LECTURE 5 FRI 9/13

<u>Last Time</u>: Minkowski Sums, zonotopes.

<u>Graphical zonotopes</u>: $G=(V,E)$, graph on $V=[n]$

$$Z_G := \sum_{\{i,j\}\in E} [\vec{e}_i, \vec{e}_j] \quad \leftarrow \text{segment between coord vectors in } \mathbb{R}^n$$

<u>Hypergraphical Polytopes</u>: H <u>hypergraph</u> on V (i.e. a multiset on non-empty subsets $I \subseteq V$).

$$P_H := \sum_{I\in H} \Delta_I, \quad \text{where } \Delta_I = \text{conv}\{\vec{e}_i \mid i\in I\}$$

<u>Q</u>: What are vertices of Z_G?

Pick a generic lin. fcn $a(x) = a_1x_1 + \dots + a_nx_n = (\vec{a}, \vec{x})$

Supporting face (vertex)

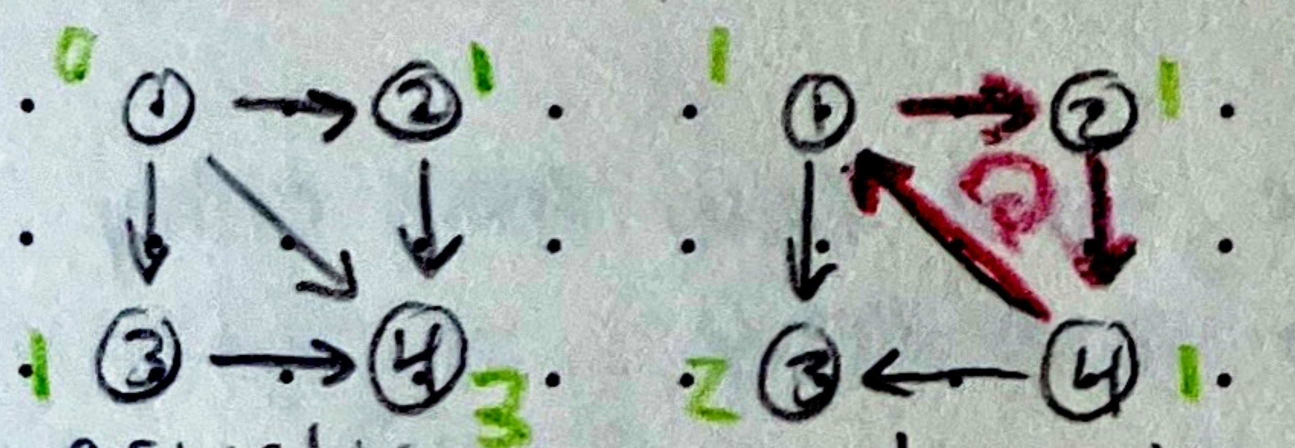
$$F_{\vec{a}, Z_G} := \sum_{\{i,j\}\in E} F_{\vec{a}, [\vec{e}_i, \vec{e}_j]} \quad F_{\vec{a}, [\vec{e}_i, \vec{e}_j]} = \begin{cases} \vec{e}_i & \text{if } a_i > a_j \\ \vec{e}_j & \text{if } a_i < a_j \end{cases}$$

$\rightsquigarrow$ Orientation of edges in G $\qquad i \xrightarrow{\text{if } a_i<a_j} j$

This gives us an:

<u>Def</u>: <u>acyclic orientation</u>: orient all edges of G so there are no directed cycles.

<u>Ex.</u>



acyclic $\qquad$ not acyclic

$v_G = (0,1,1,3)$ $\qquad$ $v_G = (1,1,2,1)$ $\leftarrow$ In degree vector

<u>Prop</u>: Vertices of Z_G correspond to acyclic orient. of G.

acyclic orient $\rightsquigarrow \vec{v}_G = (v_1, \dots, v_n)$ where

$v_i :=$ in degree of i in G

<u>Notation</u>: Will say "node" for graphs and "vertex" for polytopes so we don't get them confused.

Exercise: Prove that all lattice points ($Z_G \cap \mathbb{Z}$) of Z_G are in-degree vectors of some orientations of G.

For today: Talk about hypergraphical case

Vertices of $P_H \longleftrightarrow$ "acyclic orientations" of hypergraph H

Ex. The associohedron

$$A_n := \sum_{1 \le i \le j \le n} \Delta_{[i,j]} \qquad [i,j] = \{i, i+1, \dots, j\}$$

Note: This is one of many different non-equivalent constructions

Pick generic $a(x)$

$$F_{\vec{a}, A_n} = \sum_{i \le j} F_{\vec{a}, \Delta_{[i,j]}}$$

$\vec{e}_k$ where $k \in [i,j]$ s.t. $a_k = \max\{a_i, a_{i+1}, \dots, a_j\}$

Ex. $\vec{a} = (1, 3, 2, 7, 4, 5)$

a_4 max entry

For any consecutive interval containing 4, ($i \le 4 \le j$)

$$F_{\vec{a}, \Delta_{[i,j]}} = \vec{e}_4$$

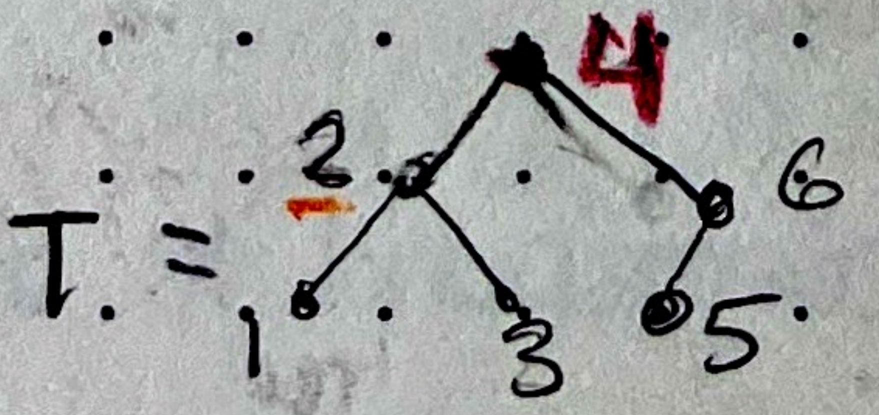
\# such intervals is $4 \cdot 3 = 12$

$\Rightarrow$ 4^{th} coord of corresponding vertex is 12

Next for intervals only on one side, find new max entry
e.g. interval on left has $a_2 = 3$ as max entry

$\Rightarrow$ get binary tree

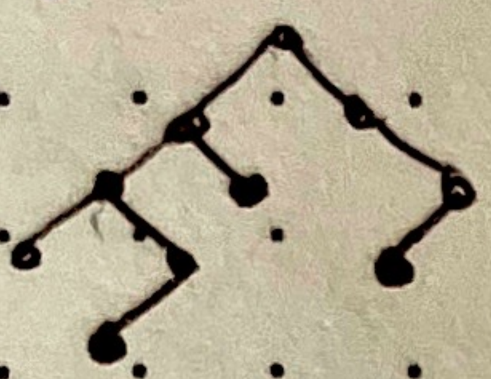
T =



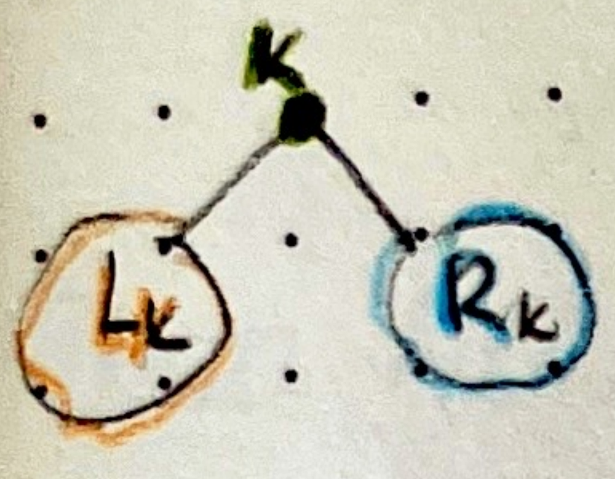
$\rightsquigarrow \vec{v}_T = (1, 4, 1, 12, 1, 2)$

In spot i, multiply the (1 + # left descendents of node i)(1 + # right descendents of node i)

In general, T an unlabelled tree on n nodes.

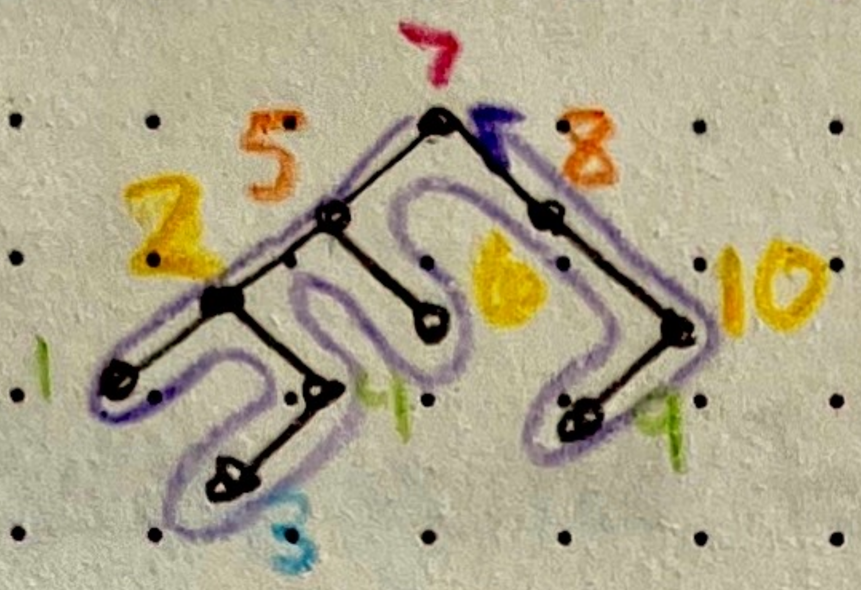
E.x. $n=10$ 

Label the nodes of T by $1, 2, \dots, n$ s.t. $\forall$ nodes



label k of node is $>$ all labels on left branch
$<$ all labels on right branch

$\Rightarrow$ uniquely determines tree labelling
(Label from top & go down)



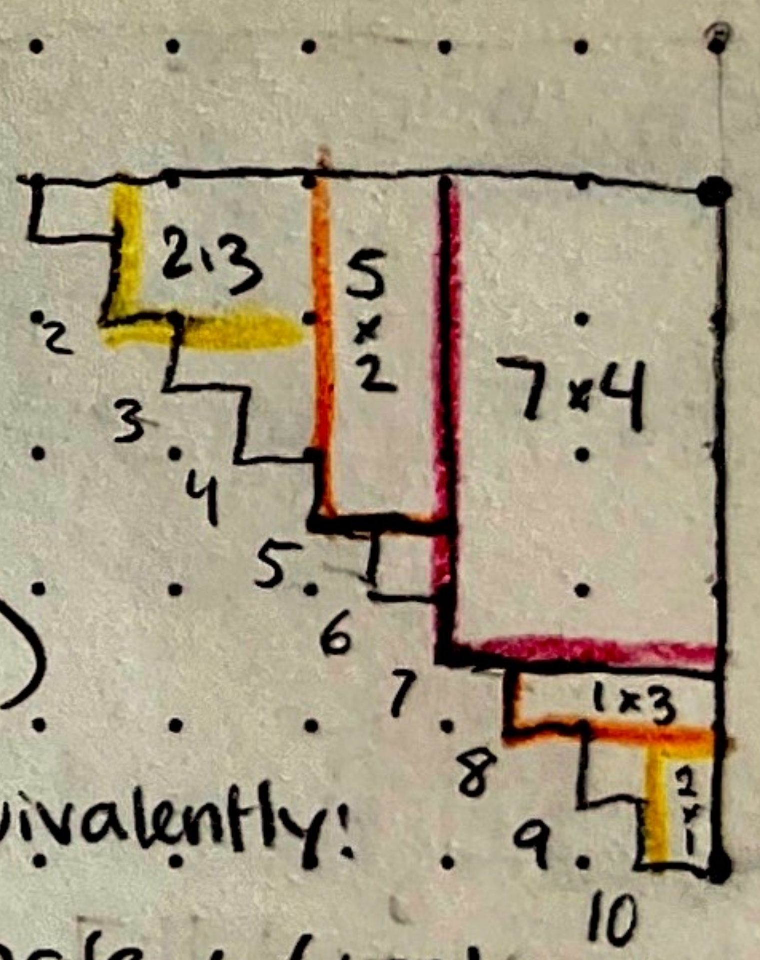
The binary search labelling of T

Equivalently:
Go around tree (as in purple line).
Label any node (w/ the next #) when we leave it going up or right.

Thrm: Vertices of A_n are in bijection w/ binary trees T on n nodes. (so Catalan many of them)

$$T \rightsquigarrow \vec{v}_T = (v_1, \dots, v_n)$$

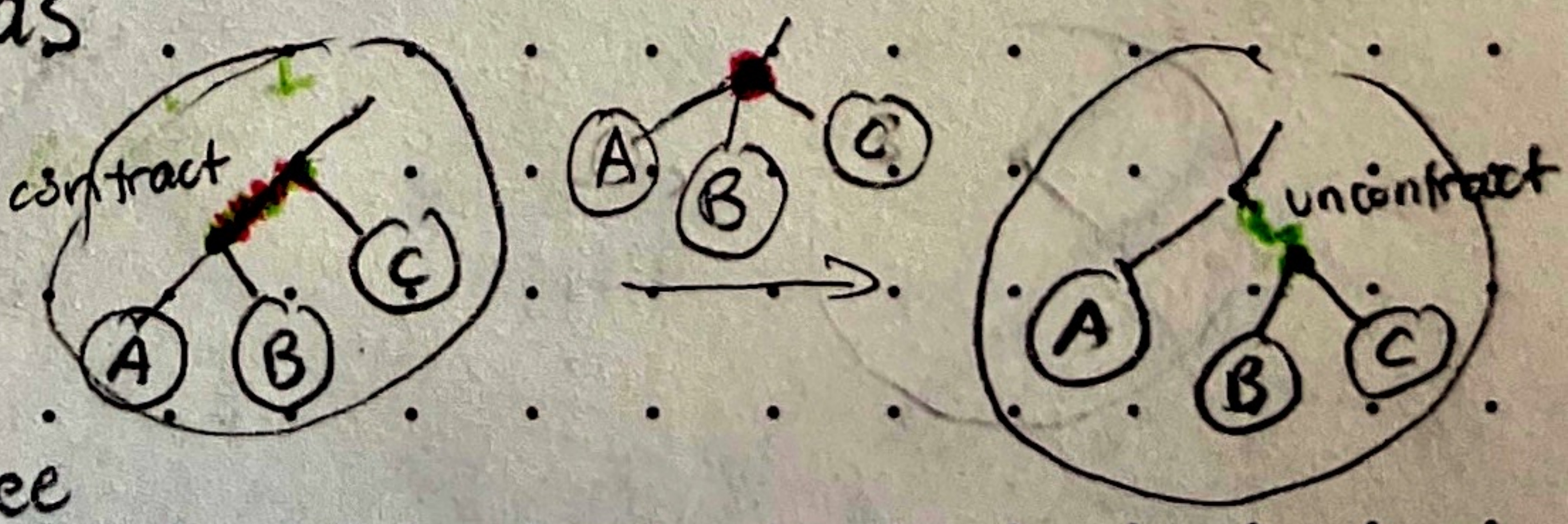
where $v_k = (|L_k|+1)(|R_k|+1)$



E.x. In tree above,
$\vec{v}_T = (1, 2\cdot 3, 1, 1\cdot 2, 5\cdot 2, 1, 7\cdot 4, 1\cdot 3, 1\cdot 1, 2\cdot 1)$
$= (1, 6, 1, 2, 10, 1, 28, 3, 1, 2)$

Equivalently:
Fill triangle w/ rectangles
v_i is the area of i^{th} rectangle

Every edge in 1-skeleton of A_n has the form $[\vec{v}_T, \vec{v}_{T'}]$ where T & T' are related by a flip



$\Rightarrow$ # edges out of $\vec{v}_T$ = # edges in tree

$h_k(A_n)$ = # binary trees on n nodes w/ k right edges
= Narayana numbers