18.212 Lecture 31 05/03/2021

Alternating Permutations

Recall from the last lecture:

Def. A permutation $w = w_1 \dots w_n \in S_n$ is <u>alternating</u> if

$$w_1 < w_2 > w_3 < w_4 > \dots$$

$A_n :=$ # of alternating permutations in S_n

n	0	1	2	3	4	5	6	7	8	...
A_n	1	1	1	2	5	16	61	272	1385	...

Recurrence Relation:

$$(**) \quad A_n = \sum_{\substack{k \in [n] \\ k \text{ even}}} \binom{n-1}{k-1} A_{k-1} \cdot A_{n-k}, \quad \text{for } n \geq 2$$

$$A_0 = A_1 = 1$$

Proof. $w = w_1 < w_2 > w_3 < \dots < w_{2\ell} > \dots w_n$

$(w' = w_1 < w_2 > \dots > w_{2\ell-1})$ $(w'' = w_{2\ell+1} \dots w_n)$

alt. perm of size $k-1$ alt. perm of size $n-k$

$k = 2\ell$

Exercise. Show that A_n also satisfies the rec. relation

$$(***) \quad 2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}, \quad n \geq 1$$

$$A_0 = A_1 = 1$$

Theorem $A_n = I_n(-1)$, where

$$I_n(x) := \sum_{\substack{T \text{ labelled} \\ \text{tree on} \\ n+1 \text{ vertices}}} x^{\mathrm{inv}(T)}$$

$$= \sum_{\substack{(f_1, \dots, f_n) \\ \text{parking} \\ \text{function}}} x^{\frac{n(n+1)}{2} - (f_1 + \dots + f_n)}$$

(the tree inversion polynomials)

Exp. Generating functions

Labelled trees & alternating permutations are <u>labelled</u> objects. So we need to use <u>exponential</u> generating functions.

Let

$$A_n(x) := \sum_{n \geq 0} A_n \frac{x^n}{n!}$$

Let's express the recur. rel. (**) in terms of $A(x)$.

(**) $\Longleftrightarrow$

$$A_n \cdot \frac{x^{n-1}}{(n-1)!} = \sum_{\substack{k \in [n] \\ k \text{ even}}} A_{k-1} \frac{x^{k-1}}{(k-1)!} \cdot A_{n-k} \frac{x^{n-k}}{(n-k)!}$$

Sum this over all $n \geq 2$

$$\underbrace{\sum_{n \geq 2} A_n \frac{x^{n-1}}{(n-1)!}}_{= A'(x) - 1} = \underbrace{\left(\sum_{\substack{k \geq 1 \\ \text{even}}} A_{k-1} \frac{x^{k-1}}{(k-1)!} \right)}_{= \text{the odd part of } A(x)} \underbrace{\left(\sum_{m \geq 0} A_m \frac{x^m}{m!} \right)}_{= A(x)}$$

Let's consider the even and odd parts of $A(x)$

$$A(x) = A^{\text{even}}(x) + A^{\text{odd}}(x),$$

where

$$A^{\text{even}}(x) := \sum_{\substack{n \geq 0 \\ n \text{ even}}} A_n \frac{x^n}{n!}$$

$$A^{\text{odd}}(x) := \sum_{\substack{n \geq 1 \\ n \text{ odd}}} A_n \frac{x^n}{n!}$$

$(**) \iff$

$$A'(x) = A^{\text{odd}}(x) \cdot A(x) + 1$$

Equivalently,

Proposition.

$$\left(A^{\text{even}}(x)\right)' = A^{\text{odd}}(x) \cdot A^{\text{even}}(x)$$

$$\left(A^{\text{odd}}(x)\right)' = A^{\text{odd}}(x) \cdot A^{\text{odd}}(x) + 1$$

Initial conditions:

$$A^{\text{even}}(0) = 1, \quad A^{\text{odd}}(0) = 0.$$

Secant & Tangent Numbers

Theorem (André 1879)

$$A^{even}(x) = \sec(x) := \frac{1}{\cos(x)}$$

$$A^{odd}(x) = \tan(x).$$

Equivalently,

$$\boxed{A(x) = \sec(x) + \tan(x)}$$

Proof Check that $\sec(x)$ & $\tan(x)$ satisfy the same differential equations:

$$\sec(x)' = \tan(x) \cdot \sec(x)$$

$$\tan(x)' = \tan(x)^2 + 1$$

and $\sec(0) = 1$, $\tan(0) = 0$.

$$\sec(x)' = \left(\frac{1}{\cos(x)}\right)' = \frac{\sin(x)}{\cos^2(x)} = \tan(x) \cdot \frac{1}{\cos(x)}$$

$$\tan(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\sin(x)' \cdot \cos(x) - \sin(x) \cdot \cos(x)'}{\cos(x)^2} =$$

$$= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = 1 + \tan(x)^2.$$

Remark. This is why the numbers A_{2n-1} are also called the tangent numbers and A_{2n} are called the secant numbers.

- The secant numbers A_{2n} are also called the Euler numbers.
- The tangent numbers A_{2n-1} are related to the Bernoulli numbers B_m.

The Bernoulli numbers are defined by the Taylor series

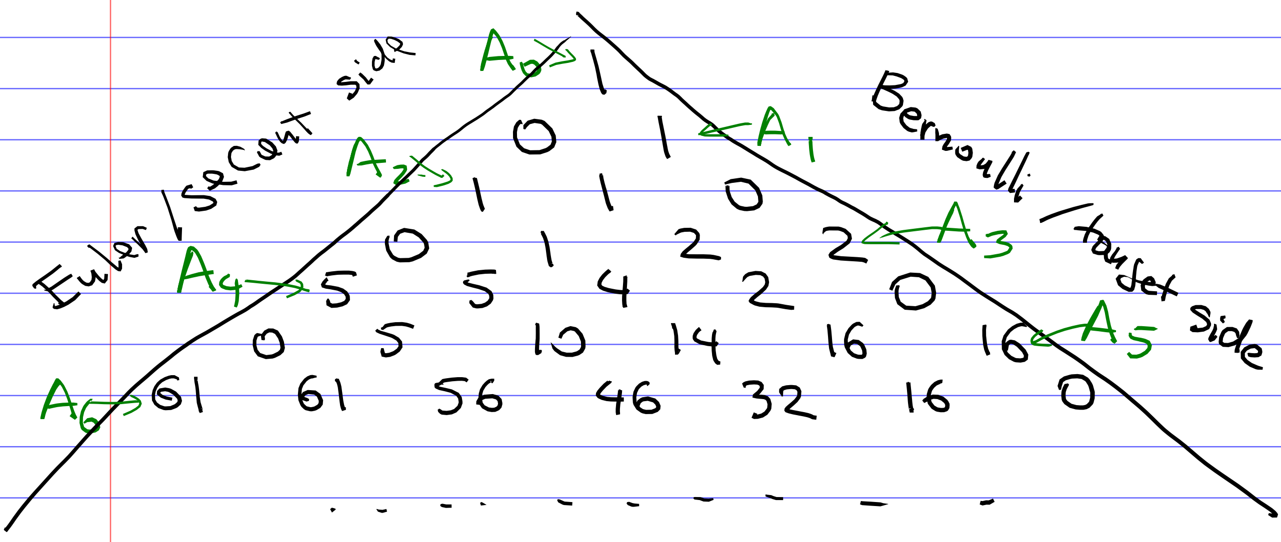
$$\frac{x}{1-e^{-x}} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}.$$

Then $B_{2n} = (-1)^{n-1} \frac{2n}{4^{2n} - 2^{2n}} A_{2n-1}$

and $B_{2n-1} = 0$, for all $n \geq 2$ except $B_1 = \frac{1}{2}$.

The Euler – Bernoulli triangle

a.k.a. the Seidel – Entringer – Arnold triangle.



<u>Rule</u>: To get the i^{th} row add the entries of the $(i-1)^{st}$ row <u>from left to right</u> if i is <u>even</u>, or <u>from right to left</u> if i is <u>odd</u>.

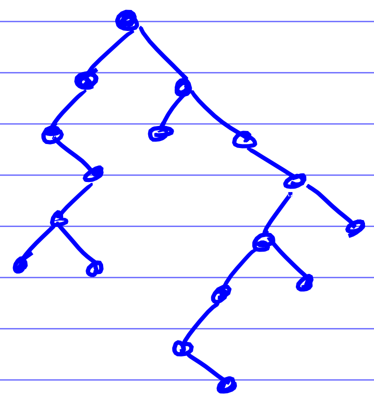
All entries in this triangle are called the <u>Entringer numbers</u>.

<u>Exercise</u> Show that this triangle contains the numbers A_n of alternating permutations on its sides, as shown above

Other combinatorial interpretations of the numbers A_n

Recall

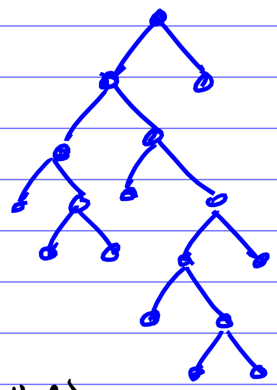
- <u>binary trees</u>



each vertex has either

- 2 children (the left & the right child),
- 1 child, which is designated as the left or the right child
- 0 children

- <u>complete binary trees</u>



each vertex has either

- 2 children (the left & the right child),
- 0 children

Notice that any complete binary tree as <u>odd</u> # of vertices.

<u>Theorem.</u> # binary trees on n vertices

$$= \text{\# complete binary trees on } 2n+1 \text{ vertices}$$

$$= C_n := \frac{1}{n+1}\binom{2n}{n}$$

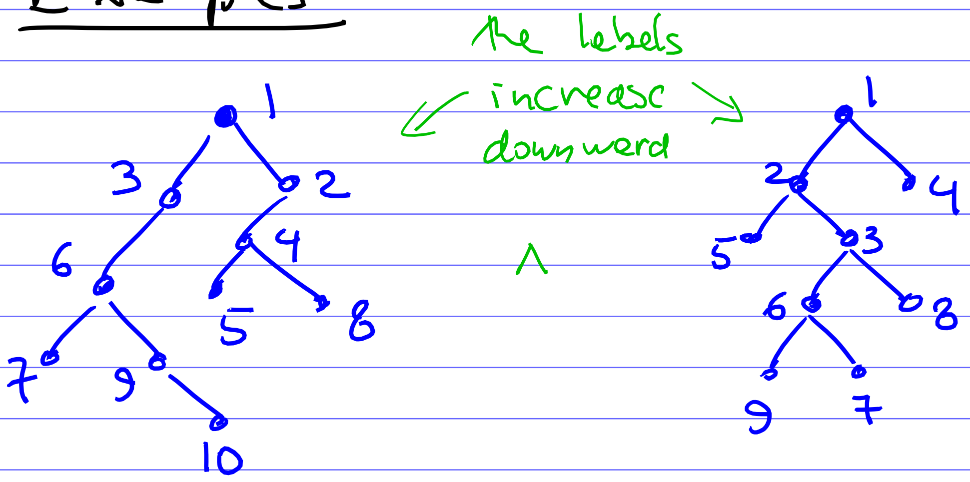
the Catalan number

These (complete) binary trees are <u>unlabelled</u> combinatorial objects.

Let's consider <u>labelled</u> versions of these objects.

Def. A (complete) increasing binary tree is a (complete) binary tree with vertices labelled by $1, 2, \ldots, n$ (without repetitions) such that the labels increase as we go away from the root.

Examples



an increasing binary tree

a complete increasing binary tree

Theorem.

- # increasing binary trees on n vertices equals $n!$
- # complete increasing binary on n vertices equals

$$= \begin{cases} A_n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(So we get only the tangent numbers.)

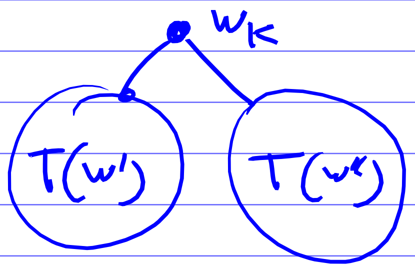
Proof Bijection between permutations an increasing binary trees:

$$\left\{\begin{matrix}\text{permutations}\\ w = w_1 \dots w_n\end{matrix}\right\} \xleftrightarrow{\text{bij.}} \left\{\begin{matrix}\text{increasing}\\ \text{binary trees } T\\ \text{on } n \text{ vertices}\end{matrix}\right\}$$

$$w \longmapsto T(w)$$

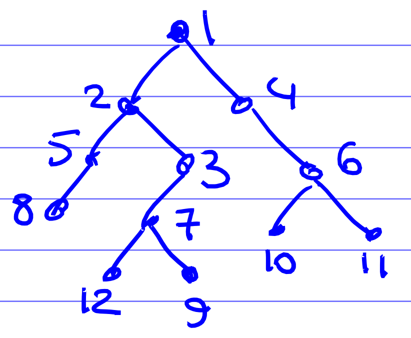
Construction by induction:

If $w = \underbrace{w_1 \dots w_{k-1}}_{w'} \; \overset{\text{the minimal entry in } w}{w_k} \; \underbrace{w_{k+1} \dots w_n}_{w''}$

then $T(w) =$ 

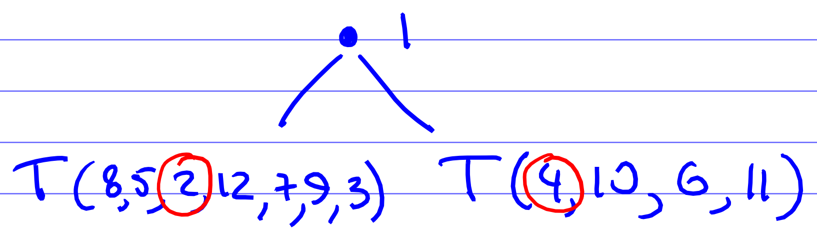
Example.

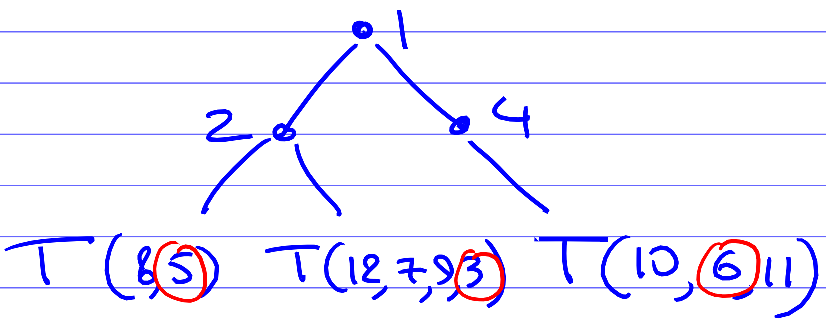
$w = 8, 5, 2, 12, 7, 9, 3, 1, 4, 10, 6, 11$

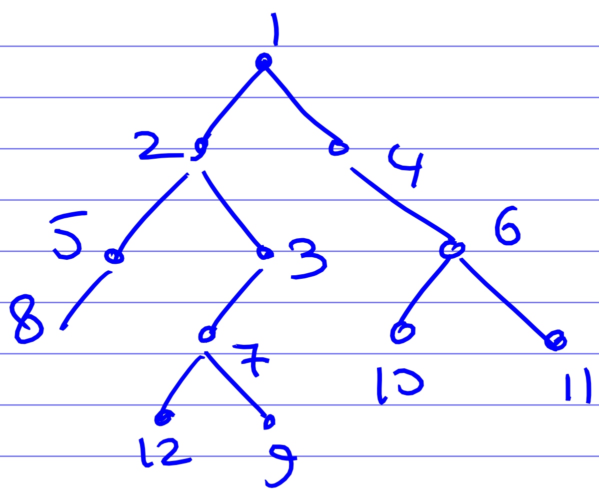
$T =$ 

Indeed,

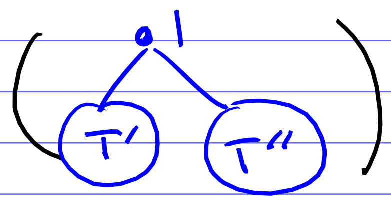
$w = \underbrace{8, 5, 2, 12, 7, 9, 3}_{w'}, 1, \underbrace{4, 10, 6, 11}_{w''}$

$T(w) =$ 

$=$ 

$=$ 

Clearly, the inverse map is given by

w  $= w(T'), 1, w(T'')$.

So this construction gives a bijection between all permutations in S_n and all increasing binary trees on n vertices.

This proves the 1st claim.

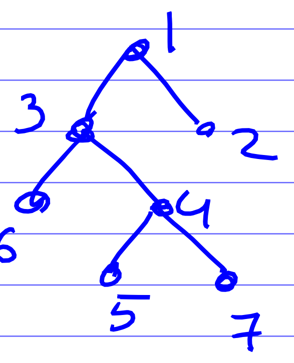
To prove the 2^{nd} claim, observe that this bijection has the property:

Let $T = T(w)$.

- T is a complete increasing binary tree iff
- $w = w_1 > w_2 < w_3 > \dots > w_{n-1} < w_n$ with odd n.

Example

$w = 6 > 3 < 5 > 4 < 7 > 1 < 2$

$T(w) =$ 

Indeed, this follows from the following more general observation:

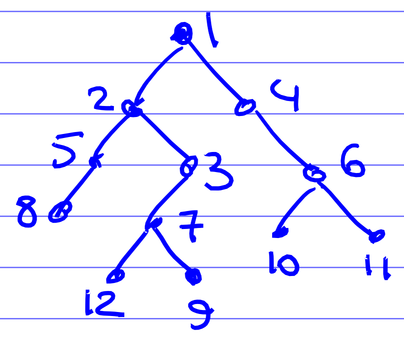
Lemma Let $T = T(w)$.

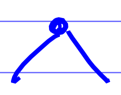
- Valleys: If $w_{i-1} > w_i < w_{i+1}$, then the vertex w_i in T has 2 children.
- Double descents: If $w_{i-1} > w_i > w_{i+1}$ then the vertex w_i in T has only the left child
- Double ascents: If $w_{i-1} < w_i < w_{i+1}$ then the vertex w_i in T has only the right child
- Peaks: If $w_{i-1} < w_i > w_{i+1}$ the the vertex w_i has no children.

(Here we assume that $w_0 = w_{n+1} = 0$.)

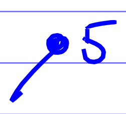
Example

$w = 8, 5, 2, 12, 7, 9, 3, 1, 4, 10, 6, 11$

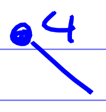
$T =$ 

$5 > 2 < 13$ $\Rightarrow$ 

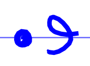
(2 children)
valley

$8 > 5 > 2$ $\Rightarrow$ 

(only left child)
double descent

$1 < 4 < 10$ $\Rightarrow$ 

(only right child
double ascent

$7 < 9 > 3$ $=$ 

(no children, i.e. a leaf)
peak

Corollary

permutations in S_n with a valleys, b double descents, c double ascent, and d peaks $(a+b+c+d=n)$

equals # increasing binary trees on n vertices among which

- a vertices have 2 children
- b vertices have only left child
- c vertices have only right child
- d vertices have no children.

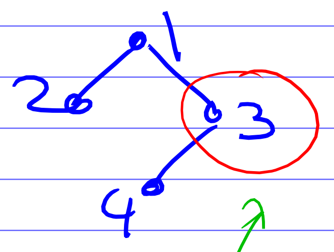
How about alternating permutations with even # entries?

$w \longmapsto T = T(w)$

$w_1 > w_2 < w_3 > \dots > w_{2m}$ iff T is an almost complete increasing binary tree.

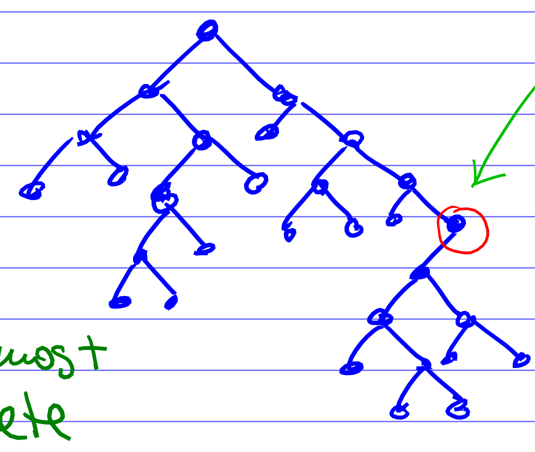
Example

$2 > 1 < 4 > 3 \rightsquigarrow$



no left child

Def. A binary tree is almost complete if all vertices have either 2 or 0 children, except one vertex which is located in the end of the chain of right edges starting at 1



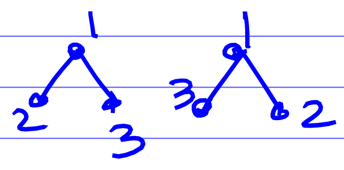
the only vertex that has 1 child

an almost complete binary tree

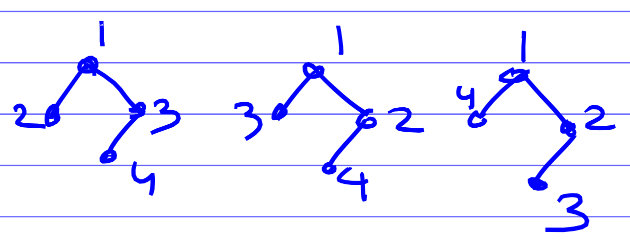
Theorem

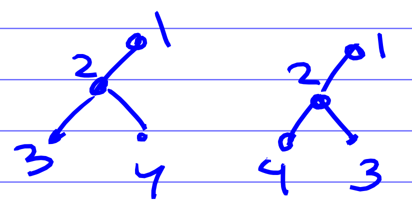
$$A_n = \begin{cases} \text{\# complete increasing binary trees on } n \text{ vertices}, & n \text{ odd} \\ \text{\# almost complete increasing binary trees on } n \text{ vertices} & n \text{ even} \end{cases}$$

Examples. $A_3 = 2$



$A_4 = 5$

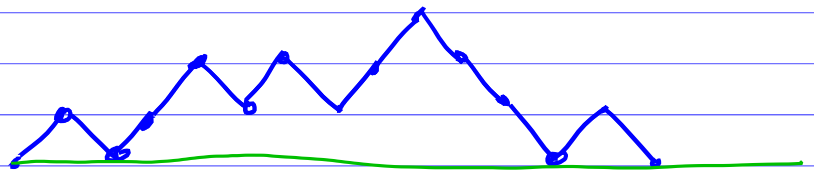




almost complete increasing binary trees on 4 vertices

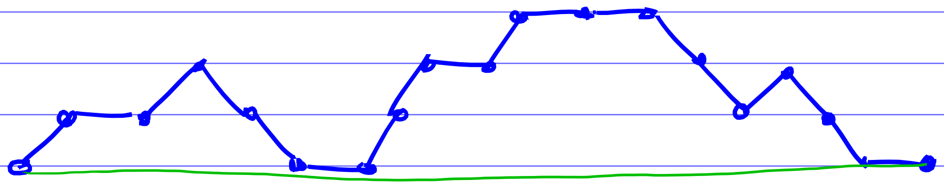
Weighted path enumeration & Françon-Viennot bijection

Recall, Dyck paths



Motzkin Paths are similar to Dyck paths, but they might also contain horizontal steps (in addition to up and down steps).

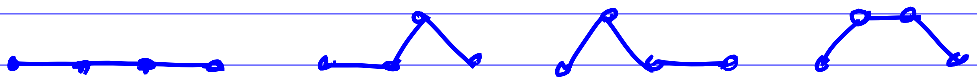
Example



a Motzkin path with 18 steps

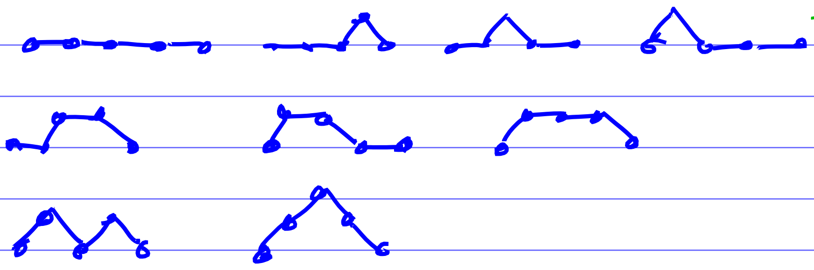
The Motzkin number M_n is # of Motzkin paths with n steps.

Example. $M_3 = 4$



4 Motzkin paths with 3 steps.

$M_4 = 9$



9 Motzkin paths with 4 steps

We will now consider weighted versions of Dyck & Motzkin paths...

Define the height $ht(s)$ of a step s in a Motzkin or Dyck path as the y-coordinate of its initial point $+1$

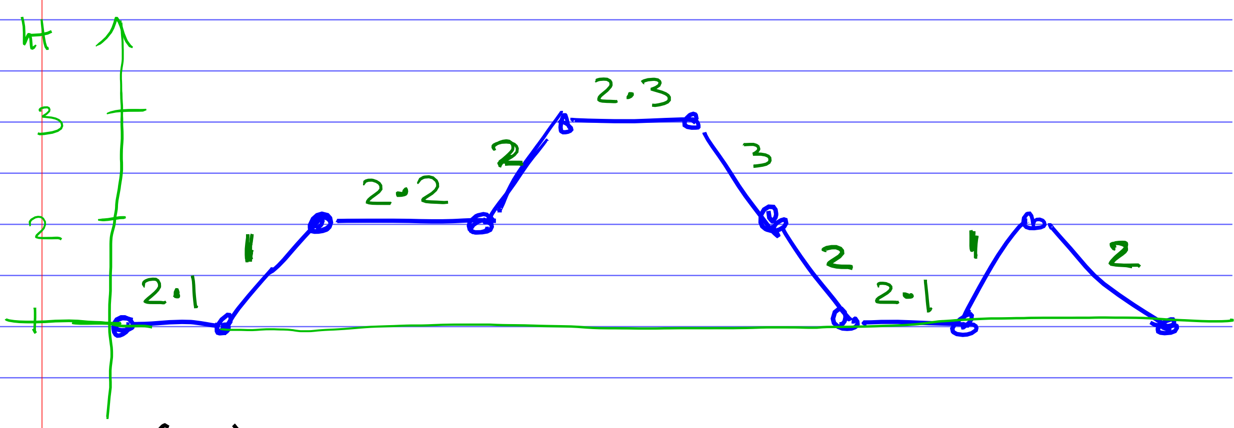
Now define the weight $wt(s)$ of a step s, as follows:

- up or down step : $wt(s) = ht(s)$
- horizontal step : $wt(s) = 2 \cdot ht(s)$.

Define the weight of path P as

$$wt(P) := \prod_{\substack{s \text{ is a} \\ \text{step in } P}} wt(s)$$

Example



$$wt(P) = 2 \cdot 1 \cdot 4 \cdot 2 \cdot 6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 2$$

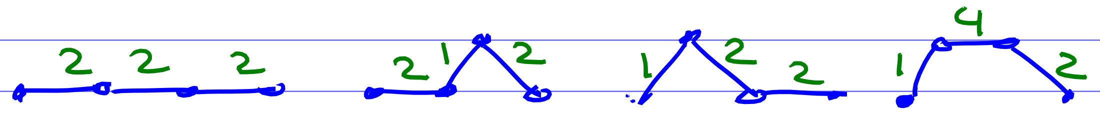
Theorem

- $$\sum_{\substack{P:\text{ Motzkin} \\ \text{path w/} \\ n \text{ steps}}} wt(P) = (n+1)!$$

- $$\sum_{\substack{P:\text{ Dyck} \\ \text{path with} \\ 2n \text{ steps}}} = A_{2n+1}$$

Examples

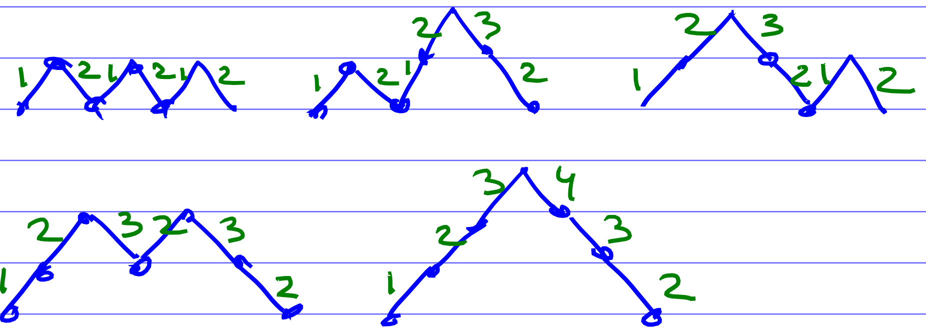
- weighted Motzkin paths with 3 steps:



$2\cdot2\cdot2 + 2\cdot1\cdot2 + 1\cdot2\cdot2 + 1\cdot4\cdot2$

$= 24 = 4!$

- weighted Dyck paths with $2\cdot3$ steps:



$1\cdot2\cdot1\cdot2\cdot1\cdot2 + 1\cdot2\cdot1\cdot2\cdot3\cdot2 + 1\cdot2\cdot3\cdot2\cdot1\cdot2$

$+ 1\cdot2\cdot3\cdot2\cdot3\cdot2 + 1\cdot2\cdot3\cdot4\cdot3\cdot2$

$= 272 = A_7$

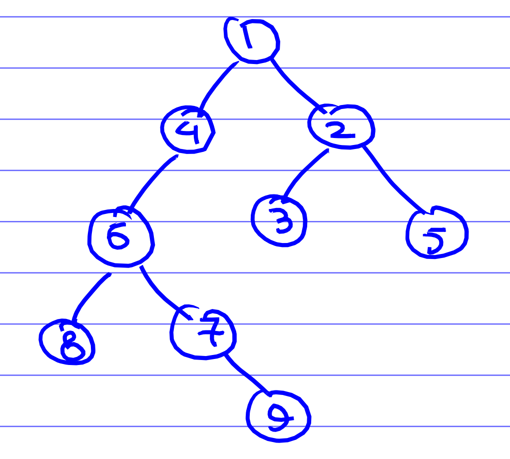
Françon – Viennot bijection gives a bijective way to prove this theorem.

We'll construct a bijection between increasing binary trees and weighted Motzkin paths

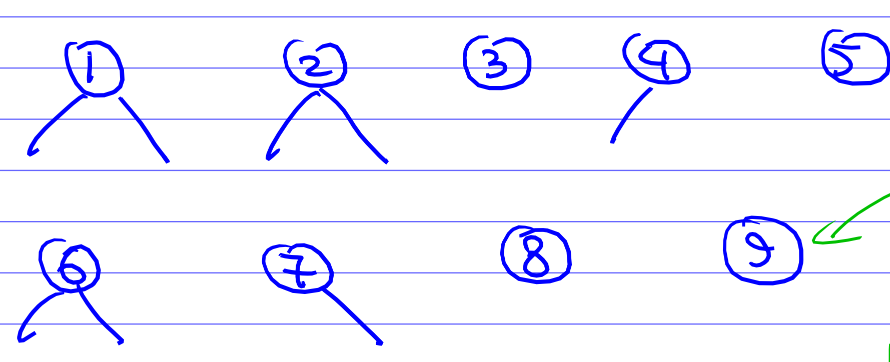
$$\left\{\begin{array}{l}\text{increasing} \\ \text{binary} \\ \text{trees on} \\ n+1 \text{ vertices}\end{array}\right\} \xleftrightarrow[\text{bij.}]{\text{Françon Viennot}} \left\{\begin{array}{l}\text{weighted} \\ \text{Motzkin} \\ \text{paths with} \\ n \text{ steps}\end{array}\right\}$$

Example

T =

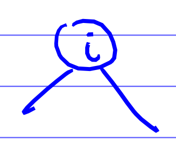
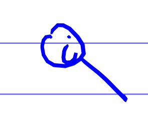


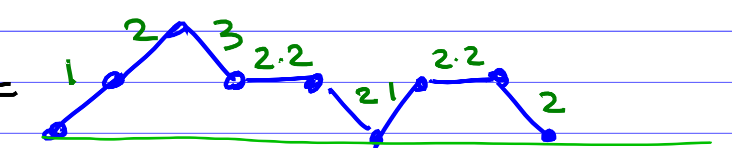
Let's "decompose" T into pieces consisting of vertices with placeholders for their children:



We'll ignore the last vertex n+1 because it will always be like this

In the corresponding Motzkin path P, for $i = 1, \dots, n$,

- if  ⟹ i^{th} step is "up"
- if i ⟹ i^{th} step is "down"
- if i or  ⟹ i^{th} step is horizontal

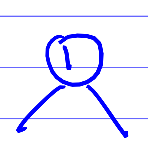
P = 

$wt(P) = 1 \cdot 2 \cdot 3 \cdot (2 \cdot 2) \cdot 2 \cdot 1 \cdot (2 \cdot 2) \cdot 2$

counts # ways to combine the pieces into a binary tree.

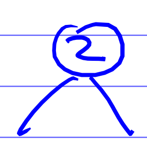
In our example:

1. 1^{st} step in P is "up"



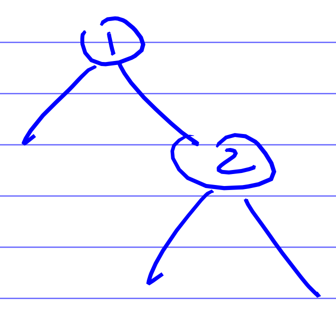
wt = 1

2. 2^{nd} step is "up", wt = 2



2 ways to attach it to one of the placeholders.

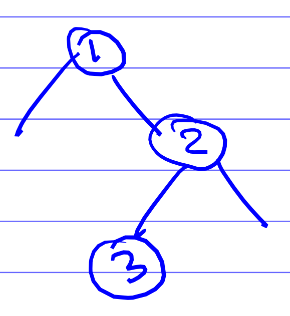
eg.



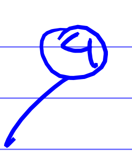
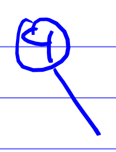
3. 3^{rd} step is "down", wt = 3

3 ways to attach ③ to one of the placeholders

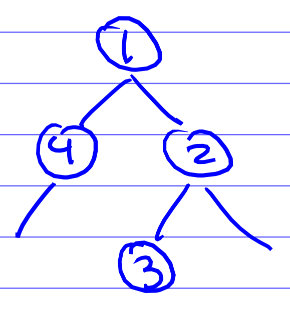
e.g.



4. 4^{th} step is horizontal, wt = 2·2

2·2 ways  or  to attach to a placeholder

eg



etc.

This gives the needed bijection & proves the 1^{st} claim.

To get the second claim about Dyck paths & A_{2n+1} we just restrict this construction to complete increasing binary trees. □

There are many other special cases of the Françon-Viennot bijection.

Corollary

\# permutations in S_{n+1} without double ascents $w_i < w_{i+1} < w_{i+2}$ (assuming that $w_0 = 0$)

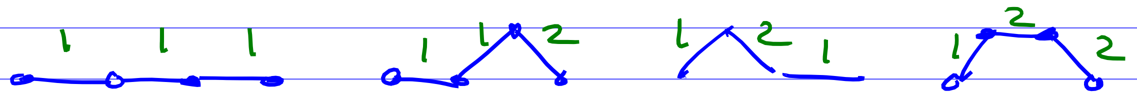
= \# increasing binary trees on $n+1$ vertices s.t. $\nexists$ vertices that have only the right child.

= \# weighted Motzkin paths with n steps and the weights given by
$wt(s) = ht(s)$ for any s

(no factor 2 for horizontal steps.)

Example $n = 3$

\# permutations in S_4 w/o double descent (& no initial descent) equals:



$1 + 2 + 2 + 4 = 9.$

Here are these 9 permutations:

2143, 3142,
3241, 3214,
4132, 4213, 4231
4312, 4321