

10.217 Lec 8.1

Eulerian Numbers

Euler 1755:

$$f_0(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$f_1(x) = 1 + 2x + 3x^2 + 4x^3 + \dots = \frac{1}{(1-x)^2}$$

$$f_2(x) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots = \frac{1+x}{(1-x)^3}$$

In general $f_n(x) = \sum_{k=0}^n (k+1)^n x^k$

$$f_2(x) = (x f_1(x))' = \left(\frac{x}{(1-x)^2} \right)' = \frac{1+x}{(1-x)^3}$$

$$f_3(x) = (x f_2(x))' = \left(\frac{x(1+x)}{(1-x)^3} \right)' = \frac{1+4x+x^2}{(1-x)^4}$$

$$f_n(x) = (x f_{n-1}(x))'$$

... $f_4(x) = \frac{1+11x+11x^2+x^3}{(1-x)^5}$, $f_5(x) = \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}$

Observations: $f_n(x) = \frac{A_n(x)}{(1-x)^{n+1}}$ where $A_n(x)$ is a polynomial in x
 of degree $n-1$, with positive coeffs, palindromic, sum of coeffs = $n!$

$$A_n(x) = \sum_{k=0}^{n-1} A(n,k) x^k$$

$A_n(x)$ are called Eulerian polynomials
 $A(n,k)$ are called Eulerian numbers

Theorem: $A(n,k) = \#$ permutations $w \in S_n$ with exactly k descents
 $= \#$ permutations with exactly k exceedances.

des(w) = $\# i$ s.t. $w_i > w_{i+1}$

exc(w) = $\# i$: $w_i > i$

18.212 Lec 8.2

Symmetry: $A(n, k) = A(n, n-1-k)$

$\tilde{w} = w_n w_{n-1} \dots w_1 \Rightarrow \text{des}(\tilde{w}) = n-1 - \text{des}(w)$

Thm: The two statistics, $\text{des}(w)$ and $\text{exc}(w)$ on permutations $w \in S_n$ are equidistributed

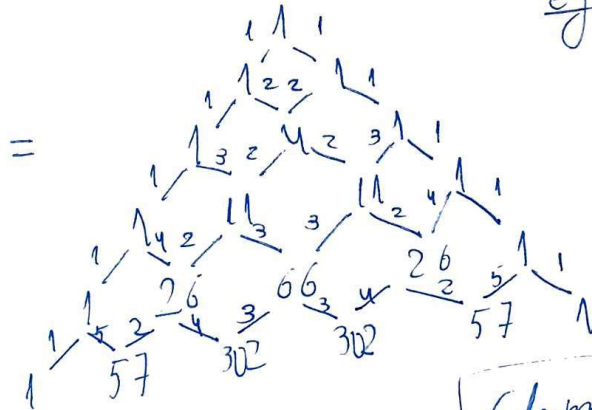
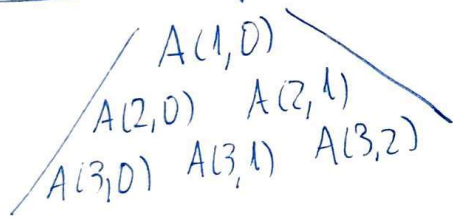
Recall: Cyc & rec are equidistributed

Bijection $f: S_n \leftrightarrow S_n$
 $w \mapsto \tilde{w}$

Example $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix} = (1, \underline{2}, \underline{5})(3, \underline{7}, \underline{8}, \underline{4})(\underline{6})$
 $= (5, \underline{1}, \underline{2})(6)(8, \underline{4}, \underline{3}, \underline{7}) \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 2 & 6 & 8 & 4 & 3 & 7 \end{pmatrix}$

Claim: $\text{des}(\tilde{w}) = \# \text{anti-excedence} = \text{exc}(w^{-1})$
 $= \# \{i \mid w_i < i\}$

Eulerian triangle



eg: $302 = 4 \cdot 26 + 3 \cdot 66$

Claim 1: This relation is equivalent to the one for f_n .

Recurrence relation:

$A(n, k) = (n-k)A(n-1, k-1) + (k+1)A(n-1, k)$

for $n \geq 1$ & $A(n, k) = 0$ if $k < 0$ or $k \geq n$
 $A(1, 0) = 1$

Claim 2: This relation holds for #tree of Exc