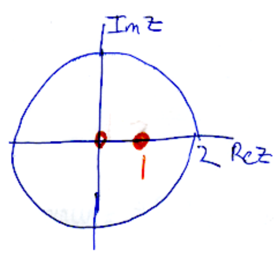
Example 10.19. Compute

$$\int_{|z|=2} \frac{5z-2}{z(z-1)}\, dz$$

Solution: Let

$$f(z) = \frac{5z-2}{z(z-1)}$$

The poles of f are at $z = 0, 1$ and the contour encloses them both.



At $z = 0$:

$$g(z) = zf(z) = \frac{5z-2}{(z-1)}$$

is analytic at 0 so the pole is simple and

$$\operatorname{Res}(f, 0) = g(0) = 2$$

At $z = 1$:

$$g(z) = (z-1)f(z) = \frac{5z-2}{z}$$

is analytic at 1 so the pole is simple and

$$\mathrm{Res}(f, 1) = g(1) = 3$$

Finally

$$\int_C \frac{5z-2}{z(z-1)}\, dz = 2\pi i \left[\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\right] = 10\pi i$$

.

Example 10.20. Compute

$$\int_{|z|=1} z^2 \sin(1/z)\, dz$$

Solution: Let

$$f(z) = z^2 \sin(1/z)$$

f has an isolated singularity at $z = 0$. Using the Taylor series for $\sin(w)$ we get

$$z^2 \sin(1/z) = z^2 \left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots \right) = z - \frac{1/6}{z} + \ldots$$

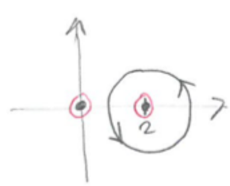
So, $\mathrm{Res}(f, 0) = b_1 = -1/6$. Thus the residue theorem gives

$$\int_{|z|=1} z^2 \sin(1/z)\, dz = 2\pi i \,\mathrm{Res}(f, 0) = -\frac{i\pi}{3}.$$

Example 10.21. Compute

$$\int_C \frac{dz}{z(z-2)^4}\, dz$$

where $C : |z-2| = 1$.



Solution: Let

$$f(z) = \frac{1}{z(z-2)^4}$$

The singularity at $z = 0$ is outside the contour of integration so it doesn't contribute to the integral.

To use the residue theorem we need to find the residue of f at $z = 2$. There are a number of ways to do this. Here's one:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2 + (z-2)} \\ &= \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2} \\ &= \frac{1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \ldots\right)\end{aligned}$$

This is valid on $0 < |z-2| < 2$. So,

$$f(z) = \frac{1}{(z-2)^4} \cdot \frac{1}{z} = \frac{1}{2(z-2)^4} - \frac{1}{4(z-2)^3} + \frac{1}{8(z-2)^2} - \frac{1}{16(z-2)} + \ldots$$

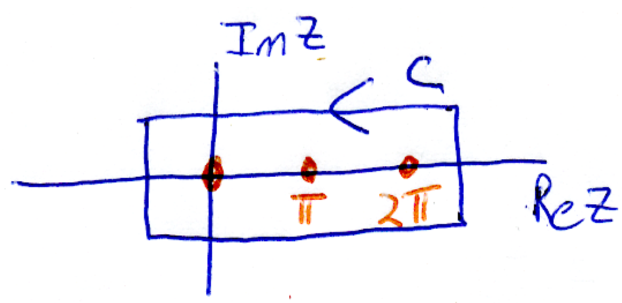
Thus, $\text{Res}(f, 2) = -1/16$ and

$$\int_C f(z)\, dz = 2\pi i \,\text{Res}(f, 2) = -\frac{\pi i}{8}$$

Example 10.22. Compute

$$\int_C \frac{1}{\sin(z)}\, dz$$

over the contour C shown.



Solution: Let

$$f(z) = 1/\sin(z)$$

There are 3 poles of f inside C at 0, π and 2π. We can find the residues by taking the limit of $(z - z_0)f(z)$. Each of the limits is computed using L'Hospital's rule. (This is valid, since the rule is just a statement about power series. We could also have used Property 5 from the section on residues of simple poles above.)
At $z = 0$:

$$\lim_{z\to 0} \frac{z}{\sin(z)} = \lim_{z\to 0} \frac{1}{\cos(z)} = 1$$

Since the limit exists, $z = 0$ is a simple pole and

$$\text{Res}(f, 0) = 1$$

At $z = \pi$:

$$\lim_{z\to\pi} \frac{z-\pi}{\sin(z)} = \lim_{z\to\pi} \frac{1}{\cos(z)} = -1$$

Since the limit exists, $z = \pi$ is a simple pole and

$$\text{Res}(f, \pi) = -1$$

At $z = 2\pi$: The same argument shows

$$\text{Res}(f, 2\pi) = 1$$

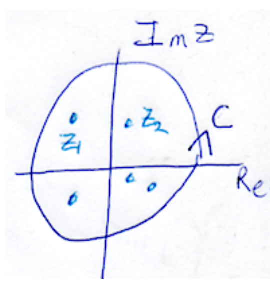
Now, by the residue theorem

$$\int_C f(z)\, dz = 2\pi i \left[\text{Res}(f, 0) + \text{Res}(f, \pi) + \text{Res}(f, 2\pi)\right] = 2\pi i.$$

10.6 Residue at ∞

The residue at ∞ is a clever device that can sometimes allow us to replace the computation of many residues with the computation of a single residue.

Suppose that f is analytic in $\mathbf{C}$ except for a finite number of singularities. Let C be a positively oriented curve that is large enough to contain all the singularities.



All the poles of f are inside C

Definition. We define the residue of f at infinity by

$$\text{Res}(f, \infty) = -\frac{1}{2\pi i} \int_C f(z)\, dz.$$

We should first explain the idea here. The interior of a simple closed curve is everything to left as you traverse the curve. The curve C is oriented counterclockwise, so its interior contains all the poles of f. The residue theorem says the integral over C is determined by the residues of these poles.

On the other hand, the interior of the curve $-C$ is everything outside of C. There are no poles of f in that region. If we want the residue theorem to hold (which we do –it's that important) then the only option is to have a residue at ∞ and define it as we did.

The definition of the residue at infinity assumes all the poles of f are inside C. Therefore the residue theorem implies

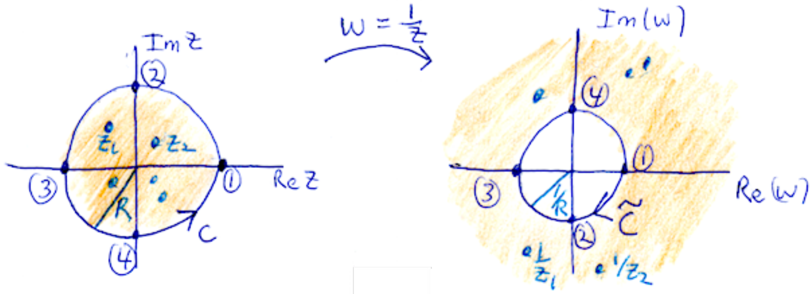
$$\text{Res}(f, \infty) = -\sum \text{ the residues of } f.$$

To make this useful we need a way to compute the residue directly. This is given by the following theorem.

Theorem. If f is analytic in $\mathbf{C}$ except for a finite number of singularities then

$$\text{Res}(f, \infty) = -\text{Res}\left(\frac{1}{w^2} f(1/w), 0\right).$$

Proof. The proof is just a change of variables: $w = 1/z$.



Change of variable: $w = 1/z$

First note that $z = 1/w$ and

$$dz = -(1/w^2)\, dw$$

Next, note that the map $w = 1/z$ carries the positively oriented z-circle of radius R to the negatively oriented w-circle of radius $1/R$. (To see the orientiation, follow the circled points 1, 2, 3, 4 on C in the z-plane as they are mapped to points on $\tilde{C}$ in the w-plane.) Thus,

$$\text{Res}(f, \infty) = -\frac{1}{2\pi i}\int_C f(z)\, dz = \frac{1}{2\pi i}\int_{\tilde{C}} f(1/w)\frac{1}{w^2}\, dw$$

Finally, note that $z = 1/w$ maps all the poles inside the circle C to points outside the circle $\tilde{C}$. So the only possible pole of $(1/w^2)f(1/w)$ that is inside $\tilde{C}$ is at $w = 0$. Now, since $\tilde{C}$ is oriented clockwise, the residue theorem says

$$\frac{1}{2\pi i}\int_{\tilde{C}} f(1/w)\frac{1}{w^2}\, dw = -\text{Res}\left(\frac{1}{w^2} f(1/w), 0\right)$$

Comparing this with the equation just above finishes the proof.

Example 10.23. Let

$$f(z) = \frac{5z - 2}{z(z-1)}$$

Earlier we computed

$$\int_{|z|=2} f(z)\, dz = 10\pi i$$

by computing residues at $z = 0$ and $z = 1$. Recompute this integral by computing a single residue at infinity.

Solution:

$$\frac{1}{w^2} f(1/w) = \frac{1}{w^2} \frac{5/w - 2}{(1/w)(1/w - 1)} = \frac{5 - 2w}{w(1 - w)}$$

We easily compute that

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{w^2} f(1/w), 0\right) = -5.$$

Since $|z| = 2$ contains all the singularities of f we have

$$\int_{|z|=2} f(z)\, dz = -2\pi i \operatorname{Res}(f, \infty) = 10\pi i.$$

This is the same answer we got before!

11 Definite integrals using the residue theorem

In this topic we'll use the residue theorem to compute some real definite integrals.

$$\int_a^b f(x)\, dx$$

The general approach is always the same

1. Find a complex analytic function $g(z)$ which either equals f on the real axis or which is closely connected to f, e.g. $f(x) = \cos(x)$, $g(z) = e^{iz}$.
2. Pick a closed contour C that includes the part of the real axis in the integral.
3. The contour will be made up of pieces. It should be such that we can compute $\int g(z)\, dz$ over each of the pieces except the part on the real axis.
4. Use the residue theorem to compute $\int_C g(z)\, dz$.
5. Combine the previous steps to deduce the value of the integral we want.

11.1 Integrals of functions that decay

The theorems in this section will guide us in choosing the closed contour C described in the introduction.

The first theorem is for functions that decay faster than $1/z$.

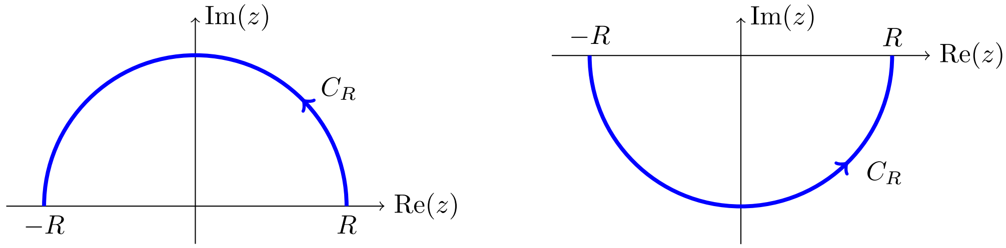
Theorem 11.1. (a) Suppose $f(z)$ is defined in the upper half-plane. If there is an $a > 1$ and $M > 0$ such that

$$|f(z)| < \frac{M}{|z|^a}$$

for $|z|$ large then

$$\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0,$$

where C_R is the semicircle shown below on the left.



Semicircles: left: $Re^{i\theta}$, $0 < \theta < \pi$ right: $Re^{i\theta}$, $\pi < \theta < 2\pi$.

(b) If $f(z)$ is defined in the lower half-plane and

$$|f(z)| < \frac{M}{|z|^a},$$

where $a > 1$ then

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0,$$

where C_R is the semicircle shown above on the right.

Proof. We prove (a), (b) is essentially the same. We use the triangle inequality for integrals and the estimate given in the hypothesis. For R large

$$\left| \int_{C_R} f(z)\, dz \right| \leq \int_{C_R} |f(z)|\, |dz| \leq \int_{C_R} \frac{M}{|z|^a}\, |dz| = \int_0^{\pi} \frac{M}{R^a} R\, d\theta = \frac{M\pi}{R^{a-1}}.$$

Since $a > 1$ this clearly goes to 0 as $R \to \infty$. $\square$

The next theorem is for functions that decay like $1/z$. It requires some more care to state and prove.

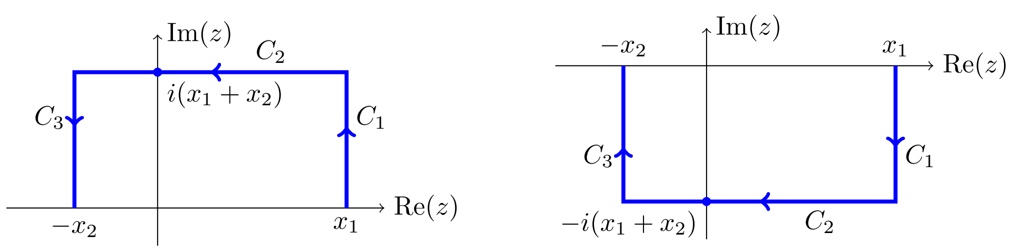
Theorem 11.2. (a) Suppose $f(z)$ is defined in the upper half-plane. If there is an $M > 0$ such that

$$|f(z)| < \frac{M}{|z|}$$

for $|z|$ large then for $a > 0$

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} f(z) e^{iaz}\, dz = 0,$$

where $C_1 + C_2 + C_3$ is the rectangular path shown below on the left.



Rectangular paths of height and width $x_1 + x_2$.

(b) Similarly, if $a < 0$ then

$$\lim_{x_1 \to \infty,\, x_2 \to \infty} \int_{C_1+C_2+C_3} f(z) e^{iaz}\, dz = 0,$$

where $C_1 + C_2 + C_3$ is the rectangular path shown above on the right.

Note. In contrast to Theorem 11.1 this theorem needs to include the factor e^{iaz}.

Proof. **(a)** We start by parametrizing C_1, C_2, C_3.

C_1: $\gamma_1(t) = x_1 + it$, t from 0 to $x_1 + x_2$

C_2: $\gamma_2(t) = t + i(x_1 + x_2)$, t from x_1 to $-x_2$

C_3: $\gamma_3(t) = -x_2 + it$, t from $x_1 + x_2$ to 0.

Next we look at each integral in turn. We assume x_1 and x_2 are large enough that

$$|f(z)| < \frac{M}{|z|}$$

on each of the curves C_j.

$$\begin{aligned}
\left| \int_{C_1} f(z) e^{iaz}\, dz \right| &\le \int_{C_1} |f(z) e^{iaz}|\, |dz| \le \int_{C_1} \frac{M}{|z|} |e^{iaz}|\, |dz| \\
&= \int_0^{x_1+x_2} \frac{M}{\sqrt{x_1^2 + t^2}} |e^{iax_1 - at}|\, dt \\
&\le \frac{M}{x_1} \int_0^{x_1+x_2} e^{-at}\, dt \\
&= \frac{M}{x_1} (1 - e^{-a(x_1+x_2)})/a.
\end{aligned}$$

Since $a > 0$, it is clear that this last expression goes to 0 as x_1 and x_2 go to ∞.

$$\begin{aligned}
\left| \int_{C_2} f(z) e^{iaz}\, dz \right| &\le \int_{C_2} |f(z) e^{iaz}|\, |dz| \le \int_{C_2} \frac{M}{|z|} |e^{iaz}|\, |dz| \\
&= \int_{-x_2}^{x_1} \frac{M}{\sqrt{t^2 + (x_1 + x_2)^2}} |e^{iat - a(x_1+x_2)}|\, dt \\
&\le \frac{M e^{-a(x_1+x_2)}}{x_1 + x_2} \int_0^{x_1+x_2} dt \\
&\le M e^{-a(x_1+x_2)}
\end{aligned}$$

Again, clearly this last expression goes to 0 as x_1 and x_2 go to ∞.

The argument for C_3 is essentially the same as for C_1, so we leave it to the reader.

The proof for part (b) is the same. You need to keep track of the sign in the exponentials and make sure it is negative.

Example. See Example 11.16 below for an example using Theorem 11.2.

11.2 Integrals $\int_{-\infty}^{\infty}$ and $\int_{0}^{\infty}$

Example 11.3. Compute

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx$$

Solution: Let

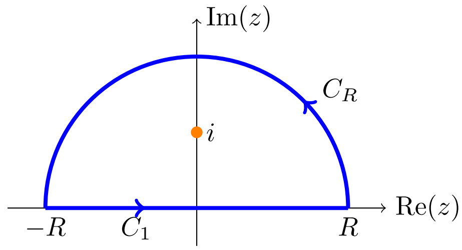
$$f(z) = 1/(1+z^2)^2$$

It is clear that for z large

$$f(z) \approx 1/z^4$$

In particular, the hypothesis of Theorem 11.1 is satisfied. Using the contour shown below we have, by the residue theorem,

$$\int_{C_1+C_R} f(z)\,dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.} \tag{48}$$



We examine each of the pieces in the above equation.

$\int_{C_R} f(z)\,dz$: By Theorem 11.1(a),

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

$\int_{C_1} f(z)\,dz$: Directly, we see that

$$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = I.$$

So letting $R \to \infty$, Eq. (48) becomes

$$I = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

Finally, we compute the needed residues: $f(z)$ has poles of order 2 at $\pm i$. Only $z = i$ is inside the contour, so we compute the residue there. Let

$$g(z) = (z-i)^2 f(z) = \frac{1}{(z+i)^2}$$

Then

$$\operatorname{Res}(f, i) = g'(i) = -\frac{2}{(2i)^3} = \frac{1}{4i}$$

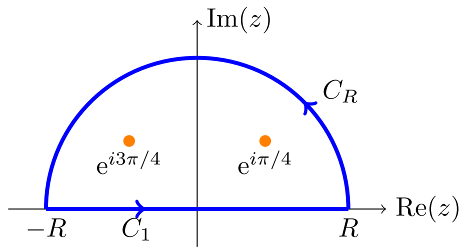
So,

$$I = 2\pi i \operatorname{Res}(f, i) = \boxed{\frac{\pi}{2}}$$

Example 11.4. Compute

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx$$

Solution: Let $f(z) = 1/(1 + z^4)$. We use the same contour as in the previous example



As in the previous example,

$$\lim_{R\to\infty} \int_{C_R} f(z) \, dz = 0$$

and

$$\lim_{R\to\infty} \int_{C_1} f(z) \, dz = \int_{-\infty}^{\infty} f(x) \, dx = I$$

So, by the residue theorem

$$I = \lim_{R\to\infty} \int_{C_1 + C_R} f(z) \, dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

The poles of f are all simple and at

$$e^{i\pi/4}, \; e^{i3\pi/4}, \; e^{i5\pi/4} \; e^{i7\pi/4}$$

Only $e^{i\pi/4}$ and $e^{i3\pi/4}$ are inside the contour. We compute their residues as limits using L'Hospital's rule. For $z_1 = e^{i\pi/4}$:

$$\operatorname{Res}(f, z_1) = \lim_{z\to z_1} (z - z_1) f(z) = \lim_{z\to z_1} \frac{z - z_1}{1 + z^4} = \lim_{z\to z_1} \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$

and for $z_2 = e^{i3\pi/4}$:

$$\operatorname{Res}(f, z_2) = \lim_{z\to z_2} (z - z_2) f(z) = \lim_{z\to z_2} \frac{z - z_2}{1 + z^4} = \lim_{z\to z_2} \frac{1}{4z^3} = \frac{1}{4e^{i9\pi/4}} = \frac{e^{-i\pi/4}}{4}$$

So,

$$I = 2\pi i(\text{Res}(f, z_1) + \text{Res}(f, z_2)) = 2\pi i \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \right) = 2\pi i \left(-\frac{2i}{4\sqrt{2}} \right) = \boxed{\pi \frac{\sqrt{2}}{2}}$$

Example 11.5. Suppose $b > 0$. Show

$$\int_0^\infty \frac{\cos(x)}{x^2 + b^2}\, dx = \frac{\pi \mathrm{e}^{-b}}{2b}$$

Solution: The first thing to note is that the integrand is even, so

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + b^2}.$$

Also note that the square in the denominator tells us the integral is absolutely convergent.

We have to be careful because $\cos(z)$ goes to infinity in either half-plane, so the hypotheses of Theorem 11.1 are not satisfied. The trick is to replace $\cos(x)$ by e^{ix}, so

$$\tilde{I} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x^2 + b^2}\, dx \qquad \text{with} \qquad I = \frac{1}{2}\text{Re}(\tilde{I})$$

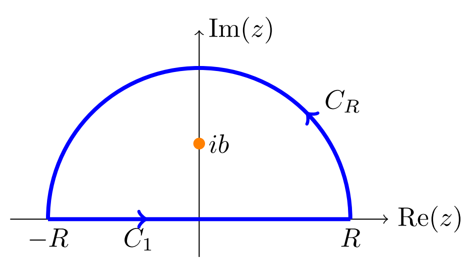
Now let

$$f(z) = \frac{\mathrm{e}^{iz}}{z^2 + b^2}$$

For $z = x + iy$ with $y > 0$ we have

$$|f(z)| = \frac{|\mathrm{e}^{i(x+iy)}|}{|z^2 + b^2|} = \frac{\mathrm{e}^{-y}}{|z^2 + b^2|}.$$

Since $\mathrm{e}^{-y} < 1$, $f(z)$ satisfies the hypotheses of Theorem 11.1 in the upper half-plane. Now we can use the same contour as in the previous examples



We have

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

and

$$\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = \tilde{I}$$

So, by the residue theorem

$$\tilde{I} = \lim_{R\to\infty} \int_{C_1+C_R} f(z)\,dz = 2\pi i \sum \text{ residues of } f \text{ inside the contour.}$$

The poles of f are at $\pm bi$ and both are simple. Only bi is inside the contour. We compute the residue as a limit using L'Hospital's rule

$$\text{Res}(f, bi) = \lim_{z\to bi} (z - bi)\frac{e^{iz}}{z^2 + b^2} = \frac{e^{-b}}{2bi}.$$

So,

$$\tilde{I} = 2\pi i\, \text{Res}(f, bi) = \frac{\pi e^{-b}}{b}$$

Finally,

$$I = \frac{1}{2}\,\text{Re}(\tilde{I}) = \frac{\pi e^{-b}}{2b},$$

as claimed.

Warning: Be careful when replacing $\cos(z)$ by e^{iz} that it is appropriate. A key point in the above example was that $I = \frac{1}{2}\,\text{Re}(\tilde{I})$. This is needed to make the replacement useful.

11.3 Trigonometric integrals

The trick here is to put together some elementary properties of $z = e^{i\theta}$ on the unit circle.

1. $e^{-i\theta} = 1/z$.
2. $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + 1/z}{2}$.
3. $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - 1/z}{2i}$.

We start with an example. After that we'll state a more general theorem.

Example 11.6. Compute

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos(\theta)}$$

Assume that $|a| \neq 1$.

Solution: Notice that $[0, 2\pi]$ is the interval used to parametrize the unit circle as $z = e^{i\theta}$.

We need to make two substitutions:

$$\cos(\theta) = \frac{z + 1/z}{2}$$

$$dz = ie^{i\theta}\,d\theta \qquad \Leftrightarrow \qquad d\theta = \frac{dz}{iz}$$

Making these substitutions we get

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos(\theta)} \\ &= \int_{|z|=1} \frac{1}{1+a^2-2a(z+1/z)/2} \cdot \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{1}{i((1+a^2)z-a(z^2+1))}\, dz \end{aligned}$$

So, let

$$f(z) = \frac{1}{i((1+a^2)z-a(z^2+1))}$$

The residue theorem implies

$$I = 2\pi i \sum \text{ residues of } f \text{ inside the unit circle.}$$

We can factor the denominator:

$$f(z) = \frac{-1}{ia(z-a)(z-1/a)}$$

The poles are at a, $1/a$. One is inside the unit circle and one is outside.

If $|a| > 1$ then $1/a$ is inside the unit circle and $\mathrm{Res}(f, 1/a) = \dfrac{1}{i(a^2-1)}$

If $|a| < 1$ then a is inside the unit circle and $\mathrm{Res}(f, a) = \dfrac{1}{i(1-a^2)}$

We have

$$I = \begin{cases} \frac{2\pi}{a^2-1} & \text{if } |a| > 1 \\ \frac{2\pi}{1-a^2} & \text{if } |a| < 1 \end{cases}$$

The example illustrates a general technique which we state now.

Theorem 11.7. Suppose $R(x, y)$ is a rational function with no poles on the circle

$$x^2 + y^2 = 1$$

then for

$$f(z) = \frac{1}{iz} R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)$$

we have

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta = 2\pi i \sum \text{ residues of } f \text{ inside } |z| = 1.$$

Proof. We make the same substitutions as in Example 11.6. So,

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta = \int_{|z|=1} R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right) \frac{dz}{iz}$$

The assumption about poles means that f has no poles on the contour $|z| = 1$. The residue theorem now implies the theorem.

11.4 Integrands with branch cuts

Example 11.8. Compute

$$I = \int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx$$

Solution: Let

$$f(x) = \frac{x^{1/3}}{1+x^2}$$

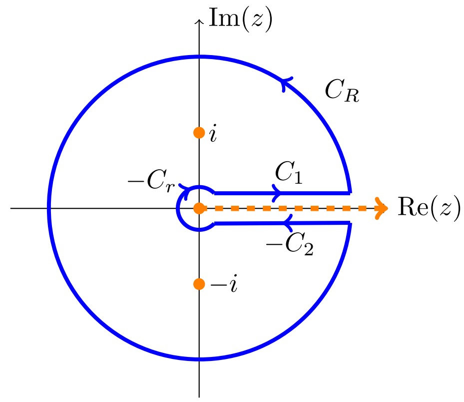
Since this is asymptotically comparable to $x^{-5/3}$, the integral is absolutely convergent. As a complex function

$$f(z) = \frac{z^{1/3}}{1+z^2}$$

needs a branch cut to be analytic (or even continuous), so we will need to take that into account with our choice of contour.

First, choose the following branch cut along the positive real axis. That is, for $z = re^{i\theta}$ not on the axis, we have $0 < \theta < 2\pi$.

Next, we use the contour $C_1 + C_R - C_2 - C_r$ shown below.



Contour around branch cut: inner circle of radius r, outer of radius R.

We put convenient signs on the pieces so that the integrals are parametrized in a natural way. You should read this contour as having r so small that C_1 and C_2 are essentially on the x-axis. Note well, that, since C_1 and C_2 are on opposite sides of the branch cut, the integral

$$\int_{C_1 - C_2} f(z)\,dz \neq 0$$

First we analyze the integral over each piece of the curve.

On C_R: Theorem 11.1 says that

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

On C_r: For concreteness, assume $r < 1/2$. We have $|z| = r$, so

$$|f(z)| = \frac{|z^{1/3}|}{|1+z^2|} \le \frac{r^{1/3}}{1-r^2} \le \frac{(1/2)^{1/3}}{3/4}.$$

Call the last number in the above equation M. We have shown that, for small r, $|f(z)| < M$. So,

$$\left| \int_{C_r} f(z)\, dz \right| \le \int_0^{2\pi} |f(r\mathrm{e}^{i\theta})||ir\mathrm{e}^{i\theta}|\, d\theta \le \int_0^{2\pi} Mr\, d\theta = 2\pi Mr.$$

Clearly this goes to zero as $r \to 0$.

On C_1:

$$\lim_{r\to 0,\, R\to\infty} \int_{C_1} f(z)\, dz = \int_0^\infty f(x)\, dx = I$$

On C_2: We have (essentially) $\theta = 2\pi$, so $z^{1/3} = \mathrm{e}^{i2\pi/3}|z|^{1/3}$. Thus,

$$\lim_{r\to 0,\, R\to\infty} \int_{C_2} f(z)\, dz = \mathrm{e}^{i2\pi/3} \int_0^\infty f(x)\, dx = \mathrm{e}^{i2\pi/3} I.$$

The poles of $f(z)$ are at $\pm i$. Since f is meromorphic inside our contour the residue theorem says

$$\int_{C_1 + C_R - C_2 - C_r} f(z)\, dz = 2\pi i(\mathrm{Res}(f, i) + \mathrm{Res}(f, -i)).$$

Letting $r \to 0$ and $R \to \infty$ the analysis above shows

$$(1 - \mathrm{e}^{i2\pi/3})I = 2\pi i(\mathrm{Res}(f, i) + \mathrm{Res}(f, -i))$$

All that's left is to compute the residues using the chosen branch of $z^{1/3}$

$$\begin{aligned} \mathrm{Res}(f, -i) &= \frac{(-i)^{1/3}}{-2i} = \frac{(\mathrm{e}^{i3\pi/2})^{1/3}}{2\mathrm{e}^{i3\pi/2}} = \frac{\mathrm{e}^{-i\pi}}{2} = -\frac{1}{2} \\ \mathrm{Res}(f, i) &= \frac{i^{1/3}}{2i} = \frac{\mathrm{e}^{i\pi/6}}{2\mathrm{e}^{i\pi/2}} = \frac{\mathrm{e}^{-i\pi/3}}{2} \end{aligned}$$

A little more algebra gives

$$(1 - \mathrm{e}^{i2\pi/3})I = 2\pi i \cdot \frac{-1 + \mathrm{e}^{-i\pi/3}}{2} = \pi i(-1 + 1/2 - i\sqrt{3}/2) = -\pi i \mathrm{e}^{i\pi/3}.$$

Continuing

$$I = \frac{-\pi i \mathrm{e}^{i\pi/3}}{1 - \mathrm{e}^{i2\pi/3}} = \frac{\pi i}{\mathrm{e}^{i\pi/3} - \mathrm{e}^{-\pi i/3}} = \frac{\pi/2}{(\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3})/2i} = \frac{\pi/2}{\sin(\pi/3)} = \frac{\pi}{\sqrt{3}}$$

Whew! (Note: a sanity check is that the result is real, which it had to be.)

Example 11.9. Compute
$$I = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

Solution: Let
$$f(z) = \frac{1}{z\sqrt{z^2-1}}$$

The first thing we'll show is that the integral
$$\int_1^\infty f(x)\,dx$$
is absolutely convergent. To do this we split it into two integrals
$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

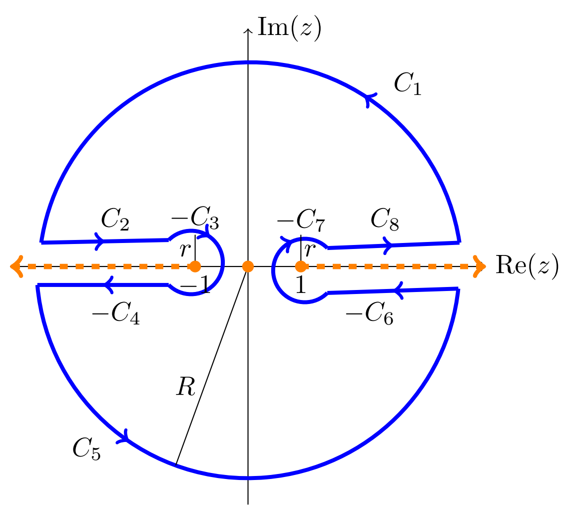
The first integral on the right can be rewritten as
$$\int_1^2 \frac{1}{x\sqrt{x+1}} \cdot \frac{1}{\sqrt{x-1}}\,dx \le \int_1^2 \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x-1}}\,dx = \frac{2}{\sqrt{2}}\sqrt{x-1}\Big|_1^2$$

This shows the first integral is absolutely convergent.

The function $f(x)$ is asymptotically comparable to $1/x^2$, so the integral from 2 to ∞ is also absolutely convergent.

We can conclude that the original integral is absolutely convergent.

Next, we use the following contour. Here we assume the big circles have radius R and the small ones have radius r.



We use the branch cut for square root that removes the positive real axis. In this branch
$$0 < \arg(z) < 2\pi \qquad \text{and} \qquad 0 < \arg(\sqrt{w}) < \pi$$

For $f(z)$, this necessitates the branch cut that removes the rays $[1, \infty)$ and $(-\infty, -1]$ from the complex plane.

The pole at $z = 0$ is the only singularity of $f(z)$ inside the contour. It is easy to compute that

$$\text{Res}(f, 0) = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i.$$

So, the residue theorem gives us

$$\int_{C_1+C_2-C_3-C_4+C_5-C_6-C_7+C_8} f(z)\, dz = 2\pi i \, \text{Res}(f, 0) = 2\pi. \tag{49}$$

In a moment we will show the following limits

$$\lim_{R\to\infty} \int_{C_1} f(z)\, dz = \lim_{R\to\infty} \int_{C_5} f(z)\, dz = 0$$
$$\lim_{r\to 0} \int_{C_3} f(z)\, dz = \lim_{r\to 0} \int_{C_7} f(z)\, dz = 0.$$

We will also show

$$\begin{aligned} \lim_{R\to\infty,\, r\to 0} \int_{C_2} f(z)\, dz &= \lim_{R\to\infty,\, r\to 0} \int_{-C_4} f(z)\, dz \\ &= \lim_{R\to\infty,\, r\to 0} \int_{-C_6} f(z)\, dz = \lim_{R\to\infty,\, r\to 0} \int_{C_8} f(z)\, dz = I. \end{aligned}$$

Using these limits, Equation (49) implies $4I = 2\pi$, i.e.

$$\boxed{I = \pi/2}$$

All that's left is to prove the limits asserted above.

The limits for C_1 and C_5 follow from Theorem 11.1 because

$$|f(z)| \approx 1/|z|^{3/2}$$

for large z.

We get the limit for C_3 as follows. Suppose r is small, say much less than 1. If

$$z = -1 + re^{i\theta}$$

is on C_3 then,

$$|f(z)| = \frac{1}{|z\sqrt{z-1}\sqrt{z+1}|} = \frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}\sqrt{r}} \le \frac{M}{\sqrt{r}}.$$

where M is chosen to be bigger than

$$\frac{1}{|-1+re^{i\theta}|\sqrt{|-2+re^{i\theta}|}}$$

for all small r.

Thus,

$$\left|\int_{C_3} f(z)\,dz\right| \le \int_{C_3} \frac{M}{\sqrt{r}}\,|dz| \le \frac{M}{\sqrt{r}} \cdot 2\pi r = 2\pi M\sqrt{r}.$$

This last expression clearly goes to 0 as $r \to 0$.

The limit for the integral over C_7 is similar.

We can parameterize the straight line C_8 by

$$z = x + i\epsilon$$

where ϵ is a small positive number and x goes from (approximately) 1 to ∞. Thus, on C_8, we have

$$\arg(z^2 - 1) \approx 0 \qquad \text{and} \qquad f(z) \approx f(x)$$

All these approximations become exact as $r \to 0$. Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{C_8} f(z)\,dz = \int_1^\infty f(x)\,dx = I.$$

We can parameterize $-C_6$ by

$$z = x - i\epsilon$$

where x goes from ∞ to 1. Thus, on C_6, we have

$$\arg(z^2 - 1) \approx 2\pi,$$

so

$$\sqrt{z^2 - 1} \approx -\sqrt{x^2 - 1}$$

This implies

$$f(z) \approx -\frac{1}{x\sqrt{x^2 - 1}} = -f(x)$$

Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{-C_6} f(z)\,dz = \int_\infty^1 -f(x)\,dx = \int_1^\infty f(x)\,dx = I.$$

We can parameterize C_2 by $z = -x + i\epsilon$ where x goes from ∞ to 1. Thus, on C_2, we have

$$\arg(z^2 - 1) \approx 2\pi$$

so

$$\sqrt{z^2 - 1} \approx -\sqrt{x^2 - 1}$$

This implies

$$f(z) \approx \frac{1}{(-x)(-\sqrt{x^2 - 1})} = f(x)$$

Thus,

$$\lim_{R\to\infty,\, r\to 0} \int_{C_2} f(z)\, dz = \int_{\infty}^{1} f(x)\,(-dx) = \int_{1}^{\infty} f(x)\, dx = I.$$

The last curve $-C_4$ is handled similarly.

11.5 Cauchy principal value

First an example to motivate defining the principal value of an integral. We'll actually compute the integral in the next section.

Example 11.10. Let

$$I = \int_0^{\infty} \frac{\sin(x)}{x}\, dx$$

This integral is not absolutely convergent, but it is conditionally convergent. Formally, of course, we mean

$$I = \lim_{R\to\infty} \int_0^{R} \frac{\sin(x)}{x}\, dx$$

We can proceed as in Example 11.5. First note that $\sin(x)/x$ is even, so

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx.$$

Next, to avoid the problem that $\sin(z)$ goes to infinity in both the upper and lower half-planes we replace the integrand by $\frac{e^{ix}}{x}$.

We've changed the problem to computing

$$\tilde{I} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx.$$

The problems with this integral are caused by the pole at 0. The biggest problem is that the integral doesn't converge! The other problem is that when we try to use our usual strategy of choosing a closed contour we can't use one that includes $z = 0$ on the real axis. This is our motivation for defining principal value. We will come back to this example below.

Definition. Suppose we have a function $f(x)$ that is continuous on the real line except at the point x_1, then we define the Cauchy principal value as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty,\, r_1\to 0} \int_{-R}^{x_1 - r_1} f(x)\, dx + \int_{x_1 + r_1}^{R} f(x)\, dx.$$

Provided the limit converges. You should notice that the intervals around x_1 and around ∞ are symmetric. Of course, if the integral

$$\int_{-\infty}^{\infty} f(x)\, dx$$

converges, then so does the principal value and they give the same value. We can make the definition more flexible by including the following cases.

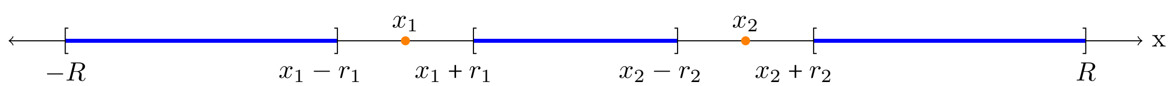
1. If $f(x)$ is continuous on the entire real line then we define the principal value as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

2. If we have multiple points of discontinuity, $x_1 < x_2 < x_3 < \ldots < x_n$, then

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim \int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{x_2-r_2} + \int_{x_2+r_2}^{x_3-r_3} + \ldots \int_{x_n+r_n}^{R} f(x)\,dx.$$

Here the limit is taken as $R \to \infty$ and each of the $r_k \to 0$.



Intervals of integration for principal value are symmetric around x_k and ∞

The next example shows that sometimes the principal value converges when the integral itself does not. The opposite is never true. That is, we have the following theorem.

Theorem 11.11. If $f(x)$ has discontinuities at $x_1 < x_2 < \ldots < x_n$ and $\int_{-\infty}^{\infty} f(x)\,dx$ converges then so does p.v. $\int_{-\infty}^{\infty} f(x)\,dx$.

Proof. The proof amounts to understanding the definition of convergence of integrals as limits. The integral converges means that each of the limits

$$\begin{aligned} &\lim_{R_1\to\infty,\, a_1\to 0} \int_{-R_1}^{x_1-a_1} f(x)\,dx \\ &\lim_{b_1\to 0,\, a_2\to 0} \int_{x_1+b_1}^{x_2-a_2} f(x)\,dx \\ &\qquad\qquad\qquad\ldots \\ &\lim_{R_2\to\infty,\, b_n\to 0} \int_{x_n+b_n}^{R_2} f(x)\,dx \end{aligned} \tag{50}$$

converges. There is no symmetry requirement, i.e. R_1 and R_2 are completely independent, as are a_1 and b_1 etc.

The principal value converges means

$$\lim \int_{-R}^{x_1-r_1} + \int_{x_1+r_1}^{x_2-r_2} + \int_{x_2+r_2}^{x_3-r_3} + \ldots \int_{x_n+r_n}^{R} f(x)\,dx \tag{51}$$

converges. Here the limit is taken over all the parameter $R \to \infty$, $r_k \to 0$. This limit has symmetry, e.g. we replaced both a_1 and b_1 in Equation (50) by r_1 etc. Certainly if the limits in Equation (50) converge then so do the limits in Equation (51). □

Example 11.12. Consider both

$$\int_{-\infty}^{\infty} \frac{1}{x}\,dx \qquad \text{and} \qquad \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x}\,dx$$

The first integral diverges since

$$\int_{-R_1}^{-r_1} \frac{1}{x}\,dx + \int_{r_2}^{R_2} \frac{1}{x}\,dx = \ln(r_1) - \ln(R_1) + \ln(R_2) - \ln(r_2).$$

This clearly diverges as R_1, $R_2 \to \infty$ and r_1, $r_2 \to 0$.

On the other hand the symmetric integral

$$\int_{-R}^{-r} \frac{1}{x}\,dx + \int_{r}^{R} \frac{1}{x}\,dx = \ln(r) - \ln(R) + \ln(R) - \ln(r) = 0.$$

This clearly converges to 0.

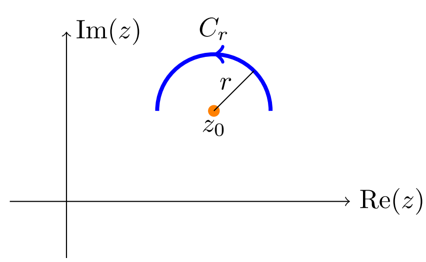
We will see that the principal value occurs naturally when we integrate on semicircles around points. We prepare for this in the next section.

11.6 Integrals over portions of circles

We will need the following theorem in order to combine principal value and the residue theorem.

Theorem 11.13. Suppose $f(z)$ has a **simple** pole at z_0. Let C_r be the semicircle $\gamma(\theta) = z_0 + r\mathrm{e}^{i\theta}$, with $0 \le \theta \le \pi$. Then

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = \pi i \,\mathrm{Res}(f, z_0) \tag{52}$$



Small semicircle of radius r around z_0

Proof. Since we take the limit as r goes to 0, we can assume r is small enough that $f(z)$ has a Laurent expansion of the punctured disk of radius r centered at z_0. That is, since the pole is simple,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots \qquad \text{for } 0 < |z - z_0| \le r.$$

Thus,

$$\int_{C_r} f(z)\,dz = \int_0^{\pi} f(z_0 + r\mathrm{e}^{i\theta})\, ri\mathrm{e}^{i\theta}\,d\theta = \int_0^{\pi} \left(b_1 i + a_0 ir\mathrm{e}^{i\theta} + a_1 ir^2\mathrm{e}^{i2\theta} + \ldots\right) d\theta$$

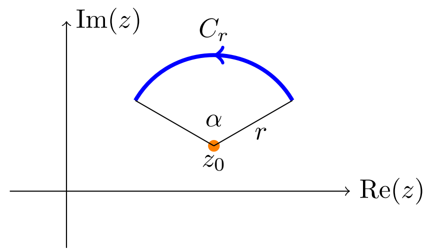
The b_1 term gives $\pi i b_1$. Clearly all the other terms go to 0 as $r \to 0$. $\square$

If the pole is not simple the theorem doesn't hold and, in fact, the limit does not exist.

The same proof gives a slightly more general theorem.

Theorem 11.14. Suppose $f(z)$ has a **simple** pole at z_0. Let C_r be the circular arc $\gamma(\theta) = z_0 + re^{i\theta}$, with $\theta_0 \le \theta \le \theta_0 + \alpha$. Then

$$\lim_{r\to 0} \int_{C_r} f(z)\, dz = \alpha i \operatorname{Res}(f, z_0)$$



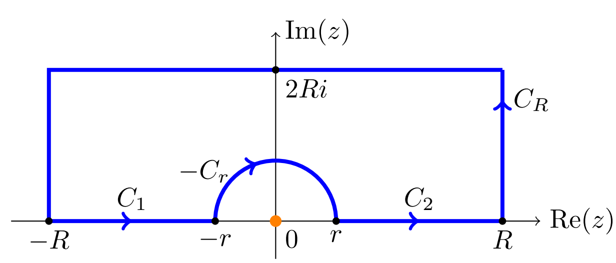
Small circular arc of radius r around z_0

Example 11.15. (Return to Example 11.10.) A long time ago we left off Example 11.10 to define principal value. Let's now use the principal value to compute

$$\tilde{I} = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$$

Solution: We use the indented contour shown below. The indentation is the little semicircle the goes around $z = 0$. There are no poles inside the contour so the residue theorem implies

$$\int_{C_1 - C_r + C_2 + C_R} \frac{e^{iz}}{z}\, dz = 0.$$



Next we break the contour into pieces.

$$\lim_{R\to\infty,\, r\to 0} \int_{C_1+C_2} \frac{e^{iz}}{z}\, dz = \tilde{I}.$$

Theorem 11.2(a) implies

$$\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{e}^{iz}}{z}\, dz = 0$$

Equation (52) in Theorem 11.13 tells us that

$$\lim_{r\to 0} \int_{C_r} \frac{\mathrm{e}^{iz}}{z}\, dz = \pi i \operatorname{Res}\left(\frac{\mathrm{e}^{iz}}{z}, 0\right) = \pi i$$

Combining all this together we have

$$\lim_{R\to\infty,\, r\to 0} \int_{C_1 - C_r + C_2 + C_R} \frac{\mathrm{e}^{iz}}{z}\, dz = \tilde{I} - \pi i = 0,$$

so $\tilde{I} = \pi i$. Thus, looking back at Example 52, where $I = \displaystyle\int_0^\infty \frac{\sin(x)}{x}\, dx$, we have

$$I = \frac{1}{2}\operatorname{Im}(\tilde{I}) = \frac{\pi}{2}.$$

There is a subtlety about convergence we alluded to above. That is, I is a genuine (conditionally) convergent integral, but $\tilde{I}$ only exists as a principal value. However since I is a convergent integral we know that computing the principle value as we just did is sufficient to give the value of the convergent integral.

11.7 Fourier transform

Definition. The Fourier transform of a function $f(x)$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ix\omega}\, dx$$

This is often read as 'f-hat'.

Theorem. (Fourier inversion formula.) We can recover the original function $f(x)$ with the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\mathrm{e}^{ix\omega}\, d\omega.$$

So, the Fourier transform converts a function of x to a function of ω and the Fourier inversion converts it back. Of course, everything above is dependent on the convergence of the various integrals.

Proof. We will not give the proof here.

Example 11.16. Let

$$f(t) = \begin{cases} \mathrm{e}^{-at} & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where $a > 0$. Compute $\widehat{f}(\omega)$ and verify the Fourier inversion formula in this case.

Solution: Computing $\widehat{f}$ is easy: For $a > 0$

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,dt = \int_{0}^{\infty} \mathrm{e}^{-at}\mathrm{e}^{-i\omega t}\,dt = \frac{1}{a+i\omega}$$

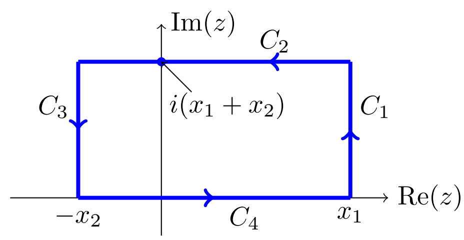
We should first note that the inversion integral converges. To avoid distraction we show this at the end of this example.

Now, let

$$g(z) = \frac{1}{a+iz}$$

Note that $\widehat{f}(\omega) = g(\omega)$ and $|g(z)| < \dfrac{M}{|z|}$ for large $|z|$.

To verify the inversion formula we consider the cases $t > 0$ and $t < 0$ separately. For $t > 0$ we use the standard contour.



Theorem 11.2(a) implies that

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_1+C_2+C_3} g(z)\mathrm{e}^{izt}\,dz = 0 \tag{53}$$

Clearly

$$\lim_{x_1\to\infty,\, x_2\to\infty} \int_{C_4} g(z)\mathrm{e}^{izt}\,dz = \int_{-\infty}^{\infty} \widehat{f}(\omega)\,d\omega \tag{54}$$

The only pole of $g(z)\mathrm{e}^{izt}$ is at $z = ia$, which is in the upper half-plane. So, applying the residue theorem to the entire closed contour, we get for large x_1, x_2:

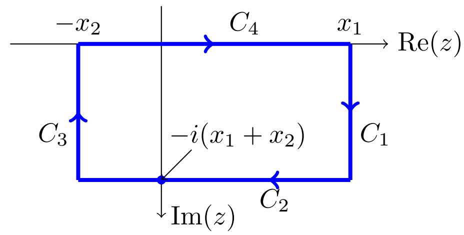
$$\int_{C_1+C_2+C_3+C_4} g(z)\mathrm{e}^{izt}\,dz = 2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{izt}}{a+iz}, ia\right) = \frac{\mathrm{e}^{-at}}{i}. \tag{55}$$

Combining the three equations 53, 54 and 55, we have

$$\int_{-\infty}^{\infty} \widehat{f}(\omega)\,d\omega = 2\pi\mathrm{e}^{-at} \qquad \text{for } t > 0$$

This shows the inversion formula holds for $t > 0$.

For $t < 0$ we use the contour



Theorem 11.2(b) implies that

$$\lim_{x_1 \to \infty,\, x_2 \to \infty} \int_{C_1 + C_2 + C_3} g(z) \mathrm{e}^{izt}\, dz = 0$$

Clearly

$$\lim_{x_1 \to \infty,\, x_2 \to \infty} \frac{1}{2\pi} \int_{C_4} g(z) \mathrm{e}^{izt}\, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, d\omega$$

Since, there are no poles of $g(z)\mathrm{e}^{izt}$ in the lower half-plane, applying the residue theorem to the entire closed contour, we get for large x_1, x_2:

$$\int_{C_1 + C_2 + C_3 + C_4} g(z) \mathrm{e}^{izt}\, dz = -2\pi i \operatorname{Res}\left(\frac{\mathrm{e}^{izt}}{a + iz}, ia \right) = 0.$$

Thus,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, d\omega = 0 \qquad \text{for } t < 0$$

This shows the inversion formula holds for $t < 0$.

Finally, we give the promised argument that the inversion integral converges. By definition

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{i\omega t}\, d\omega &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega t}}{a + i\omega}\, d\omega \\ &= \int_{-\infty}^{\infty} \frac{a\cos(\omega t) + \omega \sin(\omega t) - i\omega \cos(\omega t) + ia\sin(\omega t)}{a^2 + \omega^2}\, d\omega \end{aligned}$$

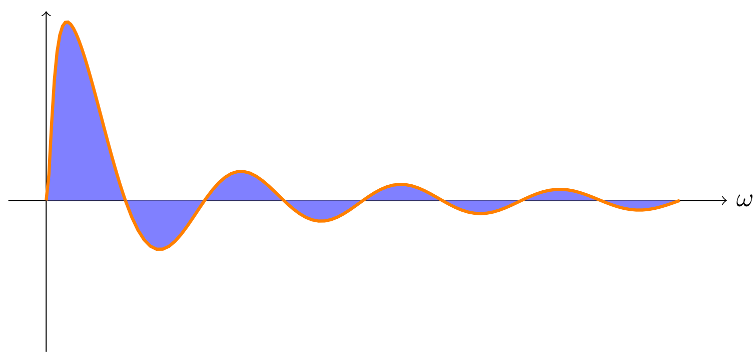
The terms without a factor of ω in the numerator converge absolutely because of the ω^2 in the denominator. The terms with a factor of ω in the numerator do not converge absolutely. For example, since

$$\frac{\omega \sin(\omega t)}{a^2 + \omega^2}$$

decays like $1/\omega$, its integral is not absolutely convergent. However, we claim that the integral does converge conditionally. That is, both limits

$$\lim_{R_2 \to \infty} \int_0^{R_2} \frac{\omega \sin(\omega t)}{a^2 + \omega^2}\, d\omega \quad \text{and} \quad \lim_{R_1 \to \infty} \int_{-R_1}^{0} \frac{\omega \sin(\omega t)}{a^2 + \omega^2}\, d\omega$$

exist and are finite. The key is that, as $\sin(\omega t)$ alternates between positive and negative arches, the function $\dfrac{\omega}{a^2+\omega^2}$ is decaying monotonically. So, in the integral, the area under each arch adds or subtracts less than the arch before. This means that as R_1 (or R_2) grows the total area under the curve oscillates with a decaying amplitude around some limiting value.



Total area oscillates with a decaying amplitude.

11.8 Solving ODEs using the Fourier transform

Let

$$D = \frac{d}{dt}$$

Our goal is to see how to use the Fourier transform to solve differential equations like

$$P(D)y = f(t)$$

Here $P(D)$ is a polynomial operator, e.g.

$$D^2 + 8D + 7I$$

We first note the following formula:

$$\widehat{Df}(\omega) = i\omega\widehat{f}. \tag{56}$$

Proof. This is just integration by parts:

$$\begin{aligned}
\widehat{Df}(\omega) &= \int_{-\infty}^{\infty} f'(t)\mathrm{e}^{-i\omega t}\,dt \\
&= f(t)\mathrm{e}^{-i\omega t}\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega\mathrm{e}^{-i\omega t}\,dt \\
&= i\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,dt \\
&= i\omega\widehat{f}(\omega)
\end{aligned}$$

In the third line we assumed that f decays so that $f(\infty) = f(-\infty) = 0$.

It is a simple extension of Equation (56) to see

$$(\widehat{P(D)f})(\omega) = P(i\omega)\widehat{f}.$$

We can now use this to solve some differential equations.

Example 11.17. Solve the equation

$$y''(t) + 8y'(t) + 7y(t) = f(t) = \begin{cases} \mathrm{e}^{-at} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Solution: In this case, we have

$$P(D) = D^2 + 8D + 7I$$

so

$$P(s) = s^2 + 8s + 7 = (s+7)(s+1)$$

The ODE

$$P(D)y = f(t)$$

transforms to

$$P(iw)\widehat{y} = \widehat{f}$$

Using the Fourier transform of f found in Example 11.16 we have

$$\widehat{y}(\omega) = \frac{\widehat{f}}{P(i\omega)} = \frac{1}{(a+i\omega)(7+i\omega)(1+i\omega)}.$$

Fourier inversion says that

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\mathrm{e}^{i\omega t}\,d\omega$$

As always, we want to extend $\widehat{y}$ to be function of a complex variable z. Let's call it $g(z)$:

$$g(z) = \frac{1}{(a+iz)(7+iz)(1+iz)}.$$

Now we can proceed exactly as in Example 11.16. We know $|g(z)| < M/|z|^3$ for some constant M. Thus, the conditions of Theorem 11.2 are easily met. So, just as in Example 11.16, we have:

For $t > 0$, e^{izt} is bounded in the upper half-plane, so we use the contour below on the left.

$$\begin{aligned} y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\mathrm{e}^{i\omega t}\,d\omega &= \frac{1}{2\pi}\lim_{x_1\to\infty,\, x_2\to\infty}\int_{C_4} g(z)\mathrm{e}^{izt}\,dz \\ &= \frac{1}{2\pi}\lim_{x_1\to\infty,\, x_2\to\infty}\int_{C_1+C_2+C_3+C_4} g(z)\mathrm{e}^{izt}\,dz \\ &= i\sum \text{residues of } \mathrm{e}^{izt}g(z) \text{ in the upper half-plane} \end{aligned}$$

The poles of $\mathrm{e}^{izt}g(z)$ are at

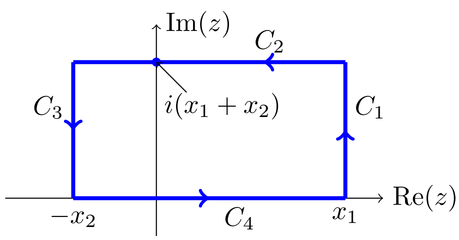
$$ia, \quad 7i, \quad i$$

These are all in the upper half-plane. The residues are respectively,

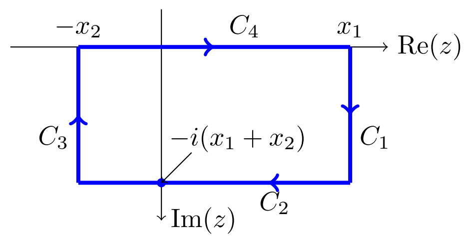
$$\frac{\mathrm{e}^{-at}}{i(7-a)(1-a)}, \quad \frac{\mathrm{e}^{-7t}}{i(a-7)(-6)}, \quad \frac{\mathrm{e}^{-t}}{i(a-1)(6)}$$

Thus, for $t > 0$ we have

$$y(t) = \frac{\mathrm{e}^{-at}}{(7-a)(1-a)} - \frac{\mathrm{e}^{-7t}}{(a-7)(6)} + \frac{\mathrm{e}^{-t}}{(a-1)(6)}.$$



Contour for $t > 0$



Contour for $t < 0$

More briefly, when $t < 0$ we use the contour above on the right. We get the exact same string of equalities except the sum is over the residues of $\mathrm{e}^{izt}g(z)$ in the lower half-plane. Since there are no poles in the lower half-plane, we find that

$$\widehat{y}(t) = 0$$

when $t < 0$.

Conclusion (reorganizing the signs and order of the terms):

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{\mathrm{e}^{-at}}{(7-a)(1-a)} + \frac{\mathrm{e}^{-7t}}{(7-a)(6)} - \frac{\mathrm{e}^{-t}}{(1-a)(6)} & \text{for } t > 0. \end{cases}$$

Note. Because $|g(z)| < M/|z|^3$, we could replace the rectangular contours by semicircles to compute the Fourier inversion integral.

Example 11.18. Consider

$$y'' + y = f(t) = \begin{cases} \mathrm{e}^{-at} & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Find a solution for $t > 0$.

Solution: We work a little more quickly than in the previous example.

Taking the Fourier transform we get

$$\widehat{y}(\omega) = \frac{\widehat{f}(\omega)}{P(i\omega)} = \frac{\widehat{f}(\omega)}{1-\omega^2} = \frac{1}{(a+i\omega)(1-\omega^2)}.$$

(In the last expression, we used the known Fourier transform of f.)

As usual, we extend $\widehat{y}(\omega)$ to a function of z:

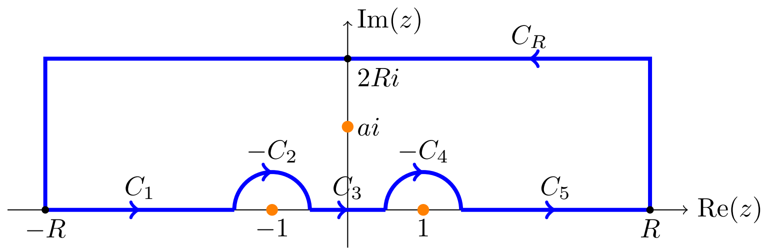
$$g(z) = \frac{1}{(a+iz)(1-z^2)}$$

This has simple poles at

$$-1, \quad 1, \quad ai$$

. Since some of the poles are on the real axis, we will need to use an indented contour along the real axis and use principal value to compute the integral.

The contour is shown below. We assume each of the small indents is a semicircle with radius r. The big rectangular path from $(R,0)$ to $(-R,0)$ is called C_R.



For $t > 0$ the function $\mathrm{e}^{izt}g(z) < M/|z|^3$ in the upper half-plane. Thus, we get the following limits:

$$\lim_{R\to\infty} \int_{C_R} \mathrm{e}^{izt}g(z)\,dz = 0 \qquad \text{(Theorem 11.2(b))}$$

$$\lim_{R\to\infty,\,r\to 0} \int_{C_2} \mathrm{e}^{izt}g(z)\,dz = \pi i\,\mathrm{Res}(\mathrm{e}^{izt}g(z), -1) \qquad \text{(Theorem 11.14)}$$

$$\lim_{R\to\infty,\,r\to 0} \int_{C_4} \mathrm{e}^{izt}g(z)\,dz = \pi i\,\mathrm{Res}(\mathrm{e}^{izt}g(z), 1) \qquad \text{(Theorem 11.14)}$$

$$\lim_{R\to\infty,\,r\to 0} \int_{C_1+C_3+C_5} \mathrm{e}^{izt}g(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty} \widehat{y}(t)\mathrm{e}^{i\omega t}\,dt$$

Putting this together with the residue theorem we have

$$\begin{aligned}
&\lim_{R\to\infty,\,r\to 0} \int_{C_1-C_2+C_3-C_4+C_5+C_R} \mathrm{e}^{izt}g(z)\,dz \\
&\quad = \text{p.v.} \int_{-\infty}^{\infty} \widehat{y}(t)\mathrm{e}^{i\omega t}\,dt - \pi i\,\mathrm{Res}(\mathrm{e}^{izt}g(z), -1) - \pi i\,\mathrm{Res}(\mathrm{e}^{izt}g(z), 1) \\
&\quad = 2\pi i\,\mathrm{Res}(\mathrm{e}^{izt}, ai).
\end{aligned}$$

All that's left is to compute the residues and do some arithmetic. We don't show the calculations, but give the results

$$\begin{aligned}
\operatorname{Res}(\mathrm{e}^{izt}g(z),-1) &= \frac{\mathrm{e}^{-it}}{2(a-i)} \\
\operatorname{Res}(\mathrm{e}^{izt}g(z),1) &= -\frac{\mathrm{e}^{it}}{2(a+i)} \\
\operatorname{Res}(\mathrm{e}^{izt}g(z),ai) &= -\frac{\mathrm{e}^{-at}}{i(1+a^2)}
\end{aligned}$$

We get, for $t > 0$,

$$\begin{aligned}
y(t) &= \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty}\widehat{y}(t)\mathrm{e}^{i\omega t}\,dt \\
&= \frac{i}{2}\operatorname{Res}(\mathrm{e}^{izt}g(z),-1) + \frac{i}{2}\operatorname{Res}(\mathrm{e}^{izt}g(z),1) + i\operatorname{Res}(\mathrm{e}^{izt}g(z),ai) \\
&= \frac{e^{-at}}{1+a^2} + \frac{a}{1+a^2}\sin(t) - \frac{1}{1+a^2}\cos(t)
\end{aligned}$$