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Eulerian Numbers

Euler 1755:

$$f_0(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$f_1(x) = 1 + 2x + 3x^2 + 4x^3 + \dots = \frac{1}{(1-x)^2}$$

$$f_2(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots = \frac{1+x}{(1-x)^3}$$

In general $f_n(x) = \sum_{k \geq 0} (k+1)^n x^k$

$$f_2(x) = (x f_1(x))' = \left(\frac{x}{(1-x)^2}\right)' = \frac{1+\cancel{x}}{(1-x)^3}$$

$$f_3(x) = (x f_2(x))' = \left(\frac{x(1+x)}{(1-x)^3}\right)' = \boxed{\frac{1+4x+x^2}{(1-x)^4}}$$

$$f_n(x) = (x f_{n-1}(x))'$$

$$\dots \quad f_4(x) = \frac{1+11x+11x^2+x^3}{(1-x)^5}, \quad f_5(x) = \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}$$

Observations: $f_n(x) = \dfrac{A_n(x)}{(1-x)^{n+1}}$ where $A_n(x)$ is a polynomial in x of degree $n-1$, with positive int coeffs, palindromic, sum of coeffs $= n!$

$$A_n(x) = \sum_{k=0}^{n-1} A(n,k)\, x^k,$$

$A_n(x)$ are called Eulerian polynomials

$A(n,k)$ —"— Eulerian numbers

Theorem: $A(n,k)$ = # permutations $w \in S_n$ with exactly k descents

= # —"— —"— with exactly k exceedances.

$des(w) = \# i$ s.t. $w_i > w_{i+1}$

$exc(w) = \# i : w_i > i$

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Symmetry: $A(n,k) = A(n, n-1-k)$

$\tilde{w} = w_n w_{n-1} \dots w_1 \quad \Rightarrow des(\tilde{w}) = n-1-des(w)$

Thm: The two statistics, $des(w)$ and $exc(w)$ on permutations $w \in S_n$ are equidistributed

Recall: Cyc & rec are equidistributed

Bijection $f: S_n \longleftrightarrow S_n$

$w \longmapsto \tilde{w}$

Example

$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix} = (1,2,\underline{5})(3,7,\underline{8},4)(\underline{6})$

$= (5,1,2)(6)(8,4,3,7) \longmapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 2 & 6 & 8 & 4 & 3 & 7 \end{pmatrix}$

Claim: $des(\tilde{w}) = aexc(\tilde{w}) = exc(w^{-1})$

#anti-excedence

$= \#\{i \mid w_i < i\}$

Eulerian triangle

$$\begin{matrix} & & A(1,0) & & \\ & A(2,0) & & A(2,1) & \\ A(3,0) & & A(3,1) & & A(3,2) \end{matrix}$$

=

$$\begin{matrix} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 4 & & 1 & & & \\ & & 1 & & 11 & & 11 & & 1 & & \\ & 1 & & 26 & & 66 & & 26 & & 1 & \\ 1 & & 57 & & 302 & & 302 & & 57 & & 1 \end{matrix}$$

eg: $302 = 4 \cdot 26 + 3 \cdot 66$

Claim 1: This relation is equivalent to the one for f_n.

Recurrence relation:

$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)$

for $n \geq 1$ & $A(n,k) = 0$ if $k < 0$ or $k \geq n$

$A(1,0) = 1$

Claim2: This relation holds for #rec and #exc