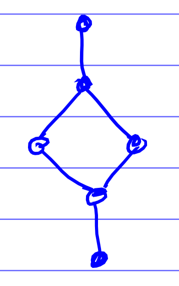
last time: posets & lattices

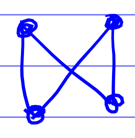
A lattice is

- a poset for which two operations "$\vee$" (join) and "$\wedge$" (meet) are well defined.
- a set with two binary operations "$\vee$" & "$\wedge$" satisfying several axioms (the commutative, associative & absorption laws).

Examples



a lattice



a poset which is not a lattice

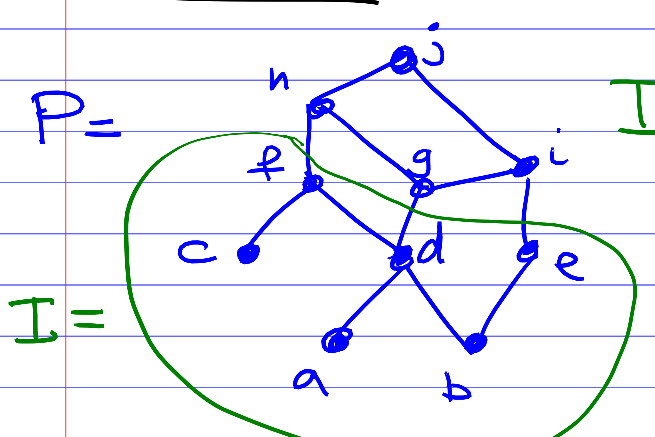
The lattice of order ideals

Let P be any poset (not necessarily a lattice)

An order ideal in P is a subset $I \subseteq P$ of elements of P such that

$$x \in I,\ y \leq_P x \Rightarrow y \in I.$$

Example:



$I = \{a, b, c, d, e, f\}$ is an order ideal

Definition. Let $J(P)$ be the poset of all order ideals in P ordered by containment:

$$I \leq J \text{ iff } I \subseteq J$$

(as sets)

$J(P)$ is called the lattice of order ideals of P.

Lemma $J(P)$ is a lattice.

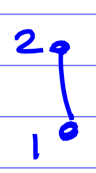
Proof: For order ideals I, J,

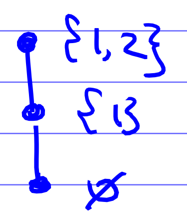
$I \wedge J = I \cap J$ ← intersection

$I \vee J = I \cup J$ ← union of sets

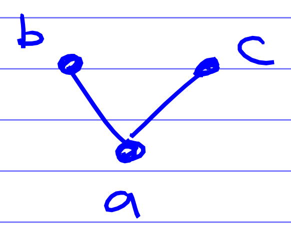
Is is easy to see that the set theoretic intersection/union of two order ideals is an order ideal. $\square$

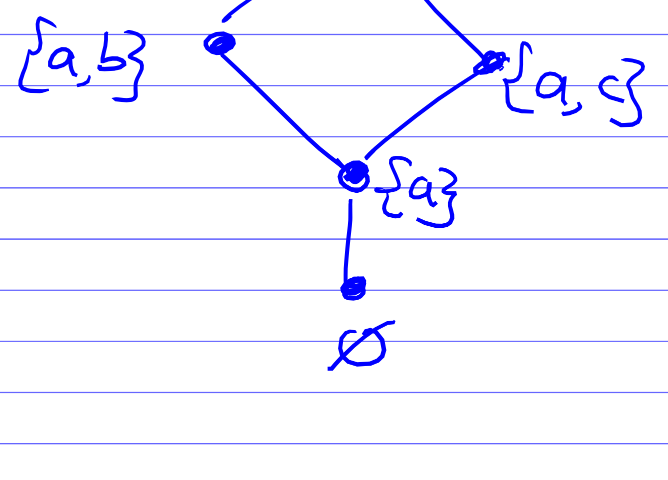
Examples.

$P =$ 

$J(P) =$ 

In general, $J(n\text{-chain}) = (n+1)\text{-chain}$.

$P =$ 

$J(P) =$ 

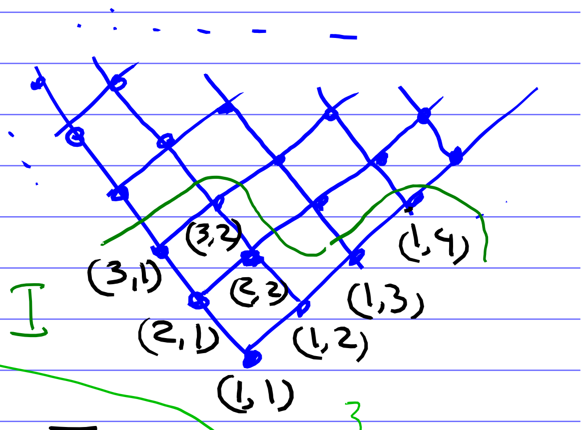
Example. Let $\mathbb{Z}_{>0} = \{1,2,3,\dots\}$ be the poset of positive integers with the usual order $1 < 2 < 3 < \cdots$ (an infinite chain).

Then $J(\mathbb{Z}_{>0}) \simeq \mathbb{Z}_{>0}$

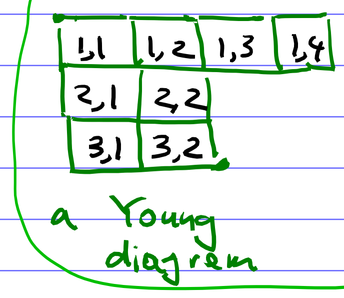
$$J(\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}) \simeq \mathbb{Y}$$

Young's lattice.

$\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} =$



Identify an element $(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ with a box in position (i,j). Then an order ideal in $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ corresponds to a collection of boxes in a Young diagram.



Example. Let O_n be the "empty poset" on n elements, i.e., any two elements of O_n are incomparable. ← That is O_n consists of a single antichain

$$O_n = \overset{\bullet}{1} \quad \overset{\bullet}{2} \quad \overset{\bullet}{3} \quad \overset{\bullet}{4} \quad \cdots \quad \overset{\bullet}{n}$$

Then $B_n = J(O_n)$

the Boolean lattice

Indeed, any subset $I \subseteq \{1,\dots,n\}$ is an order ideal of O_n.

Distributive Lattices

Actually, $J(P)$ is not only just a lattice, but also it belongs to an especially nice class of lattices.

Definition. A lattice L is called a <u>distributive lattice</u> if it satisfies the two <u>distributive laws</u>:

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$.

Remark. For numbers (say, in $\mathbb{R}$) and the usual operations of addition "$+$" and multiplication "$\cdot$", we have the distributive law:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z),$$

but we don't have the "second distributive law"

$$x + (y \cdot z) \neq (x + y) \cdot (x + z)$$

this is usually not correct

So $(\mathbb{R}, +, \cdot)$ is <u>not</u> a distributive lattice, but it shares some similarities with distributive lattices.

Lemma. $J(P)$ is a distributive lattice.

Proof. It is easy to check that the both distributive laws hold for the unions & intersection of sets. Recall, that for $I, J \in J(P)$

$$I \vee J = I \cup J$$
$$I \wedge J = I \cap J$$

the usual union & intersection of sets

$\square$

Corollary. The following posets are distributive lattices:

- n-chain $[n]$
- the Boolean lattice B_n
- Young's lattice $\mathbb{Y}$.

Fundamental Theorem on Finite Distributive Lattices (a.k.a Birkhoff's Representation Theorem), 1937.

For a finite poset P, $J(P)$ is a finite distributive lattice, and any finite distributive lattice L is isomorphic to $J(P)$ for some finite poset P.

Remark. Basically, this theorem says:

$$P \Longleftrightarrow J(P)$$

is a "one-to-one correspondence" between finite posets and finite distributive lattices.

We need to be a little careful here: In order to talk about 1-1 correspondences we need to have some sets. But there is no such thing as the "set of all finite posets".

Strictly speaking, the above statement does not make sense mathematically. In order to rigorously formulate it, we need to talk about the categories of finite posets & finite distributive lattices. But I want to avoid talking about category theory in this class.

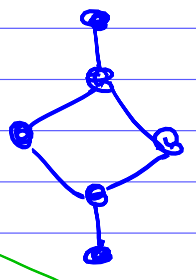
Idea of proof. For any finite distributive lattice L, we need to find a finite poset P such that $L \simeq J(P)$.

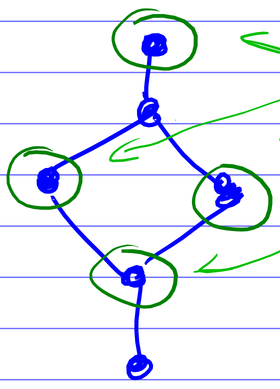
An element z of L is called *join-irreducible* if z is not a minimal element of L, and we cannot write it as $z = x \vee y$ for some $x, y \lneq z$.

Let P be the poset of all join-irreducible elements in L.

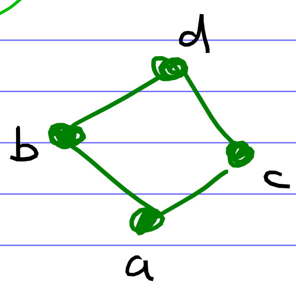
Then one can deduce, using the axioms of distributive lattices, that $L \simeq J(P)$.

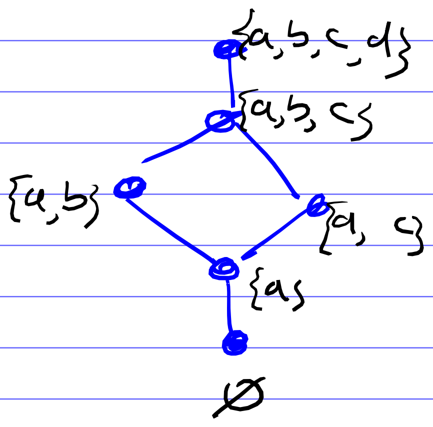
Exercise. Prove this.

Example $L =$ 



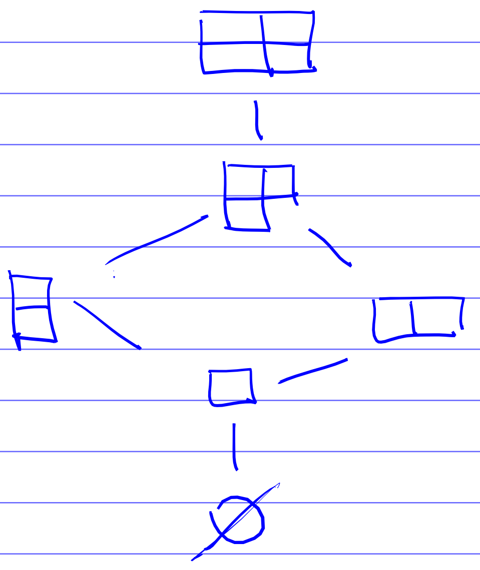
join-irreducible elements of L

$\rightsquigarrow$ $P =$ 

$J(P) =$ 

$\simeq L.$

this is the Lattice of all Young diagrams that fit inside the 2×2 square

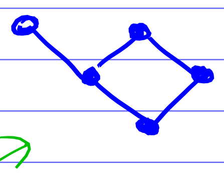


Young's lattice $\mathbb{Y}$ is not a finite lattice. But for any Young diagram λ, $\mathbb{Y}$ has a finite sublattice:

$$\mathbb{Y}_\lambda :=$$ the poset of all Young diagrams μ that fit inside λ, ordered by inclusion

$$= \text{the interval } [\varnothing, \lambda]_{\mathbb{Y}}$$ in Young's lattice $\mathbb{Y}$.

Also denote by P_λ the poset on the set of boxes of λ, ordered such that $x \leq y$ if the box y is to the South, East, or South-East of the box x.

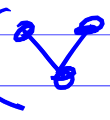
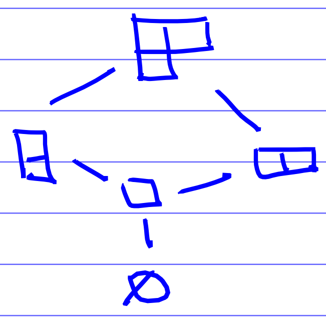
<u>Example</u>. $P_{(3,2)} =$ 

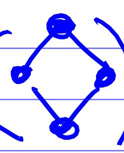
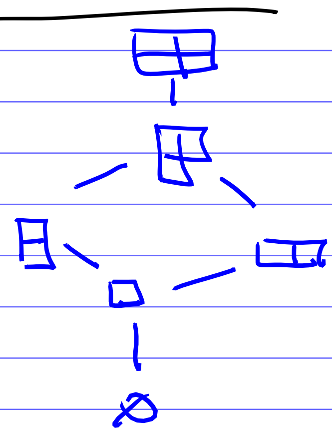
replace the boxes of λ by dots & rotate the picture by $135°$

Clearly, we have

<u>Corollary</u>. $\mathbb{Y}_\lambda = J(P_\lambda)$ is a finite distributive lattice.

<u>Examples</u>:

$\mathbb{Y}_{(2,1)} = J($  $) =$ 

$\mathbb{Y}_{(2,2)} = J($  $) =$ 

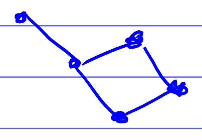
Linear extensions of posets

For a finite poset P with n elements, a <u>linear extension</u> of P is a bijective map $f : P \to \{1, 2, \ldots, n\}$ such that

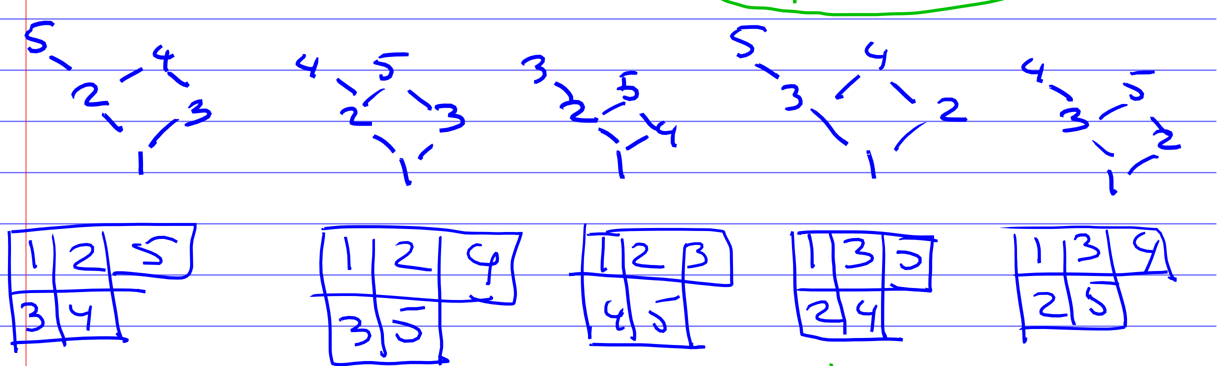
if $x \leq_P y$ then $f(x) \leq f(y)$.

Let $\mathrm{ext}(P)$ denote the number of linear extensions of P.

<u>Example</u> $P = P_{\lambda}$ (with λ the shape drawn as Young diagram) $=$



linear extensions: labellings of P to $1, 2, \ldots, n$ that increase "upwards"



In this case linear extensions are standard Young tableaux (SYT)

So $\mathrm{ext}(P) = 5$.

<u>Proposition</u>. For a finite poset P, $\mathrm{ext}(P)$ equals the number of <u>saturated</u> <u>chains</u> from the minimal element $\hat{0}$ to the maximal element $\hat{1}$ in $J(P)$.

In particular,

$f_\lambda = \mathrm{ext}(P_\lambda)$

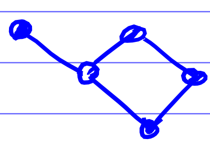
$= \#$ saturated chains from $\emptyset$ to λ in Y_λ.

SYTs of shape λ

So we can think about SYT's as

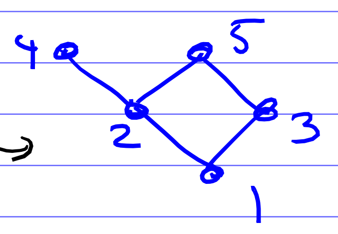
- linear extensions of P_λ
- saturated chains in Y_λ.

<u>Example</u>. $P = P_{\lambda} =$

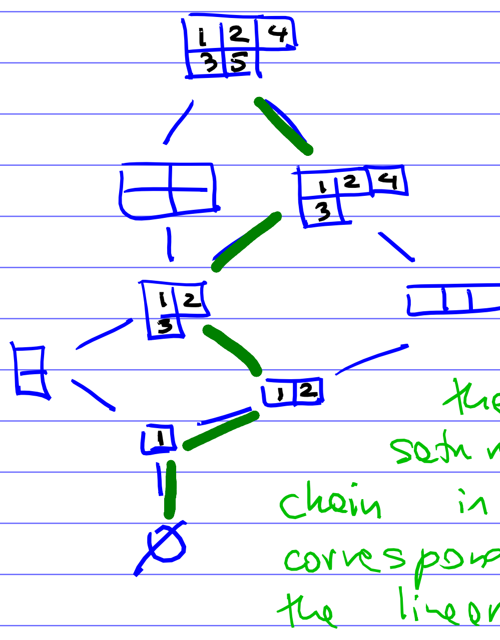


1	2	4
3	5	

$\rightsquigarrow$



$J(P) =$



the saturated chain in $J(P)$ corresponding to the linear extension (or SYT)

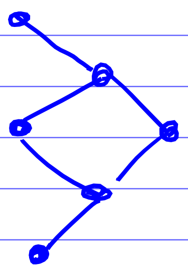
1	2	4
3	5	

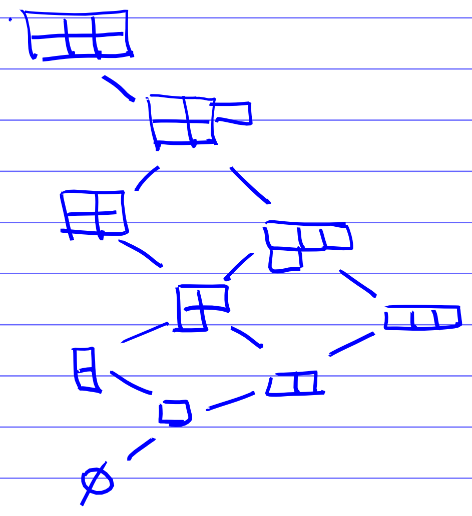
These sublattices of $\mathbb{Y}$, are also called Young's lattices

Let $L(m,n) := J([m]\times[n])$ the product of chains

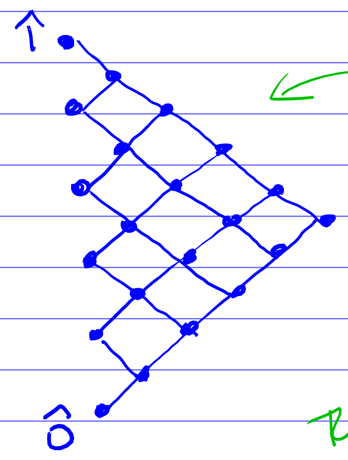
$= \mathbb{Y}_{m\times n}$ the poset of Young diagrams that fit inside the $m\times n$ rectangle.

Examples.

$L(2,2) =$ 

$L(3,2) =$ 

More generally,

$L(n,2) =$ 

these diagrams of triangular shape with $n+1$ dots on each side.

This is $L(5,2)$.

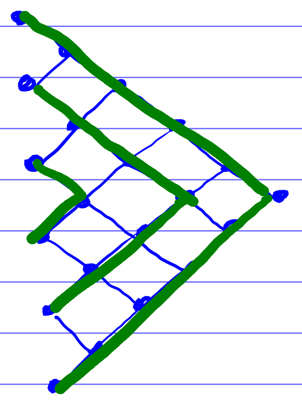
Corollary. # saturated chains from $\hat{0}$ to $\hat{1}$ in $L(n,2)$

$=$ # SYT's of shape $n\times 2$.

$=$ the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Clearly, saturated chains from $\hat{0}$ to $\hat{1}$ in $L(n,2)$ are literally Dyck paths.

Clearly, $L(n,2)$ has symmetric chain decomposition (SCD):

$L(n,2) =$ 

How about an arbitrary $L(m,n)$?

The Gaussian coefficients

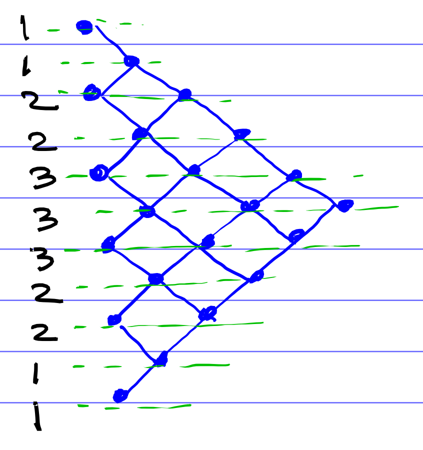
Clearly, Young's lattice $L(m,n)$ is a ranked poset with the rank function $\rho(\lambda) = |\lambda|$.

Let $r_k = r_k(L(m,n))$, $k = 0,1,\dots,m\cdot n$, denote the rank numbers of Young's lattice $L(m,n)$.

Explicitly, $r_k(L(m,n)) := \#\left\{\begin{array}{l}\text{Young diagrams } \lambda \text{ s.t.} \\ \lambda \subseteq m\times n \\ |\lambda| = k\end{array}\right\}$

The generating function
$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = r_0 + r_1 q + \dots + r_{mn} q^{m\cdot n}$$
is known as the Gaussian q-binomial coefficient.

Example:

$L(5,2) =$ 

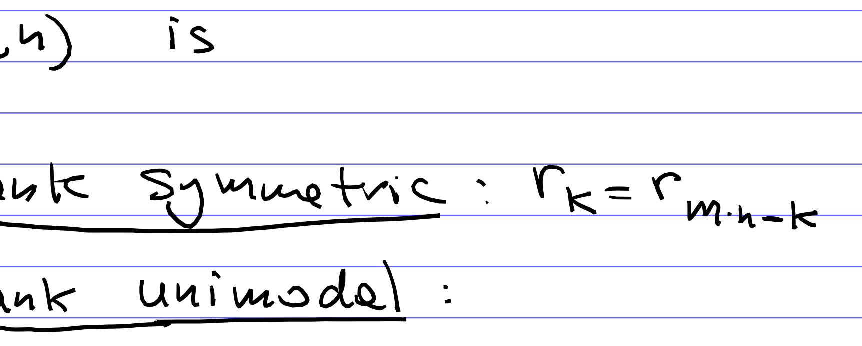
The rank numbers $r_k(L(5,2))$ are: 1, 1, 2, 2, 3, 3, 3, 2, 2, 1, 1.

The Gaussian q-binomial coeff. is
$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}.$$

Theorem. Young's lattice $L(m,n)$ is

- rank symmetric: $r_k = r_{m\cdot n-k}$
- rank unimodal:
  $r_0 \le r_1 \le \dots \le r_{\lfloor \frac{m\cdot n}{2} \rfloor} \ge \dots \ge r_{mn}$
- Sperner.

Remarks
- the rank symmetry is trivial: $\lambda \overset{1:1}{\longleftrightarrow} \lambda^\vee :=$ the complement of λ in the $m\times n$ rectangle



- the unimodality of the Gaussian coefficients is a notoriously hard result. The first proof was given by Sylvester in 1878. The first constructive proof was given more than 100 years later by K. O'Hara '1990

we will discuss Sylvesters proof

- The Sperner property of $L(m,n)$ was proved by R. Stanley '1980.

Does $L(m,n)$ has a symmetric chain decomposition?

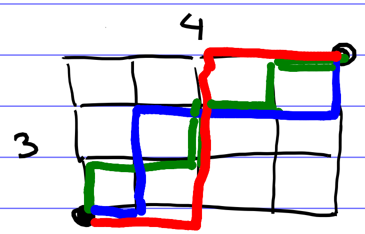
Unknown, for general m & n.

The Sperner property of $L(m,n)$ can be reformulated like this:

Let $P_1, P_2, \dots, P_N$ be any collection of lattice paths from the lower left to the upper right corners in the $m \times n$ rectangle s.t. any two paths P_i & P_j intersect

Here we meant that the paths intersect in the strict sense, not just "touch" each other

For example,



Then $N \leq r_{\lfloor \frac{m \cdot n}{2} \rfloor} := \#\left\{ \begin{array}{c} \text{Young diagr.} \\ \lambda \subseteq m \times n \\ \text{with} \\ |\lambda| = \lfloor \frac{m \cdot n}{2} \rfloor \end{array} \right\}$

q - analogs

A "q-analog" of some "classical" combinatorial number A is a polynomial $A(q) = a_0 + a_1 q + \ldots + a_d q^d$ (typically, with nonnegative integer coefficients a_i) such that $A(1) = A$ (the "classical limit").

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q-numbers

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$$
$$= \frac{1-q^n}{1-q}$$

q-factorials:

$$[n]_q! := [1]_q [2]_q \cdot \ldots \cdot [n]_q!$$

This is a bit confusing notation:
$[n]_q! \neq ([n]_q)!$

Example:

$$[3]_q! := (1+q)(1+q+q^2)$$
$$= 1 + 2q + 2q^2 + q^3$$

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q-binomial coefficients

(a.k.a the Gaussian coeffs.)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}$$

Example $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \dfrac{[4]_q [3]_q [2]_q}{[2]_q \; [2]_q}$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}$$

$$= (1+q^2) \cdot (1+q+q^2)$$

$$= 1 + q + 2q^2 + q^3 + q^4$$

This turned out to be a polynomial with positive integer coefficient. But in general, this is not obvious. Why should the numerator be divisible by the denominator?

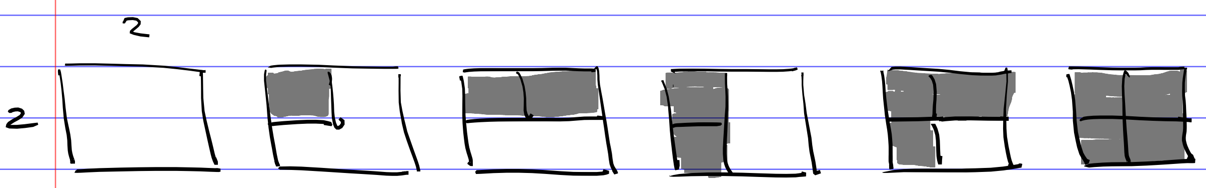
Theorem.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}.$$

The sum over Young diagr. λ that fit inside the $k\times(n-k)$ rectangle

The coefficients of this expression are exactly the rank numbers r_k of the Young's lattice $L(k, n-k)$.

Example $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q =$



$$1 + q + q^2 + q^2 + q^3 + q^4$$

Proof #1. It is not hard to prove it by induction on n. One can easily show that both L.H.S. & R.H.S. satisfy the q-Pascal's recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

Exercise Check this.

Remark This is an easy but not very conceptual proof. We'll give another more interesting proof.

q - Pascal's triangle:

$$\begin{bmatrix}0\\0\end{bmatrix}_q$$

$$\begin{bmatrix}1\\0\end{bmatrix}_q \qquad \begin{bmatrix}1\\1\end{bmatrix}_q$$

$$\begin{bmatrix}2\\0\end{bmatrix}_q \qquad \begin{bmatrix}2\\1\end{bmatrix}_q \qquad \begin{bmatrix}2\\2\end{bmatrix}_q$$

- - - - - - -

||

$$1$$

$$1 \qquad\qquad 1$$

$1 \quad q$

$$1 \qquad 1+q \qquad 1$$

$1 \quad q \qquad 1 \quad q^2$

$$1 \qquad 1+q+q^2 \qquad 1+q+q^2 \qquad 1$$

$1 \quad q \qquad 1 \quad q^2 \qquad 1 \quad q^3$

$$1 \qquad 1+q+q^2+q^3 \qquad 1+q+2q^2+q^3+q^4 \qquad 1+q+q^2+q^3 \qquad 1$$

. . . .