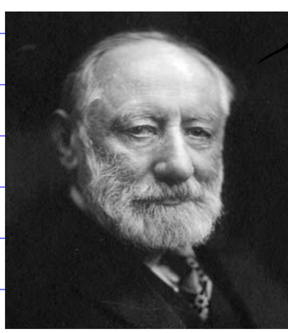
Theory of Partitions (cont'd)

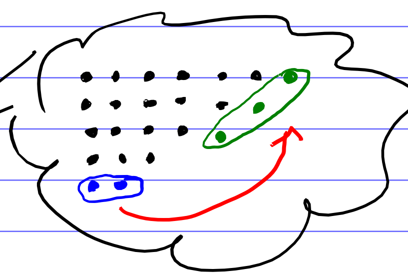
Last time: We proved Euler's

Pentagonal Theorem:

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{k=-\infty}^{+\infty} (-1)^k q^{k(3k-1)/2}$$

The proof that we gave is due to





Fabian Franklin
(1853 – 1939)

Today we'll talk about a generalization of Euler's pentagonal theorem ...

Jacobi Triple Product

Theorem [Jacobi, 1829]

Carl Gustav Jacob Jacobi (1804–1851)

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}y^{2})\left(1+\frac{x^{2n-1}}{y^{2}}\right) = \sum_{k=-\infty}^{+\infty} x^{k^2}y^{2k}.$$

Euler's pentagonal theorem is a special case of Jacobi triple product for $x=q^{3/2}$, $y^2=-q^{1/2}$.

We'll give a combinatorial proof. First, let's slightly rewrite the triple product formula by making the substitution:

$$x^2 = q \quad , \quad y^2 = z\cdot q^{1/2}$$

We get the following equivalent formula:

$$\prod_{n\geq 1}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) = \sum_{r=-\infty}^{+\infty} z^r q^{r(r+1)/2}.$$

Let's move the first term in the triple product to the R.H.S.

Theorem.
$$\left(\prod_{n\geq 1}(1+zq^n)\right)\left(\prod_{n\geq 1}(1+z^{-1}q^{n-1})\right)$$

(*)
$$=\left(\sum_{r=-\infty}^{+\infty} z^r q^{r(r+1)/2}\right)\left(\prod_{n\geq 1}\frac{1}{1-q^n}\right).$$

Proof. The coefficient of z^a in $\prod_{n\geq 1}(1+zq^n)$ equals

$$\sum_{\substack{\mu \text{ partition} \\ \text{with } a \text{ distict} \\ \text{parts}}} q^{|\mu|}.$$

Similarly, the coefficient of z^{-b} in $\prod_{n\geq 1}(1+z^{-1}q^{n-1})$ equals

$$\sum_{\substack{\nu \text{ partition} \\ \text{with } b \text{ distict} \\ \text{parts}}} q^{|\nu|-b}.$$

Of course, $\prod_{n\geq 1}\frac{1}{1-q^n}$ equals

$$\sum_{\substack{\lambda \text{ any} \\ \text{partition}}} q^{|\lambda|}.$$

In order to prove (*) we need to match all terms in the L.H.S. with all terms in the R.H.S.

We need to construct a bijection

$$\{(a, b, \mu, \nu)\} \longleftrightarrow \{(r, \lambda)\}$$

where

- LHS: $a, b \in \mathbb{Z}_{\geq 0}$ ← non-negative integers

  μ partition w/ a dist. parts

  ν partition w/ b dist. parts

- RHS: $r \in \mathbb{Z}$ ← any integer

  λ any partition

Such that the corresponding monomials are the same:

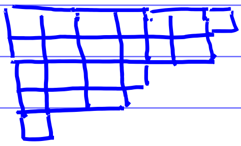
$$z^{a-b} q^{|\mu|+|\nu|-b} = z^r q^{\frac{r(r+1)}{2}+|\lambda|}$$

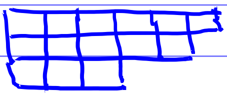
- $a - b = r$
- $|\mu| + |\nu| - b = \frac{r(r+1)}{2} + |\lambda|$.

Lets do this by example:

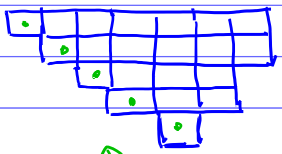
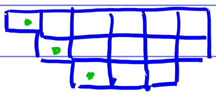
Example. $\mu = (7,6,4,3,1)$, $\nu = (6,5,3)$.

partitions w/ distinct parts

$a = 5$, $\mu = (7,6,4,3,1) =$ 

$b = 3$, $\nu = (6,5,3) =$ 

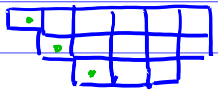
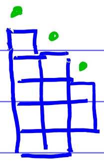
Convert μ & ν into <u>shifted Young diagrams</u> $\tilde{\mu}$ & $\tilde{\nu}$

$\tilde{\mu} =$  $\tilde{\nu} =$ 

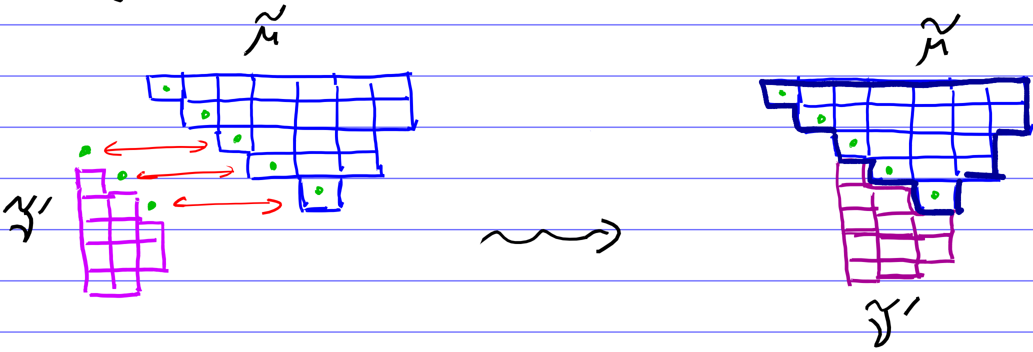
<u>shifted</u> Young diagrams

They have the same # boxes in each row as μ & ν, But the leftmost boxes in rows (marked as $\boxed{\cdot}$) form a staircase.

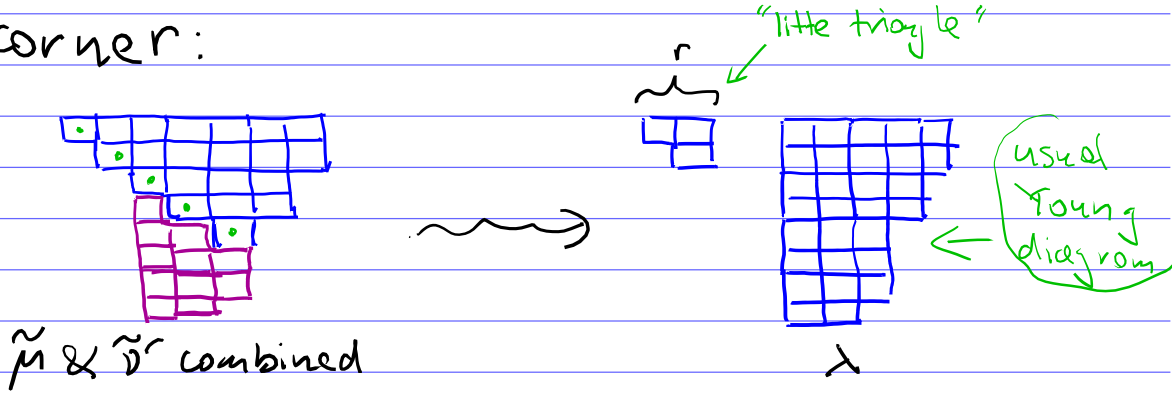
Let's transpose $\tilde{\nu}$ and remove its b diagonal boxes:

$\tilde{\nu} =$  $\rightsquigarrow$ $\tilde{\nu}' =$ 

Let's now "glue" the shapes $\tilde{\mu}$ and $\tilde{\nu}'$ into a single shape by identifying the <u>last</u> $\min(a,b)$ diagonal boxes (marked with •):



Finally, let's chop off the "little triangle" in the upper left corner:



$r = 2$

$\lambda = (5,5,4,4,3,3,3,3)$

We obtain the pair

r = # boxes on the side of the "little triangle"

$\lambda = \tilde{\mu} \cup \tilde{\nu}'$ with chopped off "little triangle"

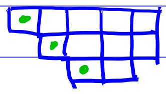
This construction

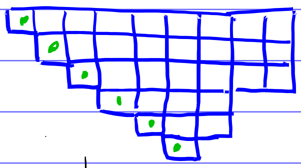
$$(a, b, \mu, \nu) \longmapsto (r, \lambda)$$

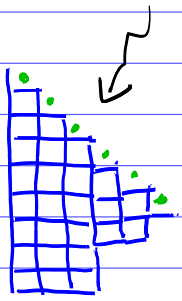
is the needed bijection.

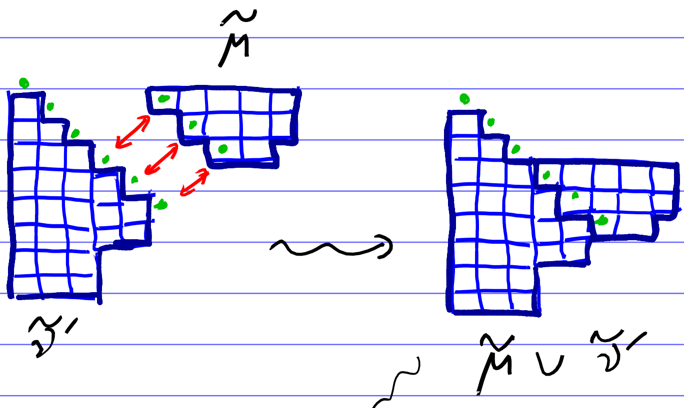
In the above example we had $a \geq b$. Let's give another example with $a < b$.

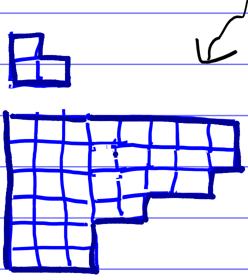
<u>Example #2</u>. $\mu = (5, 4, 2)$, $\nu = (9, 8, 7, 4, 3, 1)$

$a = 3$ $\quad \tilde{\mu} =$ 

$b = 6$ $\quad \tilde{\nu} =$ 

$\tilde{\nu}' =$ 



$\lambda =$ 

$r = a - b = -3$

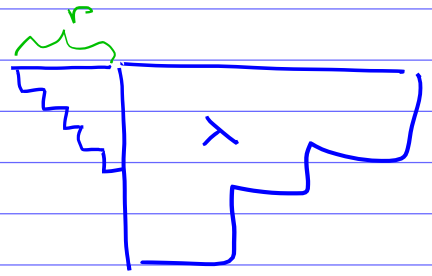
$\lambda = (8, 8, 7, 5, 3, 3)$

This operation is clearly invertible.

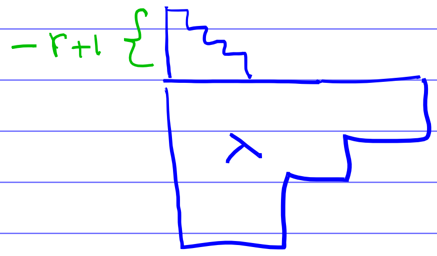
Inverse Construction

$$(r, \lambda) \longmapsto (a, b, \mu, \nu)$$

- If $r \geq 0$, then attach the "little triangle" $(r, r-1, r-2, \ldots, 1)$ to λ as follows:



- If $r < 0$, then attach the "little triangle" $(-r+1, -r+2, -r+3, \ldots)$ to λ, as follows:



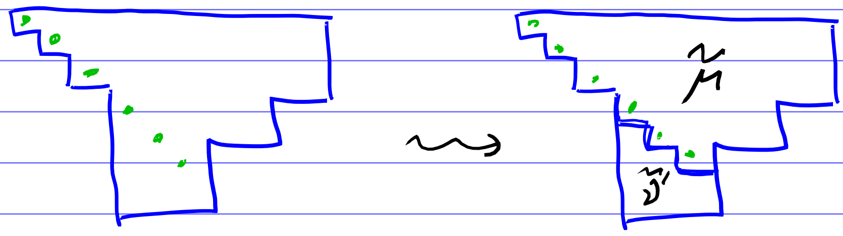
Notice that in both cases

\# boxes in the little triangle

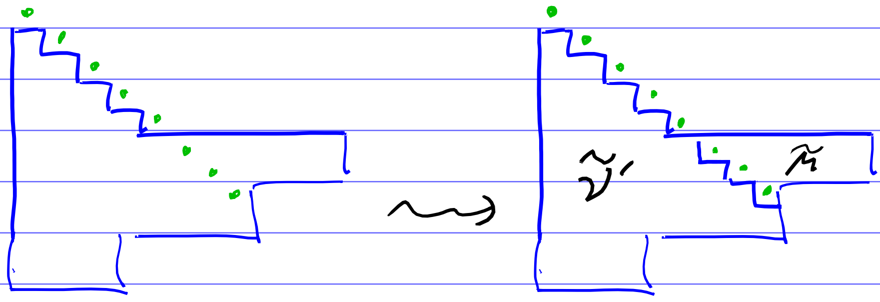
$$= \frac{r(r+1)}{2}.$$

Then cut the resulting shape along the main diagonal to get shifted shapes $\tilde{\mu}$ & $\tilde{\nu}'$:

$r \geq 0$



$r < 0$



Then get μ & ν.

So we've got the bijection

$$\{(a, b, \mu, \nu)\} \overset{\text{bij.}}{\longleftrightarrow} \{(r, \lambda)\}$$

The needed properties are clear from the construction:

- $a - b = r$
- $\underset{\substack{\uparrow \\ \#\text{ boxes} \\ \text{in } \tilde{\mu}}}{|\mu|} + \underbrace{|\nu| - b}_{\substack{\uparrow \\ \#\text{ boxes} \\ \text{in } \tilde{\nu}'}} = \underset{\substack{\uparrow \\ \#\text{ boxes} \\ \text{in the} \\ \text{"little triangle"}}}{\frac{r(r+1)}{2}} + \underset{\substack{\uparrow \\ \#\text{ boxes} \\ \text{in } \lambda}}{|\lambda|}$

(we removed b boxes on the diagonal)

Q.E.D.

Other related identities

Theorem [Gauss]

(1) $$\left(\prod_{n\geq 1}(1-q^n)\right)^3 = \sum_{k=-\infty}^{+\infty}(-1)^k k\, q^{k(k+1)/2}$$

(2) $$\prod_{n\geq 1}\frac{1-q^n}{1+q^n} = \sum_{k=-\infty}^{+\infty}(-1)^k q^{k^2}$$

Proof. (1) Specialize Jacobi triple product (*) for $z=-t$, $q=t^{1/2}$

(2) Specialize Jacobi triple product (*) for $z=-1$. $\square$

Remark. All this stuff has deep links with:

- Number theory: modular forms, Θ-functions, etc
- Representation Theory of Infinite Lie algebras; Kac-Moody algebras.

This stuff is out of scope of this course. But if later some of you will study modular forms and/or Kac-Moody theory, you might revisit Euler's pentagonal formula, Gauss identities & Jacobi triple product.

- There are generalizations of Jacobi triple product to affine Kac-Moody algebras, called

Macdonald Identities.

Remark We obtained:

Euler: $$\prod_{n\geq 1}(1-q^n) = \sum_{k=-\infty}^{+\infty}(-1)^k q^{k(3k-1)/2}$$

Gauss: $$\left(\prod_{n\geq 1}(1-q^n)\right)^3 = \sum_{k=-\infty}^{+\infty}(-1)^k k\, q^{k(k+1)/2}$$

How about $\left(\prod_{n\geq 1}(1-q^n)\right)^2$?

Actually, if the expand it we get total mess.
Unlike $\prod_{n\geq 1}(1-q^n)$ and $\left(\prod(1-q^n)\right)^3$ whose expesions contain very sparse sets of terms, the exponsion of $\left(\prod_{n\geq 1}(1-q^n)\right)^2$ contain many terms & there is no simple formula for its coefficients.
An explanation of this phenomenon is given in theory of Kac-Moody algebras.

Gaussian q-binomial coefficients

again...

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$= \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

Young diagram
that fit into
rectangle

Theorem (q-binomial formula)

$$(1+x)(1+xq)(1+xq^2)\cdots(1+xq^{n-1})$$

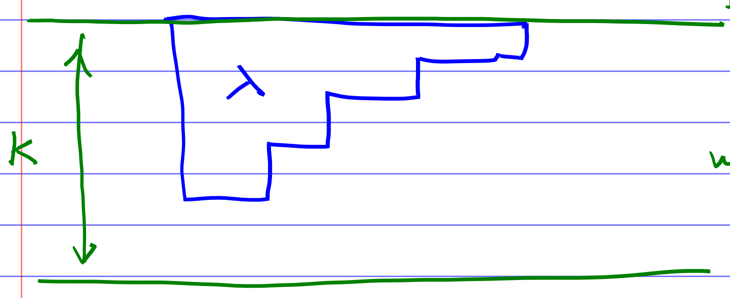
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k$$

Take the limit of this as $n \to \infty$

$$\lim_{n\to\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1}{(1-q)^k [k]_q!}$$

$$= \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

$$= \sum_{\substack{\lambda \text{ partition} \\ \text{with at most} \\ k \text{ parts}}} q^{|\lambda|}$$



encode such λ by $(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k_{\geq 0}$ where

n_i = # columns with i boxes in λ

So
$$\sum_{\substack{\lambda \text{ with} \\ \leq k \text{ rows}}} q^{|\lambda|} = \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k_{\geq 0}} q^{1\cdot n_1 + 2n_2 + \cdots + k\cdot n_k}$$

$$= \frac{1}{(1-q)} \frac{1}{(1-q^2)} \cdots \frac{1}{(1-q^k)}.$$

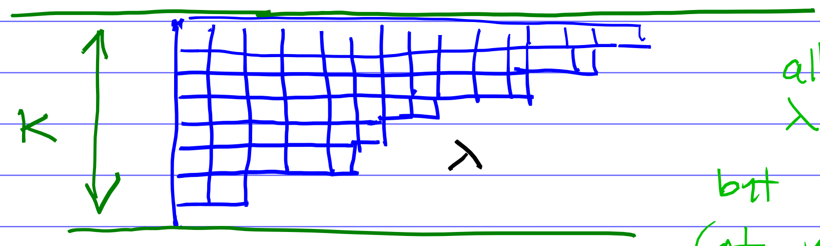
<u>Corollary</u> $\displaystyle\prod_{n\geq 0} (1 + x q^n)$

$$= \sum_{k\geq 0} \frac{q^{k(k-1)/2}}{(1-q)(1-q^2)\cdots(1-q^k)} x^k.$$

Proof. Let's give a combinatorial proof of this formula.

L.H.S. $\sum_{n\geq 0}(1+xq^n)$

$$= \sum_{k\geq 0} x^k \cdot \Big(\underbrace{\sum_{\lambda=(\lambda_1>\lambda_2>\dots>\lambda_k\geq 0)}}_{\substack{\text{partitions with } k \text{ distinct} \\ \text{parts one of which} \\ \text{can be } 0.}} q^{|\lambda|} \Big)$$

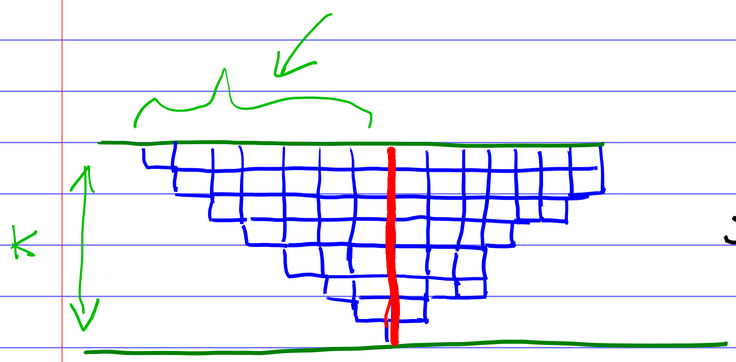


all parts of λ are distinct but we allow (at most) one zero part

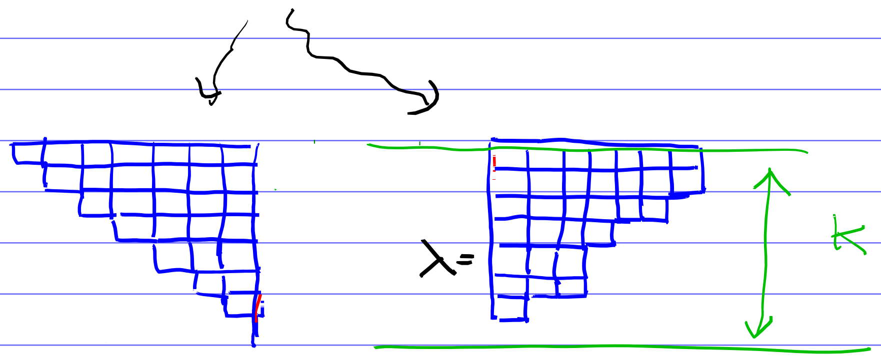
$\lambda = (14, 13, 11, 8, 6, 5, 2, 0)$

$K = 8$

Let's convert it into a shifted Young diagram $\tilde{\lambda}$ and then cut off its "staircase part"



shifted Young diagram



staircase shape with $\frac{K(K-1)}{2}$ boxes | usual Young diagram with at most K rows

So we get L.H.S.

$$= \sum_{k\geq 0} x^k \underbrace{q^{k(k-1)/2}}_{\substack{\text{comes from} \\ \text{the staircase part}}} \frac{1}{\underbrace{(1-q)\dots(1-q^k)}_{= \sum_{\substack{\lambda \text{ with} \\ \text{at most} \\ k \text{ rows}}} q^{|\lambda|}}}$$

as needed. $\square$

Here is the q-analog of the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Theorem

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{k^2} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \right)^2 .$$

It is not hard to prove the last 2 theorems, if you express them in terms of partitions.

Exercise Prove them.