18.212 Lecture 13 03/17/2021

Differential Posets

Last time: $\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$

This can be proved using Schensted insertion algorithm.

$$\underset{S_n}{\overset{\cap}{w}} \overset{RSK}{\longmapsto} (P, Q) \quad \text{pair of SYTs of the same shape} \vdash n$$

Today we'll try to understand this relation in a more general setting & break Schensted insertion steps into simpler and more elementary steps...

Let P be a graded poset with a unique minimal element $\hat{0}$.

Let $\mathbb{R}[P]$ be the vector space of all formal linear combinations of elements of P.

Define the "up" & "down" operators acting linearly on $\mathbb{R}[P]$ by

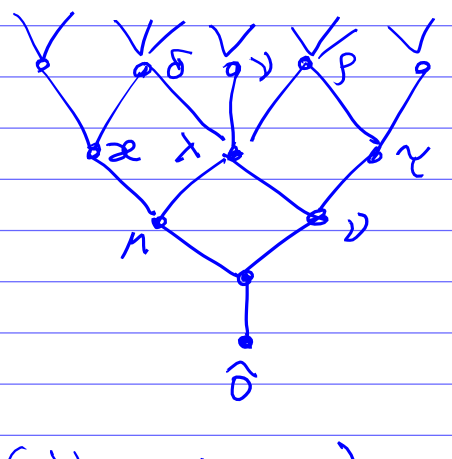
$$U : \lambda \mapsto \sum_{\mu : \mu \gtrdot \lambda} \mu$$

$$D : \lambda \mapsto \sum_{\mu : \mu \lessdot \lambda} \mu$$

for any $\lambda \in P$.

Our main example will be Young's lattice $P = \mathbb{Y}$.

Example.

$P =$ 

Here we are using symbols $\lambda, \mu, \ldots$ for elements of P, keeping in mind that in the case $P = \mathbb{Y}$ they are Young diagr.

$D(\lambda) = \mu + \nu$

$U(\lambda) = \delta + \nu + \rho$

$DU(\lambda) = (\varkappa + \lambda) + \lambda + (\lambda + \tau)$

$= 3\lambda + \varkappa + \tau$

$UD(\lambda) = (\varkappa + \lambda) + (\lambda + \tau)$

$= 2\lambda + \varkappa + \tau$

$(DU - UD)(\lambda) = \lambda.$

Definition [R. Stanley, 1988]

A ranked poset P with a unique minimal element $\hat{0}$ is a <u>differential poset</u> if its "up" & "down" operators satisfy:

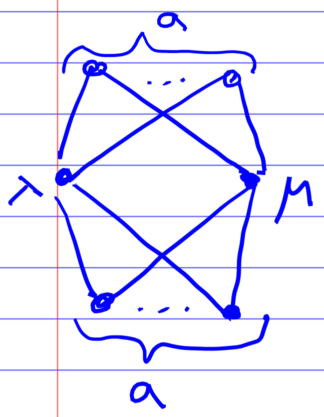
$$\boxed{D \cdot U - U \cdot D = I}$$

Here I is the identity operator.

Lemma The relation $D \cdot U - U \cdot D = I$ is equivalent to the following two combinatorial conditions on the poset P.

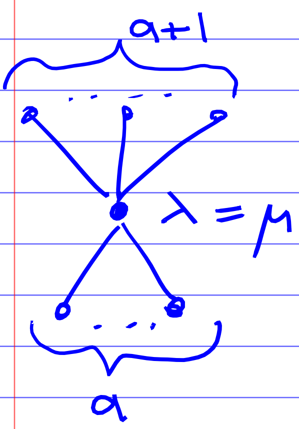
For any two elements λ & μ of P of the same rank, we have

(A) If $\lambda \neq \mu$



$$\#\{\nu : \nu \gtrdot \lambda \ \& \ \nu \gtrdot \mu\} = \#\{\gamma : \gamma \lessdot \lambda \ \& \ \gamma \lessdot \mu\}$$

(B) If $\lambda = \mu$



$$\#\{\nu : \nu \gtrdot \lambda\} = \#\{\gamma : \gamma \lessdot \lambda\} + 1$$

Proof. The coefficient of μ in $(DU - DU)(\lambda)$ should be 0 if $\lambda \neq \mu$, or 1 if $\lambda = \mu$. $\square$

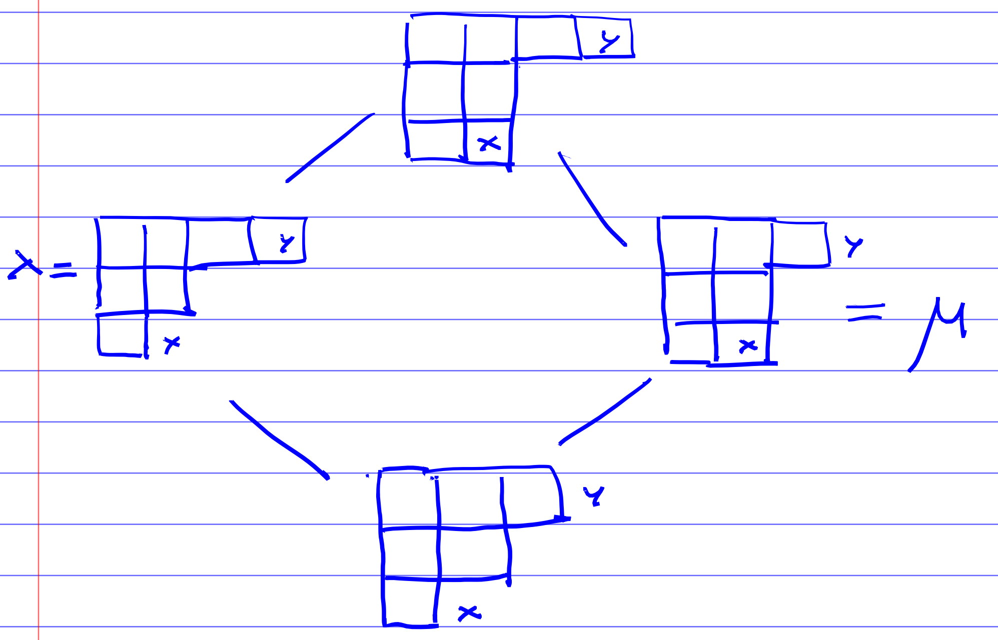
Lemma Young's lattice $\mathbb{Y}$ a differential poset.

Proof. We need to check $DU - UD = I$, or, equivalently, conditions (A) & (B) for $\mathbb{Y}$.

Let λ, μ be two Young diagrams s.t. $|\lambda| = |\mu|$.

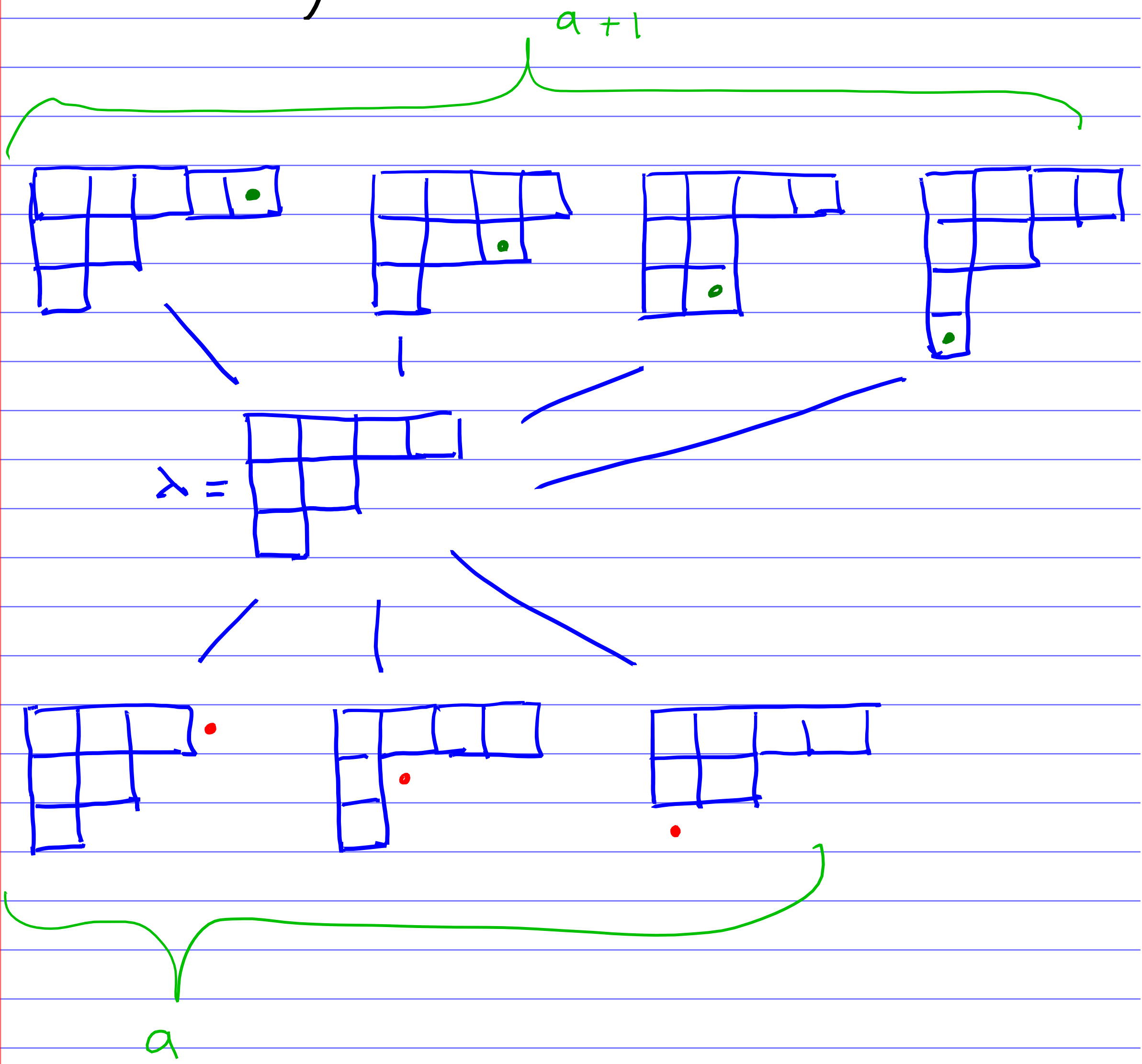
(A) $\lambda \neq \mu$

If μ can be obtained from λ by adding a box x and then removing a <u>different</u> box y, then μ can be obtained from λ by first removing box y and then adding box x, and vise versa.



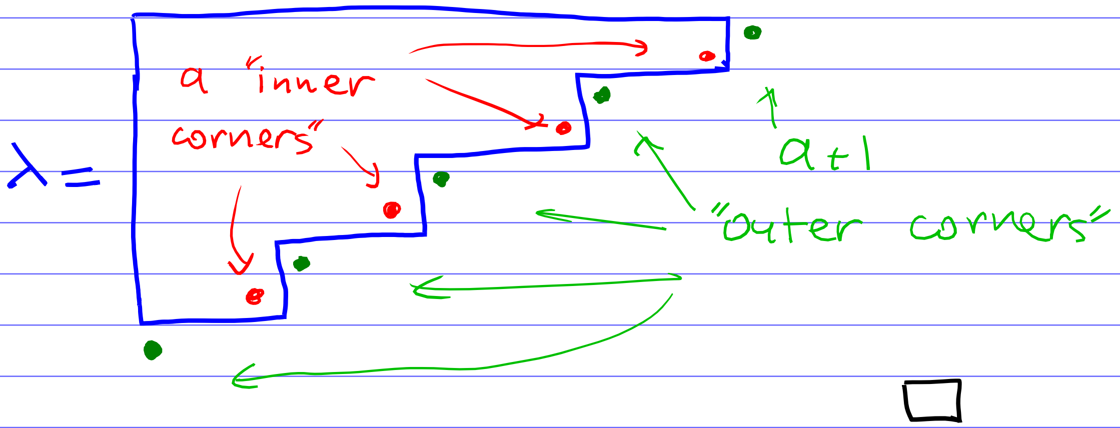
Note: for $P = \mathbb{Y}$ in case (A) $q = 0$ or 1.

(B) $\lambda = \mu$



In general, $\#\, \nu$ s.t. $\nu \gtrdot \lambda$ equals # "outer corners" of shape λ (addable boxes to shape λ).

$\#\, \gamma$ s.t. $\gamma \lessdot \lambda$ equals # "inner corners" of λ.



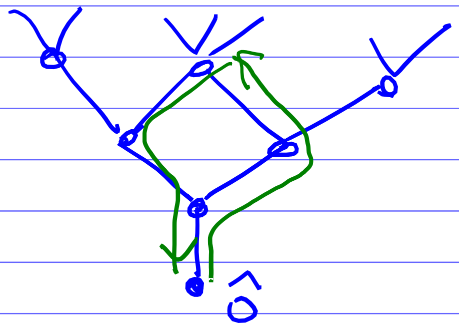
$\square$

How does all this stuff help us?

Recall $D^n U^n(\hat{0}) = N\hat{0}$, where N is the number of paths in the Hasse diagram of the poset P that

- start at the minimal elt. $\hat{0}$
- make n "up" steps
- then make n "down" steps

(and end at $\hat{0}$).



$$D^n U^n(\hat{0}) = \sum_{\substack{\lambda \in P \\ \text{rank}(\lambda) = n}} \left(\# \left\{ \begin{matrix} \text{saturated} \\ \text{chains} \\ \text{from } \hat{0} \\ \text{to } \lambda \end{matrix} \right\} \right)^2 \hat{0}.$$

For Young's lattice $P = \mathbb{Y}$ this is exactly the L.H.S of the identity $\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$

For any differential poset we have

$$(*) \quad DU - UD = I$$

$$(**) \quad D(\hat{0}) = 0$$

this is <u>not</u> zero this is zero

<u>Theorem</u> The relations (*) and (**) algebraically imply that

$$D^n U^n (\hat{0}) = n!\, \hat{0}.$$

<u>Examples:</u> <u>$n=1$</u>

$$DU(\hat{0}) = (UD + I)(\hat{0})$$
$$= 0 + \hat{0} = \hat{0}.$$

<u>$n=2$</u> $D^2U^2(\hat{0}) =$

$$= (D(DU)U)(\hat{0})$$
$$= (D(UD+I)U)(\hat{0})$$
$$= (DU)(DU)(\hat{0}) + DU(\hat{0})$$
$$= (DU)(\hat{0}) + \hat{0} = \hat{0} + \hat{0}$$
$$= 2!\,\hat{0}$$

(Boring) proof — By induction

Base: $n = 0$ ✓

Induction step — $n \geq 1$

$$D^n U^n = D^{n-1}(DU)U^{n-1}$$

$$= D^{n-1}(I + UD)U^{n-1}$$

$$= D^{n-1}U^{n-1} + D^{n-1}U(DU)U^{n-2}$$

$$= D^{n-1}U^{n-1} + D^{n-1}U(I+UD)U^{n-2}$$

$$= 2D^{n-1}U^{n-1} + D^{n-1}U^2(DU)U^{n-3}$$

$$= \dots = nD^{n-1}U^{n-1} + D^{n-1}U^n D.$$

So $D^n U^n(\hat{0})$

$$= nD^{n-1}U^{n-1}(\hat{0}) + D^{n-1}U^n D(\hat{0})$$

$$= n \cdot (n-1)!\, \hat{0} + 0.$$

$$= n!\, \hat{0} \qquad \square$$

Another proof. Notice that (*) & (**) formally imply that $D^n U^n(\hat{0}) = N\hat{0}$ for some N.

So it is enough to check that $N = n!$ for <u>any</u> pair of operators satisfying (*) & (**) (not necessarily the up & down operators acting on $\mathbb{R}[P]$).

Let us take the following pair of operators acting on the polynomial ring $\mathbb{R}[x]$

$$U: f(x) \longmapsto x f(x)$$
$$D: f(x) \longmapsto f'(x)$$

We have

$$(*) \quad DU - UD = I,$$

or equivalently,

$$(x f(x))' = x f'(x) + f(x)$$

This is Leibniz's product rule

Gottfried Wilhelm Leibniz
1646 — 1716

Here we have $1 \in \mathbb{R}[x]$ instead of $\hat{0} \in \mathbb{R}[P]$

$$(**) \quad D(1) = 0$$

or $\frac{d}{dx}(1) = 0.$

So $D^n U^n(1) =$

$$= \left(\frac{d}{dx}\right)^n (x^n) = n! \cdot 1$$

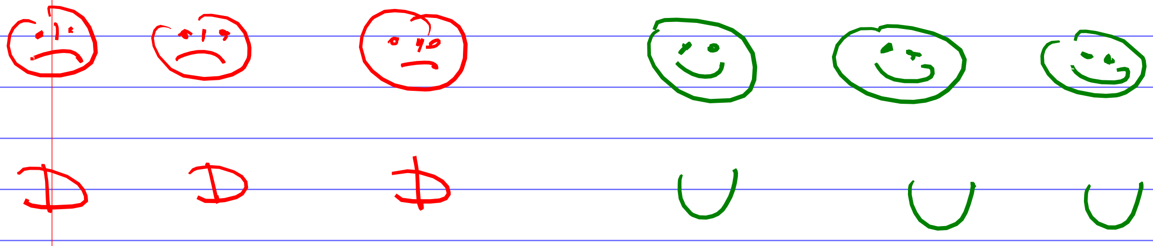
So $N = n!$ $\square$

By the way, this is the reason why differential posets are called <u>differential</u> posets.

Proof #3 Here is a more combinatorial way to see that $D^n U^n (\hat{O}) = n!\,\hat{O}$.

or rather pseudo-physical way to prove it

We can think of products like $DD\ldots DU\ldots U$ as a system of n "particles" and n "anti-particle" hopping over each other:



anti-particles particles

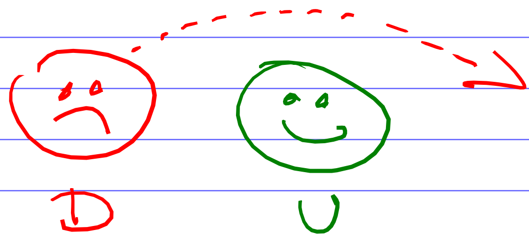
Initially, all anti-particles are to the left of the particles. The anti-particles are trying to move to the right (and the particles are trying to move to the left).

A particle U can "interact"

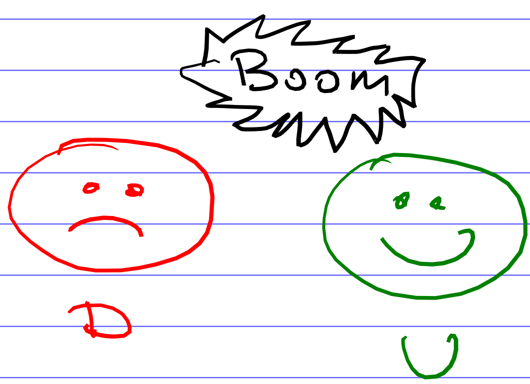
with an anti-particle D

in 2 possible ways:

- D either jumps over U to the right



- or D and U "annihilate" each other:



= nothing

Mathematically, we have the relation

$$D\,U = U\,D + I$$

D jumps over U to the right

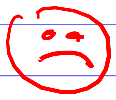
D and U annihilate each other

$$D \ldots D\,\underline{D\,U}\,U \ldots U\,(\hat{0}) =$$

$$D \ldots D\,\underline{U\,D}\,U \ldots U\,(\hat{0})$$

$$+\,D \ldots D\,(\text{nothing})\,U \ldots U\,(\hat{0})$$

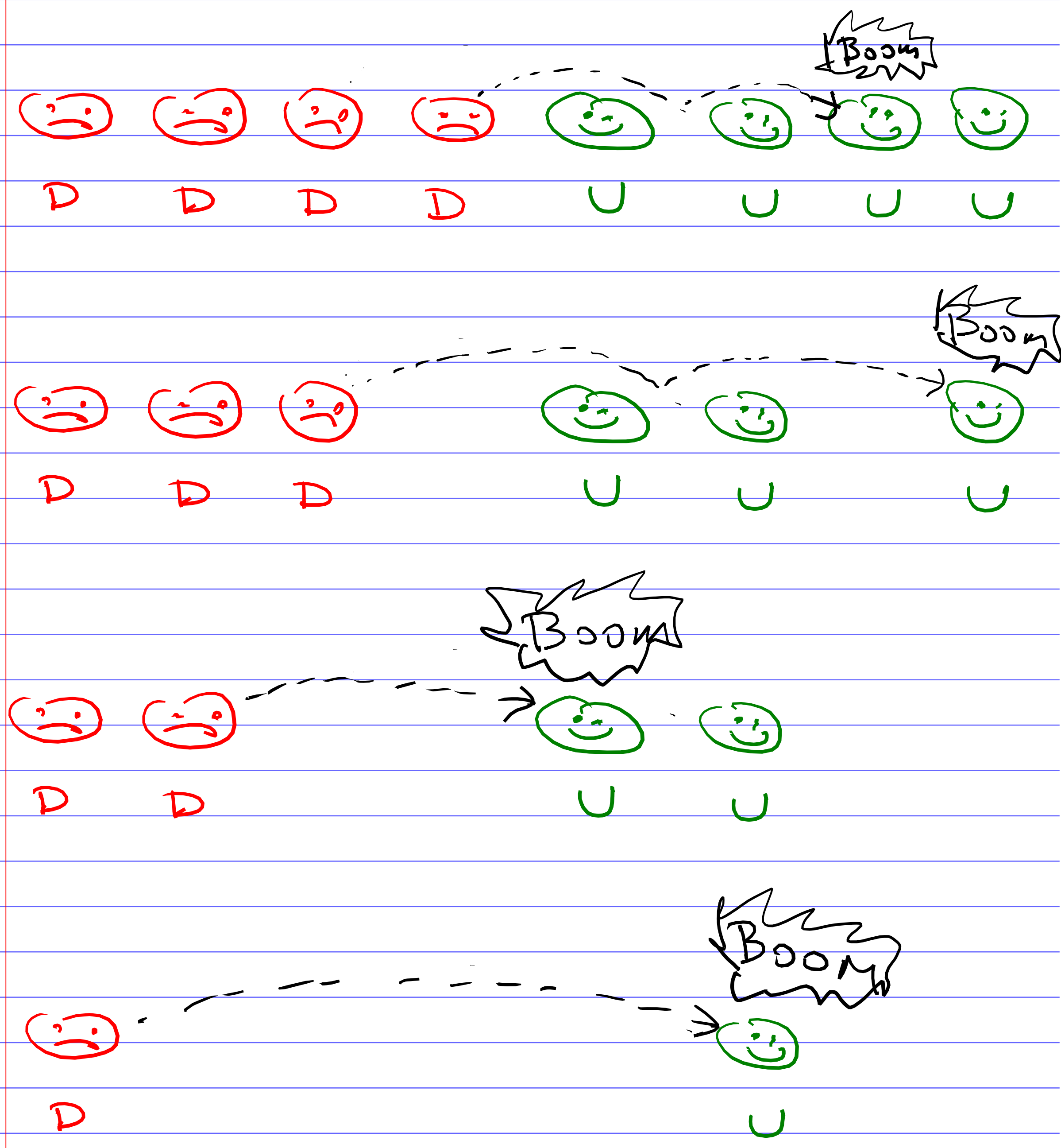
$$= \ldots\ldots$$

If one anti-particle  D survives and jums all the way to the right, we get

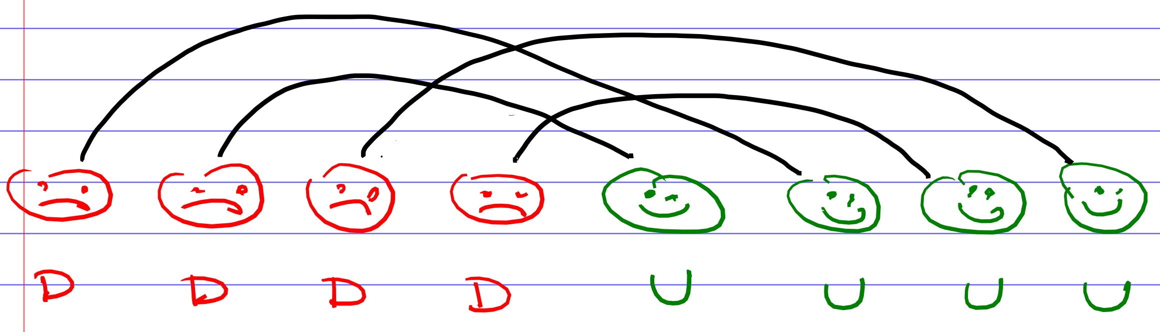
$$\ldots\ldots D\,(\hat{0}) = 0.$$

So all scenarios that give non-zero contributions to $D^n U^n (\hat{0})$ correspond to the cases when <u>each</u> anti-particle  D <u>annihilates</u> at some point with some particle  U.

Example. Here is one scenario of particles & anti-particles hopping over each other & annihilating each other:



This gives a matching between all particles  and all anti-particles  :



So $D^n U^n(\hat{0}) = N\,\hat{0}$,

where N is the number of ways to match n anti-particles ☹ with n particles ☺,

that is $N = n!$ $\square$

So we have

<u>Corollary</u> For any differential poset P,

$$\sum_{\substack{\lambda \in P \\ \mathrm{rank}(\lambda) = n}} f_\lambda(P)^2 = n!,$$

where $f_\lambda(P)$ is the number of saturated chains from $\hat{0}$ to λ in P.

In particular, (for $P = \mathbb{Y}$) we recover

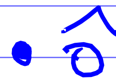
$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$$

where $f_\lambda = \#\{SYT\text{'s of shape } \lambda\}$

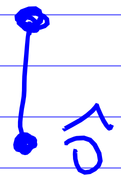
Are there any other examples of differential posets, except Young's lattice?

Let's try to construct a differential poset P rank by rank (starting from 0^{th} rank) so that it satisfies conditions (A) & (B).

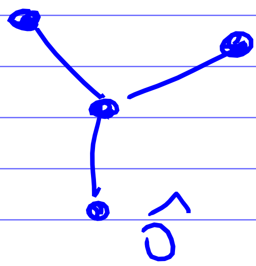
- P should have exactly one element of rank 0:



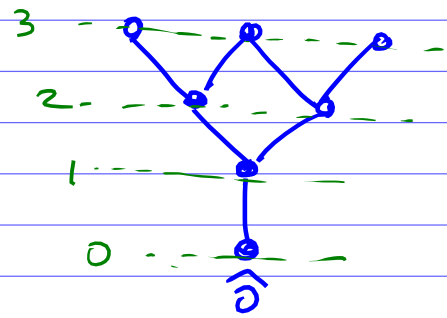
- and exactly 1 element covering $\hat{0}$ (because there are 0 elements below $\hat{0}$)



- and exactly 2 elements covering this element of rank 1, because there is exactly 1 element (namely $\hat{0}$) below it:

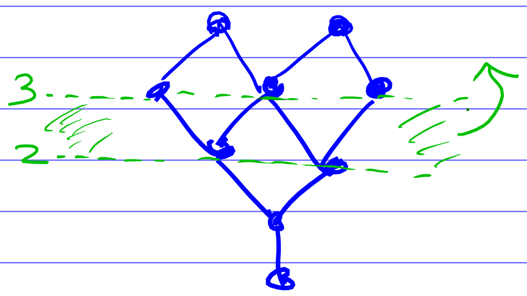


- the next level should be

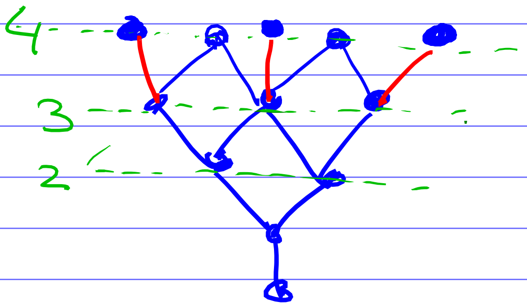


Notice that so far we don't really have any choices: Any differential poset should start like this!

Let's proceed to the next level

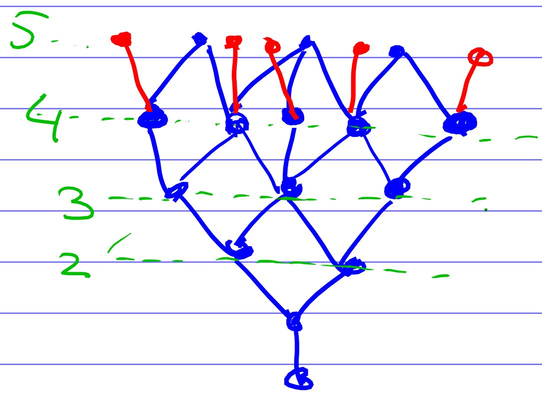


reflect the part of Hasse diagram between levels 2 & 3.



and add an additional element covering each element on level 3

- one more level (reflect & add new covering elements)



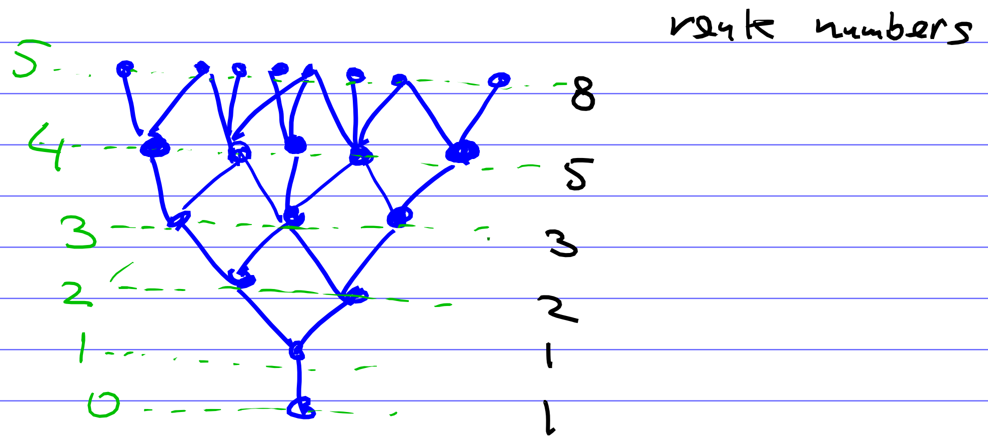
etc.

In general,
$(i+1)^{th}$ level is obtained by reflecting the part of Hasse diagram between levels i and $(i-1)$, and then adding covering elements above all elements on level i.

This way we will obtain the poset that automatically satisfies conditions (A) & (B).

What kind of poset will we get? Maybe it will be just Young's lattice $\mathbb{Y}$.

Let's calculate the rank numbers of this poset:



1, 1, 2, 5, 8, ?

The rank numbers of Young's lattice $\mathbb{Y}$ are the partition numbers

$$P(n) := \#\{\text{Young diagrams with } n \text{ boxes}\}$$

They start like this:

1, 1, 2, 3, 5, 7, 11, 15

So this poset is not Young's lattice.

Theorem. The n^{th} rank number r_n of this poset is the $(n+1)^{st}$ Fibonacci number

$$r_n = F_{n+1}.$$

Recall, $F_1 = F_2 = 1$

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 2.$$

Proof The recurrence relation $r_n = r_{n-1} + r_{n-2}$ is clear from the construction. $\square$

This is why this poset is called the <u>Fibonacci lattice</u> and denoted $\mathbb{F}$.

<u>Exercise</u>. Give a non-recursive description of the Fibonacci lattice $\mathbb{F}$.

Prove that it is indeed a <u>lattice</u>.

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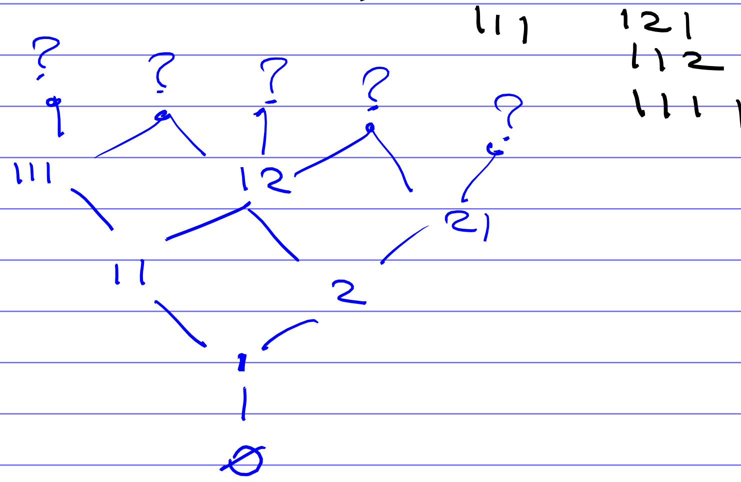
not partitions

Recall, that F_{n+1} equals # <u>compositions</u> of n with all parts equal 1 or 2.

n	0	1	2	3	4	...
F_{n+1}	1	1	2	3	5	...
compositions	$\varnothing$	1	2 11	21 12 111	22 211 121 112 1111	

$\mathbb{F} =$



Q: How to label the elements of $\mathbb{F}$ by compositions with all parts 1 & 2?

Corollary Like Young's lattice, the Fibonacci lattice $\mathbb{F}$ satisfies.

$$\sum_{\substack{x \in \mathbb{F} \\ \operatorname{rank}(\lambda)=n}} \left(f_x(\mathbb{F})\right)^2 = n!$$

What about more general paths in the Hasse diagram of a differential poset P?

Let w be any word in the letters "U" & "D" with exactly n U's and exactly n D's. Let w^{rev} be its reversed word.

Example, $n=4$

$W = U U D U U D D D$

$W^{rev} = D D D U U D U U.$

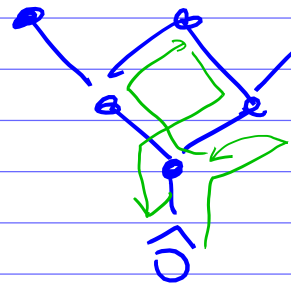
We can view w^{rev} as a product of "up" & "down" operators and define

$f_w(P)$ is the number N such that $w^{rev}(\hat{0}) = N\hat{0}.$

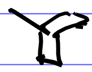
Combinatorially,

$f_w(P) :=$ # paths in the Hasse diagram of P that start and end at $\hat{0}$ with the pattern of up & down steps given by the word w.

Example $w = UUDUUDDD$

$P =$ 

$$f_w(P) = 2 \cdot 6 = 12.$$

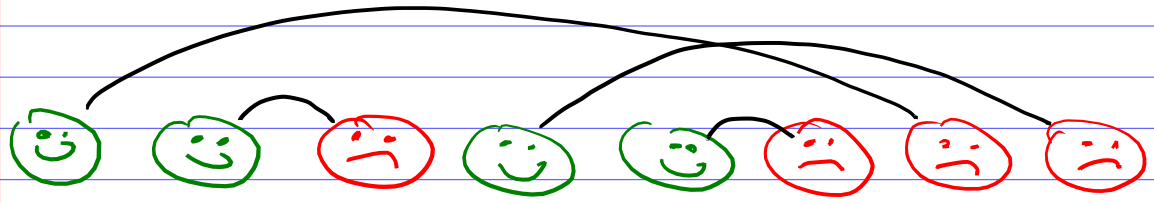
Def. For $P =$  such paths (with an arbitrary pattern of "up" & "down" steps) are called oscillating tableau.

So we are counting # oscillating tableaux that start & end at $\hat{0}$ and have a given pattern of up & down steps.

Using the same construction with hopping particles ☺ and anti-particles ☹ we can deduce that

Theorem $f_w(P)$ equals # ways to match all U's with all D's so that each U is matched with a D to the right of it (in the word w).

Example. $w = UUDUUDDD$

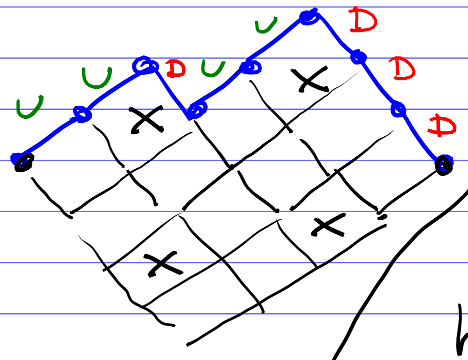


Notice that f_w is non-zero only if w is a Dyck word.

Let's convert w to a Dyck path.

Example. $w = UUDUUDDD$

λ



$= (4, 4, 3, 3)$

Let λ be the Young diag. (rotated by 135°) below this Dyck path.

Notice that a matching of U's & D's corresponds to a placement of n non-attacking rooks on the "chessboard" $\varkappa$

<u>Theorem</u> $f_w =$

$=$ # perfect matchings (as above)

$=$ # rook placements on

why is # rook placements given by this product?

$$\varkappa = (\varkappa_1 \geq \varkappa_2 \geq \ldots \geq \varkappa_n)$$

$$= \varkappa_n \cdot (\varkappa_{n-1}-1)(\varkappa_{n-2}-2)\cdots(\varkappa_1-n+1)$$

$$= \varkappa'_n \cdot (\varkappa'_{n-1}-2)(\varkappa'_{n-2}-2)\cdots(\varkappa'_1-n+1),$$

where $\varkappa' = (\varkappa'_1, \ldots, \varkappa'_n)$ is the conjugate partition to $\varkappa$.

<u>Exercise</u> Prove combinatorially that these two products (for $\varkappa$ and for $\varkappa'$) are equal to each other for any Young diagram $\varkappa \subseteq n \times n$ such that all terms in the product are positive.

We can also easily deduce that f_w is given by this product using differential operators (as in Proof #2).

Example $w = UUDUUDDD$

f_w equals $DDDUUDUU(1)$

where $U : g(x) \longmapsto x\,g(x)$

$D : g(x) \longmapsto g'(x)$.

We get

$$f_w = \left(x^2\left(x^2\right)'\right)'''$$

$$= \left(x^2 \cdot 2x\right)'''$$

$$= 2 \cdot 3 \cdot 2 \cdot 1 = 12.$$