LECTURE 16 Fri 10/11

Rota's Crosscut Theorem

↳ The last step of proof of Zaslavsky's Thm

L a finite lattice

Def: Atoms are elts. that cover $\hat{0}$.

Lower crosscut $C \subset L \setminus \{\hat{0}\}$

any subset containing all the atoms

Then $\mu_L(\hat{0}, \hat{1}) = \sum\limits_{\substack{B \subseteq C \text{ s.t.} \\ \bigvee_{b \in B} b = \hat{1}}} (-1)^{|B|}$

Note: In most applications we just take $C = \{\text{atoms}\}$

Def: Möbius algebra of lattice L

A commutative assosciative algebra $/\mathbb{R}$

Two linear bases: $\{a_x \mid x \in L\}$, $\{b_x \mid x \in L\}$

$(*)\ b_x = \sum\limits_{y \geq x} \mu(x,y)\, a_y$

$\Updownarrow$ def of $\mu(x,y)$

$(**)\ a_x = \sum\limits_{y \geq x} b_y$

Multiplication: $a_x \cdot a_y = a_{x \vee y} \quad \forall x, y \in L$

Lemma: $b_x \cdot b_y = \delta_{xy} b_x \quad \forall x, y \in L$

↖ Kronecker's δ function $\delta_{xy} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Proof: Let "$\square$" be multiplication given by

$b_x \square b_y := \delta_{xy} b_x \quad \forall x, y \in L$

$$a_x \square a_y \overset{(**)}{=} \Big(\sum_{s \geq x} b_s\Big) \square \Big(\sum_{t \geq y} b_t\Big)$$

$$= \sum_{\substack{s \geq x \\ t \geq y}} b_s \square b_t = \sum_{\underbrace{s \geq x \text{ AND } y}_{\text{i.e. } s \geq x \vee y}} b_s \overset{(**)}{=} a_{x \vee y}$$

So "$\square$" = "$\cdot$" as multiplication operations. $\boxtimes$

Proof of crosscut thrm:

$$a_{\hat{0}} \overset{(**)}{=} \sum_{y \in L} b_y = \mathbb{1} \quad \text{(the identity elt. in the algebra)}$$

$$a_{\hat{0}} - a_x = \sum_{y \in L} b_y - \sum_{y \geq x} b_y = \sum_{y \ngeq x} b_y$$

$$\prod_{x \in C} (\mathbb{1} - a_x) = \prod_{x \in C} \left(\sum_{y \ngeq x} b_y \right)$$

Lemma $\Rightarrow$ $$= \sum_{\substack{y \text{ s.t.} \\ y \ngeq x \\ \forall x \in C}} b_y \underset{\substack{\text{by def} \\ \text{of crosscut}}}{=} b_{\hat{0}} \overset{(*)}{=} \sum_{y \in L} \mu(\hat{0}, y) a_y$$

Take the coeff of $a_{\hat{1}}$ on both sides

RHS: $\mu(\hat{0}, \hat{1})$

LHS: Coeff of $a_{\hat{1}}$ in $\prod_{x \in C} (\mathbb{1} - a_x)$

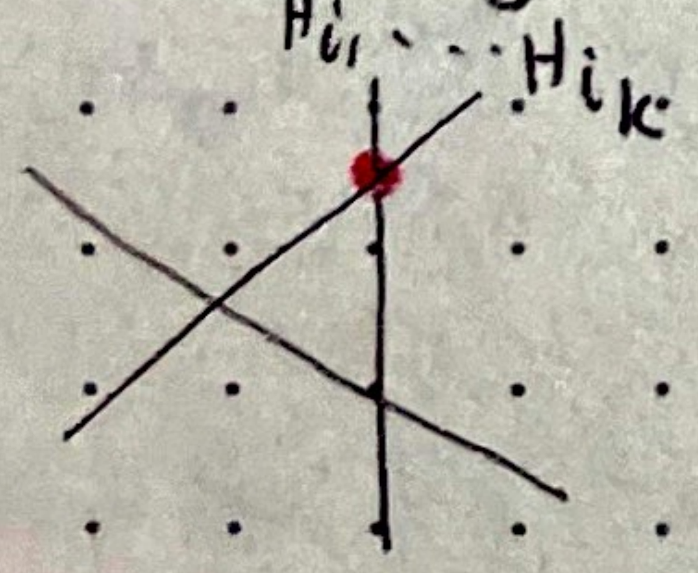
$$= \sum_{\substack{B \subseteq C \\ \text{s.t. join} = \hat{1}}} (-1)^{B}$$

Applications of Zaslvasky's Thrm

$\mathcal{A} = \{H_1, \ldots, H_N\}$ $\quad$ $H_i = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \vec{v}_i \rangle = h_i\}$

$\vec{v}_1, \ldots, \vec{v}_N$ are normal vectors to H_i's

Case of generic $h_1, \ldots, h_N \in \mathbb{R}$



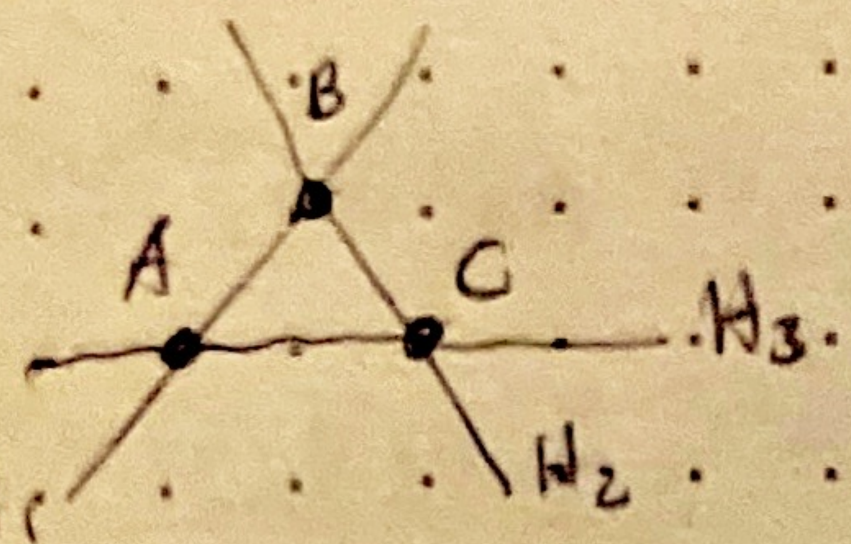
$$\forall x \in H_{i_1} \cap \ldots \cap H_{i_k} \neq 0$$
$$\Updownarrow$$
$$\vec{v}_{i_1}, \ldots, \vec{v}_{i_k} \text{ lin. ind. vectors}$$

Lemma: For generic h_i's,

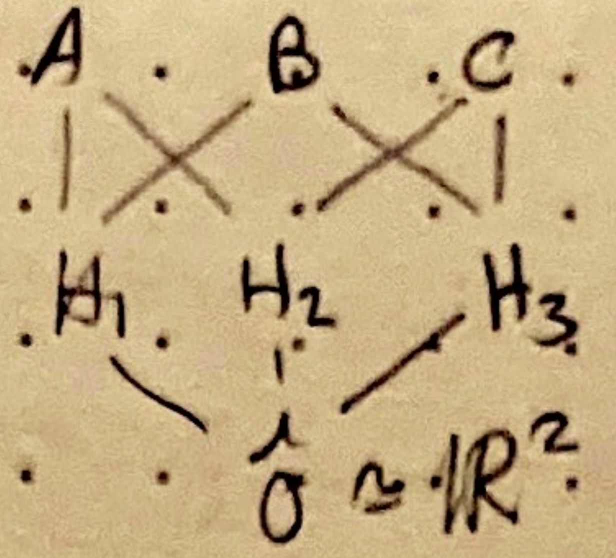
$L_{\mathcal{A}} \simeq$ semilattice of lin. ind. subsets of vectors $\vec{v}_i$ ordered by inclusion

intersection semilattice

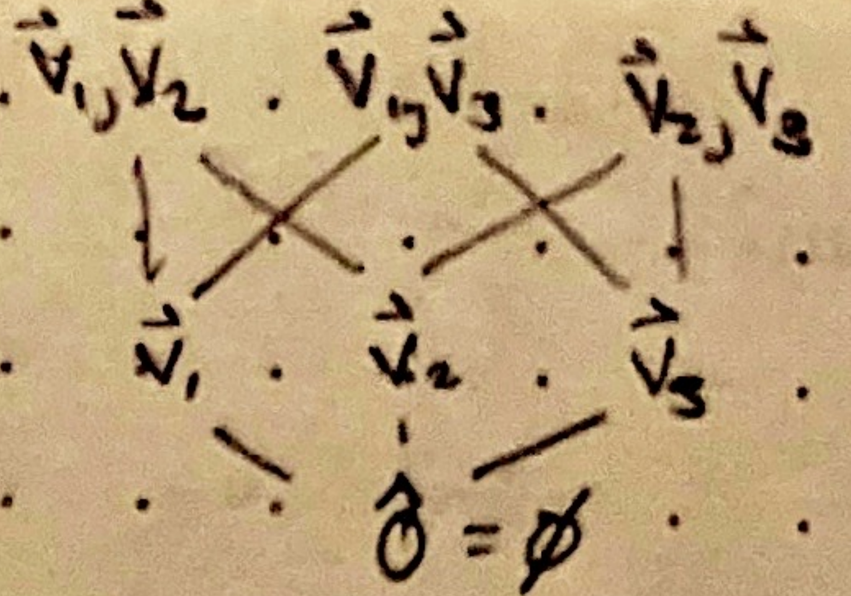
E.x.



$L_{\mathcal{A}} =$



$\simeq$



Any interval $[\hat{0}, X] \simeq$ boolean lattice
$\mu(\hat{0}, X) = (-1)^k \qquad k = \operatorname{codim}(X)$

<u>Lemma</u>: For generic h_i's,

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\{i_1, \ldots, i_k\} \subset [N] \text{ s.t.} \\ v_{i_1}, \ldots, v_{i_k} \text{ are lin. ind.}}} (-1)^k t^{n-k}$$

<u>Cor</u>: In this case
$r(\mathcal{A}) = \#$ independent subsets of $\vec{v}_i$'s
$b(\mathcal{A}) = \sum_{\substack{I \subseteq [N] \\ \text{independent}}} (-1)^{|I|}$ (assuming that $\mathcal{A}$ is essential)

<u>Recall</u>: h_i's generic $\xleftrightarrow[\text{to}]{\text{corresponds}}$ regular fine zonotopal tilings $\Rightarrow$

<u>Cor</u>: For a regular fine zonotopal tiling of
$Z = Z(\vec{v}_1, \ldots, \vec{v}_N)$
\# vertices in tiling = \# indep. subsets of $\vec{v}_i$

Comment! This is related to pset problem 4 that asks about
f vector of fine zonotopal tiling,
but problem 4 needs to work even for non-regular tilings.

<u>Finite Field Method of Athanasiadis</u>

Give another formula for characteristic poly that would be obvious generalization of chromatic poly

Assume $\{H_1, \ldots, H_N\}$ arr. in $\mathbb{R}^n$
all coefficients are integers

$\mathbb{F}_q$ a finite field with $q = p^r$ elts. for p a prime

Let $\mathcal{A}_q = \{H_1^q, \ldots, H_N^q\}$ arr. in $\mathbb{F}_q^n$ (given by same equations)

BUT be careful!

$\{2x + 2y = 1\}$
$\downarrow \mathbb{F}_2$
$\{0 = 1\}$

This hyperplane dissappears when taken in $\mathbb{F}_2$!

Luckily, this kind of thing doesn't happen too much:

Lemma: For all $q = p^r$, except for finite subset of primes, $\mathcal{A}_q$ is a valid hyperplane arr. in $\mathbb{F}_q^n$ and $L_{\mathcal{A}_q} \simeq L_{\mathcal{A}}$

Proof: Basically we need

$$\dim_{\mathbb{R}} (H_{i_1} \cap \dots \cap H_{i_k}) = \dim_{\mathbb{F}_q} (H_{i_1}^q \cap \dots \cap H_{i_k}^q)$$

for all subsets of hyperplanes

Basically putting all these things in a matrix & looking at when minors are 0 or not 0.

If $p \notin \left\{ \begin{array}{l} \text{any prime factor of} \\ \text{any minor of the} \\ \text{matrix of coeff} \end{array} \right\}$ then equality above holds

↳ And fin. matrix has finitely many minors, so fails for at most finitely many primes

$\rightsquigarrow L_{\mathcal{A}} \simeq L_{\mathcal{A}_q}$ for infinitely many primes