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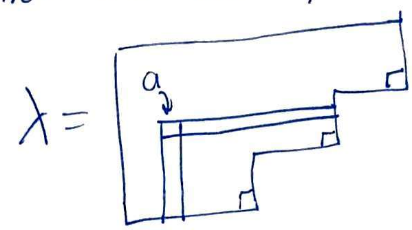
Hook Length Formula (Cont'd)

$f_\lambda := SYT(\lambda) \quad H(\lambda) := \prod_{a \in \lambda} h(a)$

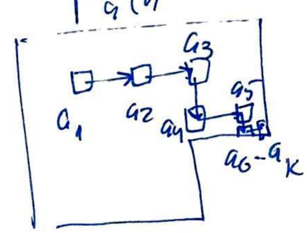
hook at box a, hook length h(a)



HLF: $f_\lambda = \dfrac{n!}{H(\lambda)}$

Recurrance: $f_\lambda = \sum_{c \text{ corner}} f_{\lambda / c}$

Prop: $\displaystyle\sum_{\lambda \vdash n,\ c \text{ corner of } \lambda} \frac{1}{n} \frac{H(\lambda)}{H(\lambda \setminus c)} = 1$



Hook walks:

(1) Pick random box
(2) If not corner, jump somewhere else in its hook uniformly till reach corner.

$\text{Prob}(P) = \frac{1}{n} \cdot \frac{1}{h(a_1)-1} \frac{1}{h(a_2)-1} \cdot \ldots \cdot \frac{1}{h(a_{k-1})-1}$

Prop: $\displaystyle\sum_{P \text{ hook walks ending at } c} \text{Prob}(P) = \frac{1}{n} \frac{H(\lambda)}{H(\lambda \setminus c)} \iff \sum wt(P) = \frac{H(\lambda)}{H(\lambda \setminus c)}$

Weights of boxes in λ and path P

$wt(a) = \dfrac{1}{h(a)-1} \qquad wt(P) = wt(a_1) \ldots wt(a_{k-1})$

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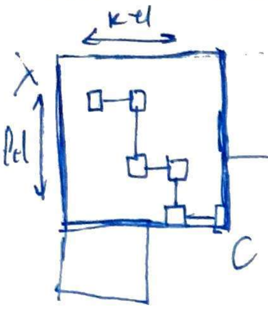
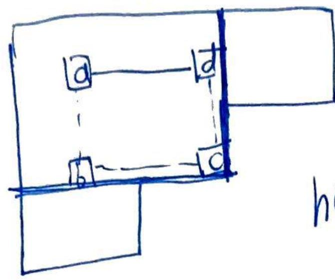
Fix λ and a corner C R_{ec}





$$h(a)+h(c)=h(b)+h(d)$$

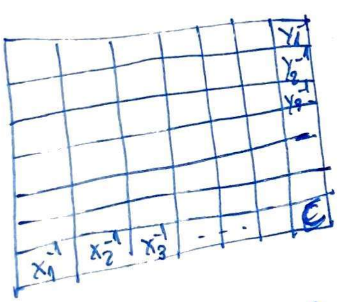
$$h(a)-1+h(c)-1=h(b)-1+h(d)-1$$

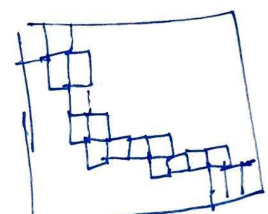
$$h(a)-1=h(b)-1+h(d)-1$$

Denote by $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_l$ $h(b)-1$ for all boxes in the same row/col as corner c.

Weight of boxes in R_{ec}

in box (i,j) have $\dfrac{1}{x_i+y_j}$





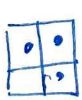
Want to calculate $\displaystyle\sum_{P \text{ hook walk}} wt(P)$

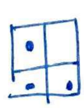
lattice paths inside Rec from the upper left to the bottom right corner

__Lemma 1:__ $\displaystyle\sum_{\substack{P \text{ lattice paths} \\ \text{in } R_{ec}}} wt(P) = \frac{1}{x_1 \dots x_k} \frac{1}{y_1 \dots y_l}$

Ex: $k=l=1$

$1/(x_1+y_1)$	$1/y_1$
$1/x_1$	1

 $\dfrac{1}{x_1+y_1} \cdot \dfrac{1}{y_1} \cdot 1$

 $\dfrac{1}{x_1+y_1} \cdot \dfrac{1}{x_1} \cdot 1$

$$\Rightarrow \frac{1}{x_1+y_1} \cdot \frac{1}{y_1} + \frac{1}{x_1+y_1} \cdot \frac{1}{x_1} = \frac{1}{x_1 y_1}$$

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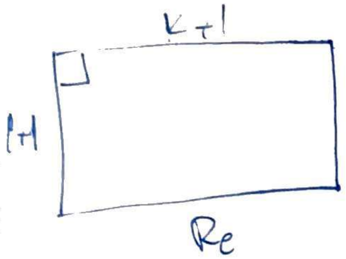
Proof of lemma 1: Induction on $k+l$

Base ✓

Induction step: $\displaystyle\sum_{P \text{ lattice path}} wt(P) = \sum_{P \text{ starts with right step}}' wt(P) + \sum_{P \text{ starts with down step}}'' wt(P)$



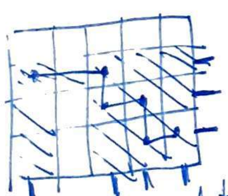
$$= \frac{1}{x_1+y_1}\left(\frac{1}{y_1 y_2 \cdots y_l}\cdot\frac{1}{x_2\cdots x_k} + \frac{1}{y_2\cdots y_l\, x_1\cdots x_k}\right) = \frac{1}{x_1+y_1}\left(\frac{1}{x_1}+\frac{1}{y_1}\right)\frac{1}{x_2\cdots x_k}\,\frac{1}{y_2\cdots y_l}$$

$$= \frac{1}{x_1\cdots x_k}\,\frac{1}{y_1\cdots y_l} \quad \square$$

Lemma 2: $\displaystyle\sum_{P \text{ hook walk in } R_e} wt(P) = \left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_l}\right)$

Pf:

$$= \sum_{\text{subgrids in } R_e}\left(\sum_{\text{lattice paths}} wt(P)\right) \overset{\text{lemma 1}}{=} \left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{y_l}\right) \quad \square$$



↑ Hook walk is a lattice path in a subgrid

However note that $\displaystyle\frac{H(\lambda)}{H(\lambda\backslash c)} = \frac{(x_1+1)}{x_1}\,\frac{(x_2+1)}{x_2}\cdots\frac{(x_k+1)}{x_k}\,\frac{(y_1+1)}{y_1}\cdots\frac{(y_l+1)}{y_l}$

$$= \prod_{a \text{ in same row or column as } c}\frac{h(a)}{h(a)-1} = \prod\frac{h_\lambda(a)}{h_{\lambda\backslash c}(a)} = \frac{H(\lambda)}{H(\lambda\backslash c)}$$