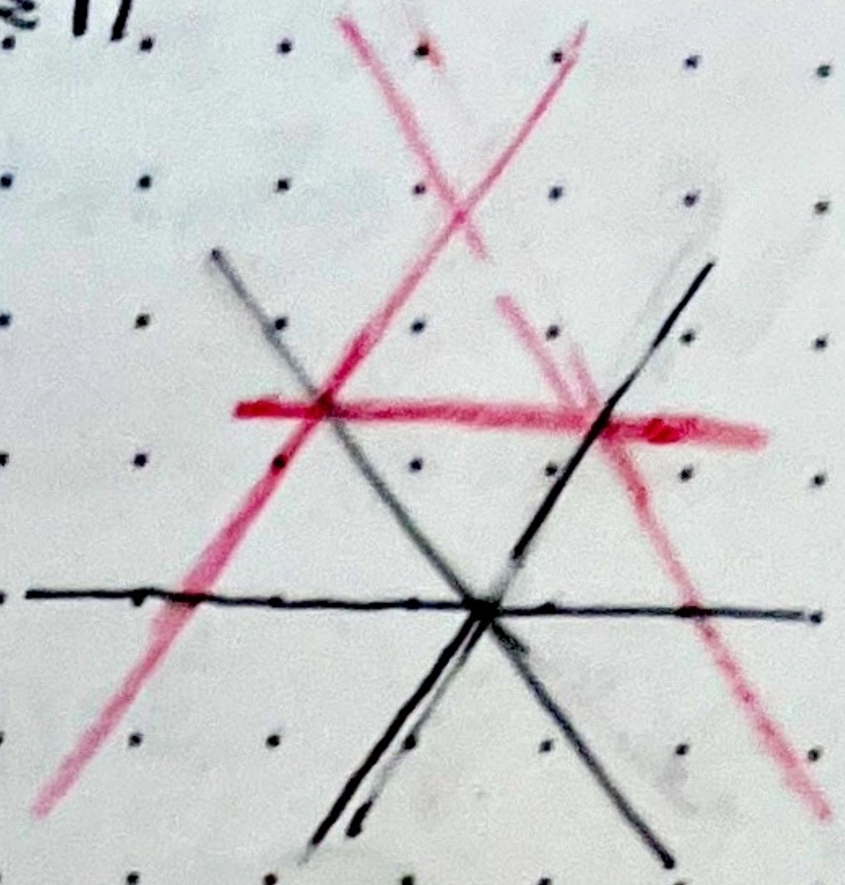
LECTURE 18 Fri 10/18

Shi Arrangement

Shi_n : $x_i - x_j = 0, 1 \qquad 1 \le i < j \le n.$



Thrm:

(1) Char. Poly $\chi_{\mathrm{Shi}_n}(t) = t \cdot (t-n)^{n-1}$

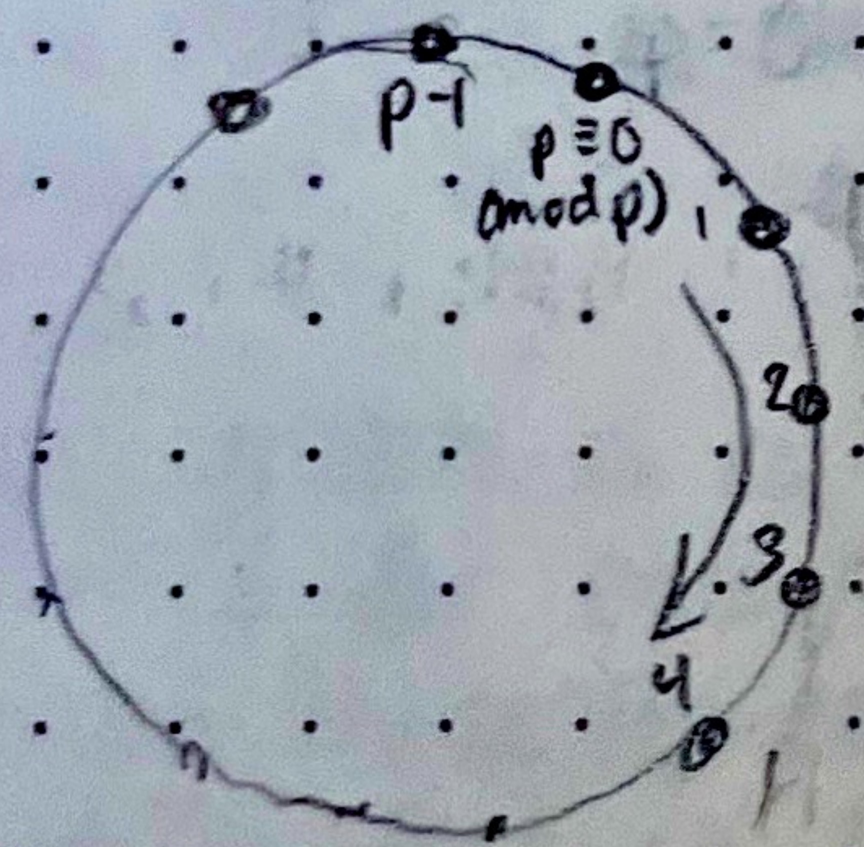
(2) # regions $= (n+1)^{n-1}$

(3) # bounded regions : $(n-1)^{n-1}$
(restricted to the hyperplane $\{x_1 + \ldots + x_n = 0\}$)

Proof: (1) $\Rightarrow$ (2), (3) by Zaslavsky's Thrm.

(1) Assume $q = p^1$ (for sufficiently large $p >$ all primes in P_{bad}).

$\mathbb{F}_p$

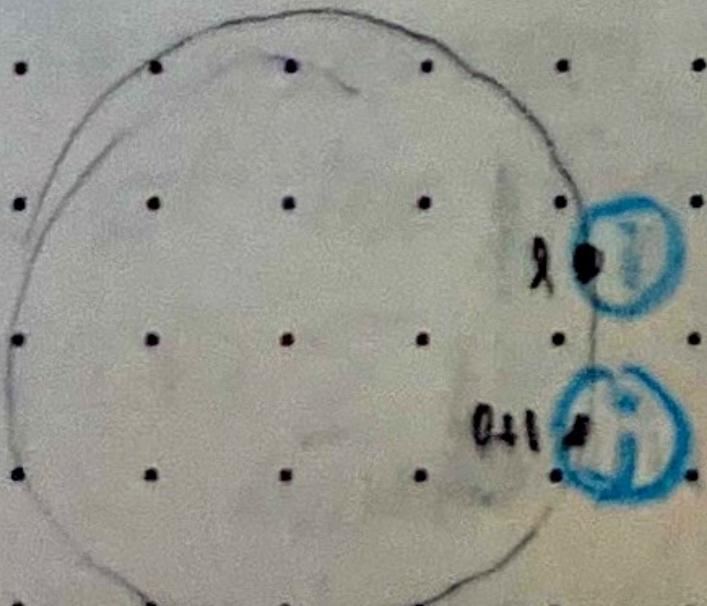


By finite field method (FFM):

$\chi_{\mathrm{Shi}_n}(p) = \#\left\{ (x_1, \ldots, x_n) \in \mathbb{F}_p^n \text{ s.t. } x_i \neq x_j,\ x_i - x_j \neq 1 \text{ for } i<j \right\}$

Can represent $(x_1, \ldots, x_n)$ by n labelled balls at positions x_i, $i = 1, \ldots, n$ on the circle.

If have 2 adjacent balls.

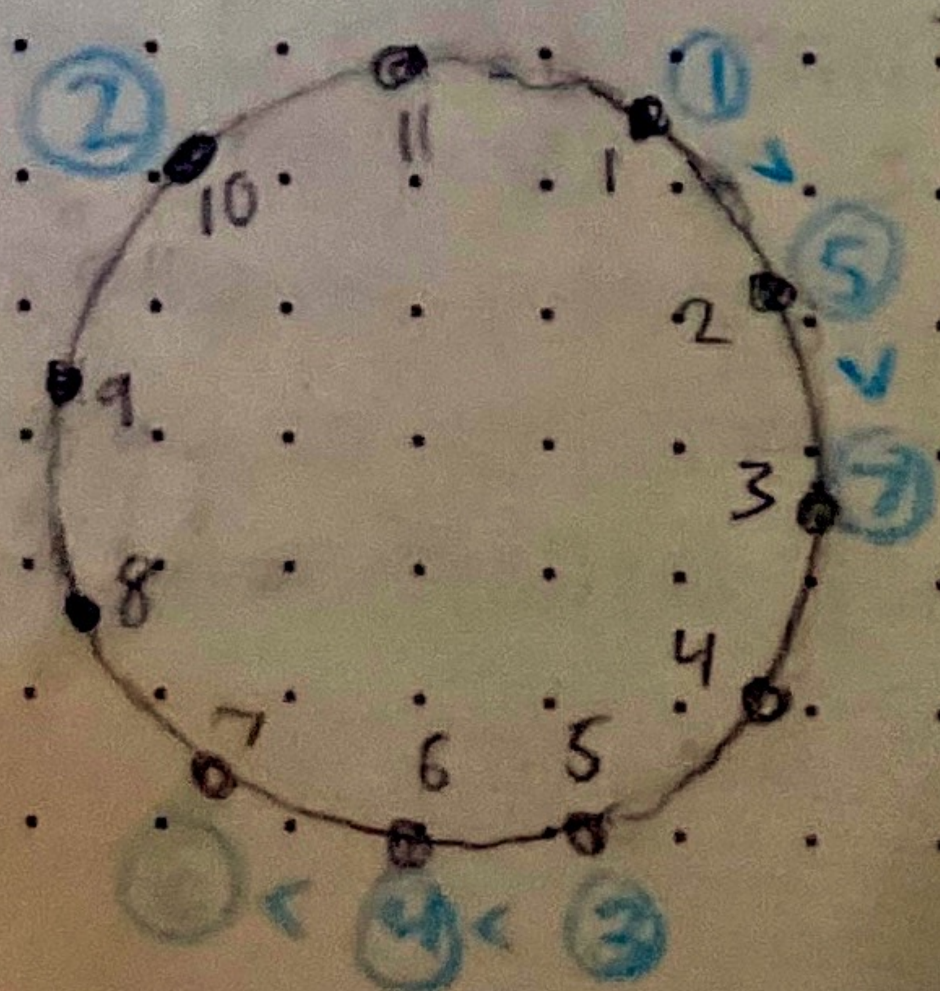


$\Rightarrow i < j$

By symmetry, can put ball 1 in spot 1. ($x_1 = 1$)

$\Rightarrow$ #placements $= p \cdot$ #placement where ball 1 in position 1

Ex. $n=7, p=11$.



$\rightsquigarrow$ Represent it by weak ordered set partition $\pi = (B_1 | B_2 | \ldots | B_{p-n})$ of $[n]$ with $p-n$ (possibly empty) blocks s.t. $1 \in B_1$.

$\rightsquigarrow \pi = (157 | 346 | \emptyset | 2)$

B_i = set of labels of balls between $(i-1)^{st}$ and i^{th} empty spots.

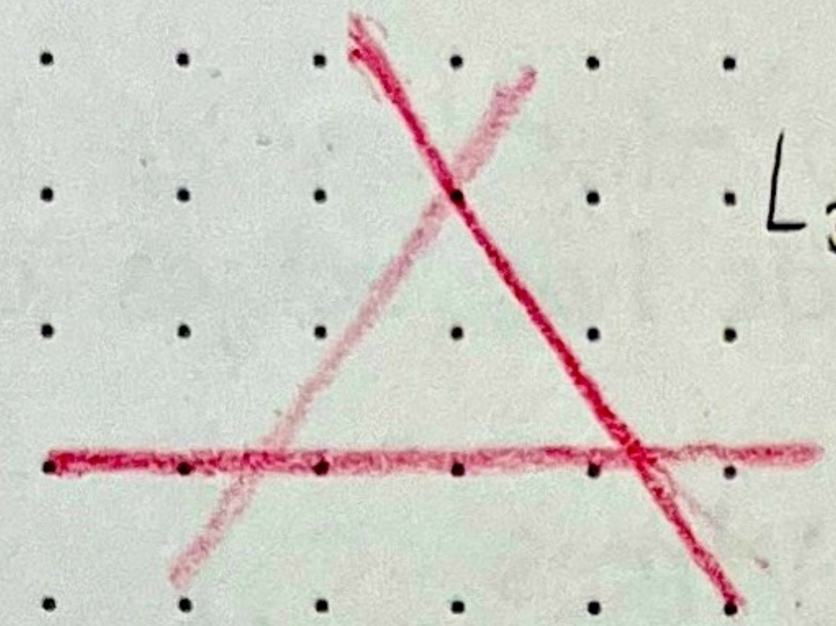
such π's $= (p-n)^{n-1}$ b/c any $i \in \{2, \ldots, n\}$ can be in any of $p-n$ blocks

$\Rightarrow \chi(p) = p \cdot (p-n)^{n-1}$
Should hold for all values.

Liniel Arrangement

L_n: $x_i - x_j = 1$

$1 \leq i < j \leq n$



L_3

Thrm. # Regions of L_n is $f_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$.
(Conjectured by Liniel, first proved by Postnikov).

n	1	2	3	4	5	6
f_n	1	1	7	36	246	2104

Could try to use FFM again, but now could have multiple balls in one spot.
Actually Postnikov proved it w/ No-broken-circuit thrm.

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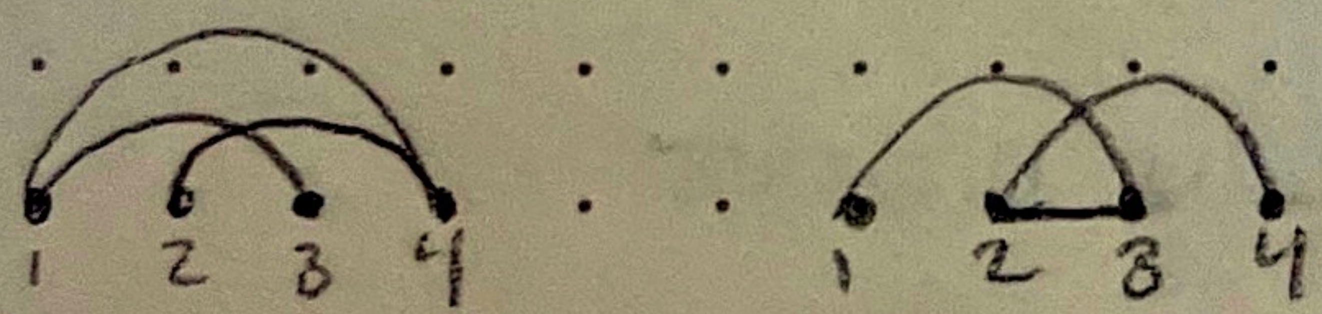
Alternating trees

Def: Labelled trees on $n+1$ node s.t. the labels in any path alternate.
(Equivalently: no pair of edges $i < j < k$).

Ex. All alternating trees on 4 nodes (n=3).

All non-crossing trees:

crossing trees:



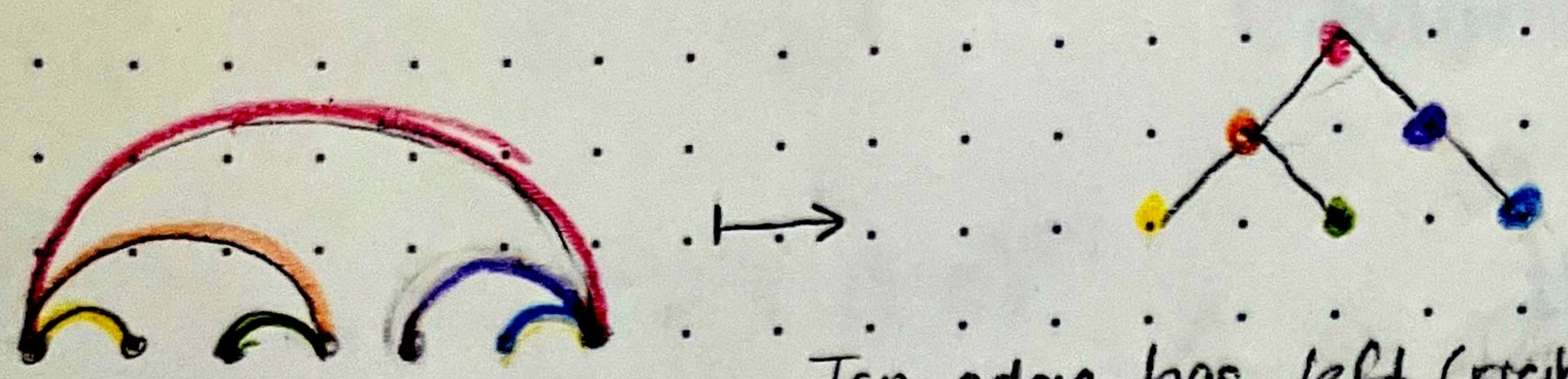
Thrm: (1) # all alternating trees on $n+1$ vertices $= f_n$ (from above)

(2) # non-crossing alt. trees on n vertices $= C_n = \frac{1}{n+1}\binom{2n}{n}$
catalan #s

Exercise: Prove (1)

Proof of (2): Bijection

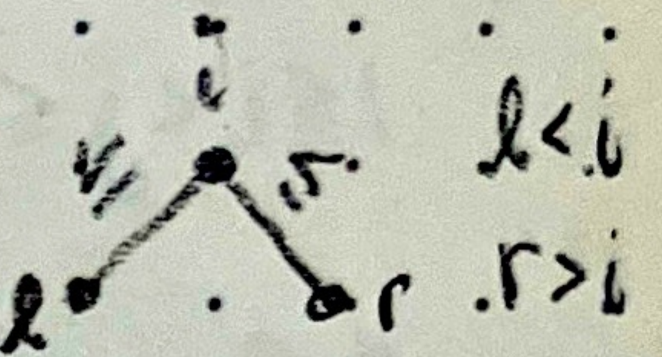
$\left\{ \begin{array}{c} \text{non-crossing alt.} \\ \text{trees on ntl. verts} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(unlabelled)} \\ \text{binary trees} \end{array} \right\}$



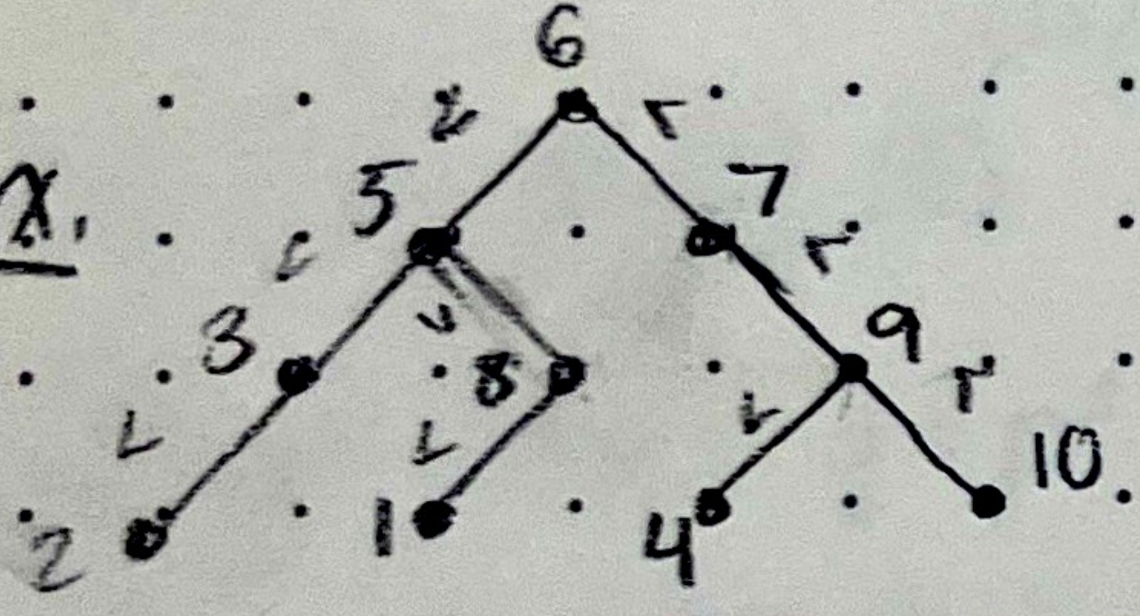
Top edge has left (righ) child if left (right) side has edges underneath

Local binary search (LBS) trees

Labelled binary trees on n verts s.t. $\forall$ vertex i



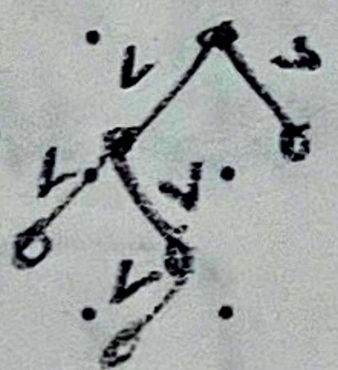
Ex.



Thrm: f_n = # LBS trees on n nodes

Thrm: (1) # labelled bin. trees on n nodes $= n!\,C_n$ = # regions of Cat_n.

(2) # labelled bin trees on n nodes s.t. left child > its parent $= (n+1)^{n-1}$ = # regions of Shi_n.

(3) # LBS trees on n nodes  $= f_n$ = # regions of L_n

(4) # increasing binary trees on n verts (label of any child) > (label of its parent) $= n!$ = # regions of braid arrangement

Proof of (4): $\underbrace{5\ 3\ 6\ 2\ 10}_{\text{left branch}}\ 1\ \underbrace{7\ 8\ 4\ 9}_{\text{right branch}}$ turn into incr. bin. tree

Then repeat the procedure within the branches