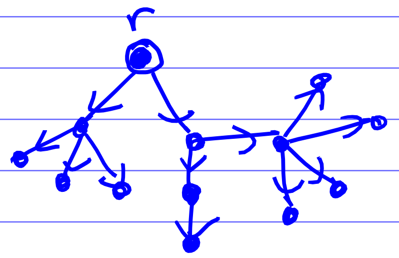
Directed Matrix Tree Theorem (cont'd)

$G = (V, E)$ is a digraph on $V = [n]$.

Arborescence rooted at vertex $r \in V$ is a subgraph $T \subseteq G$ s.t.

- T is a spanning tree of G (regarded as a undirected graph)
- $\forall$ vertex $v \in V$, there exist a (unique) directed path from r to v in T.



an arborescence

The **directed Laplacian matrix** $L = (l_{ij})$ of G: ($n \times n$ matrix)

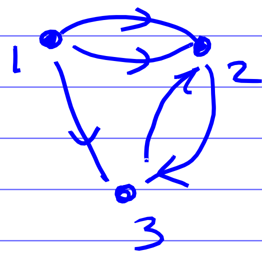
$$l_{ij} = \begin{cases} -\#\text{edges from } i \text{ to } j, & i \neq j \\ \text{indeg}_G(i), & \text{if } i = j \end{cases}$$

the indegree of vertex i

Theorem $\forall\, r, k \in [n]$

arborescences of G rooted at r equals the cofactor

$$L^{kr} := (-1)^{k+r} \det\left(L \text{ without } k^{th} \text{ row and } r^{th} \text{ column}\right)$$

Example. $G =$ 

$$L = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

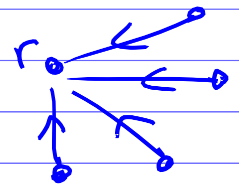
aborescences of G rooted at $r=1$

$$= L^{11} = L^{21} = L^{31} = 5.$$

Proof #1. Induction on the number $|E|$ of edges in G.

Let $Arb_r(G)$ be the number of arborescences of G rooted at r.

Base of induction: G contains only edges ending at root r.
(Then $Arb_r(G) = L^{kr} = 0$ for $n \geq 2$.)

$G =$ 

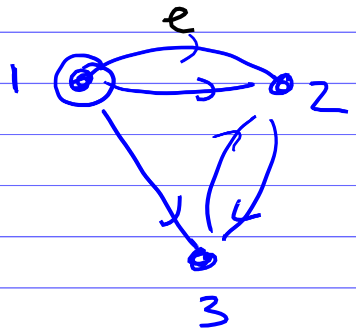
Induction Step:

Pick any edge $i \xrightarrow{e} j$ of G such that $j \neq r$.

Let $G_1 = G$ with edge e removed

$G_2 = G$ with all edges $e' \neq e$ ending at the same vertex j removed

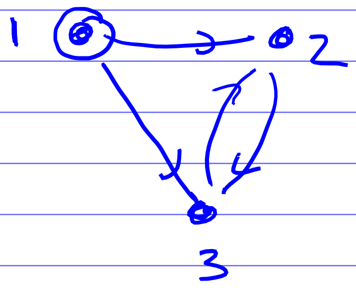
Example

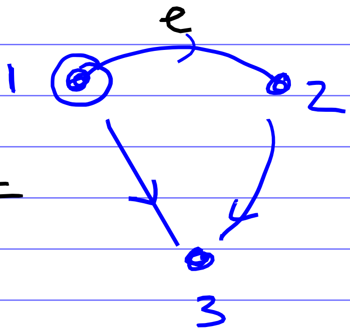
$G =$ 

$r = 1$

$e : 1 \xrightarrow{e} 2$

$j = 2$

$G_1 =$ 

$G_2 =$ 

Lemma

$$\mathrm{Arb}_r(G) = \mathrm{Arb}_r(G_1) + \mathrm{Arb}_r(G_2)$$

Proof Any abrorescence T of G rooted at r contains exactly 1 edge $\xrightarrow{f} j$ ending at vertex j. Consider 2 cases:

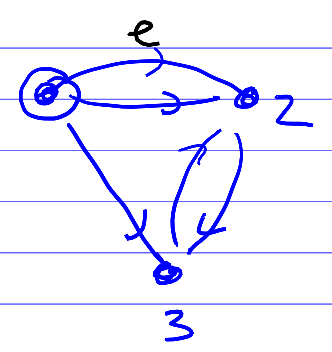
I. $f = e$

Then T is an arborescence of graph G_1

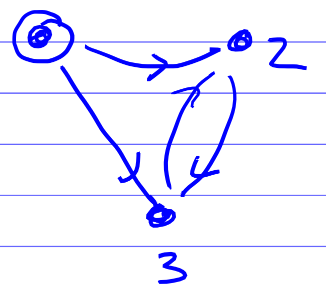
II. $f \neq e$. Then T is an abrorescence of G_2

Let's now look at directed Laplacians of the graphs G, G_1, G_2:

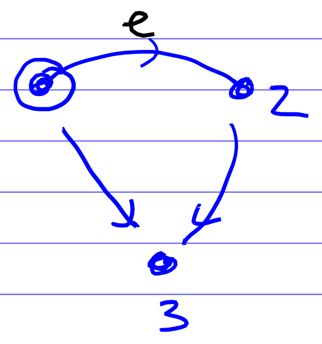
Example (G, e as above)

$G =$ 

$$L_G = \begin{array}{c} \text{j^{th} column} \\ \begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array}$$

$G_1 =$ 

$$L_{G_1} = \begin{array}{c} \text{j^{th} col.} \\ \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array}$$

$G_2 =$ 

$$L_{G_2} = \begin{array}{c} \text{j^{th} col.} \\ \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

Lemma. All entries of the matrices L_G, L_{G_2}, L_{G_2} are the same, except the entries in the j^{th} column.

For j^{th} columns, we have

$$L_G^j = L_{G_1}^j + L_{G_2}^j$$

j^{th} colums of L_G, L_{G_1}, L_{G_2}

By multilinearity of det we have

$$L_G^{kr} = L_{G_1}^{kr} + L_G^{kr}$$ for any cofactor s.t. $r \neq j$.

Notice that these cofactors are obtained by removing r^{th} column $(r \neq j)$ and some row.

Now we have

$$\mathrm{Arb}_r(G) = \mathrm{Arb}_r(G_1) + \mathrm{Arb}_r(G_2)$$

$$L^{kr}_{G} = L^{kr}_{G_1} + L^{kr}_{G_2}$$

Clearly, # edges in G_1 is less that # edges in G.

It is possible that $G = G_2$. This happens only if e is a unique edge of G ending at j. In this case, let G' be the graph G with edge e contracted. It is easy to see that

$$\mathrm{Arb}_r(G) = \mathrm{Arb}_r(G')$$

$$L^{kr}_{G} = L^{kr}_{G'}$$

By induction, we have

$$\mathrm{Arb}_r(G_1) = L^{kr}_{G_1}, \quad \mathrm{Arb}_r(G_2) = L^{kr}_{G_2}$$

(or, in the above special case, $\mathrm{Arb}_r(G') = L^{kr}_{G'}$)

In all cases, we deduce by induction on $|E|$ that

$$\mathrm{Arb}_r(G) = L^{kr}. \qquad \square$$

<u>Remark.</u> It is harder to prove the original (undirected) MTT by induction.

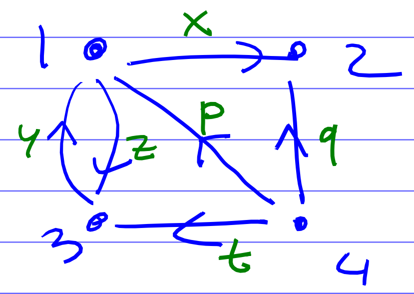
But it is easy to prove the directed MTT (which is a stronger theorem) by induction.

Let's us give another proof of directed MTT, base on the <u>involution</u> <u>principle</u>.

We'll consider the weighted version of the theorem, but we'll prove it only for principal cofactors of L.

$G = (V, E)$ digraph with weights x_e assigned to the directed edges $e \in E$.

<u>Example</u>

$G =$ 

For an arborescence $T \subseteq G$

$$\text{weight}(T) = \prod_{e \in T} x_e$$

Weighted directed Laplacian:

$L = (\ell_{ij})$

$$\ell_{ij} = \begin{cases} -\displaystyle\sum_{\substack{e \text{ edge} \\ \text{from } i \text{ to } j}} x_e & i \neq j \\ \displaystyle\sum_{\substack{f \text{ an edge} \\ \text{ending at } i}} x_f & i = j \end{cases}$$

<u>Example</u> (G as above)

$$L = \begin{bmatrix} y+p & -x & -z & 0 \\ 0 & x+q & 0 & 0 \\ -y & 0 & z+t & 0 \\ -p & -q & -t & 0 \end{bmatrix}$$

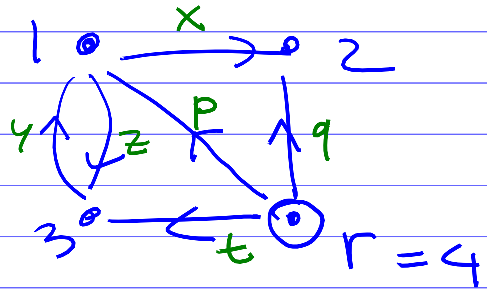
Theorem. Fix root $r \in V$.

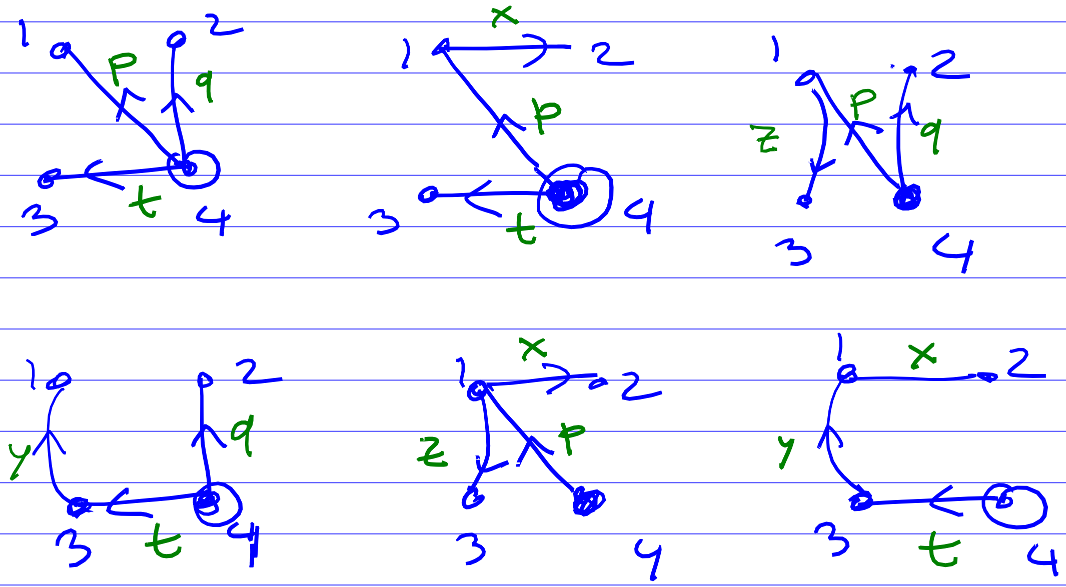
$$\sum_{\substack{T \text{ arborescence} \\ \text{of } G \text{ rooted} \\ \text{at } r}} \text{weight}(T) = L^{rr}$$

principal cofactor of the Laplacian.

Example

G as above, $r = 4$

$G =$ 



$$pqt + xpt + zpq + yqt + xzp + xyt$$

$$= \begin{vmatrix} y+p & -x & -z \\ 0 & x+q & 0 \\ -y & 0 & z+t \end{vmatrix}.$$

Proof. WLOG, assume $r = n$.

$$L^{nn} \stackrel{\text{def}}{=} \sum_{w \in S_{n-1}} (-1)^{inv(w)} \prod_{i=1}^{n-1} \ell_{i w_i}$$

sum over all permutations $w \in S_{n-1}$.

$$\text{sign}(w) = (-1)^{inv(w)}$$

inversions in w.

Cyclic notation for permutation w:

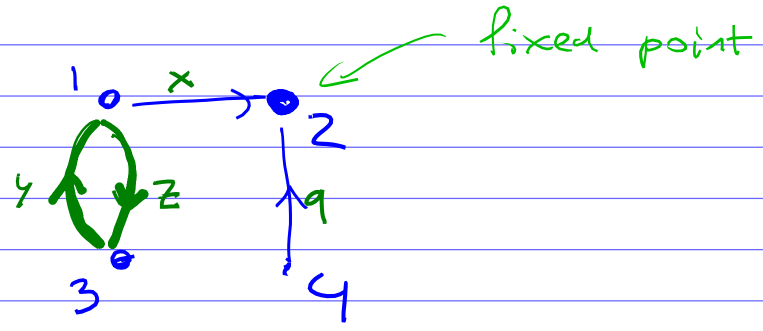
$$w = (\ldots)(\ldots)\ldots(\cdot)(\cdot)\ldots(\cdot)$$

cycles of size ≥ 2 fixed points

So $L^{nn} =$

(*)
$$= \sum_{\substack{\text{subdivision of } [n-1] \\ \text{into disjoint union} \\ \text{of cycles } C_1, C_2, \ldots \text{ of } G \\ \text{and "fixed points"}}} \pm \left(\prod_{\substack{e \text{ edge} \\ \text{of some} \\ \text{cycle}}} x_e\right) \cdot \prod_{\substack{i \\ \text{fixed} \\ \text{point}}} \left(\sum_{f:\, i \to \cdot} x_f\right)$$

Example. One term in the expansion of L^{nn}:



$$\pm\, y \cdot z \cdot (x + q)$$

Let's calculate the sign of this term.

<u>Lemma</u> For a permutation $w = (\underbrace{\cdots}_{k})$ that consists of a single cycle of size k (and some fixed points),

$$\mathrm{sign}(w) = (-1)^{k-1}.$$

<u>Examples:</u> $\mathrm{sign}((ij)) = -1$ a transposition

$\mathrm{sign}((ijk)) = 1$ a 3-cycle

etc.

Also each off diagonal entry of Laplacian matrix comes with "$-$" sign.

So the overall sign of each term in the expansion

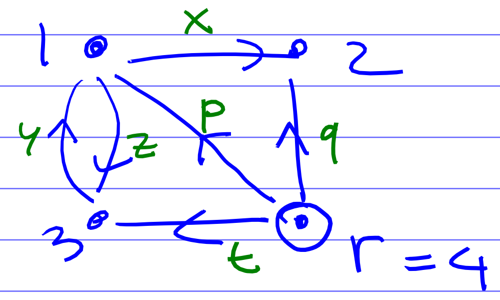
(*) is $(-1)^{\#\{\text{cycles of length} \geq 2\}}$

Putting all this together, we get $L^{nn} =$

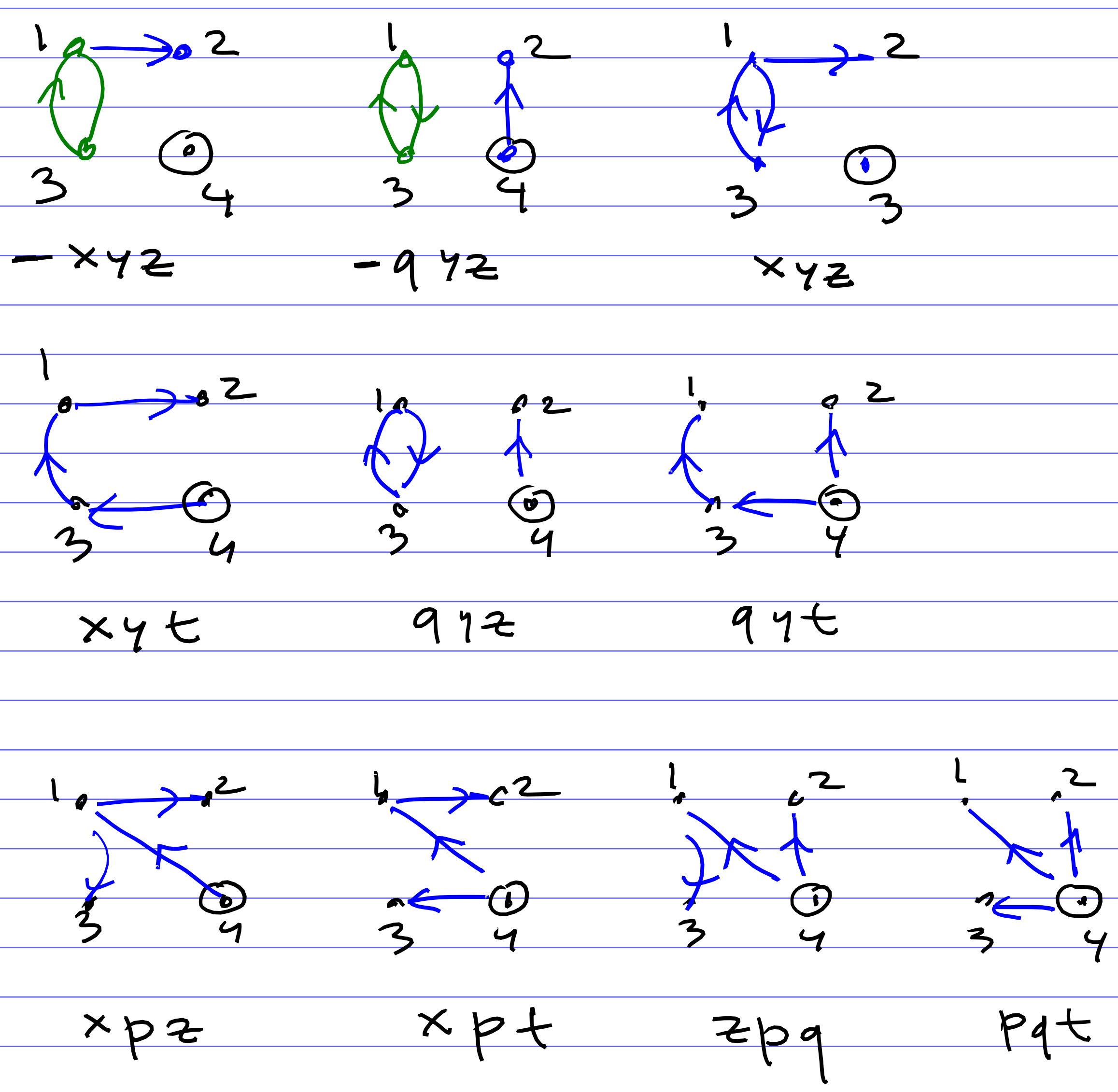
$$= \sum_{H \subset G} (-1)^{\#\{\text{green cycles}\}} \prod_{e \in H} x_e$$

over all subgraphs $H \subset G$ with edges colored in blue and green such that

- All green edges form a collection of disjoint cycles on the set $[n-1] = \{1, 2, \ldots, n-1\}$.
- For any $i \in [n-1]$ which is not covered by a green cycle, there is exactly 1 blue edge entering vertex i.
  All blue edges have this form.

Example. $G =$ 

Colored subgraphs H:



This is the expansion of L^{nn} obtained from the definition of determinant.

Observe that arborescences of G rooted at $r=n$ are exactly graphs H as above that have no directed cycles.
(In particular, they don't have green edges.)

Let's construct a sign reversion involution on graphs H as above that contain at least 1 cycle:

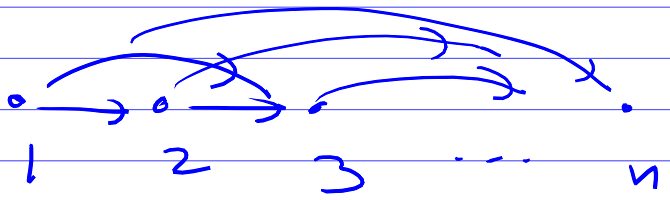
- Pick the lex. minimal cycle C of H & switch the color (green $\leftrightarrow$ blue) of all edges in C.

Notice that this operation changes the parity of the number of green cycles. So the corresponding monomials cancel each other.

The "surviving" monomials correspond to arborescences of G and come with "+" signs. $\square$

An example of directed MTT.

Let G be the digraph on $1, 2, \ldots, n$ with edges $i \to j$ for all $i < j$

$G =$ 

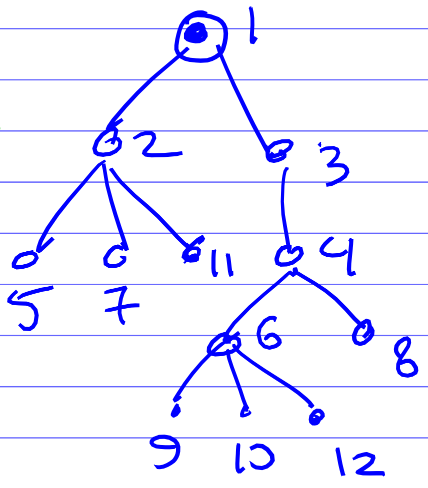
$$L = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots & -1 \\ & & 2 & \cdots & -1 \\ & \Large 0 & & \ddots & \vdots \\ & & & & n-1 \end{bmatrix}$$

arborescences of L rooted at $r = 1$ equals

$$L^{11} = 1 \cdot 2 \cdot \ldots \cdot (n-1) = (n-1)!$$

Such arborescences are called <u>increasing trees</u>

The labels increase as we go away from vertex 1.



an increasing tree

<u>Corollary.</u>

increasing trees on n vertices equals $(n-1)!$

Also not hard to prove bijectively.

One can use spanning trees (and arborescences) to prove some identities.

Exercise. Prove

Abel's binomial formula 1826

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y(y+kz)^{k-1}(x-kz)^{n-k}$$

Niels Henrik Abel
(1802 - 1829)