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Physica A 374 (2007) 559–572
PHYSICA A
www.elsevier.com/locate/physa

One-dimensional non-relativistic and relativistic Brownian motions: a microscopic collision model

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Received 4 May 2006
Available online 1 August 2006

Abstract

We study a simple microscopic model for the one-dimensional stochastic motion of a (non-)relativistic Brownian particle, embedded into a heat bath consisting of (non-)relativistic particles. The stationary momentum distributions are identified self-consistently (for both Brownian and heat bath particles) by means of two coupled integral criteria. The latter follow directly from the kinematic conservation laws for the microscopic collision processes, provided one additionally assumes probabilistic independence of the initial momenta. It is shown that, in the non-relativistic case, the integral criteria do correctly identify the Maxwellian momentum distributions as stationary (invariant) solutions. Subsequently, we apply the same criteria to the relativistic case. Surprisingly, we find here that the stationary momentum distributions differ slightly from the standard Jüttner distribution by an additional prefactor proportional to the inverse relativistic kinetic energy.

Keywords: Random walk; Lattice models; Relativistic Brownian motion; Relativistic collision processes

1. Introduction

The implementation of the Brownian motion concept [1–6] into special relativity [7,8] represents a longstanding issue in mathematical and statistical physics (classical references are [9–11]; more recent contributions include [12–22]; for a kinetic theory approach, see Refs. [23–25]). In two recent papers [26,27] we have discussed in detail how one can construct Langevin equations for relativistic Brownian motions (see Debbasch et al. [28,29] and Zygadlo [30] for similar approaches, and also Dunkel and Hänggi [31]). Thereby, it was demonstrated that, in general, the relativistic Langevin equation per se cannot uniquely determine the corresponding Fokker–Planck equation (FPE). This dilemma is caused by the fact that the relativistic Langevin equations, e.g., if written in laboratory coordinates, may exhibit a multiplicative coupling between a function of the momentum coordinate and a Gaussian white noise process (laboratory frame ≡ rest frame of

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2006.07.013

the heat bath). Thus, depending on the choice of the discretization rule, one obtains different forms of relativistic FPEs characterized by different stationary solutions.

In Refs. [26,27] we have analyzed the three most popular discretization rules for Langevin equations with multiplicative noise, namely, Ito's pre-point discretization rule [32,33], the Fisk–Stratonovich mid-point rule [34–37], and the Hänggi–Klimontovich (HK) post-point rule [38–41]. As a main result it was found that only the HK interpretation of the Langevin equation yields a FPE, whose stationary solution coincides with the one-dimensional relativistic Jüttner–Maxwell distribution, as known from Jüttner's early work on the relativistic gas [42,43] and also from the relativistic kinetic theory [23,24]. Thus, in absence of other qualifying criteria, one may conclude that the post-point discretization rule is favorable. However, it naturally arises the question, if one can gain additional insights by studying the microscopic collision processes, which cause the stochastic motion of a Brownian particle.

The present paper intends to partially address this question for the one-dimensional (1D) case; i.e., instead of focusing on the "macroscopic" Langevin formulation, we will study here Brownian motions by means of a simple microscopic 1D model. Since, on the level of the Brownian motion approach, the exact nature of the microscopic interaction forces is usually negligible we shall restrict ourselves to purely elastic collisions between a Brownian particle and the constituents of a surrounding heat bath. As will be demonstrated below, this simple model suffices to identify stationary (invariant) momentum distributions for both non-relativistic and relativistic 1D collision processes.[1]

The paper is organized as follows: Section 2 is dedicated to the non-relativistic case, serving as the test example for our approach. After briefly summarizing the basic model assumptions, we will derive two coupled integral criteria (Section 2.3), which can be used to identify the invariant momentum distributions for the heat bath particles and the Brownian particles, respectively. It is shown that these integral criteria do indeed yield the correct stationary momentum distribution of the non-relativistic Brownian motion, namely, the well-known Maxwellian momentum distribution (Section 2.4). Afterwards, the same method is applied to the relativistic case (Section 3). Remarkably, we find that, under exactly same preconditions, the invariant relativistic distributions do *not* exactly correspond to the standard Jüttner distributions, but rather to modified Jüttner functions including an additional prefactor $1/E$, where E is the relativistic kinetic energy of the particle under consideration. The paper concludes with a summary and a discussion of the results (Section 4).

2. Non-relativistic Brownian motions

In this part, we review non-relativistic 1D-Brownian motions. Later on, we will pursue an analogous approach to identify the stationary momentum distribution of a relativistic Brownian particle, embedded into a relativistic heat bath.

2.1. Basic model assumptions

In the three-dimensional (3D) case a simple idealized model for Brownian motions can be imagined as follows: Consider a (infinitely heavy) box of volume $\mathscr{V}$, possessing diathermal [44] walls and being at rest in the inertial laboratory frame Σ_0. Let this box contain a homogeneous, quasi-ideal (weakly interacting) gas, consisting of approximately point-like particles with identical masses m. Further, assume that the gas (or liquid) particles—referred to as 'heat bath', hereafter—surround a Brownian particle of mass $M \gg m$. Then, due to frequent elastic collisions with heat bath particles, the Brownian particle performs 3D random motions. Given the distribution of the heat bath, the stochastic dynamics of the Brownian particle is determined by the collision kinematics (cross-sections) governing the interaction with the heat bath particles.

If, as in the present paper, one wishes to study the 1D case, slight modifications of the above model are necessary. The reason is that, typically, two particles cannot simply exchange positions if their motions are confined to one dimension. To circumvent this problem, we shall therefore imagine the heat bath particles as having *fixed positions* on a 1D lattice but *non-vanishing momenta*. Correspondingly, in this 1D (lattice) model

[1] More precisely, we shall additionally require the probabilistic independence of the initial momenta during each single collision—but this is a rather reasonable, weak restriction.

the Brownian particle may jump from one lattice point to the next during one time step. Additionally, we will impose that at each lattice point there does indeed occur an elastic interaction in accordance with the laws of momentum and energy conservation. Mathematically, the latter assumption corresponds to considering distributions conditional on the event "a collision has occurred".

Based on this idealized 1D (lattice) model, it is our primary objective to determine self-consistently the invariant (i.e., stationary) momentum distributions for both heat bath and Brownian particles. To this end, we shall next summarize the simple kinematic equations governing the collisions in this model. By interpreting the momentum coordinates as coupled random variables, we then derive in Section 2.3 two general integral criteria which have to be satisfied by the stationary distributions. As shown in Section 2.4, in the non-relativistic case the stationary solutions are given by the well-known Maxwell distributions. The integral criteria apply to both non-relativistic and relativistic collisions; hence, we can use them later to also identify the stationary distributions of the corresponding relativistic model (cf. Section 3).

2.2. Kinematics of single collision events

The momentum and energy balance per (elastic) collision reads

$$E + \varepsilon = \hat{E} + \hat{\varepsilon}, \tag{1a}$$

$$P + p = \hat{P} + \hat{p}. \tag{1b}$$

Here and below, capital letters refer to the Brownian particle and small letters to heat bath particles; quantities without (with) hat-symbols refer to the state before (after) the collision. In the non-relativistic case, we have, e.g., before the collision

$$P = MV, \quad p = mv, \tag{2a}$$

$$E = \frac{P^2}{2M}, \quad \varepsilon = \frac{p^2}{2m}, \tag{2b}$$

where v and V denote the velocities with respect to the laboratory frame Σ_0. Taking into account both conservation of momentum and (kinetic) energy, one finds for a single collision the elementary results

$$\hat{P}(p, P) = \left(\frac{2M}{M+m}\right)p + \left(\frac{M-m}{M+m}\right)P, \tag{3a}$$

$$\hat{p}(P, p) = \left(\frac{2m}{M+m}\right)P + \left(\frac{m-M}{M+m}\right)p. \tag{3b}$$

We again stress that Eqs. (3) do implicitly assume that a collision has indeed occurred (otherwise, the momenta would remain unchanged); i.e., in mathematical terms, any results obtained by employing Eqs. (3) are valid conditional on the information that a collision event has taken place.

Now let us suppose we know the joint two-particle PDF $\psi_2(p, P)$ for the particle momenta *before* the collision. Then, the kinematic laws (3) determine uniquely the marginal momentum PDFs $\hat{\Phi}(\hat{P})$ and $\hat{\phi}(\hat{p})$ after the collision, formally defined by

$$\hat{\Phi}(\hat{P}) = \int \mathrm{d}\hat{p}\,\hat{\psi}_2(\hat{p}, \hat{P}), \tag{4a}$$

$$\hat{\phi}(\hat{p}) = \int \mathrm{d}\hat{P}\,\hat{\psi}_2(\hat{p}, \hat{P}), \tag{4b}$$

where $\hat{\psi}_2(\hat{p}, \hat{P})$ is the joint momentum PDF *after* the collision (here and below integrals with unspecified boundaries range from $-\infty$ to $+\infty$.) Our main objective will be to identify *stationary* (*invariant*) momentum distributions, satisfying *by definition*

$$\hat{\psi}_2(\hat{p}, \hat{P}) \equiv \psi_2(\hat{p}, \hat{P}). \tag{5}$$

The latter condition just means that a stationary PDF must remain invariant in microscopic collisions. In particular, we shall look for stationary solutions which can be written in the product form

$$\psi_2(p,P)=\phi(p)\,\Phi(P). \tag{6}$$

Mathematically, this corresponds to the assumption that, *in the stationary state, the momenta P and p can be considered as independently distributed random variables*. For stationary PDFs of form (6) the stationarity criterion (7) reduces to

$$\hat{\Phi}(\hat{P})=\Phi(\hat{P}),\quad \hat{\phi}(\hat{p})=\phi(\hat{p}), \tag{7}$$

where $\hat{\psi}_2(\hat{p},\hat{P})=\hat{\phi}(\hat{p})\,\hat{\Phi}(\hat{P})$ is the joint distribution after the collision. From a physical point of view, the independence assumption for (p,P) or, alternatively, of $(\hat{p},\hat{P})$ is guided by the experience that the well-known equilibrium momentum distributions of quasi-ideal non-relativistic and relativistic N particle gases (i.e., the Maxwell and Jüttner distributions) can be written as products of one-particle momentum distributions.

In order to be able to determine the stationary PDFs for a given collision kinematics, we next derive general integral criteria. It comes as no surprise that, in the non-relativistic case, the stationary solutions will be given by a pair of Maxwellians.

2.3. Integral criteria for stationary momentum distributions

Consider two *independently* distributed random variables Y and Z, and a derived random variable $X=X(Y,Z)$. The corresponding PDFs are denoted by $\Phi_X(X)$, $\Phi_Y(Y)$ and $\Phi_Z(Z)$. The average of some test function $g(X)$ with respect to Φ_X can then be written as

$$\int \mathrm{d}X g(X)\Phi_X(X)=\int \mathrm{d}Y\int \mathrm{d}Z g(X(Y,Z))\Phi_Y(Y)\,\Phi_Z(Z). \tag{8}$$

Assuming that the (partially) inverse transformation $Z=Z(Y,X)$ is well-defined (i.e., strictly monotonous for each fixed Y), and that Fubini's theorem [45] is applicable, we can rewrite the last equation in the form

$$\begin{aligned}\int \mathrm{d}X g(X)\Phi_X(X) &= \int \mathrm{d}Y\int \mathrm{d}X\left|\frac{\partial Z}{\partial X}\right| g(X)\Phi_Y(Y)\,\Phi_Z(Z(Y,X))\\ &= \int \mathrm{d}X g(X)\int \mathrm{d}Y\left|\frac{\partial Z}{\partial X}\right|\Phi_Y(Y)\,\Phi_Z(Z(Y,X)).\end{aligned} \tag{9}$$

Since the latter equation holds for any test function g, one obtains the well-known transformation law

$$\Phi_X(X)=\int \mathrm{d}Y\left|\frac{\partial Z}{\partial X}\right|\Phi_Y(Y)\Phi_Z(Z(Y,X)). \tag{10}$$

For completeness, we note that Eq. (10) can also be obtained by starting from

$$\Phi_X(x)=\int \mathrm{d}Y\int \mathrm{d}Z\,\delta(x-X(Y,Z))\Phi_Y(Y)\Phi_Z(Z) \tag{11}$$

and performing the Z-integration (with δ denoting the Dirac delta-function). We next consider an explicit example, which will be investigated in detail in the remainder.

Example ($\hat{P}=\hat{P}(p,P)$ *and* $\hat{p}=\hat{p}(P,p)$). The idea is that we express the final momenta in terms of the initial momenta, cf. Eq. (3). That is, setting $X=\hat{P}$, $Y=p$, $Z=P$ and, correspondingly, $\Phi_X\equiv\hat{\Phi}$, $\Phi_Y\equiv\phi$, $\Phi_Z\equiv\Phi$, we can write Eq. (10) as

$$\hat{\Phi}(\hat{P})=\int \mathrm{d}p\left|\frac{\partial P}{\partial \hat{P}}\right|\phi(p)\,\Phi(P(p,\hat{P})). \tag{12a}$$

For a given pair of initial distributions (Φ,ϕ), this equation can be used to calculate the PDF of the Brownian particle after the collision. Analogously, by setting $X=\hat{p}$, $Y=P$, $Z=p$ and $\Phi_X\equiv\hat{\phi}$, $\Phi_Y\equiv\Phi$,

$\Phi_Z \equiv \phi$, we find for the PDF of the heat bath particles

$$\hat{\phi}(\hat{p}) = \int \mathrm{d}P \left|\frac{\partial p}{\partial \hat{p}}\right| \Phi(P)\phi(p(P,\hat{p})). \tag{12b}$$

In particular, for *stationary* distributions satisfying Eqs. (7) we have $\hat{\phi} \equiv \phi$ and $\hat{\Phi} \equiv \Phi$ and, hence, obtain from Eqs. (12) the integral criteria

$$\Phi(\hat{P}) = \int \mathrm{d}p \left|\frac{\partial P}{\partial \hat{P}}\right| \phi(p)\Phi(P(p,\hat{P})), \tag{13a}$$

$$\phi(\hat{p}) = \int \mathrm{d}P \left|\frac{\partial p}{\partial \hat{p}}\right| \Phi(P)\phi(p(P,\hat{p})). \tag{13b}$$

Given a certain microscopic kinematic law, any pair (Φ, ϕ) satisfying Eqs. (13) provides a self-consistent stationary distribution. For completeness we mention that mathematically equivalent criteria are obtained by exchanging the positions of p and P as functional arguments, i.e., by considering $\hat{P} = \hat{P}(P,p)$ and/or $\hat{p} = \hat{p}(p,P)$, respectively.

Before discussing solutions of Eqs. (13) for the non-relativistic Brownian motion, it is worthwhile to stress the following fact: Since the derivation of Eqs. (13) is based on rather general assumptions, these integral equations can be applied to find the stationary PDF Φ not only in the non-relativistic but also in the relativistic case (as will be done in Section 3). The additional mathematical assumption, underlying the derivation of Eqs. (13), is that the initial momenta P and p can be viewed as independently distributed random variables; i.e., loosely speaking, this postulate is the only point leaving some freedom for potential modifications, all other parts are dictated by the physical conservation laws. As stated before, from a physical point of view, the independence assumption for (p,P) or, alternatively, of $(\hat{p},\hat{P})$ is guided by the experience that the well-known equilibrium momentum distributions of quasi-ideal non-relativistic and relativistic N particle gases (i.e., the Maxwell and Jüttner distributions) can usually be written as products of one-particle momentum distributions.

2.4. Stationarity of the Maxwell distribution

In the last part of this section, we briefly outline that the integral criteria Eqs. (13) are satisfied by the normalized Maxwell distributions

$$\Phi(P) = \left(\frac{1}{2\pi M k_B T}\right)^{1/2} \exp\left(-\frac{P^2}{2Mk_B T}\right), \tag{14a}$$

$$\phi(p) = \left(\frac{1}{2\pi m k_B T}\right)^{1/2} \exp\left(-\frac{p^2}{2mk_B T}\right), \tag{14b}$$

where k_B denotes the Boltzmann constant, and T is the temperature parameter.

In order to apply Eqs. (13), we require functions $P(p,\hat{P})$ and $p(P,\hat{p})$. From Eqs. (3), we find

$$P(p,\hat{P}) = \left(\frac{M+m}{M-m}\right)\hat{P} - \left(\frac{2M}{M-m}\right)p, \tag{15a}$$

$$p(P,\hat{p}) = \left(\frac{M+m}{m-M}\right)\hat{p} - \left(\frac{2m}{m-M}\right)P, \tag{15b}$$

and, thus, Eqs. (13) take the explicit form

$$\Phi(\hat{P}) = \left|\frac{M+m}{M-m}\right| \int \mathrm{d}p\,\phi(p)\,\Phi(P(p,\hat{P})), \tag{16a}$$

$$\phi(\hat{p}) = \left|\frac{M+m}{m-M}\right| \int \mathrm{d}P\,\Phi(P)\,\phi(p(P,\hat{p})). \tag{16b}$$

As one can now easily verify by insertion, the Maxwell distributions (14) do indeed satisfy Eqs. (16). Consequently, Eqs. (14) provide a pair of self-consistent stationary solutions for the non-relativistic collision kinematics, conditional on the information that a collision has occurred.

3. Relativistic Brownian motions

Since the integral equations (12) and (13) apply to the relativistic case as well, we can, in principle, proceed exactly analogous to the non-relativistic case. However, some purely technical difficulties arise due to the facts that: (i) the relativistic collision kinematics is more complex than the non-relativistic one, and (ii) the potential candidates for stationary distributions do not allow for the solving Eqs. (12) and (13) analytically. Hence, after having specified all required transformation formulae (Section 3.1), we will evaluate the PDFs on the left-hand sides of the non-stationary integral equations (12) numerically,[2] probing different types of candidate distributions (Section 3.2).

3.1. Kinematics of a single collision event

In the relativistic case, the momentum and energy balance per (elastic) collision can again be written in the form

$$E + \varepsilon = \hat{E} + \hat{\varepsilon}, \tag{17a}$$

$$P + p = \hat{P} + \hat{p}. \tag{17b}$$

Compared with the non-relativistic case, the only difference is that we now use the relativistic expressions for momentum and kinetic energy, respectively. Specializing to the laboratory frame Σ_0(= rest frame of the heat bath), using units such that the speed of light $c = 1$, we have [compare Eqs. (2)]

$$P = MV\gamma(V), \qquad p = mv\gamma(v), \tag{18a}$$

$$E = (M^2 + P^2)^{1/2}, \quad \varepsilon = (m^2 + p^2)^{1/2}, \tag{18b}$$

where V and v denote the particles' velocities, and

$$\gamma(v) = (1 - v^2)^{-1/2}. \tag{18c}$$

Suppose we are given the information that a single collision has occurred. Then, solving Eqs. (17) for $\hat{P}$, we find the explicit representations [compare Eqs. (3)]

$$\hat{P}(p, P) = \frac{2v_0 E - (1 + v_0^2)P}{1 - v_0^2}, \tag{19a}$$

$$\hat{p}(P, p) = \frac{2v_0 \varepsilon - (1 + v_0^2)p}{1 - v_0^2}, \tag{19b}$$

where the velocity

$$v_0 = \frac{p + P}{\varepsilon + E} = \frac{\hat{p} + \hat{P}}{\hat{\varepsilon} + \hat{E}} = \hat{v}_0 \tag{19c}$$

corresponds to the Lorentz boost from Σ_0 to the center-of-mass frame (see Appendix A for details of the calculation). As one may easily check, in the non-relativistic limit case Eqs. (19) reduce to Eqs. (3).

In order to be able to apply the integral criteria (12), we also need to determine the (partially) inverse transformations $P(p, \hat{P})$ and $p(P, \hat{p})$, respectively. In the non-relativistic case, this task was rather simple, see

[2]Since integrals (12) are one-dimensional they can be solved numerically with e.g., the software package Mathematica [46].

Eqs. (15). In the relativistic case, however, a bit of extra care is required. To illustrate this, let us first consider the momentum equation (19a) for Brownian particle. For any fixed value p with $|p|<\infty$, one finds

$$\hat{P}_+(p) := \lim_{P\to+\infty} \hat{P}(p,P) = \frac{(m^2+M^2)p+(M^2-m^2)\varepsilon}{2m^2}, \tag{20a}$$

$$\hat{P}_-(p) := \lim_{P\to-\infty} \hat{P}(p,P) = \frac{(m^2+M^2)p-(M^2-m^2)\varepsilon}{2m^2}. \tag{20b}$$

Hence, at finite p, the inverse transformation $P(p,\hat{P})$ has a finite support, corresponding to the interval (without loss of generality, we will assume here and below that $M>m$)

$$I(p) = [\hat{P}_-(p), \hat{P}_+(p)].$$

Hence, we find the following explicit form of the inverse transformation:

$$P(p,\hat{P}) = \frac{Q+R}{S}, \quad \hat{P}\in I(p),\ \ p\in(-\infty,\infty), \tag{21a}$$

with abbreviations

$$Q = [m^2 - M^2 + 2\,(p-\hat{P})\,p][M^2 p - m^2\hat{P} + M^2\,(p-\hat{P})], \tag{21b}$$

$$R = 2\,(p-\hat{P})(m^2-M^2)\varepsilon\hat{E}, \tag{21c}$$

$$S = (m^2-M^2)^2 - 4(p-\hat{P})(M^2 p - m^2\hat{P}). \tag{21d}$$

Note that the limits in Eqs. (20) correspond to the curves $S=0$. For later use, we also give the formal inversion of Eqs. (20):

$$p_+(\hat{P}) := \frac{(m^2+M^2)\hat{P}+(M^2-m^2)\hat{E}}{2M^2}, \tag{22a}$$

$$p_-(\hat{P}) := \frac{(m^2+M^2)\hat{P}-(M^2-m^2)\hat{E}}{2M^2}, \tag{22b}$$

which allow us to rewrite Eq. (21) equivalently as

$$P(p,\hat{P}) = \frac{Q+R}{S}, \quad \hat{P}\in(-\infty,\infty),\ \ p\in[p_-(\hat{P}), p_+(\hat{P})]. \tag{23}$$

Eq. (23) is in such a form that it can directly be inserted into the integral equation (12a).

Finally, going through an analogous analysis for the heat bath particle yields

$$p(P,\hat{p}) = \frac{q+r}{s}, \quad \hat{p}\in[\hat{p}_-(P), \hat{p}_+(P)],\ \ P\in(-\infty,\infty), \tag{24a}$$

where

$$q = [M^2 - m^2 + 2\,(P-\hat{p})\,P][m^2 P - M^2\hat{p} + m^2\,(P-\hat{p})], \tag{24b}$$

$$r = 2\,(P-\hat{p})(M^2-m^2)\hat{\varepsilon}E, \tag{24c}$$

$$s = (m^2-M^2)^2 - 4(P-\hat{p})(m^2 P - M^2\hat{p}), \tag{24d}$$

and

$$\hat{p}_+(P) := \lim_{p\to+\infty} \hat{p}(P,p) = \frac{(m^2+M^2)P+(M^2-m^2)E}{2M^2}, \tag{25a}$$

$$\hat{p}_-(P) := \lim_{p\to-\infty} \hat{p}(P,p) = \frac{(m^2+M^2)P-(M^2-m^2)E}{2M^2}. \tag{25b}$$

Note that the limits in Eq. (25) correspond to the curves $s = 0$, and their formal inversions can be defined by

$$P_+(\hat{p}) := \frac{(m^2 + M^2)\hat{p} + (M^2 - m^2)\hat{\varepsilon}}{2m^2}, \tag{26a}$$

$$P_-(\hat{p}) := \frac{(m^2 + M^2)\hat{p} - (M^2 - m^2)\hat{\varepsilon}}{2m^2}, \tag{26b}$$

which allows us to rewrite Eq. (21) equivalently as

$$p(P, \hat{p}) = \frac{q + r}{s}, \quad \hat{p} \in (-\infty, \infty), \quad P \in [P_-(\hat{p}), P_+(\hat{p})]. \tag{27}$$

Eq. (27) is given in such a form that it can directly be inserted into the integral equation (12b).

3.2. Testing the integral criterion

Analogous to the non-relativistic case, we aim to exploit the integral criteria (12) in order to determine the invariant momentum distributions. To this end, we proceed as follows: For functions $P(\hat{p}, \hat{P})$ and $p(\hat{P}, \hat{p})$, given in Eqs. (23) and (27), we evaluate numerically integrals (12):

$$\hat{\Phi}(\hat{P}) = \int_{p_-(\hat{P})}^{p_+(\hat{P})} \mathrm{d}p \left|\frac{\partial P}{\partial \hat{P}}\right| \phi(p) \Phi(P(p, \hat{P})), \tag{28a}$$

$$\hat{\phi}(\hat{p}) = \int_{P_-(\hat{p})}^{P_+(\hat{p})} \mathrm{d}P \left|\frac{\partial p}{\partial \hat{p}}\right| \Phi(P) \phi(p(P, \hat{p})). \tag{28b}$$

for a number specified grid-points $\hat{P}$ and $\hat{p}$, respectively. The boundaries of the integration are chosen in accordance with the support intervals of the functions $P(\hat{p}, \hat{P})$ and $p(\hat{P}, \hat{p})$; cf. Eqs. (23) and (27), respectively. We use different pairs of initial PDFs (Φ, ϕ), and test if

$$\hat{\Phi}(\hat{P}) \equiv \Phi(\hat{P}), \quad \hat{\phi}(\hat{p}) \equiv \phi(\hat{p}) \tag{29}$$

hold simultaneously. If the answer is positive, we conclude that the candidate functions (Φ, ϕ) fulfill the stationarity criteria (13) and, thus, are invariant solutions for the relativistic collision process.

We next discuss our choice of the initial PDFs. Guided by the results of Refs. [26,27], we consider

$$\Phi(P) = \frac{\mathcal{N}_\eta(M)}{E^\eta} \exp\left(-\frac{E}{k_B T}\right), \tag{30a}$$

$$\phi(p) = \frac{\mathcal{N}_\eta(m)}{\varepsilon^\eta} \exp\left(-\frac{\varepsilon}{k_B T}\right), \tag{30b}$$

where $\eta \in [0, 1]$ is a free parameter, and ε and E denote the relativistic kinetic energies, respectively. By choosing the symmetric candidate distributions (30), we automatically specialize to the rest frame of the heat bath. For a given set of parameters (m, M, T, η) the normalization constants $\mathcal{N}_\eta(m)$ and $\mathcal{N}_\eta(M)$ are determined by the conditions

$$1 = \int \mathrm{d}P \Phi(P), \quad 1 = \int \mathrm{d}p \phi(p). \tag{31}$$

Let us briefly recall how different values of η may arise in the context of the Langevin description of relativistic Brownian motions, as developed in Ref. [26]. Specializing to the rest frame of the heat bath, one can derive the following Langevin equation for the stochastic motion of a relativistic Brownian particle [26]:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -vP + \left[\frac{E_\eta(P)}{M}\right]^{1/2} \xi(t). \tag{32}$$

Here, ν is the friction parameter and ξ ordinary Gaussian white noise with amplitude $\mathscr{D}$, i.e., ξ is characterized by

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2\mathscr{D}\delta(t-t'). \tag{33}$$

On the rhs of Eq. (32) the noise ξ couples multiplicatively to (a function of) the momentum coordinate P. Hence, different discretization rules may yield different stationary momentum distributions [47]. It is convenient to parameterize discretization rules as follows:

$$E_\eta(P) = E[\eta\, P(t) + (1-\eta)P(t+\mathrm{d}t)], \quad \eta \in [0,1], \tag{34}$$

with $E(P)$ denoting the relativistic kinetic energy. In this notation, e.g., $\eta = 0$ corresponds to the HK post-point discretization rule [38–41], $\eta = \frac{1}{2}$ to the Fisk–Stratonovich mid-point discretization [34–36] and $\eta = 1$ to Ito's pre-point discretization [32]. Introducing the temperature via the Einstein relation

$$k_B T = \frac{\mathscr{D}}{m\nu}, \tag{35}$$

the candidate PDFs in Eqs. (30) represent the stationary distributions associated with different values of η [26]. In particular, only for $\eta = 0$ (HK rule) the standard relativistic Jüttner–Maxwell distribution is recovered.

We are now in the position to discuss the numerical results. Figs. 1(a)–(c) and 1(e)–(f) show functions $\hat{\Phi}(\hat{P})$ and $\hat{\phi}(\hat{p})$, respectively, as obtained for different values of η. In each diagram the solid lines correspond to the initial momentum distributions from Eqs. (30). The triangles indicate the distributions resulting after the collision, $\hat{\Phi}(\hat{P})$ and $\hat{\phi}(\hat{p})$, obtained by numerically integrating Eqs. (12) at 50 different values of $\hat{P}$ and $\hat{p}$,

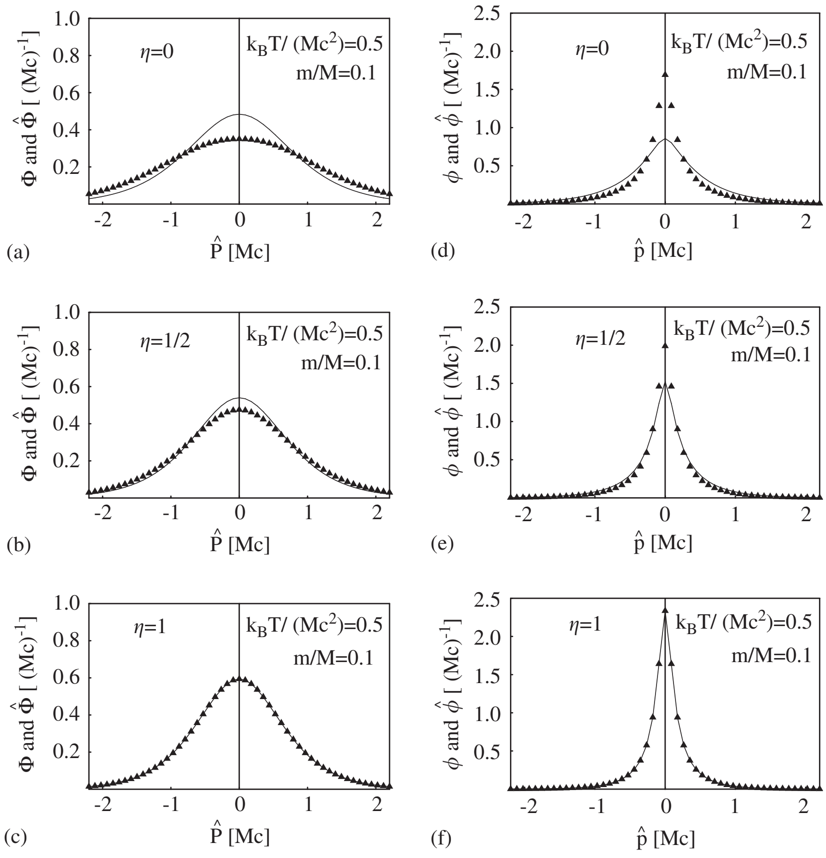

Fig. 1. Initial distributions (solid line) and numerical solutions (triangles) of Eqs. (28) for the momentum PDFs of the Brownian particle (a)–(c) and the heat bath particles (d)–(f). As one readily observes, see diagrams (c) and (f), only for $\eta = 1$ the initial distributions are left invariant by the elastic collision process.

respectively. As one readily observes, for $\eta = 1$, corresponding to diagrams (c) and (f), the initial distributions remain invariant in the course the elastic collision process. Hence, according to these results, the modified Jüttner functions (30) with $\eta = 1$ are the relativistic analogs of the non-relativistic Maxwell distribution. For completeness, we mention that we have tested the integral criteria over a wide range of temperature and mass parameters and always found that only the distributions with $\eta = 1$ are invariant (all numerical integrations were performed with the function `NIntegrate` of the computer algebra program Mathematica [46]). Furthermore, we note that the normalization was conserved with high accuracy during the numerical integration, i.e., the numerically found distributions $(\hat{\Phi}, \hat{\phi})$ remained normalized to unity with high accuracy.

4. Summary and discussion

We have studied a simple microscopic model for 1D non-relativistic and relativistic Brownian motions. It was our main objective to identify the (invariant) stationary momentum distributions for both Brownian and heat bath particles on the basis of the underlying microscopic collision processes. To this end we have formulated two integral criteria, which relate the initial momentum distributions to the momentum distributions resulting after an elastic collision of a Brownian particle with a heat bath particle (Section 2.3). The assumptions (postulates) underlying derivation of the integral criteria can be summarized as follows:

- validity of the standard kinematic conservation laws,
- occurrence of the collision event,
- independence of the initial momenta.

It was then demonstrated that, under these assumptions, the integral criteria do correctly reproduce the Maxwellian distributions as the stationary solutions for the non-relativistic Brownian motions (Section 2.4).

Subsequently, the integral criteria were applied to the relativistic case (Section 3.2). Here, we found that *the standard Jüttner distributions, corresponding to $\eta = 0$ in Eqs.* (30), *are not stationary with respect to the integral criteria*; i.e., given the information that the collision has indeed occurred, the standard Jüttner distributions do not remain invariant in the course of an elastic relativistic collision. Instead, *the invariant distributions are given by modified Jüttner functions*, *corresponding to $\eta = 1$* in Eqs. (30). This result is quite surprising, since initially we had expected that the invariant solutions are given by standard Jüttner functions with $\eta = 0$. However, as known from earlier work [26], modified Jüttner distributions with $\eta \neq 0$ may also appear as stationary solutions in the 1D relativistic Langevin theory, with the value of η depending on the discretization scheme that is used. In this context, we mention a recent paper by Lehmann [48], who argues that Jüttner's original approach [42] is non-covariant. Furthermore, we note that the invariant solution with $\eta = 1$ can also be interpreted as a simple exponential (canonical) distribution with respect to the Lorentz-invariant volume element of momentum space, $\mathrm{d}^D p/p^0 = \mathrm{d}^D p/(p^2 + m^2)^{1/2}$, where D is the number of spatial dimensions [49].

Due to the fact that our results are based on only three basic assumptions, there is very little freedom for modifications such that one could hope to recover the standard Jüttner distributions as invariant solutions. Hence, according to our opinion, this problem deserves further consideration in the future. For example, as the next step, it would be desirable to perform similar studies for simple 2D and 3D models.[3] Then the kinematics of a single collision process becomes more complicated, because—even for the simplest hard-sphere models—momentum may be redistributed in different directions (depending on the impact parameter). Hence, if one wishes to formulate analogous integral criteria for identifying the stationary (invariant) 2D/3D momentum distributions, one will have to include additional equations takingg into account the cross-sections.[4] However, if it should turn out that deviations from the standard Jüttner–Maxwell distribution

[3]Very recently, the authors were informed by L.O. Silva that Marti et al. have implemented a 3D relativistic Monte Carlo model for charged particle collisions, and that their numerically obtained equilibrium solutions correspond to modified Jüttner distributions with $\eta = 1$ as well (conference poster by M. Marti, R.A. Fonseca, L.O. Silva "A collision module for OSIRIS", P5.015, 32nd European Physical Society Conference on Plasma Physics, Rome, June 2006).

[4]We note, the applicability of simple kinematic models as discussed here is, in principle, limited to situations where high-energies quantum processes, as e.g., creation and annihilation of particles, can be neglected.

persist in higher space dimensions as well, then this might be of relevance for calculating relativistic corrections in high-energy physics [50] and astrophysics (e.g., to the Sunyaev–Zeldovich effect [51,52]).

Acknowledgments

The authors are grateful to J. Chluba, A. Fingerle, and, in particular, to S. Hilbert for numerous helpful discussions.

Appendix A. Relativistic collisions

Our goal is to validate Eqs. (19). To this end, consider two particles having positions x and X and relativistic momenta p and P, as defined by Eq. (18), with respect to (wrt.) the 1D inertial lab-frame Σ_0. Assume that the particles collide at, say, $t = 0$, which implies that:

(i) if $v(t) > V(t)$ at $t < 0$, then $x(t) \leqslant X(t)$ or, alternatively,
(ii) if $v(t) < V(t)$ at $t < 0$, then $x(t) > X(t)$.

In order calculate the momenta after the collision, $\hat{p}$ and $\hat{P}$, as functions of the initial momenta, p and P, it is convenient to perform a Lorentz transformation to the center-of-mass frame. Before doing this, we briefly establish some notations. We define $(1+1)$-momenta wrt. Σ_0 by

$$\mathfrak{p} = (\varepsilon, p), \quad \mathfrak{P} = (E, P). \tag{A.1}$$

Assume that some inertial frame Σ' moves with velocity v_0 in Σ_0, then the Lorentz transformation matrix Λ_{v_0} can be parametrized as follows

$$\Lambda_{v_0} = \frac{1}{\sqrt{1-v_0^2}} \begin{pmatrix} 1 & -v_0 \\ -v_0 & 1 \end{pmatrix}. \tag{A.2}$$

Its inverse is obtained by replacing v_0 with $-v_0$, i.e.,

$$\Lambda_{v_0}^{-1} = \Lambda_{-v_0} = \frac{1}{\sqrt{1-v_0^2}} \begin{pmatrix} 1 & v_0 \\ v_0 & 1 \end{pmatrix}. \tag{A.3}$$

For example, given $\mathfrak{P} = (E, p)$, the corresponding $(1+1)$-momentum vector wrt. Σ', denoted by $\mathfrak{P}' = (E', P')$, is obtained via matrix multiplication

$$\mathfrak{P}' = \Lambda_{v_0} \mathfrak{P}. \tag{A.4}$$

Center-of-mass frame. In the following let us assume that Σ' is an inertial center-of-mass frame of the colliding particles; i.e., in Σ' we have, by definition,

$$p' + P' = 0. \tag{A.5}$$

This condition determines the Lorentz transformation parameter as

$$v_0 = \frac{p+P}{\varepsilon + E} = \frac{\hat{p} + \hat{P}}{\hat{\varepsilon} + \hat{E}} = \hat{v}_0. \tag{A.6}$$

Furthermore, for elastic collisions, the energy and momentum balance become particularly simple in Σ':

$$\hat{E}' = E', \quad \hat{\varepsilon}' = \varepsilon', \tag{A.7a}$$

$$\hat{P}' = -P' = p' = -\hat{p}', \tag{A.7b}$$

where, as before, hat-symbols refer to the momenta after the collision. It is convenient to express Eqs. (A.7) in matrix form, e.g., writing

$$\hat{\mathfrak{P}}' = \sigma' \mathfrak{P}', \quad \hat{\mathfrak{p}}' = \sigma' \mathfrak{p}', \tag{A.8a}$$

where the momentum transfer matrix σ' of the elastic collision is defined by

$$\sigma' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma'^{-1}. \tag{A.8b}$$

Lab frame. It is now straightforward to convert results (A.8) to the laboratory frame by applying the corresponding Lorentz transformations; e.g., for the Brownian particle we find

$$\hat{\mathfrak{P}} = \Lambda_{v_0}^{-1} \hat{\mathfrak{P}}' = \Lambda_{v_0}^{-1} \sigma' \mathfrak{P}' = \Lambda_{v_0}^{-1} \sigma' \Lambda_{v_0} \mathfrak{P} = \sigma \mathfrak{P}, \tag{A.9a}$$

where

$$\sigma = \Lambda_{v_0}^{-1} \sigma' \Lambda_{v_0} = \frac{1}{1 - v_0^2} \begin{pmatrix} 1 + v_0^2 & -2v_0 \\ 2v_0 & -(1 + v_0^2) \end{pmatrix}. \tag{A.9b}$$

is the momentum transfer matrix wrt. Σ_0, and v_0 is given by Eq. (A.6). Analogously, we find for the heat bath particle

$$\hat{\mathfrak{p}} = \sigma \mathfrak{p}. \tag{A.10}$$

An explicit evaluation of Eqs. (A.9a) and (A.10) yields

$$\begin{pmatrix} \hat{E} \\ \hat{P} \end{pmatrix} = \frac{1}{1 - v_0^2} \begin{pmatrix} (1 + v_0^2)E - 2v_0 P \\ 2v_0 E - (1 + v_0^2)P \end{pmatrix}, \tag{A.11a}$$

$$\begin{pmatrix} \hat{\varepsilon} \\ \hat{p} \end{pmatrix} = \frac{1}{1 - v_0^2} \begin{pmatrix} (1 + v_0^2)\varepsilon - 2v_0 p \\ 2v_0 \varepsilon - (1 + v_0^2) p \end{pmatrix}, \tag{A.11b}$$

which contains the desired result, cf. Eqs. (19).

Finally, we also calculate the inverse momentum transfer matrix

$$\sigma^{-1} = (\Lambda_{v_0}^{-1} \sigma' \Lambda_{v_0})^{-1} = \Lambda_{v_0}^{-1} \sigma'^{-1} \Lambda_{v_0} = \Lambda_{v_0}^{-1} \sigma' \Lambda_{v_0} = \sigma, \tag{A.12}$$

allowing us to write

$$\begin{pmatrix} E \\ P \end{pmatrix} = \frac{1}{1 - \hat{v}_0^2} \begin{pmatrix} (1 + \hat{v}_0^2)\, \hat{E} - 2\hat{v}_0\, \hat{P} \\ 2\hat{v}_0\, \hat{E} - (1 + \hat{v}_0^2)\, \hat{P} \end{pmatrix}, \tag{A.13a}$$

$$\begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \frac{1}{1 - \hat{v}_0^2} \begin{pmatrix} (1 + \hat{v}_0^2)\, \hat{\varepsilon} - 2\hat{v}_0\, \hat{p} \\ 2\hat{v}_0\, \hat{\varepsilon} - (1 + \hat{v}_0^2) \hat{p} \end{pmatrix}. \tag{A.13b}$$

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