18.212 lec 1.1 Algebraic Combinatorics

web: math.mit.edu/~apost/courses/18.212/

Catalan Numbers C_n, $n \geq 0$

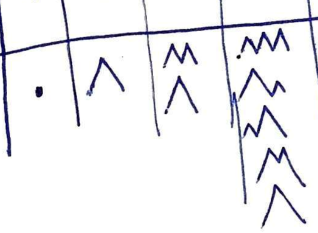
Def: $C_n = \#$ sequences $(\varepsilon_1, \dots, \varepsilon_{2n})$
- $(\varepsilon_i = \pm 1)$
- $\varepsilon_1 + \dots + \varepsilon_i \geq 0 \ \forall i$
- $\varepsilon_1 + \dots + \varepsilon_{2n} = 0$

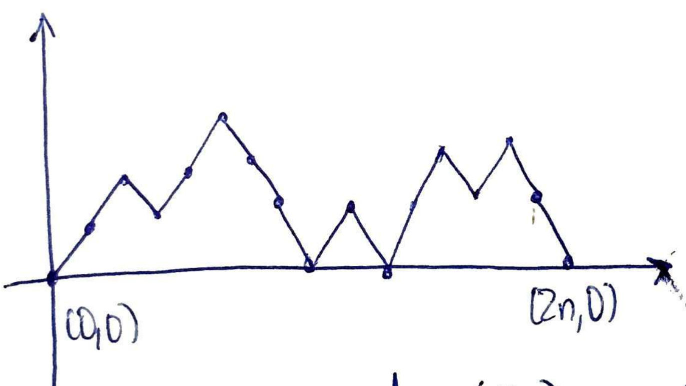
Dyck paths

Ex: $n=8$

$1, 1, -1, 1, 1, -1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1)$

n	0	1	2	3	4	~~4~~ 5
C_n	1	1	2	5	14	42





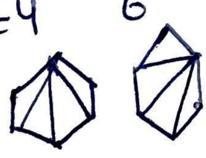
Ex: $C_3 = \frac{1}{4}\binom{6}{3} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}}{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 1\cdot 2\cdot 3}} = 5$

Thm 1: $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

Triangulations

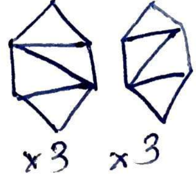
Thm 2: $C_n = \#$ triangulations on a convex $(n+2)$-gon

Ex $n=4$ 6



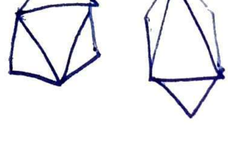
... + rotations

6



$\times 3$ $\times 3$

+ rotations



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Paranthesizations

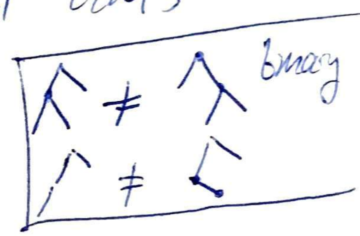
Thm 3 $C_n = \#$ paranthesization of $n+1$ letters

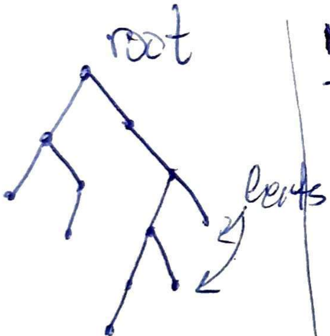
Ex $n=5$ (((((a b) c) d) e) f) ((a((b c)(d e)))f)

Binary Trees

Thm 4: $C_n = \#$ plane binary trees with n vertices

Thm 4,5: $C_n = \#$ Complete plane binary trees with $n+1$ leafs



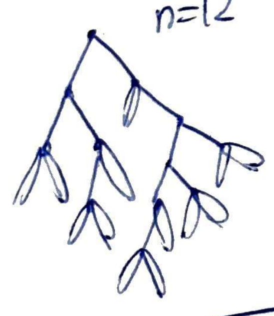


Plane binary trees

$n=3$

Def: A plane tree is complete if every non-leaf vertex has exactly 2 children

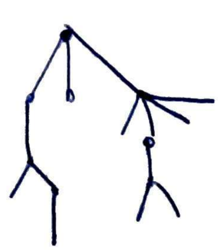
Ex: $n=12$

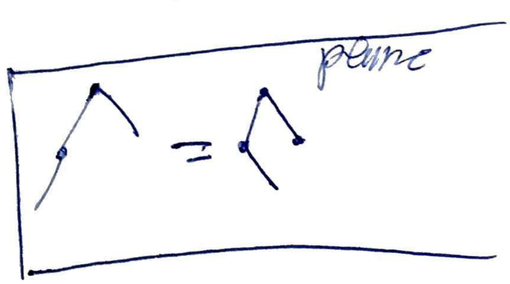


Plane Trees

Thm 5: $C_n = \#$ plane trees with $n+1$ vertices

Ex: $n=14$





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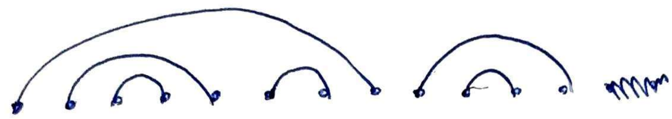
Thm $C_n = \#$ non-crossing perfect matchings on $2n$ vertices

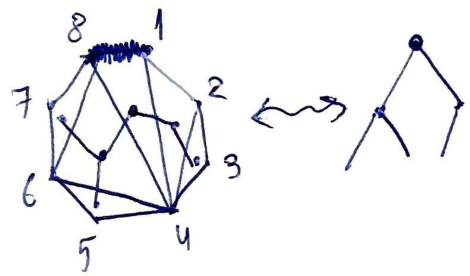
Ex $n=6$



Some bijections

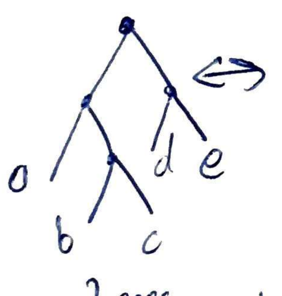
$\left\{\begin{array}{l}\text{triangulations} \\ \text{on } (n+2)\text{ gon}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{plane binary} \\ \text{trees with} \\ n \text{ vertices}\end{array}\right\}$

Ex $n=6$



$\left\{\begin{array}{l}\text{complete plane} \\ \text{binary trees with} \\ n+1 \text{ leaves}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{paranthesization} \\ \text{of } n+1 \text{ letters}\end{array}\right\}$

$n=4$



$\Longleftrightarrow \big(\big(a(bc)\big)(de)\big)$

$\left\{\begin{array}{l}\text{plane trees} \\ \text{with } n+1 \text{ vertices}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Dyck path} \\ \text{w/ } 2n \text{ steps}\end{array}\right\}$ do DFS from root