Random Combinatorial Billiards

Colin Defant
Harvard

November 22, 2024

Alex's Class

Combinatorial Billiards

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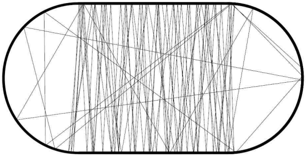
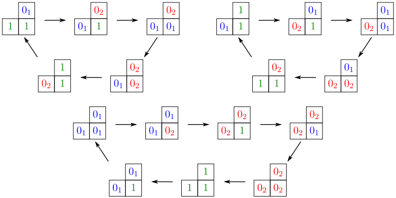
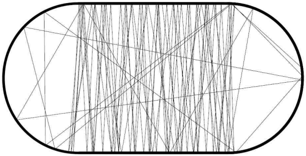


Mathematical Billiards



Dynamical Algebraic Combinatorics

Combinatorial Billiards



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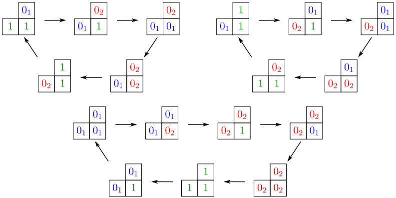

Dynamical Algebraic Combinatorics

Combinatorial billiards combines these topics.

- Combinatorial billiard systems are rigid and discretized. They can be modeled combinatorially or algebraically.
- We can ask precise questions about combinatorial billiard systems in high-dimensional spaces.

Combinatorial Billiards

Basic Setup: Start with the Coxeter arrangement of a Coxeter group W. Shine a beam of light in some particular direction. When the light hits a hyperplane, it can change its direction according to some rule. Discretize the beam of light.

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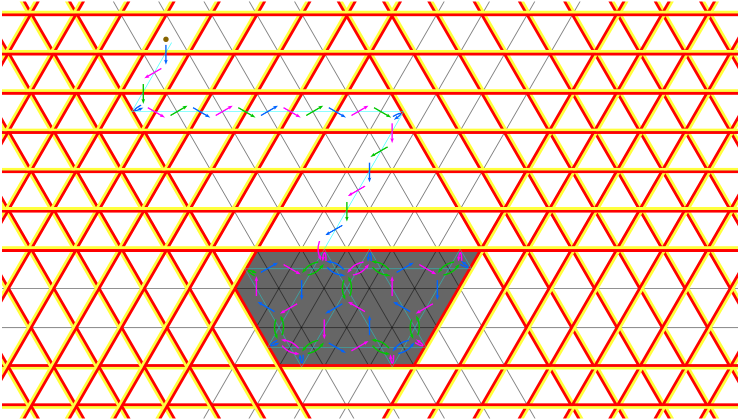

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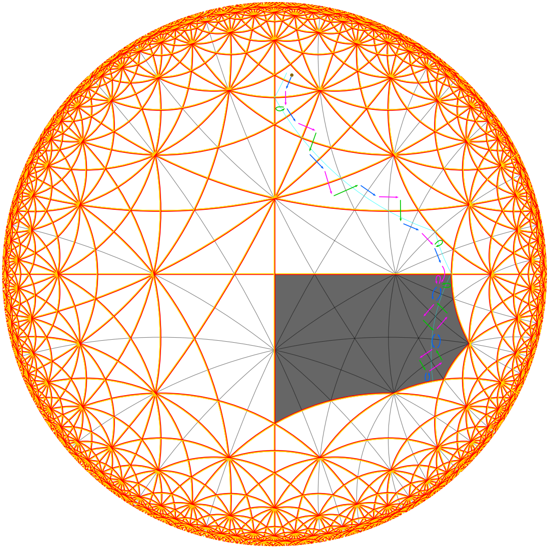
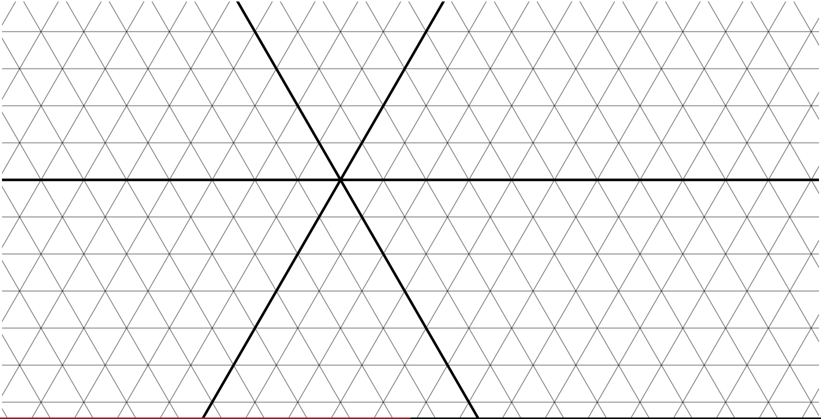


The Affine Symmetric Group

The Affine Symmetric Group

Let $V = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n : \gamma_1 + \cdots + \gamma_n = 0\}$. Let $\mathcal{H}_n = \{\mathrm{H}_{i,j}^0 : 1 \leq i < j \leq n\}$ and $\widetilde{\mathcal{H}}_n = \{\mathrm{H}_{i,j}^k : 1 \leq i < j \leq n,\, k \in \mathbb{Z}\}$, where

$$\mathrm{H}_{i,j}^k = \{(\gamma_1, \ldots, \gamma_n) \in V : \gamma_i - \gamma_j = k\}.$$



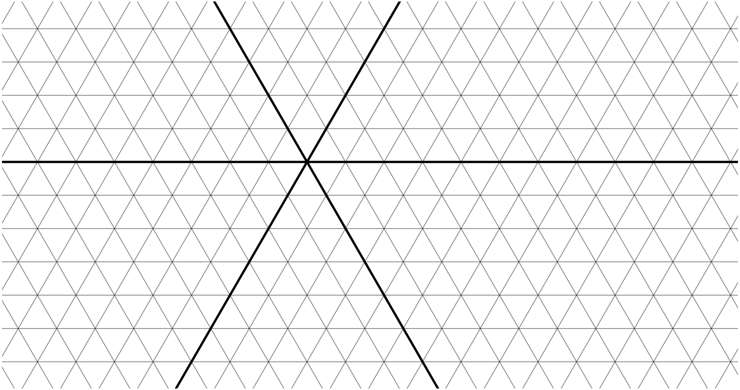
The Affine Symmetric Group

[1] Some definitions and results in this talk generalize to all affine Weyl groups.

The Affine Symmetric Group

The *affine symmetric group*[1] $\widetilde{\mathfrak{S}}_n$ is the group generated by the reflections through the hyperplanes in $\widetilde{\mathcal{H}}_n$.

The *symmetric group* $\mathfrak{S}_n$ is the subgroup of $\widetilde{\mathfrak{S}}_n$ generated by the reflections through the hyperplanes in $\mathcal{H}_n$.



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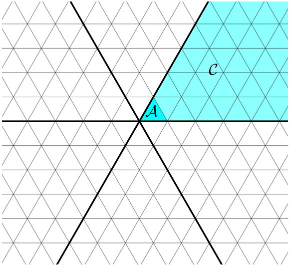
The Affine Symmetric Group

The connected components of $V \setminus \bigcup \mathcal{H}_n$ are called *chambers*.

The connected components of $V \setminus \bigcup \widetilde{\mathcal{H}}_n$ are called *alcoves*.

The *fundamental chamber* is $\mathcal{C} = \{(\gamma_1, \ldots, \gamma_n) \in V : \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$.
The *fundamental alcove* is $\mathcal{A} = \{(\gamma_1, \ldots, \gamma_n) \in \mathcal{C} : \gamma_n \geq \gamma_1 - 1\}$.

There is a bijection from $\mathfrak{S}_n$ to the set of chambers given by $u \mapsto u\mathcal{C}$.



Lam's Reduced Random Walk

Start at $\mathcal{A}$. At each time step, choose an adjacent alcove uniformly at random. If moving to that alcove causes you to pass through a hyperplane that you have already crossed, do nothing. Otherwise, move to that adjacent alcove.

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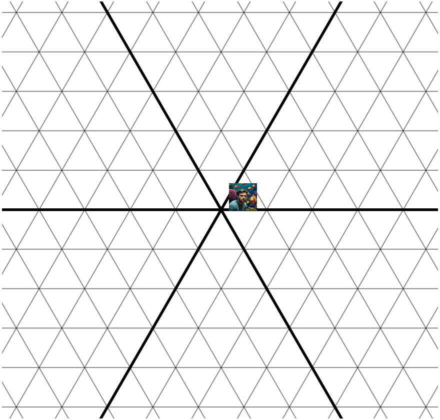

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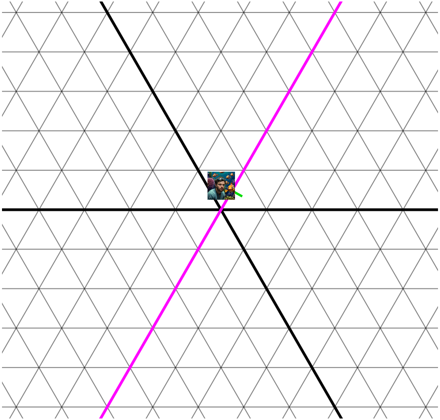

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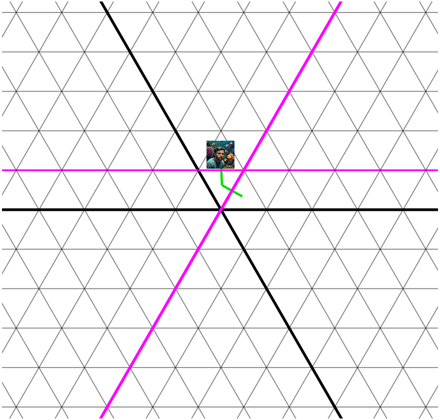

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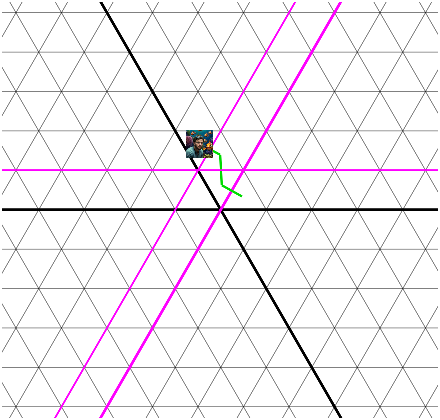

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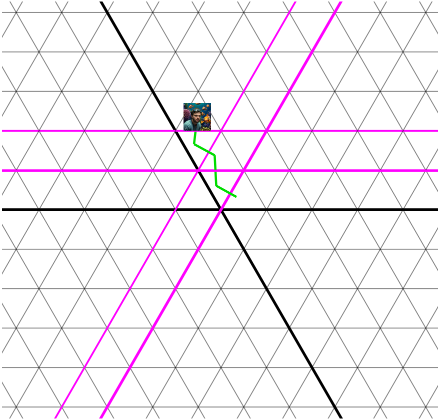

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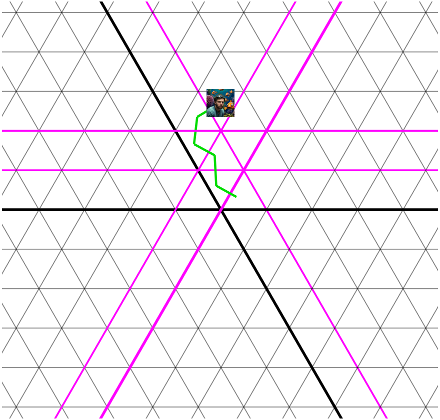

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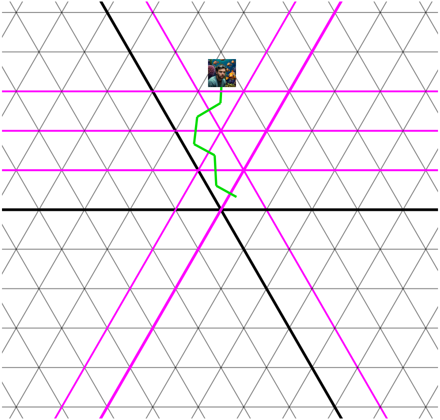

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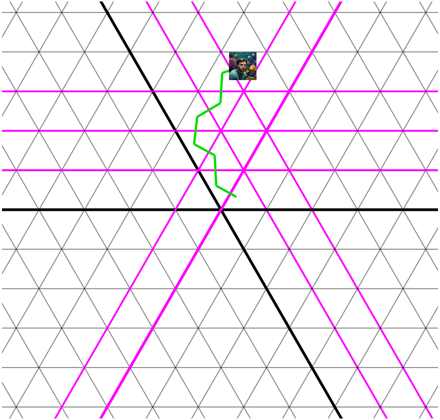

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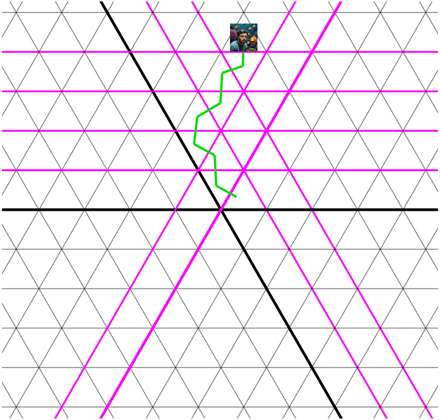

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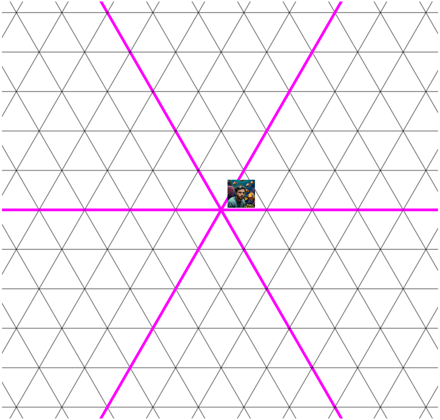

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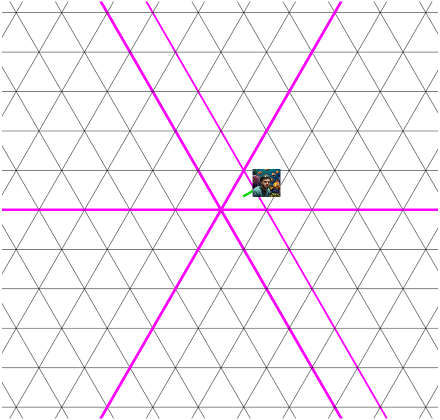

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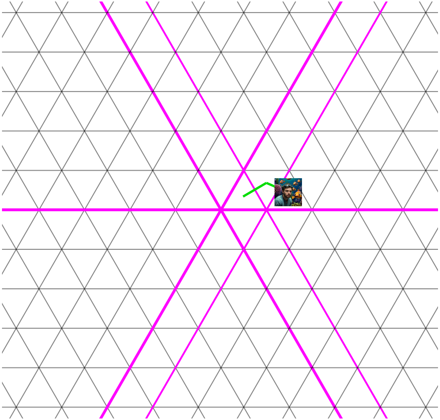

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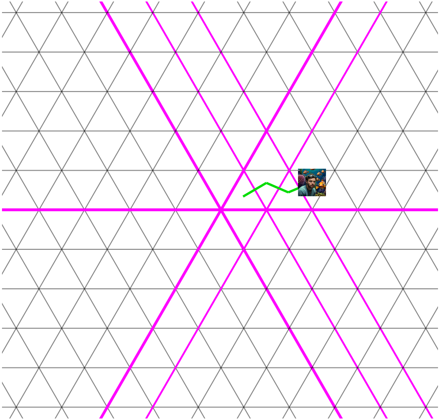

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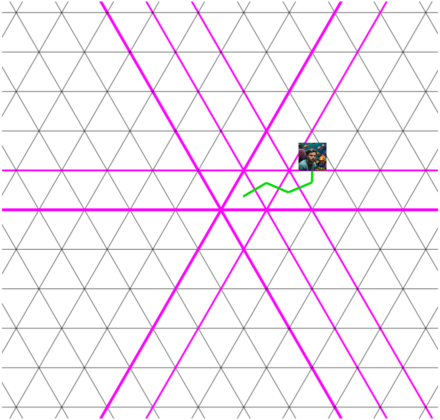

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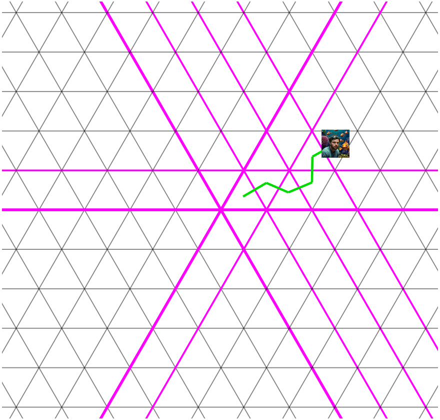

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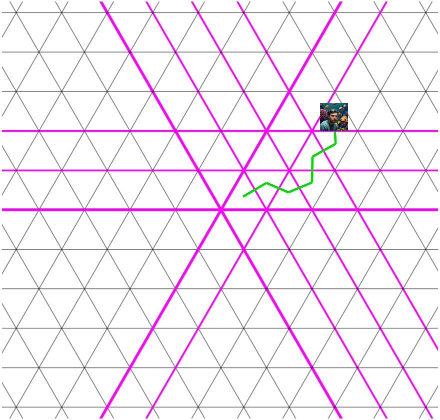

Projecting to a Torus

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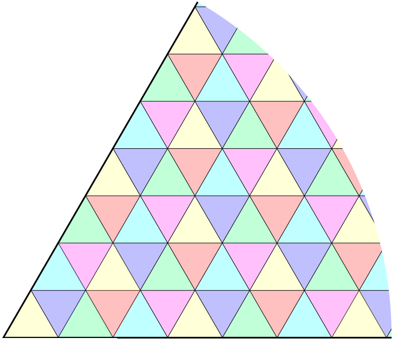
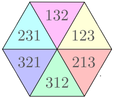
The affine Grassmannian reduced random walk projects to a Markov chain on $\mathfrak{S}_n$ called the *multispecies totally asymmetric simple exclusion process* (multispecies TASEP) on a ring (cycle).





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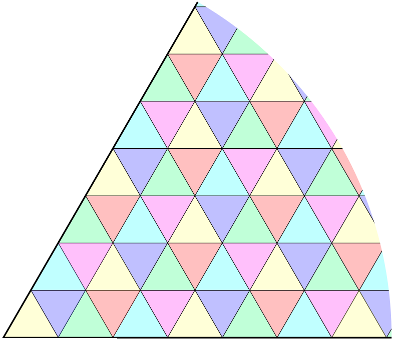

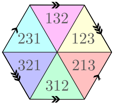

Lam's Theorem

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Let ζ be the stationary distribution of the multispecies TASEP.

Let

$$\psi_{\mathrm{Lam}} = \sum_{\substack{w \in \mathfrak{S}_n \\ w^{-1}(1) < w^{-1}(n)}} \zeta(w)(e_{w^{-1}(1)} - e_{w^{-1}(n)}),$$

where e_i is the i-th standard basis vector of $\mathbb{R}^n$.

Theorem (Lam, 2015)

With probability 1*, the affine Grassmannian reduced random walk travels asymptotically in the direction of* ψ_{Lam}*, and the reduced random walk travels asymptotically in the direction of one of the vectors in* $\mathfrak{S}_n \psi_{\mathrm{Lam}}$*.*

Computing the Directions

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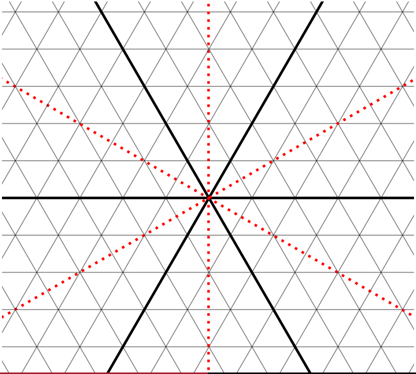
By analyzing the multispecies TASEP, Ayyer and Linusson computed ψ_{Lam}.

Theorem (Ayyer–Linusson, 2017)

The vector ψ_{Lam} is a positive scalar multiple of $\sum_{1\le i<j\le n}(j-i)(e_i-e_j)$.



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Fix $p \in (0,1)$.

Start at a point in the interior of $\mathcal{A}$. Shine a beam of light in the direction of some vector $\eta \in V$. When the beam hits a hyperplane in $\widetilde{\mathcal{H}}_n$ that it has not yet crossed, it passes through with probability p and reflects with probability $1-p$. When the beam hits a hyperplane that it has already crossed, it must reflect.

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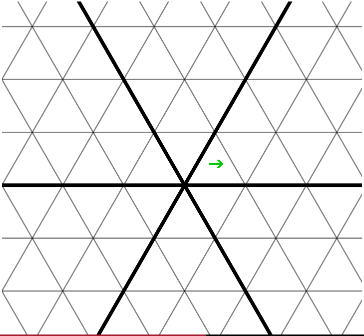

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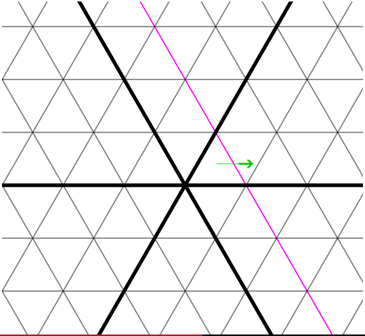

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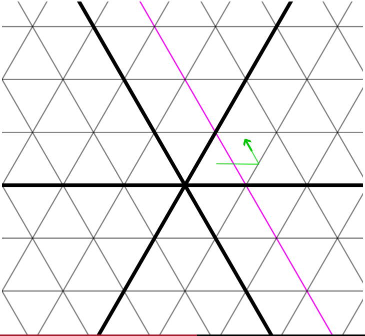

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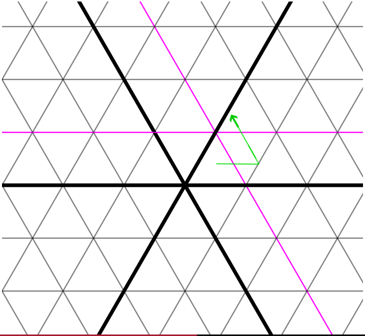

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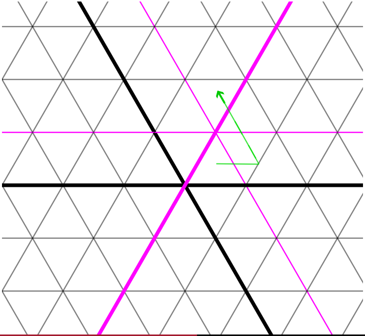

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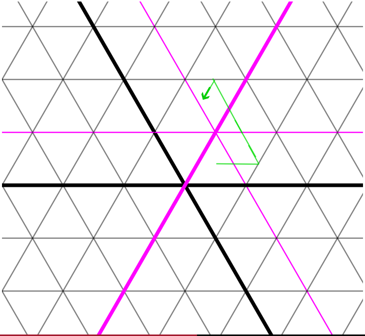

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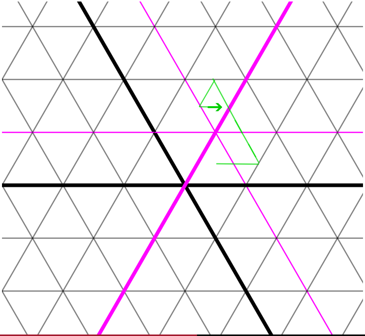

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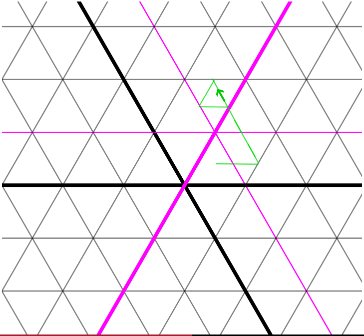

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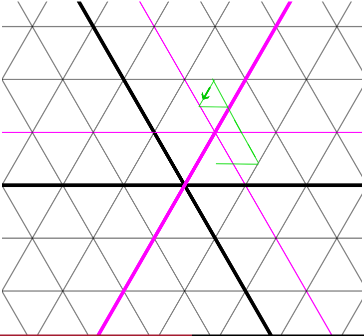

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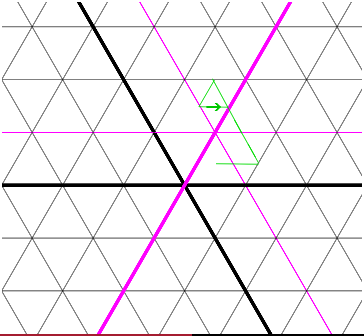

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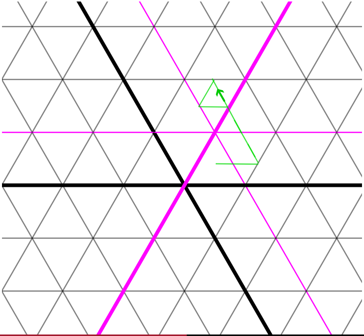

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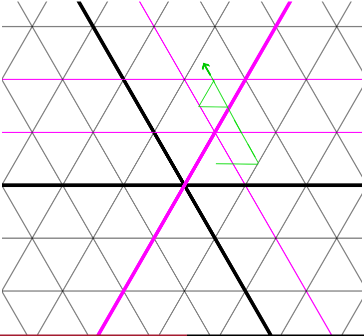

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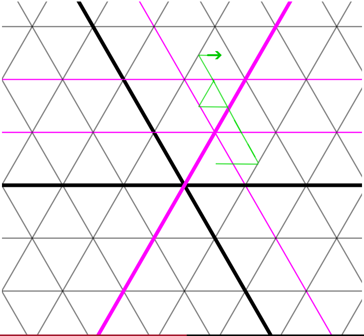

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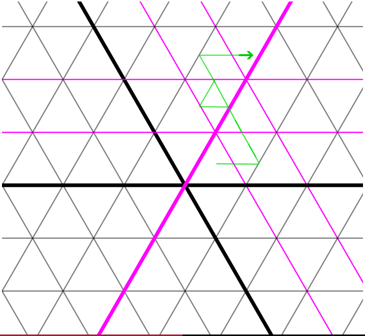

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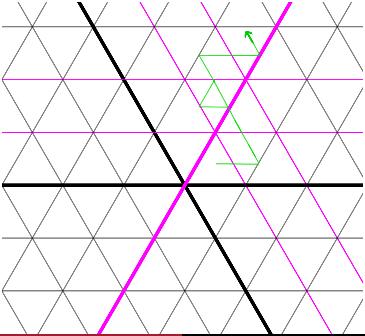

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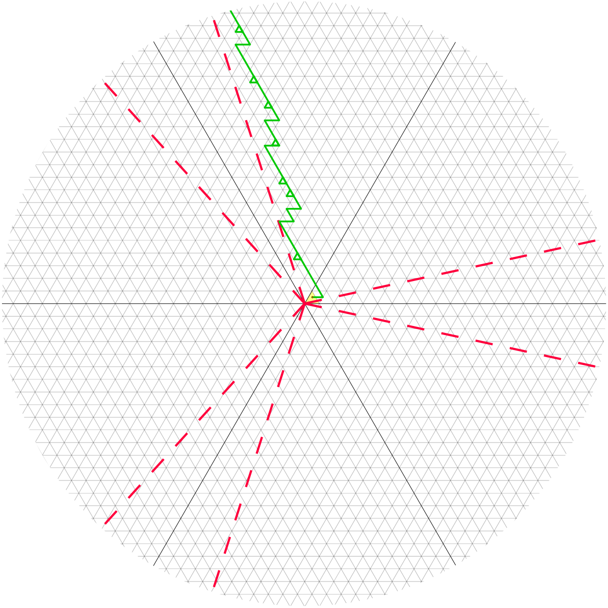

Affine Grassmannian Reduced Random Billiards

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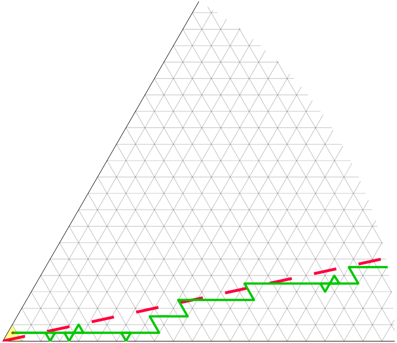

Affine Grassmannian Reduced Random Billiards

Affine Grassmannian Reduced Random Billiards

Theorem (D., 2024+)

There exists a vector $\psi_\eta^{(p)}$ (depending on p and the initial direction η) such that with probability 1,

- *the affine Grassmannian reduced random billiard trajectory travels asymptotically in the direction of $\psi_\eta^{(p)}$ and*
- *the reduced random billiard trajectory travels asymptotically in the direction of one of the vectors in $\mathfrak{S}_n\psi_\eta^{(p)}$.*

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The affine Grassmannian reduced random billiard trajectory projects to a finite Markov chain whose stationary distribution can be used to compute $\psi_\eta^{(p)}$.

A Special Initial Direction

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Assume now that the light beam initially shines in the direction of the vector $\delta = (1, 1, \ldots, 1, -n+1)$.

Theorem (D., 2024+)

The vector $\psi_\delta^{(p)}$ *is a positive scalar multiple of*

$$\sum_{1 \le i < j \le n} \frac{(j-i)(2n-(i+j-1)p)}{(n-ip)(n-(i-1)p)(n-jp)(n-(j-1)p)}(e_i - e_j).$$

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In the limit as $p \to 0$, we recover ψ_{Lam}.

Projecting to the Torus

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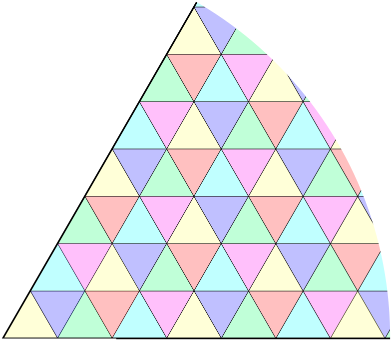
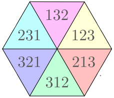
Project the affine Grassmannian reduced random billiard trajectory to the torus. A state is a pair $(w, i) \in \mathfrak{S}_n \times \mathbb{Z}/n\mathbb{Z}$, where w tells us the alcove containing the beam of light and i encodes the direction the beam is facing.





The Multispecies ASEP

Fix $t \in [0, 1)$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. For $k, k' \in \mathbb{Z}$, let

$$\mathfrak{f}_t(k, k') = \begin{cases} 1 & \text{if } k > k'; \\ t & \text{if } k < k'; \\ 0 & \text{if } k = k'. \end{cases}$$

Let S_λ be the set of permutations of λ. When $\lambda = (n, n-1, \ldots, 1)$, identify S_λ with $\mathfrak{S}_n$.

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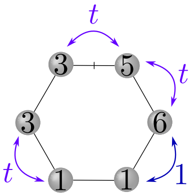
The *multispecies ASEP* is a Markov chain with state space S_λ.
Represent a state μ by placing particles of species $\mu_1, \ldots, \mu_n$ on sites $1, \ldots, n$ of a ring (cycle).

For $i \in \mathbb{Z}/n\mathbb{Z}$, particles on sites i and $i+1$ swap with rate $\mathfrak{f}_t(\mu_i, \mu_{i+1})$.



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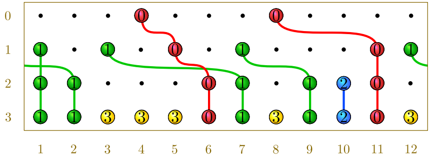

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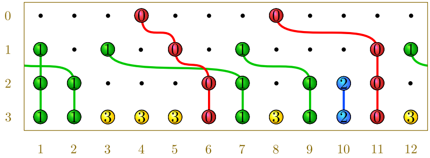
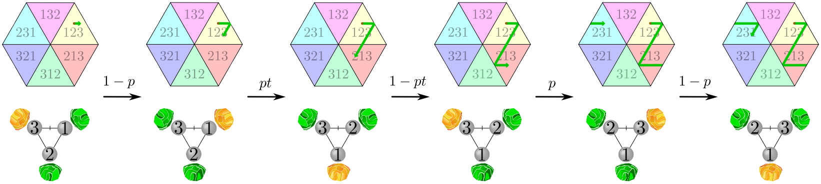


Corteel, Mandelshtam, and Williams, building off of work of Cantini, de Gier, and Wheeler, introduced *ASEP polynomials*, which are polynomials $F_\mu(x_1, \ldots, x_n; q, t) \in \mathbb{C}(q,t)[x_1, \ldots, x_n]$. They showed that the stationary probability of μ in the multispecies ASEP is $F_\mu(1, \ldots, 1; 1, t)$ (up to normalization).

The Stoned Multispecies ASEP

The state space is $S_\lambda \times \mathbb{Z}/n\mathbb{Z}$. Represent (μ, j) by placing particles of species $\mu_1, \dots, \mu_n$ on the sites of a ring, placing a gold on site j, and placing green stones on all other sites.



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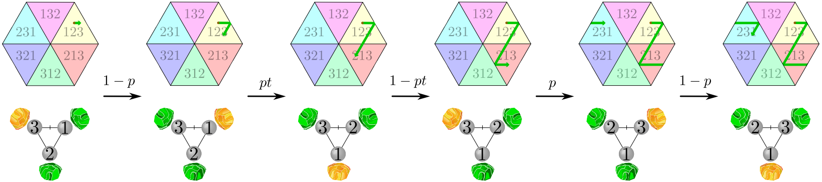

For a transition from state (μ, j), the gold stone swaps with the green stone on site $j+1$. The stones send a signal to the particles on sites j and $j+1$, telling them to swap. The signal reaches the particles with probability p. If the particles receive the signal, they follow orders with probability $\mathfrak{f}_t(\mu_j, \mu_{j+1})$.

The Stoned Multispecies ASEP

Theorem (D., 2024+)

Let $\chi = \frac{1-p}{1-pt}$. *The stationary probability of* (μ, j) *in the stoned multispecies ASEP is* $F_\mu(1, \ldots, 1, \chi, 1, \ldots, 1; 1, t)$ *(up to normalization), where the* χ *is in position* j.

When $t = 0$, this allows us to compute $\psi_\delta^{(p)}$ by analyzing multiline queues.

The Stoned Multispecies ASEP

There is a more general version of the stoned multispecies ASEP in which the green stones are numbered $1, \ldots, n-1$ and the probability of the signal reaching the particles is some probability p_i depending on the number i of the green stone that swapped with the gold stone. In this setting, the stationary distribution is given by evaluating ASEP polynomials at generic values.

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Ayyer, Martin, and Williams recently constructed a completely different Markov chain whose stationary distribution is also given by evaluating ASEP polynomials at generic values.

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- There is a stoned version whose stationary distribution is given by evaluating inhomogeneous TASEP polynomials at generic values.
- The stoned version also has a billiards interpretation.

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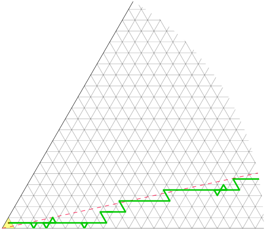
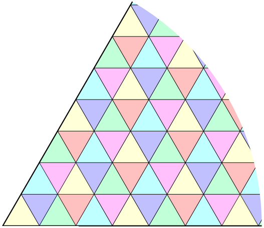
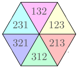
- The stoned version also has a billiards interpretation in a type-C affine Weyl group.

THANKS!



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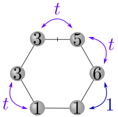
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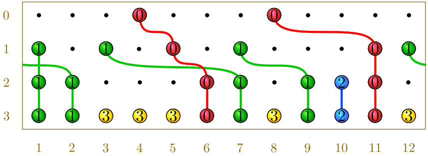
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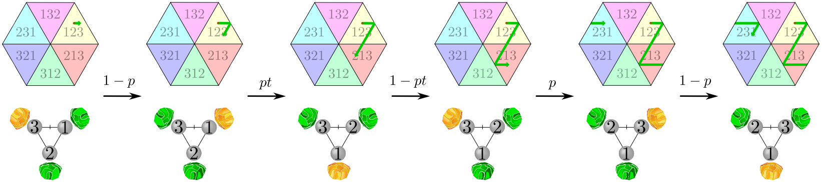
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