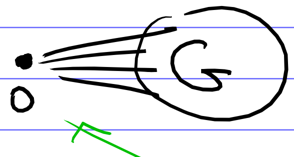
18.212 Lecture 22 04/09/2021

last time: Reciprocity formula

$G = (V, E)$ a simple graph

$V = [n]$

$\overline{G} = (V, \overline{E})$ its complementary graph.

$G^+ =$ 

one extra vertex connected with all vertices of G

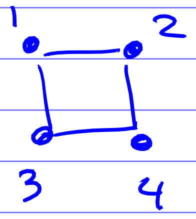
$$F_G(x) = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G^+}} x^{\deg_T(0)-1}.$$

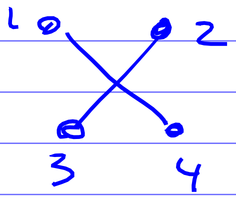
Then # spanning trees of G

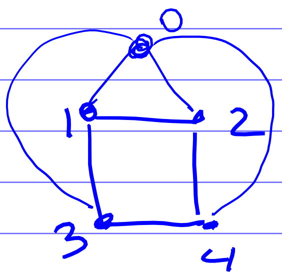
$$= \frac{1}{n} F_G(0).$$

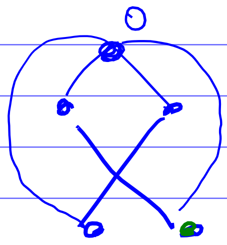
Theorem $F_G = (-1)^{n-1} F_{\overline{G}}(-x-n).$

Example

$G =$ 

$\overline{G} =$ 

$G^+ =$ 

$\overline{G}^+ =$ 

$$F_G(x) = x^3 + 8x^2 + 20x + 16$$

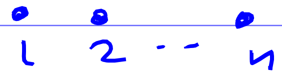
$$F_{\overline{G}}(x) = x^3 + 4x^2 + 4x + 0$$

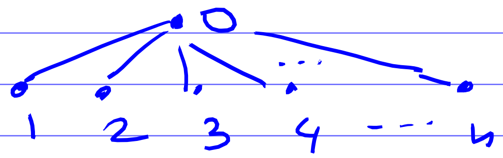
$$F_G(x) = (-1)^3 F_{\overline{G}}(-x-4).$$

We can easily deduce many formulas from this theorem.

Some Examples

1. The "empty" graph

$$O_n = ([n], \emptyset) =$$ 

$$O_n^+ =$$ 

$$F_{O_n}(x) = x^{n-1}$$

$$\overline{O_n} = K_n$$

$$F_{K_n}(x) = (-1)^{n-1}(-x-n)^{n-1}$$

$$= (x+n)^{n-1}$$

$\Rightarrow$ # spanning trees of K_n

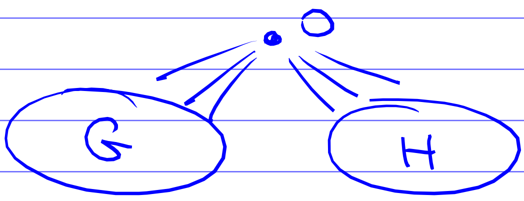
$$= \frac{1}{n} F_{K_n}(0) = \frac{1}{n} \cdot n^{n-1}$$

$$= \boxed{n^{n-2}}$$ Cayley's formula

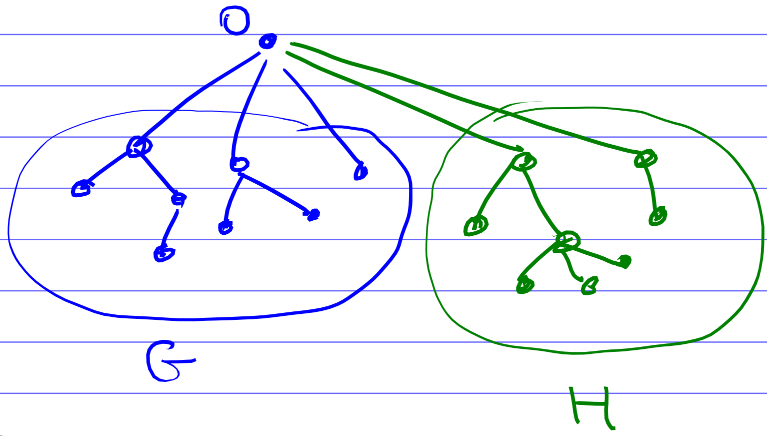
2. Let $G \dot\cup H$ be the disjoint union of two graphs:

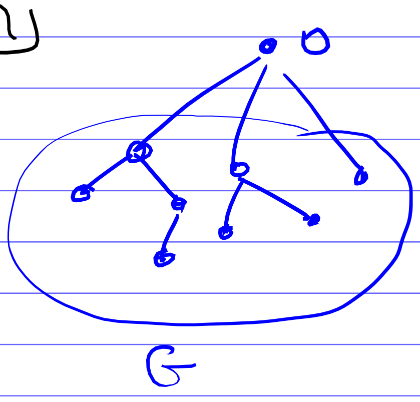
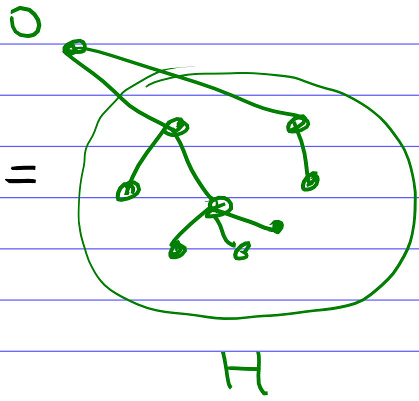
$$G \dot\cup H = G \qquad H$$

$$F_{G \dot\cup H}(x) = x \cdot F_G(x) \cdot F_H(x)$$

$(G \dot\cup H)^+ =$ 

Indeed, any spanning tree T of $(G \dot\cup H)^+$ is the union of a spanning tree T_1 of G^+ and a spanning tree T_2 of H^+.

$T =$ 

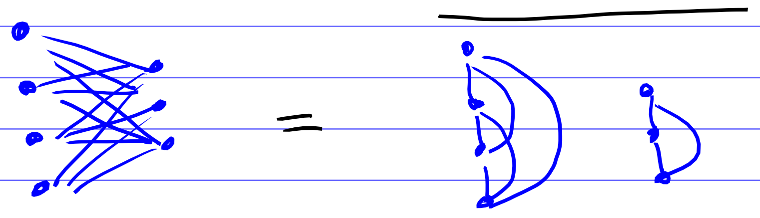
$T_1 =$  $T_2 =$ 

$$(\deg_T(0) - 1) = (\deg_{T_1}(0) - 1) + (\deg_{T_2}(0) - 1) + 1.$$

Notice that

the complete bipartite graph $\rightarrow$ $K_{m,n} = \overline{K_m \,\dot{\cup}\, K_n}$.

Example



$K_{4,3}$ $\qquad\qquad$ K_4 $\quad$ K_3

$$F_{K_m \dot{\cup} K_n}(x) = x \, F_{K_m}(x) \cdot F_{K_n}(x)$$

$$= x \cdot (x+m)^{m-1} \cdot (x+n)^{n-1}.$$

$$F_{K_{m,n}}(x) = F_{\overline{K_m \cup K_n}}(x)$$

$$= (-1)^{m+n-1} (-x-m-n) \cdot$$

$$\cdot \left((-x-m-n) + m \right)^{m-1} \left((-x-m-n) + n \right)^{n-1}$$

$$= (x+m+n) \cdot (x+n)^{m-1} \cdot (x+m)^{n-1}.$$

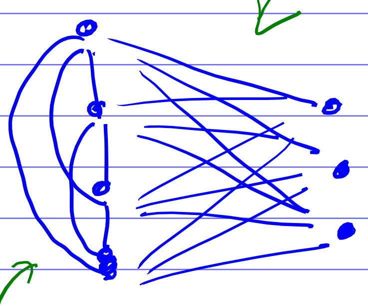
So # spanning trees of $K_{m,n}$ is $\frac{1}{m+n} F_{K_{m,n}}(0) =$

$$= \frac{1}{m+n} \cdot (m+n) \, n^{m-1} \cdot m^{n-1}$$

$$= \boxed{n^{m-1} \cdot m^{n-1}}$$

Exercise Find an exact formula for # spanning trees in the graph:

all vertices of the left part are connected with all vertices of the right part.



the complete graph K_m inside the left part

the "empty" graph O_n inside the right part

Generalizations of the Matrix Tree Theorem

Weighted MTT

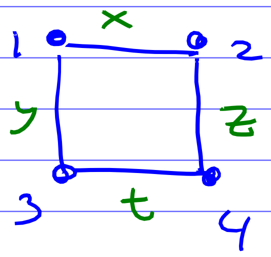
$G = (V, E)$ graph on $V = [n]$

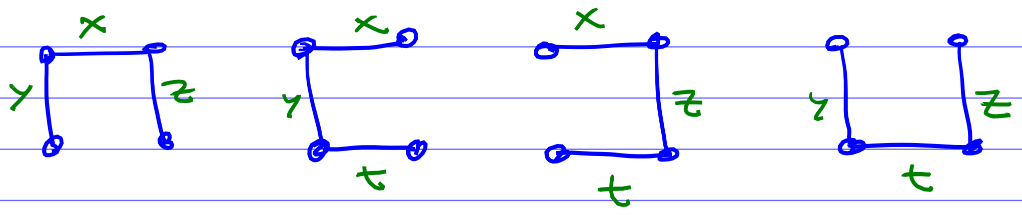
Assign weights x_e to edges $e \in E$ of G.

For a spanning tree T, let

$$wt(T) := \prod_{\substack{e \text{ edge} \\ \text{of } T}} x_e$$

Let $$\Pi_G := \sum_{\substack{T \text{ is a} \\ \text{spanning} \\ \text{tree of} \\ G}} wt(T).$$

Example. $G =$ 



$$\Pi_G = x \cdot y \cdot z + x \cdot y \cdot t + x \cdot z \cdot t + y \cdot z \cdot t.$$

Weighted Laplacian Matrix

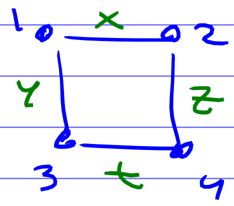
$L = (\ell_{ij})$ $\quad$ $n \times n$ matrix

$$\ell_{ij} = \begin{cases} -\displaystyle\sum_{\substack{e \text{ edge} \\ \text{between } i \,\&\, j}} x_e & i \neq j \\ \displaystyle\sum_{\substack{e \text{ edge} \\ \text{incident to } i}} x_e & i = j \end{cases}$$

$\tilde{L}$ reduced weighted Laplacian, $\quad$ $(n-1)\times(n-1)$ matrix

i.e. L with i^{th} row & i^{th} column removed.

(Assume $i = n$)

Example $\quad G =$ 

$$L = \begin{bmatrix} x+y & -x & -y & 0 \\ -x & x+z & 0 & -z \\ -y & 0 & y+z & -t \\ 0 & -z & -t & z+t \end{bmatrix}$$

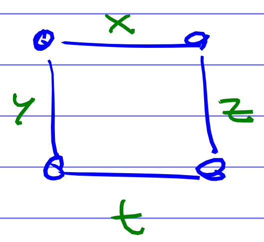
$$\tilde{L} = \begin{bmatrix} x+y & -x & -y \\ -x & x+z & 0 \\ -y & 0 & y+z \end{bmatrix}$$

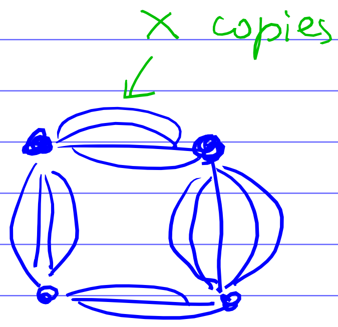
Theorem (Weighted MTT)

$\Pi_G = \det(\tilde{L})$.

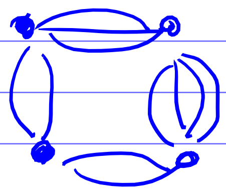
Proof Enough to prove it if all weights x_e are positive integers.

Let G^m be the graph obtained from G by replacing each edge e of G by x_e copies of e.

Ex. $G =$ 

$\rightsquigarrow$ $G^m =$ 

Then Π_G is the number of spanning trees of G^m, and the weighted Laplacian L of G is the usual Laplacian of G^m.

Example $G^m =$ 

$x = 3$
$y = 2$
$z = 4$
$t = 2$

spanning trees of G^m is

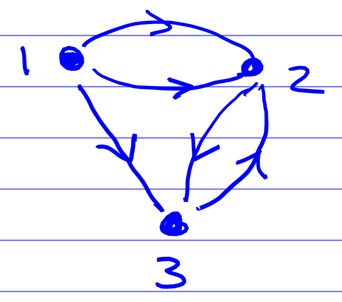
$3 \cdot 2 \cdot 4 + 3 \cdot 2 \cdot 2 + 3 \cdot 4 \cdot 2 + 2 \cdot 4 \cdot 2$.

$$L(G^m) = \begin{bmatrix} 5 & -3 & -2 & 0 \\ -3 & 7 & 0 & -4 \\ -2 & 0 & 4 & -2 \\ 0 & -4 & -2 & 6 \end{bmatrix}$$

So usual MTT $\Longrightarrow$ weighted MTT. $\square$

Directed Matrix Tree Theorem

$G=(V,E)$ a directed graph, (or digraph) on $V=[n]$

Example $G=$ 

a digraph

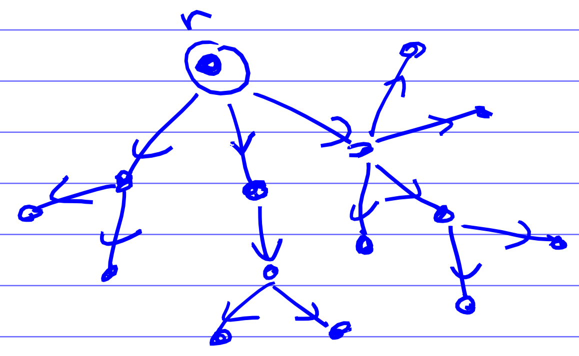
"Directed spanning trees" are called "arborescences".

Fix a root vertex $r\in V$

Definition. An arborescence T in G with root r is a subgraph of G s.t.

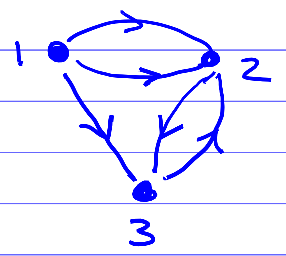
- T is a spanning tree of G (considered as an undirected graph)
- $\forall$ vertex $v\in V$, there exists a directed path in T from the root r to v.

In other words, an arborescence is a tree with all edges directed away from the root:

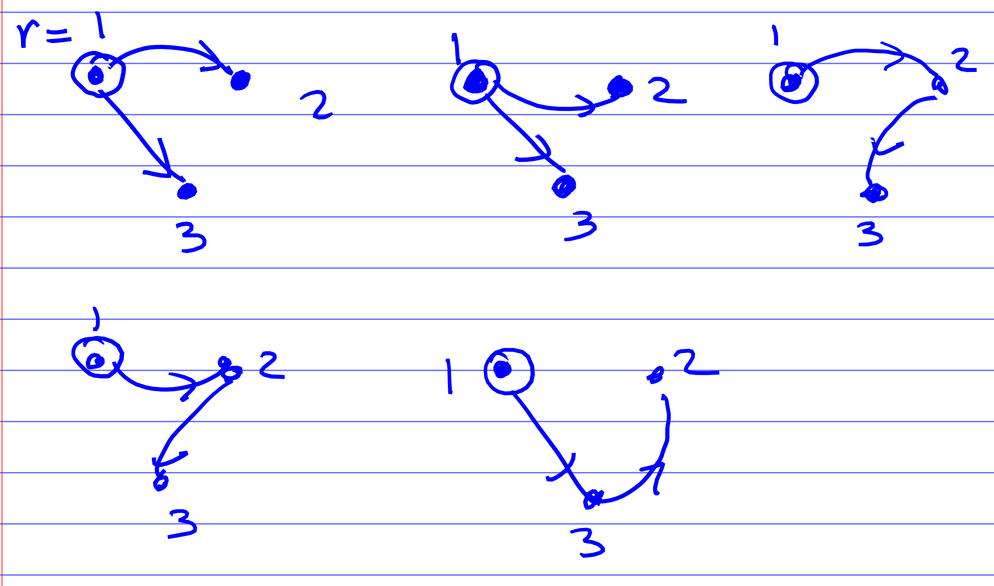


an arborescence

(Arborescences are also called "out-trees".)

Example. $G=$ 

has 5 arborescences with root $r=1$:



5 arborescences

Adjacency matrix of G is the $n \times n$ matrix $A = (a_{ij})$, where a_{ij} is the number of <u>directed</u> edges from i to j.

<u>Example</u> For G as above

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

this is no longer a symmetric matrix

Let $\text{indeg}_G(v)$ and $\text{outdeg}_G(v)$ be the <u>indegree</u> and the <u>outdegree</u> of vertex v in G.

<u>Example</u> G as above

indegrees:

$\text{indeg}(1) = 0$, $\text{indeg}(2) = 3$, $\text{indeg}(3) = 2$.

outdegrees:

$\text{outdeg}(1) = 3$, $\text{outdeg}(2) = 1$, $\text{outdeg}(3) = 1$.

Define the <u>Laplacian matrix</u> of digraph G, as follows:

$$L = L^{in} := \operatorname{diag}(d_1, \dots, d_n) - A$$

where $d_i = \operatorname{indeg}_G(i)$.

<u>Example</u> G as above

$$L = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

the diag. entries are the <u>indegrees</u>

(Notice that all <u>column</u> sums of L are zero. But row sums can be non-zero.)

<u>Def.</u> The $(i,j)^{th}$ <u>cofactor</u> of matrix L is

$$L^{ij} := (-1)^{i+j} \det\begin{pmatrix} L \text{ without} \\ i^{th} \text{ row and} \\ j^{th} \text{ column} \end{pmatrix}.$$

Directed Matrix Tree Theorem.

Fix any $r, k \in [n]$.

The number of abrorescences in G with root at r equals the cofactor L^{kr}.

Example G as above, $r = 1$.

$$L^{11} = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 3 \cdot 2 - 1 = 5$$

$$L^{21} = - \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \cdot 2 + 1 = 5$$

$$L^{31} = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 2 + 3 = 5$$

So $L^{11} = L^{21} = L^{31} =$

$=$ # arborescences rooted at 1.

Remarks.

- In this directed MTT, we count arborescences, i.e. out-trees of G and use the matrix $L = L^{in}$ with indegrees on the diagonal.

- All cofactors L^{kr} are independent of k, (i.e. $L^{kr} = L^{k'r}$ $\forall k, k'$).

But they may be different for different r's.

For example (for above G),

$$L^{12} = L^{22} = L^{32} = 0$$

and G has no arborescences rooted at $r=2$.
(G has no directed path from vertex 2 to 1.)

Special Case:

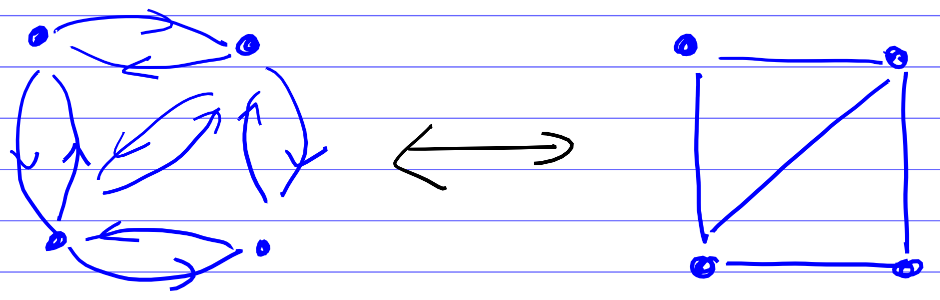
Assume that A & L are symmetric matrices, i.e.,

$a_{ij} = a_{ji} \quad \forall i, j.$

(# edges from i to j = # edges from j to i.)

$\mathrm{indeg}_G(i) = \mathrm{outdeg}_G(i) \quad \forall i$

In other words, G can be viewed as an undirected graph:



In this case, arborescences (for any fixed r) correspond to usual spanning trees in the undirected graph.

So we deduce the usual MTT from directed MTT.

Corollary Let L be the usual Laplacian matrix of an (undirected) graph G. Then all cofactors L^{ij} of L are equal to each other.

$$L^{ij} = \#\text{spanning trees of } G$$

$$\forall i, j \in [n].$$

Remark. This is a stronger claim than the MTT that we proved earlier. (Earlier we considered only the principle cofactors L^{ii} of L.)

There are several ways to prove directed MTT.

- Proof by induction on # edges in G.
- Proof by the involution principle.

It is hard to give an inductive proof for the original (undirected) version of MTT.

But we'll see next week that the directed MTT has a simple proof by induction.

We've seen a similar phenomenon when we discussed Cayley's formula:

$$n^{n-2} \rightsquigarrow (x_1+\dots+x_n)^{n-2}$$

hard to prove by induction   easy to prove by induction

Q: How is it possible?

Why is it easier to prove a stronger claim?

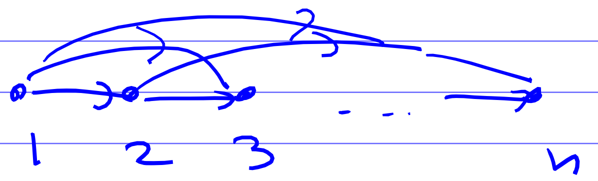
Is this a contradiction to the basic principles of logic?

A. No. This is not a logical contradiction.

If we want to prove a stronger claim by induction, that means that the induction hypothesis is a stronger claim. So it is <u>easier</u> to deduce things using a stronger hypothesis.

<u>Bottom line</u>: Sometimes It is easier to prove more difficult theorems.

Example Let's calculate # arborescences rooted at 1 in the digraph



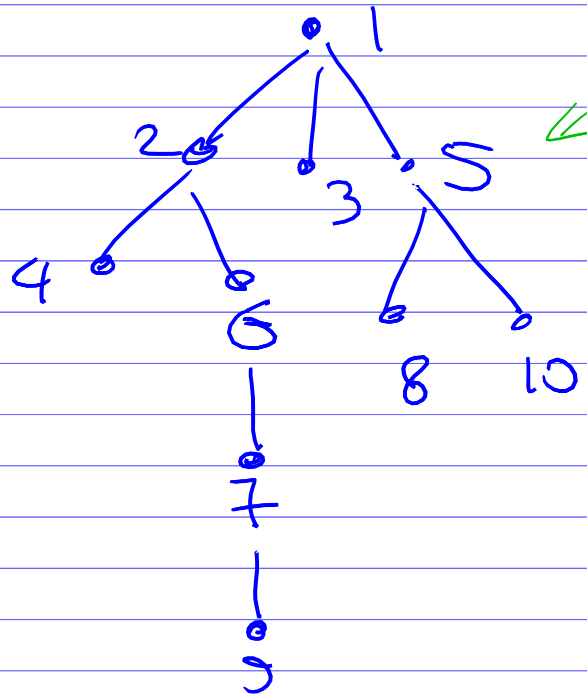
all vertices $i<j$ connected by edges $i \to j$

$$L = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots & -1 \\ & & 2 & \cdots & -1 \\ & 0 & & \ddots & \\ & & & & n-1 \end{bmatrix}$$

$$L^{11} = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) = (n-1)!$$

Avoborescences for this graph are called <u>increasing trees</u>:

An increasing tree:



the labels increase as we go down

<u>Corollory</u> # increasing trees on n vertices equals $(n-1)!$