18.212 lec 41

Young Tableaux and Hook length formula

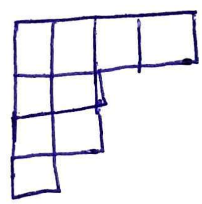
Some notation:

Partitions: $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n ($\lambda \vdash n$) if

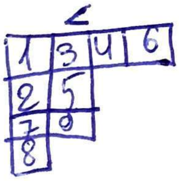
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0 \quad \text{and} \quad \lambda_1 + \dots + \lambda_k = n$$

Ex $5 = 3 + 2 = 2 + 3$ same partition of 5

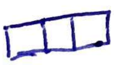
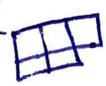
Young diagrams: Ex $\lambda = (4,2,2,1) \vdash 9$

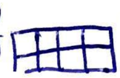
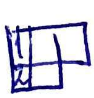
$\lambda =$ 

Def: A Std Young Tableau (SYT) of shape λ is a way to fill boxes of λ with integers $1, 2, \dots, n$ (w/o repetitions) s.t. the entries increase in rows & columns of λ.

Ex:  this is SYT of shape $(4,2,2,1)$

Denote $f_\lambda := \#$ SYT's of shape λ.

Ex f  $= 1$, f  $= 2$, f  $= 2$, f  $= 2 + 3 = 5$

f  $= 5$, f  $= 14$?  or 

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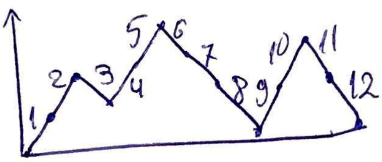
Prop: $f_{(n,n)} = f_{\underbrace{\square\square\cdots\square\square}_{n}} = C_n$

Pf: bij between $\left\{ \text{SYT of shape } (n,n) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Dyck paths} \\ \text{w/ } 2n \text{ steps} \end{array} \right\}$

$T =$

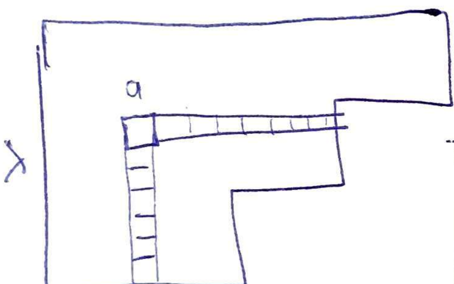
1	2	4	5	9	10
3	6	7	8	11	12

$\xrightarrow{n=6}$



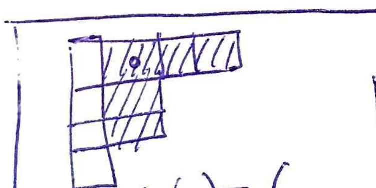
(go up on i-th step iff i in upper row.)

The Hook Length Formula



the hook at box a.

hook length $(:= h(a)) :=$ # boxes in the hook at a



$h(a) = 6$

$$H(\lambda) = \prod_{a \in \lambda} h(a)$$

Hook length formula (Frame – Robinson – Thrall 1953)

$\forall$ partition $\lambda \vdash n$: $f_\lambda = \dfrac{n!}{H(\lambda)}$

Ex: $f_{(3,2)} = f_{\begin{smallmatrix} 4 & 2 & 1 \\ 2 & 1 & \end{smallmatrix}} = \dfrac{5!}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 5$

Cor $C_n = \dfrac{1}{1+n}\binom{2n}{n}$

Hook Walk Proof of the HLF (by Greene – Nijenhuis – Wilf 1979) "probabilistic proof"

recurrence Rels for f_λ. **Lemma:** $f_\lambda = \sum_{c \text{ is a corner of } \lambda} f_{\lambda \setminus c}$

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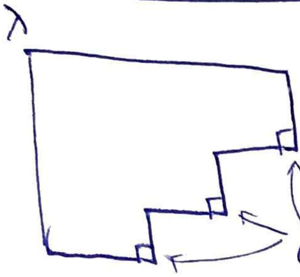


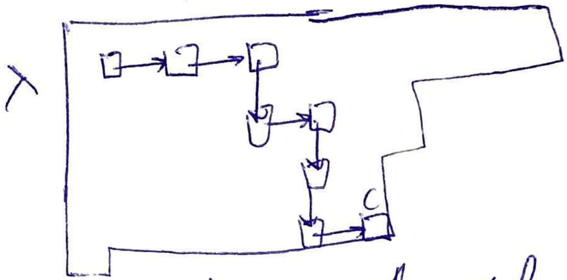
Ex: $f_{\square} = f_{\square} + f_{\square} = 2+3=5$

Need to prove that: $\dfrac{n!}{H(\lambda)} \overset{?}{=} \sum\limits_{c \text{ corner of } \lambda} \dfrac{(n-1)!}{H(\lambda \setminus c)} \iff 1 = \sum\limits_{\text{corner of } \lambda} \dfrac{H(\lambda)}{H(\lambda\setminus c)} \cdot \dfrac{1}{n}$

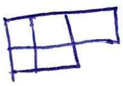
Can we interpret this as probability

Hook walk

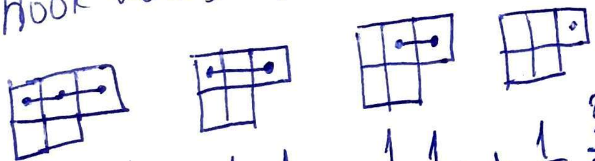
Fix $\lambda \vdash n$. Do the following steps



1. Randomly pick a box of λ
2. Stop if it is a corner
3. Jump to any other box of ~~[illegible]~~ hook of a with equal probability of $\frac{1}{h(a)-1}$.

Ex: $\lambda = (3,2) =$ 

Hook walks to c_1:



$\frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{3} + \frac{1}{5}\frac{1}{2} + \frac{1}{5} \overset{?}{=} \frac{4}{5}$

$c_2 = \frac{3}{5}$