18.212 Lecture 25 04/16/2021

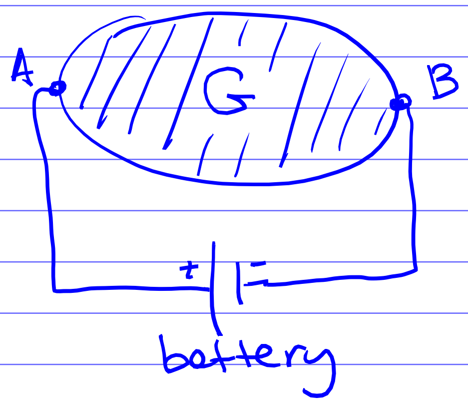
An electrical network:

$G = (V, E)$ with edge weights

$$C_e = \frac{1}{R_e}, \quad e \in E$$

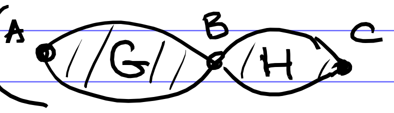
(C_e = the conductivity of edge e = $(\text{resistance})^{-1}$.)

$A, B \in V$ two selected vertices

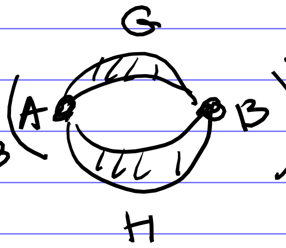


$R_{AB}(G)$ = the resistance of between vertices A, B of G.

Series connection:

$R_{AC}\Big($  $\Big) = R_{AB}(G) + R_{BC}(H)$

Parallel connection:

$R_{AB}\Big($  $\Big)^{-1} = R_{AB}(G)^{-1} + R_{AB}(H)^{-1}$

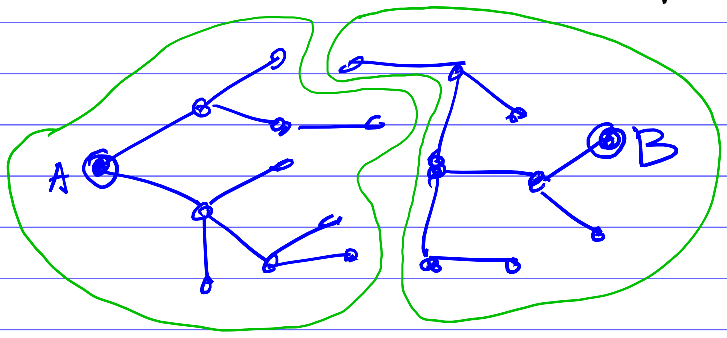
In general,

Theorem. $R_{AB}(G) =$

$$= \frac{\sum_{F \subset G} \text{weight}(F)}{\sum_{T \subset G} \text{weight}(T)} ;$$

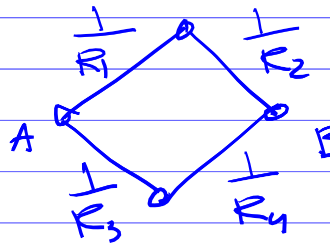
where

- The sum in the denominator is over all spanning trees T in G
- The sum in the numerator is over all forests F in G s.t.
  - F contains all vertices of G
  - F has exactly 2 connected components
  - A & B belong to different connected components of F.



- $\text{weight}(F) = \prod_{\substack{e \text{ edge} \\ \text{of } F}} \frac{1}{R_e}$

Example

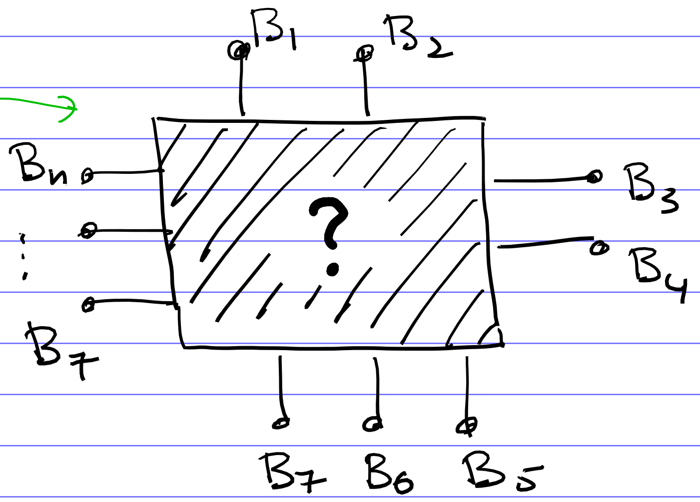
$G =$ 

$$R_{AB} = \frac{\frac{1}{R_1}\cdot\frac{1}{R_3} + \frac{1}{R_1}\cdot\frac{1}{R_4} + \frac{1}{R_3}\cdot\frac{1}{R_2} + \frac{1}{R_2}\cdot\frac{1}{R_4}}{\frac{1}{R_1}\cdot\frac{1}{R_2}\cdot\frac{1}{R_3} + \frac{1}{R_1}\frac{1}{R_2}\frac{1}{R_4} + \frac{1}{R_1}\frac{1}{R_3}\frac{1}{R_4} + \frac{1}{R_2}\frac{1}{R_3}\frac{1}{R_4}}$$

Black box problem or

Inverse Boundary Problem

black box
Some unknown network



$B_1, B_2, \ldots, B_n$ boundary vertices

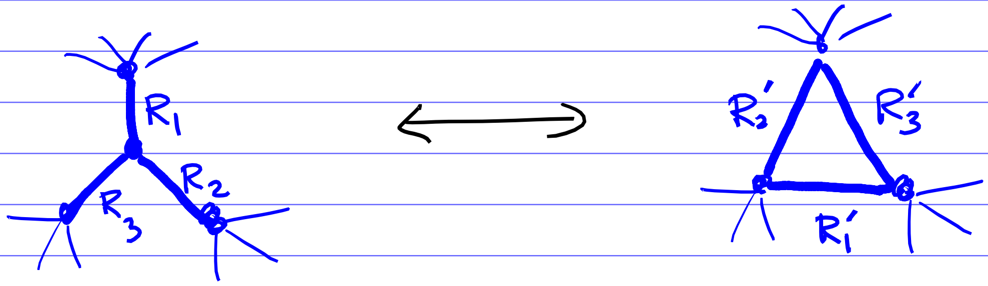
Boundary Measurements: We can measure the resistances

$$R_{ij} = R_{B_i B_j}(G).$$

Question: Can we reconstruct a network G from its boundary measurements R_{ij}? What can we say about G if we know all R_{ij}?

[Curtis, Ingerman, Morrow, 1998] solved this for planar graphs.

Y – Δ transform:



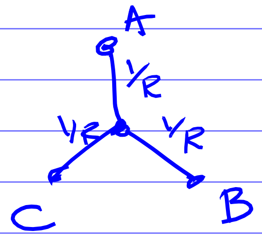
For any 3 edges in G with resistances R_1, R_2, R_3 forming "Y" (where central vertex in not a boundary vertices), we can replace them by 3 edges forming "Δ" with certain resistances R_1', R_2', R_3' s.t. all boundary measurements remain the same.

$$(R_1, R_2, R_3) \longleftrightarrow (R_1', R_2', R_3')$$

is a certain invertible transformation.

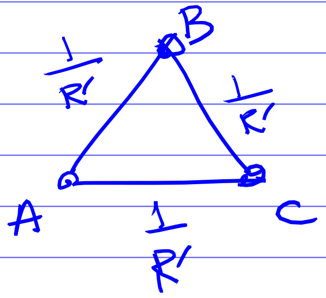
Exercise Find an explicit expression for R_1', R_2', R_3' in terms of R_1, R_2, R_3

Example $R_1 = R_2 = R_3 = R$



$$R_{AB} = \frac{2\cdot\left(\frac{1}{R}\right)^2}{\left(\frac{1}{R}\right)^3} = 2R$$

$$R_{AB} = R_{AC} = R_{BC} = 2R$$

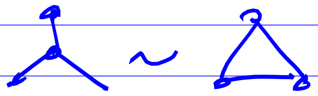
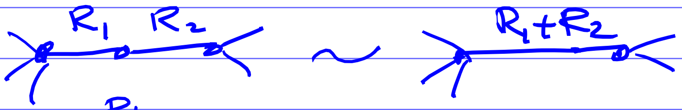


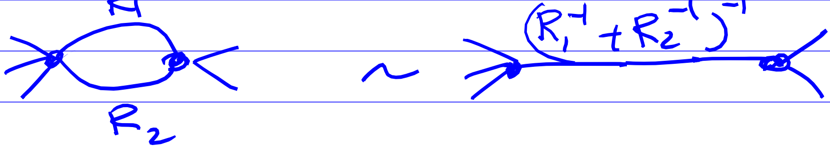
$$R_{AB} = \frac{2\left(\frac{1}{R'}\right)^2}{3\left(\frac{1}{R'}\right)^3} = \frac{2}{3}R'$$

$$2R = \frac{2}{3}R'.$$

So $R' = 3R.$

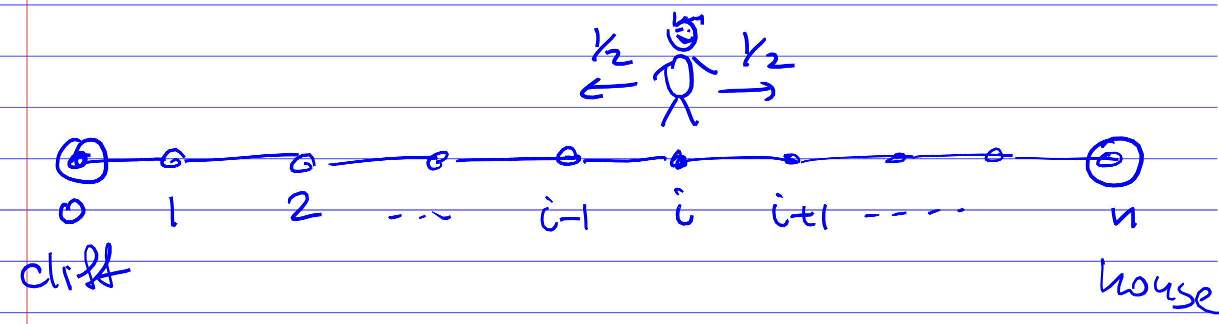
Theorem [CIM]. Let G_1, G_2 be two planar networks with n boundary vertices. Then G_1 & G_2 have the same boundary measurements iff they can be obtained from each other by a sequence of

- Y-Δ transforms 
- 

- 

Random Walks on Graphs

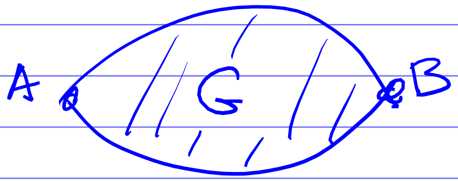
Example (Drunkard's walk, see Lecture 2)



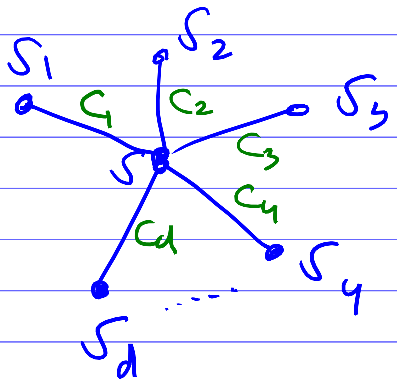
We start at vertex i and walk randomly untill we reach vertex 0 or vertex n. If we reach n first, we win; if we reach 0 then we lose.

$$\text{Prob}\left(\begin{matrix}\text{start at } i\\ \text{and win}\end{matrix}\right) = \frac{i}{n}$$

Let's generalize this to any finite graph G with edge weights $c_e > 0$ and two marked vertices A & B



Random walk on G



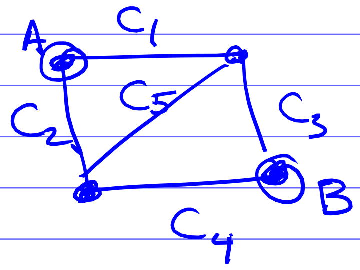
At each step we go from a vertex v to one of its neighbors v_i $(i = 1, \ldots, d = \deg(v))$ with probability

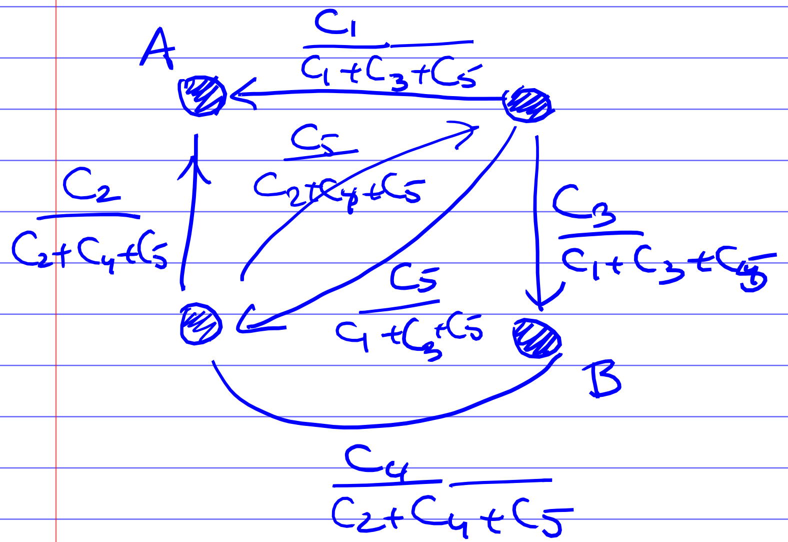
$$\frac{c_i}{c_1 + \cdots + c_d}.$$

We stop when we reach vertex A or vertex B.

We win if we reach B and we loose if we reach A.

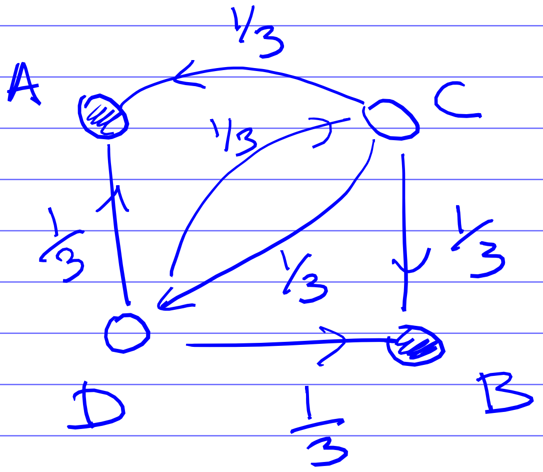
Example $G =$





Let $p(v)$ = the probability that a random walk starting at vertex v reaches B (before reaching A)

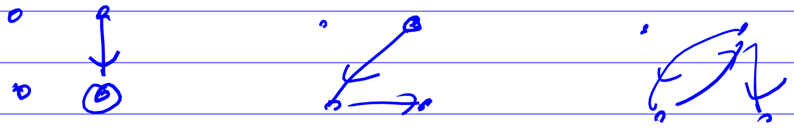
Example (Same as before with all $C_e = 1$.)



$$p(A) = 0, \quad p(B) = 1$$

$$p(C) = p(D) =$$

$$= \frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \dots$$



$$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{2}.$$

In general, $p(A) = 0$

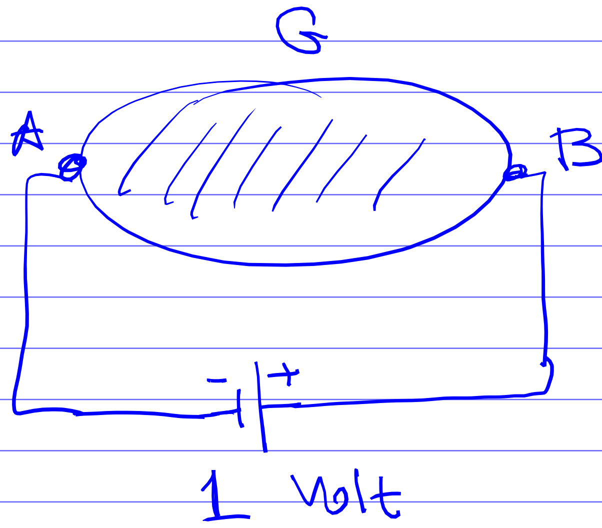
$p(B) = 1$ and

$0 \le p(s) \le 1$ for any vertex s.

Interpretation of this random walk on G in terms of electrical network

Let's view G as an electrical network with resistances $R_e = \frac{1}{c_e}$.

Connect vertices A & B to the battery with voltage $= 1$ (Volt)



Let $v \mapsto U_v$ be the resulting potential function on vertices v of G.

$U_A = 0$, $U_B = 1$

$0 \le U_v \le 1 \quad \forall$ vertex v.

Theorem

$\text{Prob}(v) = U_v$.

the probability to a random walk starting at v reaches B before reaching A

the potential of vertex v in the electrical network

Proof. In last lecture, we've seen that the potential can be found by solving the system of linear equations with Kirchhoff's matrix K (same as Laplacian matrix L).

$$(*) \qquad K \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

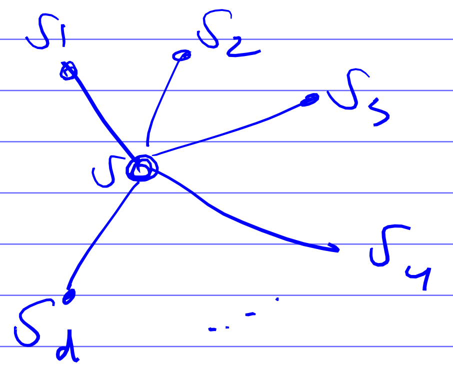
$$U_1 = 0, \quad U_n = 1$$

Unknowns: $U_2, \dots, U_{n-1}, I$

(Here G has vertices $1, \dots, n$; $A = 1$ and $B = n$.)

Let's show that the probabilities $p_v = \text{prob}(v)$ satisfy the same system $(*)$ of linear equations.

Indeed



$v \neq A, B$

$d = \deg(v)$, $v_1, \dots, v_d$ all neighbors of v.

$e_1, \dots, e_d$ edges adjacent to v

$v \overset{e_i}{\text{———}} v_i$

$$p_v = \sum_{i=1}^{d} \frac{c_{e_i}}{c_{e_1} + \dots + c_{e_d}} \cdot p_{v_i}$$

or, equivalently,

$$(c_{e_1} + \dots + c_{e_d})\, p_v + \sum_{i=1}^{d} (-c_{e_i})\, p_{v_i} = 0$$

a diagonal entry of Kirchhoff's (Laplacian) matrix K

an off diagonal entry of K

In matrix form we get

$$K \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$p_1 = 0$, $p_n = 1$

unknowns: $p_2, \dots, p_{n-1}, I$

(Here we assume that $A = 1$ and $B = n$)

This is exactly the same linear system as $(*)$.

So $p_v = u_v$ for any vertex v. $\square$

Exercise. Two people Alice & Bob play the following game:

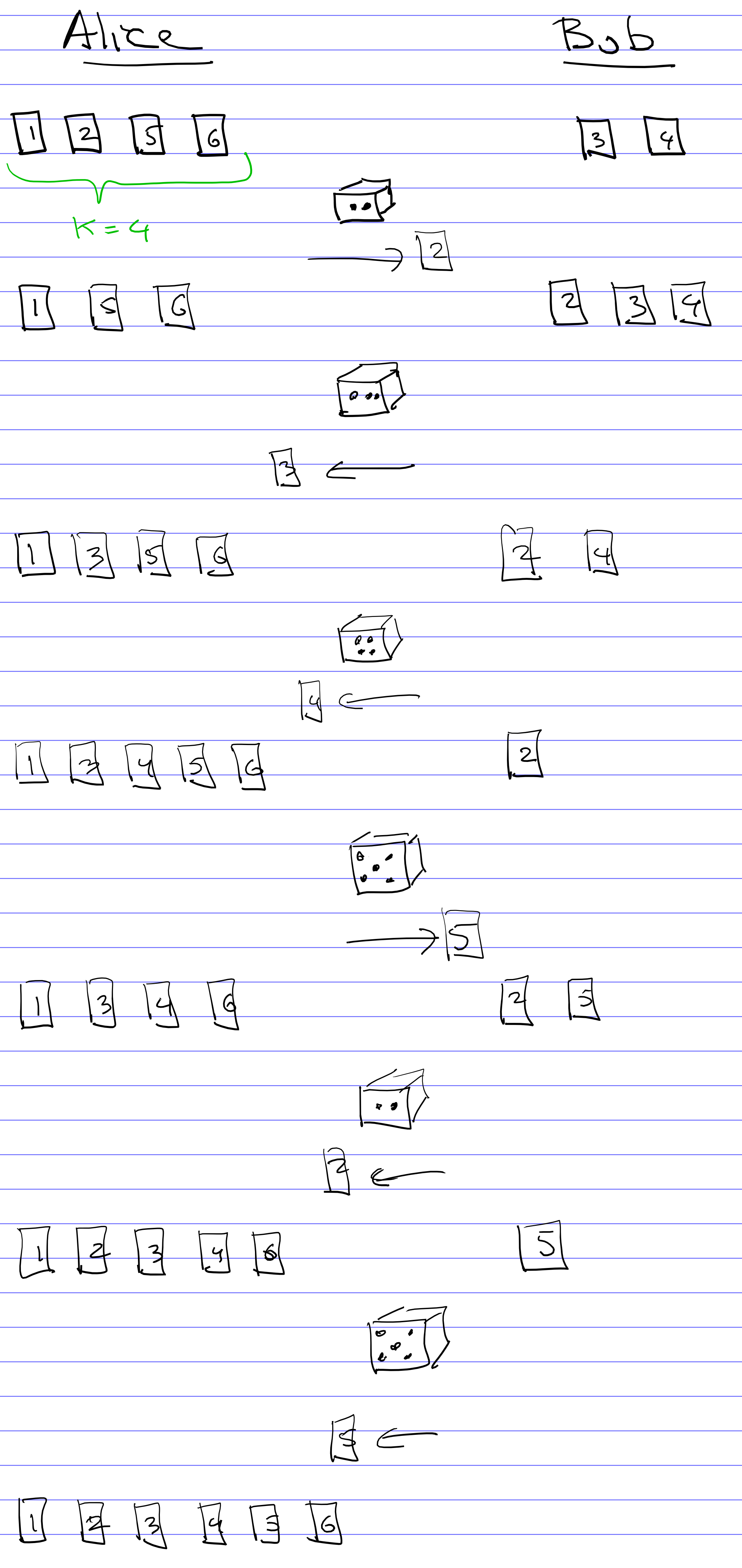
There are 6 cards labelled $1, 2, \ldots, 6$.
Initially, Alice has k cards and Bob has the remaining $6-k$ cards.

Each turn they roll a dice which randomly (and uniformly) generates a number $i \in \{1, 2, \ldots, 6\}$.
Then the person who has the i^{th} card gives this card to the opponent.

The game continues until one person collects all 6 cards and wins the game.

Find the probability that Alice wins the game.

Example



Alice wins.

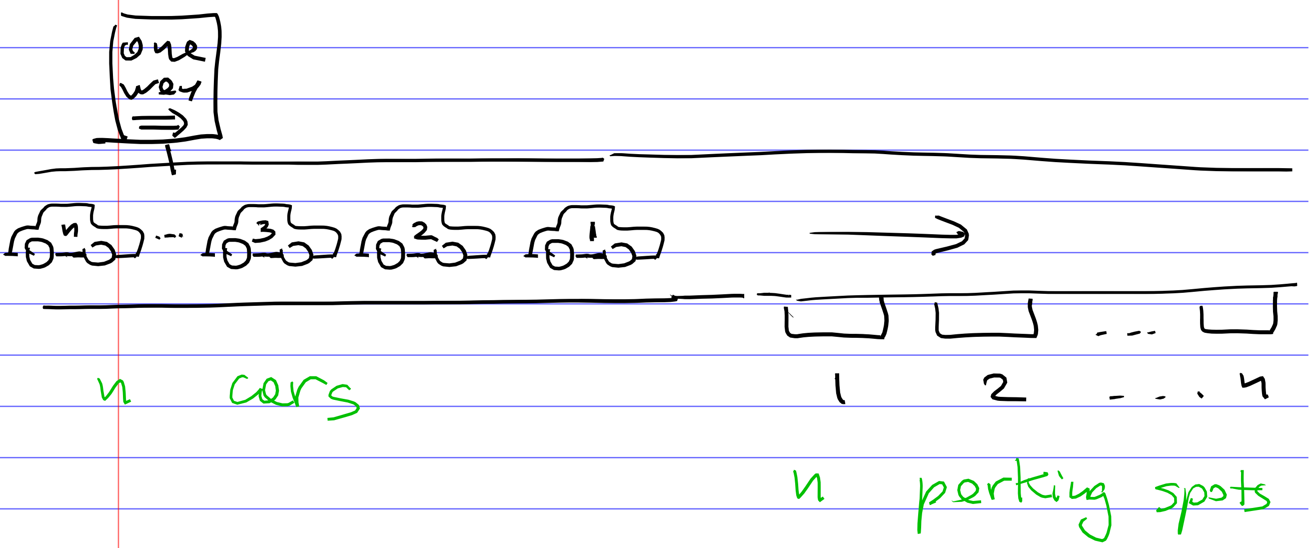
More on Cayley's formula

\# spanning trees in K_n = n^{n-2}

Other interpretations of the number n^{n-2} (or $(n+1)^{n-1}$)

Parking functions:

- n cars & n parking spots on a one way road.



Each driver prefers some parking spot. Let $f_i \in [n]$ be the favorite parking spot of the driver of the i^{th} car.

Each car goes directly to its favorite spot f_i. If the spot is empty, the car parks there. Otherwise it keep driving and takes the first available spot.

$(f_1, \ldots, f_n)$ is called a parking function if all n cars can park.

Example $n = 3$

parking functions:

$(1,2,3)$ & all other permutations of $1, 2, 3$

$(1,1,2)$, $(1,1,3)$, $(1,2,2)$ permutations.

$(1,1,1)$

Not parking functions:

$(1,3,3)$, $(2,2,2)$ $(2,2,3)$
$(2,3,3)$, $(3,3,3)$

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parking functions

$$6 + 3 + 3 + 3 + 1 = 16.$$

Theorem # parking functions for n cars is $(n+1)^{n-1}$.

Exercise. Find a bijection between parking functions for n cars and spanning trees in K_{n+1}.

Shi hyperplane arrangement

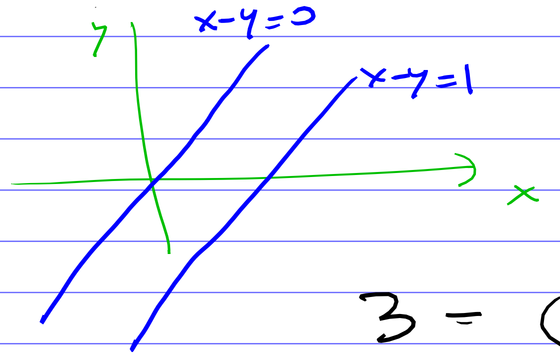
$2\binom{n}{2}$ hyperplanes in $\mathbb{R}^n$ given by the equations

$$\left.\begin{array}{l} x_i - x_j = 0 \\ x_i - x_j = 1 \end{array}\right. \quad \text{for } 1 \le i < j \le n$$

Theorem These hyperplanes subdivide $\mathbb{R}^n$ into $(n+1)^{n-1}$ regions.

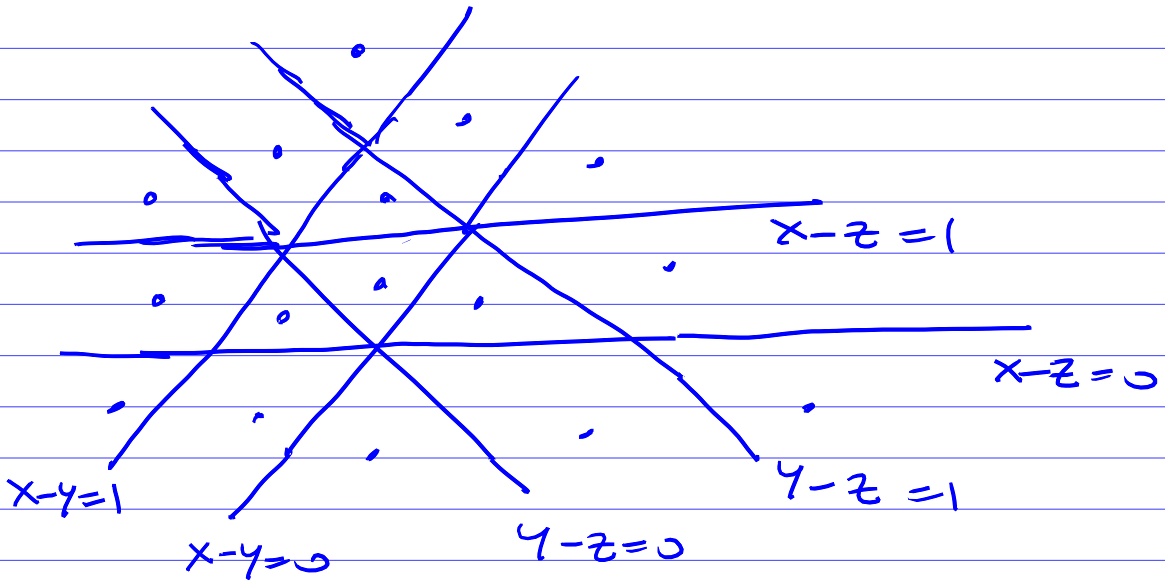
Examples: $n = 2$

$\mathbb{R}^2$



$3 = (2+1)^{2-1}$ regions

$n=3$ We'll draw 2-dim section of $\mathbb{R}^3$ by the hyperplane $x+y+z=0$



$16 = (3+1)^{3-1}$ regions.

The permutohedron

Π_n = the convex hull of $n!$ points $(w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ for all permutations $w_1, \dots, w_n$

Π_n is an $(n-1)$ dimensional polytope in $\mathbb{R}^n$, because it lies in the hyperplane $x_1 + x_2 + \quad + x_n = \text{Const.}$

Let $\Pi_n' = p(\Pi_n) \subset \mathbb{R}^{n-1}$ be the projection of Π_n under

$$p : (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1}).$$

Theorem. $\mathrm{Vol}_{n-1}(\Pi_n') = n^{n-2}$

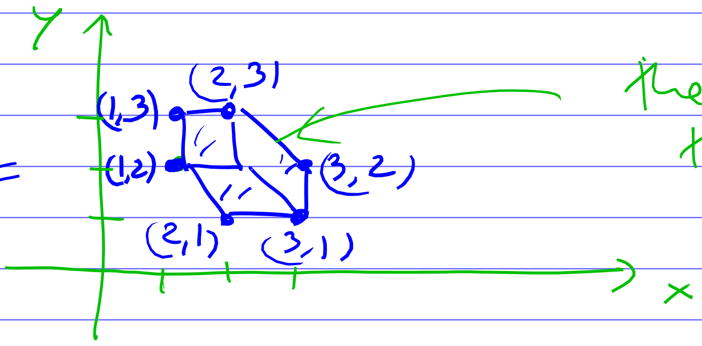
Example $n = 3$

Π_3 is the convex hull of 6 points:

$(1,2,3), (1,3,2), \dots, (3,2,1)$

projecting onto $\mathbb{R}^2$, we get

$\Pi_3' =$



the area of this hexagon is $3 = (2+1)^{2-1}$.