THE UNIVERSITY OF CHICAGO

ON THE GLOBAL BEHAVIOR OF WAVE MAPS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2013

To Elysa

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ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Professor Wilhelm Schlag. Each time I try to thank Professor Schlag for anything, big or small, he brushes it off saying that he is just doing his job. But "just doing his job" cannot possibly cover the countless hours that he has spent teaching and guiding me, the way that he has motivated me without making me feel pressured, or the fact that he never made me feel dumb even when I said something dumb, but rather explained things to me with patience and humility. Every time I walked into his office – even the times when I was completely miserable about the progress of my work – I left feeling more confident and with hope that things would work out as long as I kept working. Professor Schlag did not simply give me problems to work on and tell me to which papers or books to read. He created a path for me to follow for the rest of my career. He taught me to be a mathematician. I am honored to be his student.

Next, I would like to thank Professor Carlos Kenig, who has also been a wonderful mentor to me. I have learned so much from his classes, and recently I had the great opportunity of working with him on three successful research projects that make up a large part of this thesis. It was a pleasure working with him and I appreciate the time and patience that he invested in our collaboration.

I am also extremely grateful to Raphaël Côte, with whom I had the privilege of working with and learning from during his year-long visit to Chicago. The fruits of our collaboration make up the final two chapters of this thesis.

I would never have made it through my time here at Chicago, in particular the first-year, without my close friends and classmates, Jared Bass, Strom Borman, Rita Jimenez-Rolland, Mike Miller, Mohammad Rezaei, Daniele Rosso, and Brent Werness.

To my family – my wife Elysa, my Mom and Dad, my brother Willie, and our companion Adeline – I don't really know what to say, since nothing could come close to touching what you guys mean to me, and it seems silly for me to try to address this here. I should actually

just apologize for being so annoying and for making you deal with all of the worrying that I go through. Thank you for putting up with me, and for *everything*.

Elysa, my moon and my stars, this is for you…

ABSTRACT

We study wave maps equation in three distinct settings.

First, we prove a small data result for wave maps on a curved background. To be specific, we consider the Cauchy problem for wave maps $u : \mathbb{R} \times M \to N$, for Riemannian manifolds (M, g) and (N, h). We prove global existence and uniqueness for initial data, (u_0, u_1), that is small in the critical norm $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(M; TN)$, in the case $(M, g) = (\mathbb{R}^4, g)$, where g is a small perturbation of the Euclidean metric. This part of the thesis has appeared in print in [52].

Next, we establish relaxation of an arbitrary 1-equivariant wave map from $\mathbb{R}^{1+3}_{t,x} \setminus (\mathbb{R} \times B(0,1)) \to \mathbb{S}^3$ of finite energy and with a Dirichlet condition at $r = 1$, to the unique stationary harmonic map in its degree class. This settles a recent conjecture of Bizoń, Chmaj, and Maliborski who observed this asymptotic behavior numerically, and can be viewed as a verification of the *soliton resolution conjecture* for this particular model. The chapters concerning these results are based on joint work with Wilhelm Schlag [53], and with Carlos Kenig and W. Schlag, [35].

In the final two chapters, we consider 1-equivariant wave maps from $\mathbb{R}^{1+2} \to \mathbb{S}^2$. For wave maps of topological degree zero we prove global existence and scattering for energies below twice the energy of harmonic map, Q, given by stereographic projection. This gives a proof in the equivariant case of a refined version of the *threshold conjecture* adapted to the degree zero theory where the true threshold is $2\mathcal{E}(Q)$, not $\mathcal{E}(Q)$. The aforementioned global existence and scattering statement can also be deduced by considering the work of Sterbenz and Tataru in the equivariant setting.

For wave maps of topological degree one, we establish a classification of solutions blowing up in finite time with energies less than three times the energy of Q. Under this restriction on the energy, we show that a blow-up solution of degree one decouples as it approaches the blow-up times into the sum of a rescaled Q plus a remainder term of topological degree zero

of energy less than twice the energy of Q. This result reveals the universal character of the known blow-up constructions for degree one, 1-equivariant wave maps of Krieger, Schlag, and Tataru as well as Raphaël and Rodnianski.

Lastly, we establish a classification of all degree one global solutions whose energies are less than three times the energy of the harmonic map Q. In particular, for each global energy solution of topological degree one, we show that the solution asymptotically decouples into a rescaled harmonic map plus a radiation term. Together with the degree one finite time blow-up result, this gives a characterization of all 1-equivariant, degree one wave maps in the energy regime $[E(Q), 3E(Q))$. The last two chapters are based on joint work with Raphaël Côte, C. Kenig, and W. Schlag, [15, 16].

CHAPTER 1

INTRODUCTION

1.1 The Wave Maps Equation

This thesis consists of various results on the wave maps equation. In physics, wave maps arise as a model in both particle physics as what are called nonlinear σ–models, see [27], [56], and in general relativity, see [11]. From a purely mathematical perspective, wave maps are the natural hyperbolic analogs of *harmonic maps* in the elliptic case, and *harmonic map heat flow* in the parabolic case.

We begin with a definition. Let (M, g) be a Riemannian manifold of dimension d. Denote by $(\tilde{M}, \eta)$ the Lorentzian manifold $\tilde{M} = \mathbb{R} \times M$, with the metric η represented in local coordinates by $\eta = (\eta_{\alpha\beta}) = \mathrm{diag}(-1, g_{ij})$. Let (N, h) be a complete Riemannian manifold without boundary of dimension n.

A map $u : (\tilde{M}, \eta) \longrightarrow (N, h)$ is called a wave map if it is, formally, a critical point of the functional

$$\mathcal{L}(u) = \frac{1}{2} \int_{\tilde{M}} \langle du, du \rangle_{T^*\tilde{M} \otimes u^*TN} \, \mathrm{dvol}_\eta.$$

Here we view the differential, du, of the map u as a section of the vector bundle $(T^*\tilde{M} \otimes u^*TN, \eta \otimes u^*h)$, where u^*TN is the pullback of TN by u and u^*h is the pullback metric. In local coordinates this becomes

$$\mathcal{L}(u) = \frac{1}{2} \int_{\tilde{M}} \eta^{\alpha\beta}(z) h_{ij}(u(z)) \partial_\alpha u^i(z) \partial_\beta u^j(z) \sqrt{|\eta|} \, dz.$$

The Euler-Lagrange equations for $\mathcal{L}$ are given by

$$\frac{1}{\sqrt{|\eta|}} D_\alpha \left(\sqrt{|\eta|}\eta^{\alpha\beta}\partial_\beta u \right) = 0, \tag{1.1.1}$$

where D is the pull-back covariant derivative on u^*TN. In local coordinates on N, writing $u = (u^1, \dots, u^n)$, we can rewrite (1.1.1) as

$$\Box_\eta u^k = -\eta^{\alpha\beta}\Gamma^k_{ij}(u)\partial_\alpha u^i \partial_\beta u^j, \tag{1.1.2}$$

where $\Box_\eta u := -\partial_{tt} u + \Delta_g u$, and

$$\Delta_g u := \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta u)$$

is the Laplace-Beltami operator on M. $\Gamma^k_{ij} = \frac{1}{2}h^{k\ell}(\partial_i h_{\ell j} + \partial_j h_{i\ell} - \partial_\ell h_{ij})$ are the Christoffel symbols associated to the metric connection on N. We will often study the Cauchy problem for wave maps in local coordinates. That is, given smooth, finite energy initial data

$$\begin{aligned} &u_0 : M \to N, \\ &u_1 : M \to u_0^* TN, \text{ such that } \forall x,\ u_1(x) \in T_{u_0(x)}N. \end{aligned} \tag{1.1.3}$$

a solution to the Cauchy problem is a smooth map $u(t)$ satisfying (1.1.2) with

$$\vec{u}(0) := (u(0), \partial_t u(0)) = (u_0, u_1).$$

We remark that we often use the notation $\vec{u}(t)$ to denote the pair $\vec{u}(t) = (u(t), \partial_t u(t))$.

Wave maps can also be defined extrinsically. This approach is equivalent to the intrinsic approach, see for example [68, Chapter 1]. By the Nash-Moser embedding theorem there exists $m \in \mathbb{N}$ large enough so that we can isometrically embed $(N, h) \hookrightarrow (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$, where

$\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. We can thus consider maps $u : (\tilde{M}, \eta) \to (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ such that $u(t,x) \in N$ for every $(t,x) \in \tilde{M}$. Wave maps can then be defined formally as critical points of the functional

$$\mathcal{L}(u) = \frac{1}{2} \int_{\tilde{M}} \eta^{\alpha\beta} \left\langle \partial_\alpha u, \partial_\beta u \right\rangle \sqrt{|\eta|}\, dz.$$

One can show that u is a wave map if and only if

$$\Box_\eta u \perp T_u N. \tag{1.1.4}$$

From this we can deduce that u satisfies

$$\Box_\eta u = -\eta^{\alpha\beta} S(u)(\partial_\alpha u, \partial_\beta u), \tag{1.1.5}$$

where S is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^m$. For the Cauchy problem in the extrinsic formulation, we consider initial data

$$(u_0, u_1) : (M, g) \to TN,$$

by which we mean

$$\begin{aligned} &u_0(x) \in N \hookrightarrow \mathbb{R}^m, \\ &u_1(x) \in T_{u_0(x)} N \hookrightarrow \mathbb{R}^m \; \forall \; x \in M. \end{aligned}$$

One can formally establish energy conservation from the extrinsic definition (1.1.4). Define the energy

$$\mathcal{E}(u, \partial_t u)(t) := \frac{1}{2} \int_M \left(|\partial_t u|^2 + |d_M u|^2 \right) \sqrt{|g|}\, dx, \tag{1.1.6}$$

where by $d_M u$ we mean the differential of the map $u(t) : M \to \mathbb{R}^m$. Observe that $\Box_\eta u \perp T_u N$ implies that $\left\langle \Box_\eta u, \partial_t u \right\rangle = 0$. Hence we have

$$\begin{aligned}
0 &= -\int_M \left\langle \Box_\eta u, \partial_t u \right\rangle_{u(x)} \sqrt{|g|}\, dx \\
&= \int_M \langle \partial_t \partial_t u, \partial_t u \rangle_{u(x)} \sqrt{|g|}\, dx - \int_M \left\langle \partial_\alpha (\sqrt{|g|} g^{\alpha\beta} \partial_\beta u), \partial_t u \right\rangle_{u(x)} dx \\
&= \frac{1}{2} \int_M \frac{d}{dt} |\partial_t u|^2 \sqrt{|g|}\, dx + \int_M \left\langle g^{\alpha\beta} \partial_\beta u, \partial_\alpha \partial_t u \right\rangle_{u(x)} \sqrt{|g|}\, dx \\
&= \frac{d}{dt} \left(\frac{1}{2} \int_M \left(|\partial_t u|^2 + |d_M u|^2 \right) \sqrt{|g|}\, dx \right).
\end{aligned}$$

Integrating in time then gives $\mathcal{E}(u, \partial_t u)(t) = \mathcal{E}(u, \partial_t u)(0)$ for any time t.

1.1.1 Wave Maps on a Flat Background and Criticality

The case of wave maps on a Euclidean background $(M, g) = (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ has received much attention in recent years. Here, η is the Minkowksi metric on $\mathbb{R}^{1+d}$ and the intrinsic formulation (1.1.1) simplifies to

$$\eta^{\alpha\beta} D_\alpha \partial_\beta u = 0 \tag{1.1.7}$$

For convenience we rewrite the Cauchy problem,

$$\begin{aligned}
\Box u^k &= -\eta^{\alpha\beta} \Gamma^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j, \\
\vec{u}(0) &= (u_0, u_1)
\end{aligned} \tag{1.1.8}$$

with the conserved energy given by

$$\mathcal{E}(\vec{u})(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\partial_t u|_h^2 + |\nabla u|_h^2 \; dx = \text{constant}.$$

In this setup, we note that wave maps are invariant under the scaling

$$\vec{u}(t,x) \mapsto \vec{u}_\lambda(t,x) = \left(u\left(\lambda t, \lambda x\right), \lambda \partial_t u\left(\lambda t, \lambda x\right)\right) \quad \text{for} \quad \lambda > 0$$

On the other hand, we have

$$\mathcal{E}(\vec{u}_\lambda) = \lambda^{2-d}\mathcal{E}(\vec{u})$$

In light of the above, the Cauchy problem is called energy critical when $d = 2$, since the energy is unaffected by the rescaling of the solution. When we have $d > 2$, wave maps are referred to as energy supercritical. Here it is energetically favorable for the solution to shrink to a point and hence finite time blow-up is expected. The lone energy subcritical case is $d = 1$.

The scaling critical norm in d spacial dimensions is $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ as we have

$$\|\vec{u}_\lambda(0)\|_{\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)} = \|\vec{u}(0)\|_{\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)}.$$

Thus, the Cauchy problem for (1.1.8) is called critical in $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ and called energy-critical when $d = 2$.

1.1.2 Equivariant Wave Maps

In the presence of symmetries, such as when the target manifold (N, h) is rotationally symmetric, one often singles out a special class wave maps called equivariant wave maps. For example, for the sphere $N = S^d$ one requires that $u \circ \rho = \rho^\ell \circ u$ where ℓ is a positive integer and $\rho \in SO(d)$ acts on both the domain $\mathbb{R}^d$ and target S^d by rotation, in the latter case about a fixed axis. Equivariant wave maps have been extensively studied, see for example Shatah [67], Christodoulou, Tahvildar-Zadeh [13, 12], Shatah, Tahvildar-Zadeh [70, 71]. For a summary of these developments, see the book by Shatah and Struwe [68].

In the latter chapters of this thesis we will focus on the case when the target manifold is the d-sphere, $(\mathbb{S}^d, h)$ where h is the round metric on $\mathbb{S}^d$, $d = 2, 3$, and equivariance class $\ell = 1$. However, several of the results hold in more general settings as will be explained later. To illustrate how an equivariance assumption leads to a simplification of the Cauchy problem, we outline the $2d$ case below. In spherical coordinates,

$$(\psi, \omega) \mapsto (\sin\psi\cos\omega, \sin\psi\sin\omega, \cos\psi),$$

on $\mathbb{S}^2$, the metric g is given by the matrix $g = \mathrm{diag}(1, \sin^2(\psi))$. In the 1-equivariant setting, we thus require our wave map, u, to have the form

$$u(t, r, \omega) = (\psi(t, r), \omega) \mapsto (\sin\psi(t, r)\cos\omega,\ \sin\psi(t, r)\sin\omega,\ \cos\psi(t, r)),$$

where (r, ω) are polar coordinates on $\mathbb{R}^2$. In this case, the Cauchy problem (1.1.8) reduces to an equation for the azimuth angle ψ, namely,

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1). \end{aligned} \tag{1.1.9}$$

The conservation of energy reads

$$\mathcal{E}(\vec{u})(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\,dr = \mathrm{const.} \tag{1.1.10}$$

Any $\psi(r, t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(t, 0) = m\pi$ and $\psi(t, \infty) = n\pi$ for all $t \in I$, where m, n are fixed integers. This requirement splits the energy space into disjoint classes according to this topological condition. The wave map evolution preserves these classes.

1.1.3 History

Wave maps have been studied extensively over the past few decades and we give a brief and noninclusive overview of some of the significant developments here. In the subsequent chapters we will review the relevant history in more detail.

In the energy super-critical case, $d \geq 3$, Shatah [67] showed that self-similar blow-up can occur for solutions of finite energy. In the energy critical case, $d = 2$, there is no self similar blow-up as demonstrated by Shatah and Struwe [68]. In the equivariant, energy critical setting, Struwe [76] proved that if blow-up does occur then the solution must converge, after rescaling, to a non-constant, co-rotational harmonic map. Recently, Krieger, Schlag, and Tataru [50], Rodnianski-Sterbenz [63] and Raphael Rodnianski [62] have constructed finite energy wave maps $u : \mathbb{R}^{1+2} \to \mathbb{S}^2$ that blow up in finite time.

The well-posedness theory for critical, spherically symmetric wave maps was developed by Christodoulou and Tahvildar-Zadeh [13, 12], and in the equivariant setting by Shatah and Tahvildar-Zadeh [70, 71]. In the non-equivariant case, Klainerman and Machedon [39, 40, 41, 42], and Klainerman and Selberg [44, 45], established well-posedness in the subcritical norm $H^s \times H^{s-1}(\mathbb{R}^d)$ with $s > \frac{d}{2}$ by exploiting the null-form structure present in (1.1.8).

The first breakthrough in the non-equivariant critical theory, $s = \frac{d}{2}$, was accomplished by Tataru [84, 81], where he proved global well-posedness for smooth data that is small in the scaling critical Besov space $\dot{B}^{\frac{d}{2}}_{2,1} \times \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ for $d \geq 2$. Then, in the ground-breaking works, [77, 78], Tao proved global well-posedness for wave maps $u : \mathbb{R}^{1+d} \to \mathbb{S}^k$ for smooth data that is small in the critical Sobolev norm $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ for $d \geq 2$. Later, this result was extended to more general targets by Klainerman and Rodnianski [43], Krieger [46], [47], [48], Nahmod, Stefanov and Uhlenbeck [58], Shatah and Struwe [69], and by Tataru [82], [83].

Finally, the difficult large data, energy critical case has been undertaken in a remarkable series of papers by Krieger and Schlag [49], Sterbenz and Tataru [74], [75], and Tao [79]. In particular, these papers show that all smooth finite energy data leads to a unique global and smooth evolution which scatters to zero in the energy space when the target manifold does not admit finite energy harmonic maps. Moreover, [74, 75] establish the so-called *threshold conjecture*, which states that all wave maps with energy below that of a minimal energy harmonic map are global in time and scatter. These results will be explained in more detail in Chapters 5 and Chapter 6 where a refined version of the *threshold conjecture* is considered in the case of equivariant energy critical wave maps to $\mathbb{S}^2$.

1.1.4 Outline of the Thesis

In this thesis we study the Cauchy problem for wave maps in three distinct settings. The main results have all appeared in research articles, several of which have been written in collaboration with Raphaël Côte, Carlos Kenig, and Wilhelm Schlag in different combinations.

In Chapter 2, we examine wave maps on curved backgrounds. In this case the left-hand-side of (1.1.2) involves variable coefficients and this makes the problem more challenging. Indeed, many of the main tools used in the study of dispersive equations, in particular Strichartz estimates, have their roots in sophisticated techniques from harmonic analysis and, unfortunately, these tools do not extend easily to the case of variable coefficient equations. Recently, using phase-space transformations, Metcalfe and Tataru [57] established Strichartz estimates for free waves on curved backgrounds in the case that the domain is a small perturbation of Minkowski space. After deducing a slight refinement of these estimates, we prove small data global well-posedness and scattering in the critical $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$ norm for wave maps on curved backgrounds in dimension $d = 4$. The domains considered are small perturbations of Euclidean space and our results hold for a general class of targets, namely those with bounded geometry. In particular, we prove the following theorem:

Theorem 1.1.1. *[52] Let (N, h) be a smooth, complete, n-dimensional Riemannian manifold without boundary and with bounded geometry. Let $(M, g) = (\mathbb{R}^4, g)$ with g satisfying* *the asymptotic flatness and smallness conditions to be made precise in* (2.1.1)-(2.1.4)*. Let* $(\tilde{M}, \eta) = (\mathbb{R} \times M, \eta)$ *with* $\eta = \mathrm{diag}(-1, g)$*. Then there exists an* $\varepsilon_0 > 0$ *such that for every* $(u_0, u_1) \in H^2 \times H^1((M, g); TN)$ *with*

$$\|(u_0, u_1)\|_{\dot{H}^2 \times \dot{H}^1} < \varepsilon_0, \tag{1.1.11}$$

there exists a unique global wave map, $u : (\tilde{M}, \eta) \to (N, h)$*, with initial data* $\vec{u}(0) = (u_0, u_1)$*, such that* $\vec{u} \in C^0(\mathbb{R}; H^2(M; N)) \times C^0(\mathbb{R}; H^1(M; TN))$*. Moreover,* u *satisfies the global estimates*

$$\|du\|_{L_t^\infty \dot{H}_x^1} + \|du\|_{L_t^2 L_x^8} \lesssim \varepsilon_0. \tag{1.1.12}$$

In addition, any higher regularity of the data is preserved.

The proof proceeds via the outline provided by Shatah-Struwe [69] in the case of a flat background by working in the Coulomb frame. The approach in [69] constituted a significant simplification of the method used by Tao who established the first global well-posedness results in the critical norm in high dimensions in [77]. The cases of lower dimensions, $d = 2, 3$ are significantly more difficult as the structure, as opposed to only just the size, of the nonlinearity plays a crucial role in the analysis, see [78]. The content of Chapter 2 has appeared in [52].

In the remaining chapters, we restrict our attention to equivariant wave maps. The equations in the equivariant setting are greatly simplified and provide an ideal environment in which to attempt to understand possible *large data* dynamics in the presence of stationary solutions, which are called harmonic maps.

Over the last several years, Kenig and Merle have employed their innovative version of Bourgain's *induction on energy principle*, [6], to obtain global existence and scattering results for both focusing and defocusing semi-linear wave equations, with additional conditions needed in the focusing case as finite time blow-up can occur, see [36], [37]. A fundamental part of their strategy, which has come to be known as the Kenig-Merle method, involves the use of the concentration compactness techniques and in particular the profile decomposition of Bahouri and Gérard, [1]. The concentration compactness procedure, which is rooted in the underlying symmetries of the problem, has turned out to be extremely versatile, and has been a key ingredient in the recent classification results of large data dynamics for semi-linear waves, which includes identifying blow-up mechanisms and describing the long time behavior of global solutions such as the resolution of solutions into multi-bumps plus radiation as predicted by what is loosely referred to as the *soliton resolution conjecture*; see Duyckaerts, Kenig, Merle [22, 21, 24, 23].

Concentration compactness techniques have recently been applied to the wave maps equation as well in the groundbreaking work of Krieger, Schlag [49] on the large data scattering theory for non-equivariant energy critical wave maps to $\mathbb{H}^2$.

In Chapters 3 and 4, we use concentration compactness techniques to investigate 1-equivariant wave maps from $1+3$–dimensional Minkowski space *exterior* to a ball and with $\mathbb{S}^3$ as target. To be specific, we consider wave maps $u : \mathbb{R} \times (\mathbb{R}^3 \setminus B) \to \mathbb{S}^3$ with a Dirichlet condition on ∂B, i.e., $u(\partial B) = \{N\}$ where N is the north pole. In the 1-equivariant formulation of this equation, where ψ is the azimuth angle measured from the north pole, the Cauchy problem reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \quad \forall r \geq 1, \\ &\psi(t,1) = 0 \quad \forall\, t \geq 0, \quad (\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1). \end{aligned} \tag{1.1.13}$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(t,\infty) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed, giving rise to a notion of degree. Removing the unit ball gives rise to several striking features, namely, (1) the absence of scaling symmetry renders the formerly energy-supercritical equation subcritical relative to the energy, and (2) it admits infinitely many stationary solutions $(Q_n, 0)$, which are harmonic maps indexed by their topological degree.

This exterior equation (3.1.2) was proposed by Bizon, Chmaj and Maliborski [5] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps, or the *soliton resolution conjecture.* In the physics literature, this model was introduced in [2] as an easier alternative to the Skyrmion equation. Moreover, [2] stresses the analogy with the damped pendulum which plays an important role in our analysis. Numerical simulations described in [5] indicate that in each equivariance class ℓ, and for each topological degree n, *every solution* scatters to the unique harmonic map that lies in this class. In [53], together with Schlag we verified this conjecture for the topologically trivial solutions, i.e., degree $n = 0$. These solutions start at the north-pole and eventually return there. For $n \geq 1$ we obtained a perturbative result in [53] by proving Strichartz estimates for the linearized operator around Q_n. Later, together with Kenig and Schlag, [35], we established the full conjecture for all degrees $n \geq 0$. This result can be thought of as a verification of the *stable* soliton resolution conjecture for this particular equation.

Theorem 1.1.2. *[53] [35] Let (ψ_0, ψ_1) be smooth finite energy data of degree $n \geq 0$. Then there exists a unique and global smooth solution $\vec{\psi}(t)$ to (1.1.13) of degree n with $\vec{\psi}(0) = (\psi_0, \psi_1)$. Moreover, $\vec{\psi}(t)$ scatters to $(Q_n, 0)$ as $t \to \infty$.*

The above theorem is proved using the Kenig-Merle concentration compactness/rigidity method, with the novel aspect of our implementation being the techniques we used in the rigidity argument. The Kenig-Merle framework can be compartmentalized into three independent steps. First, one establishes the theorem for initial data that is close to the ground state harmonic maps in the energy space via a perturbative method based on Strichartz estimates. For the second step, referred to as the concentration compactness argument, one assumes that the theorem fails and then uses concentration compactness type arguments based on Bahouri-Gerard type profile decompositions to construct a minimal, non-scattering solution called the critical element. The key point here is that one can show that the critical element has a pre-compact trajectory in the energy space. The final step, referred to as the rigidity argument, involves showing that the critical element cannot possibly exist. For this part of the argument, we give two completely independent proofs, one that works only in the degree zero case, and a second that holds for *all degrees*.

The first rigidity proof, which holds only in the degree 0 case, is based on virial identities, which arise from contracting the stress energy tensor with appropriate vector fields. This approach relies heavily on the precise structure of the nonlinearity and hence is extremely equation-specific. Indeed, in order to prove the degree 0 case *without any upper bound on the energy* we demonstrate that the natural virial functional is globally coercive on $\mathcal{H}$. This requires a detailed variational argument, the most delicate part of which consists of a phase-space analysis of the Euler-Lagrange equation which uses classical ODE techniques.

The second argument, which holds for all degrees is based on what has come to be known as the *channels of energy* method and has its roots in the work of Duyckaerts, Kenig, and Merle on the quintic, semi-linear wave equation as well as $3d$ super-critical semi-linear waves; see [23, 25]. As opposed to the virial approach, this argument is robust with respect to the nonlinearity at the level of the nonlinear wave under consideration, and relies instead on the underlying elliptic theory and a new class of estimates, called *exterior energy estimates* for the underlying radial *free wave*.

The idea is to provide an asymptotic lower bound on the energy of free waves exterior to the light cone with base $R > 0$ in terms of the free energy of the data outside the ball of radius R. This estimate fails as stated due to the fact that data $(f, 0)$ and $(0, g)$ with $f(r) = g(r) = r^{2-d}$ – note that this is the Newton potential – have corresponding solutions that have vanishing exterior energy for any $R > 0$. Therefore one must project away from the plane formed by this data in the energy space to establish a lower bound.

To use these estimates in the nonlinear setting, one then notes that the compactness property of the critical element implies that it has vanishing exterior energy for all cones with base $R > 0$. Choosing R large enough, the data outside a ball of radius R is small and the nonlinear and linear evolutions remain close up to a lower order term coming from the Duhamel integral. One can then use the exterior estimates for the linear evolution to show that the nonlinear evolution must, in fact, be an elliptic solution that fails to have a crucial property – here we show that the Dirichlet boundary condition is violated.

Finally, in Chapters 5 and 6, we study *energy critical* equivariant wave maps to positively curved targets, in particular to $\mathbb{S}^2$. As is the case with more general dispersive equations, the asymptotic behavior of energy critical wave maps is of particular interest. Here one can distinguish between the small data and large data theory. Energy critical wave maps with initial data that have small energy exhibit relatively simple global dynamics as the waves become asymptotically free under fairly generic assumptions on the target, a phenomena referred to as *scattering*. This was established in the non-equivariant case in the landmark work of Tao, [78] when the target is $\mathbb{S}^2$, and later extended to $\mathbb{H}^2$ by Krieger [48], and then to wide class of targets by Tataru, [83].

On the other hand the dynamical structure for large energy critical wave maps is very rich, with the geometry of the target playing a decisive role. Negatively curved targets lead to defocusing type behavior and in a remarkable series of papers global existence and scattering *for all smooth data* was established in the non-equivariant case by Krieger and Schlag, [49], Sterbenz, Tataru [74], [75], and Tao [79].

In the work of Sterbenz and Tataru, [75], the possibility of blow-up was linked to the existence of nontrivial finite energy stationary solutions, namely harmonic maps, a result that was previously seen in the simpler equivariant setting by Struwe [76]. For positively curved targets that do admit harmonic maps, they proved what is referred to as the *threshold conjecture*, which states that for all smooth data with energy below the energy of a minimal energy nontrivial harmonic map, Q, the corresponding evolution is global in time and scatters. Many questions remain in the case of positively curved targets including understanding the dynamics when one looks above the threshold, and the results in these chapters are in this direction in the case of the simpler equivariant model, where explicit blow-up solutions have been constructed in the important works of Krieger, Schlag, Tataru [50], Raphael, Rodnianski [62] and Rodnianski, Sterbenz [63].

Here, together with R. Côte, C. Kenig, and W. Schlag, [15], we give a new proof of a refined version of the threshold conjecture in the equivariant setting for wave maps $\mathbb{R}^{1+2} \to \mathbb{S}^2$, based on the concentration compactness/rigidity method of Kenig and Merle. Then, we provide a classification of the possible dynamics for all degree 1 wave maps with energy less that 3 times the energy of the unique (up to scaling) harmonic map to the sphere, $Q(r) = 2\arctan(r)$, a truly large data result; see [15, 16] for the submitted versions of these results.

We denote by $\mathcal{H}_n$ the space of finite energy data of degree n and we note that the unique harmonic map $(Q, 0) \in \mathcal{H}_1$ has *minimal* energy amongst degree 1 maps, with $\mathcal{E}(Q, 0) = 4$. The following theorem summarizes the main results in Chapters 5 and Chapter 6.

Theorem 1.1.3. *[15, 16] Let $\vec{\psi}(0) := (\psi_0, \psi_1)$ be smooth, finite energy data.*

1. ***Degree** 0**–threshold:** Let $\mathcal{E}(\psi_0, \psi_1) < 2\mathcal{E}(Q, 0)$, $\vec{\psi}(0) \in \mathcal{H}_0$. Then the solution exists globally, and scatters (energy on compact sets vanishes as $t \to \infty$). For any $\delta > 0$ there exist data of energy $< 2\mathcal{E}(Q, 0) + \delta$ which blow up in finite time.*

2. ***Degree** 1**, finite time blowup:** Let $\mathcal{E}(\psi_0, \psi_1) < 3\mathcal{E}(Q, 0)$, $\vec{\psi}(0) \in \mathcal{H}_1$. If the solution $\psi(t)$ blows up at, say, $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q, 0)$, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + o_{\mathcal{H}_0}(1) \qquad as \;\; t \to 1$$

3. ***Degree* 1*, global solutions:*** *Let* $\mathcal{E}(\psi_0, \psi_1) < 3\mathcal{E}(Q, 0)$*,* $\vec{\psi}(0) \in \mathcal{H}_1$*. If the solution* $\vec{\psi}(t) \in \mathcal{H}_1$ *exists globally in time then there exists a continuous function,* $\lambda : [0, \infty) \to (0, \infty)$ *with* $\lambda(t) = o(t)$ *as* $t \to \infty$*, a solution* $\vec{\varphi}_L(t) \in \mathcal{H}_0$ *to the linearization of* (1.1.9) *about* $\vec{0}$*, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q(\cdot/\lambda(t)), 0) + o_{\mathcal{H}_0}(1) \quad \textit{as} \quad t \to \infty$$

The degree 0 result follows from the Kenig–Merle method, [36], [37], the novel part of our implementation being the development of a robust rigidity theory for wave maps with pre-compact (up to symmetries) trajectories in the energy space. We note that one can also deduce the degree 0 theorem by considering the work of Sterbenz, Tataru [75], in the equivariant setting.

The techniques developed by Duyckaerts Kenig and Merle in [22], [24] motivated the proofs of the degree 1 results, as we used certain elements of their ideology, in particular concentration compactness techniques. We also relied explicitly on several classical results in the field of equivariant wave maps. In particular, crucial roles are played by the vanishing of the kinetic energy proved by Shatah, Tahvildar-Zadeh [70], and Struwe's bubbling off theorem, [76], in our finite time blow-up result. Another key ingredient is the new class of exterior energy estimates for the underlying even-dimensional linear wave equation proved in [18].

1.1.5 Notation

In what follows we will adopt the convention that $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le Cg$. Similarly, $f \simeq g$ will mean that there exist constants $c, C > 0$ such that $cg \le f \le Cg$. We also warn the reader that some notation may change meaning between chapters. Each chapter is meant to be read on its own and the relevant notation for each chapter will be defined within that very chapter.

CHAPTER 2

WAVE MAPS ON A CURVED BACKGROUND

2.1 Introduction

In this chapter we prove global well-posedness for wave maps on curved backgrounds that are small perturbations of Euclidean space. In the case of a nonlinear dispersive equation, one expects that data that is small in the critical norm leads to a global and smooth evolution. For wave maps in dimensions $d \geq 4$ on flat backgrounds, this was established by Tao in the breakthrough work, [77]. Later the work of Shatah and Struwe in [69] gave a significant simplification of Tao's argument in dimensions $d \geq 4$, and it is on the methods utilized in [69], that this present work is based. In [69], Shatah and Struwe consider the Cauchy problem for wave maps $u : \mathbb{R}^{1+d} \to N$ with initial data $(u_0, u_1) \in H^{\frac{d}{2}} \times H^{\frac{d}{2}-1}(\mathbb{R}^d, TN)$ that is small in the critical norm $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, TN)$ for $d \geq 4$. The target manifold N is assumed to have bounded geometry. Their main result is a proof of the existence of a unique global solution, $(u, \dot{u}) \in C^0(\mathbb{R}; H^{\frac{d}{2}}) \times C^0(\mathbb{R}; H^{\frac{d}{2}-1})$. Existence is deduced by way of the following global a priori estimates for the differential, du, of the wave map:

$$\|du\|_{L_t^\infty \dot{H}_x^{\frac{d}{2}-1}} + \|du\|_{L_t^2 L_x^{2d}} \lesssim \|u_0\|_{\dot{H}^{\frac{d}{2}}} + \|u_1\|_{\dot{H}^{\frac{d}{2}-1}}$$

In order to prove the above estimates, Shatah and Struwe exploit the gauge invariance of the wave maps system and introduce the Coulomb frame. This allows one to derive a system of wave equations for du that is amenable to a Lorentz space version of the endpoint Strichartz estimates proved in [34]. The connection form, A, associated to the Coulomb frame on the vector bundle u^*TN appears in the nonlinearity of the wave equation for du, and estimates to control its size are crucial to the argument. The Coulomb gauge condition implies that A satisfies a certain elliptic equation, and it is this structure that enables the proof, for example, of the essential $L_t^1 L_x^\infty$ estimates for A, see [69, Proposition 4.1].

Here, we consider the Cauchy problem for wave maps $u : \mathbb{R} \times M \to N$, where the background manifold (M, g) is no longer Euclidean space. We follow the same basic argument as in [69] and derive a wave equation for the u^*TN-valued 1-form, du, using the Coulomb gauge as our choice of frame on u^*TN. As the geometry of (M, g) is no longer trivial, the resulting equation for du is, in its most natural setting, an equation of 1-forms. In coordinates on M, we can rewrite the equation for du in components, obtaining a system of variable coefficient nonlinear wave equations. This is the content of Section 2.4.

The main technical ingredients in [69] are elliptic-type estimates for the connection form A, and the endpoint Strichartz estimates for the wave equation used to control the $L_t^\infty \dot{H}_x^{\frac{d}{2}-1} \cap L_t^2 L_x^{2d}$ norm of du. In order to proceed as in [69], but now in the setting of a curved background manifold, we will need replacements for each of these items.

In what follows, we restrict our attention to the case that the background manifold (M, g) is $(\mathbb{R}^4, g)$, with g a small perturbation of the Euclidean metric, as in this case we have suitable replacements for the technical tools used in [69]. Here we view the equations for the components of connection form, A, as a system of variable coefficient elliptic equations and prove elliptic estimates via a perturbative argument; see Proposition 2.3.2. We employ several tools from the theory of Lorentz spaces to prove the crucial $L_t^1 L_x^\infty$ estimates for A.

In order to have suitable Strichartz estimates, we tailor our assumptions on the metric g so that the variable coefficient wave equations for du are of the type studied by Metcalfe and Tataru in [57]. We deduce a Lorentz refinement to the Strichartz estimates in [57], see Section 2.8 below, which we use to prove global a priori estimates for du in Section 2.5.

The global-in-time Strichartz estimates for variable coefficient wave equations in [57] that we use in the proof of the a priori estimates for du have emerged from Tataru's method of using phase space transforms and microlocal analysis to prove dispersive estimates for variable coefficient dispersive equations. In the case of the variable coefficient wave equation, the Bargmann transform is used to construct a parametrix that satisfies suitable dispersive estimates. Localized energy estimates are then used to control error terms when proving estimates for the variable coefficient operator. We refer the reader to [85, 86, 88, 87, 89, 90] and of course to [57], for more details and history. A very brief summary is included in Section 2.8.

Our main theorem is a global existence and uniqueness result for the Cauchy problem for wave maps in this setting, with data (u_0, u_1) that is small in the critical norm $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(M, TN)$. The precise statement of the result is Theorem 2.1.1 below.

2.1.1 Geometric Framework

We set $(M, g) = (\mathbb{R}^4, g)$ with g a small perturbation of the Euclidean metric on $\mathbb{R}^4$, satisfying the following assumptions: Let $\varepsilon > 0$ be a small constant, to be specified later. We will require

$$\|g - g_0\|_{L^\infty} \leq \varepsilon \tag{2.1.1}$$

$$\|\partial g\|_{L^{4,1}(\mathbb{R}^4)} \lesssim \varepsilon \tag{2.1.2}$$

$$\|\partial^2 g\|_{L^{2,1}(\mathbb{R}^4)} \lesssim \varepsilon \tag{2.1.3}$$

$$\|\partial^k g\|_{L^2(\mathbb{R}^4)} < \infty \quad \text{for } k \geq 3 \tag{2.1.4}$$

where $g_0 = \mathrm{diag}(1, 1, 1, 1)$ is the Euclidean metric on $\mathbb{R}^4$ and $L^{p,q}(\mathbb{R}^4)$ denotes the Lorentz space. Assumptions (2.1.1)–(2.1.3) are needed in order to prove the elliptic estimates for the connection form, A, associated to the Coulomb frame in Section 2.3.1. Note that these assumptions are consistent with, and are, in fact, stronger than the weak asymptotic flatness conditions specified in Metcalfe and Tataru [57], namely

$$\sum_{j \in \mathbb{Z}} \sup_{|x| \simeq 2^j} |x|^2 \left|\partial^2 g(x)\right| + |x| \, |\partial g(x)| + |g(x) - g_0| \leq \varepsilon \tag{2.1.5}$$

This will justify our application in Section 2.5 of the Strichartz estimates for variable coefficient wave equations deduced in [57].

The assumptions in (2.1.4) are needed in order to establish the high regularity local theory for wave maps. This theory will be used in the existence argument in Section 2.7.

We will also record a few comments regarding the assumptions on the target manifold (N, h). We will assume that (N, h) is a smooth complete Riemannian manifold, without boundary that is isometrically embedded into $\mathbb{R}^m$. Following [69], we also assume that N has bounded geometry in the sense that the curvature tensor, R, and the second fundamental form, S, of the embedding are bounded and all of their derivatives are bounded.

In the argument that follows, we will assume that either N admits a parallelizable structure or that N is compact, as we will require a global orthonormal frame for TN in our argument. Such a frame does not, of course, exist for a general compact manifold. However if N is compact, by an argument in [32], we can avoid this inconvenience by constructing a certain isometric embedding $J : N \hookrightarrow \tilde{N}$ where $\tilde{N}$ is diffeomorphic to the flat torus $\mathbb{T}^m$ and admits an orthonormal frame. This embedding J is constructed so that u is a wave map if and only if the composition $J \circ u$ is a wave map, see [32, Lemma 4.1.2]. This allows us to work with $J \circ u : \tilde{M} \to \tilde{N}$ instead of with u. Hence we can assume without loss of generality that the target manifold N admits a global orthonormal frame $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_n)$ for the tangent space TN.

2.1.2 Main Result

The initial data for the Cauchy problem, $(u, \dot{u})|_{t=0} = (u_0, u_1)$, can either be viewed intrinsically or extrinsically. In the extrinsic formulation, we will consider initial data

$$(u_0, u_1) \in (M, g) \to TN \tag{2.1.6}$$

by which we mean $u_0(x) \in N \hookrightarrow \mathbb{R}^m$ and $u_1(x) \in T_{u_0(x)}N \hookrightarrow \mathbb{R}^m$ for almost every $x \in M$. And we say that $(u_0, u_1) \in H^s_e \times H^{s-1}_e(M; TN)$ if $u_0 \in H^s(M; \mathbb{R}^m)$ and $u_1 \in H^{s-1}(M; \mathbb{R}^m)$. The homogeneous spaces $\dot{H}^s_e \times \dot{H}^{s-1}_e(M; TN)$ are defined similarly. For the definition of the spaces $H^s(M; \mathbb{R}^m)$ we refer the reader to Section 2.9.1, or to [31].

To view the data intrinsically, we will put to use the parallelizable structure on TN. Let our initial data be given by (u_0, u_1) where $u_0 : M \to N$ and $u_1 : M \to u_0^*TN$ with $u_1(x) \in T_{u_0(x)}N$. Observe that u_0^*TN inherits a parallelizable structure from TN, see Section 2.3, and let $e = (e_1, \dots, e_n)$ be an orthonormal frame for u^*TN. Since $du_0 : TM \to u^*TN$ we can find a u^*TN-valued 1-form $q_0 = q_0^a e_a$ such that $du_0 = q_0^a e_a$. Similarly we can find $q_1^a : M \to \mathbb{R}$ such that $u_1 = q_1^a e_a$. We then say that $(u_0, u_1) \in H^s_i \times H^{s-1}_i(M; TN)$ if $q_0^a \in H^{s-1}(TM; \mathbb{R})$ and $q_1^a \in H^{s-1}(M; \mathbb{R})$ for each $1 \leq a \leq n$. These norms are further discussed in Section 2.9.1. Again, the homogeneous versions $\dot{H}^s_i \times \dot{H}^{s-1}_i(M; TN)$ are defined similarly.

In Section 2.3.2, we show that if we choose the frame e to be the Coulomb frame, see Section 2.3, then the extrinsic and intrinsic approaches to defining the homogeneous Sobolev norms of our data (u_0, u_1) are equivalent. This will allow us to use both definitions interchangeably in the arguments that follow.

Also in the appendix, Section 2.9.1, we show that the "covariant" Sobolev spaces $\dot{H}^s(M; N)$, with $(M, g) = (\mathbb{R}^4, g)$ with the metric g as in (2.1.1) – (2.1.4) are equivalent to the "flat" spaces $\dot{H}^s((\mathbb{R}^4, g_0); N)$ with the Euclidean metric g_0 on $\mathbb{R}^4$. Hence in what follows we can, when convenient, ignore the non-Euclidean metric g for the purpose of estimating Sobolev norms, replacing covariant derivatives on M with partial derivatives and the volume form dvol_g with the Euclidean volume form.

We can now re-state the main result in this chapter; see Theorem 1.1.1.

Theorem 2.1.1. *Let (N, h) be a smooth, complete, n-dimensional Riemannian manifold without boundary and with bounded geometry. Let $(M, g) = (\mathbb{R}^4, g)$ with g as in* (2.1.1)–

(2.1.4) *and let* $(\tilde{M}, \eta) = (\mathbb{R} \times M, \eta)$ *with* $\eta = \mathrm{diag}(-1, g)$*. Then there exists an* $\varepsilon_0 > 0$ *such that for every* $(u_0, u_1) \in H^2 \times H^1((M, g); TN)$ *such that*

$$\|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} < \varepsilon_0 \tag{2.1.7}$$

there exists a unique global wave map, $u : (\tilde{M}, \eta) \to (N, h)$*, with initial data* $(u, \dot{u})|_{t=0} = (u_0, u_1)$*, such that* $(u, \dot{u}) \in C^0(\mathbb{R}; H^2(M; N)) \times C^0(\mathbb{R}; H^1(M; TN))$*. Moreover,* u *satisfies the global estimates*

$$\|du\|_{L_t^\infty \dot{H}_x^1} + \|du\|_{L_t^2 L_x^8} \lesssim \varepsilon_0. \tag{2.1.8}$$

In addition, any higher regularity of the data is preserved.

We will use a bootstrap argument to prove the global estimates (2.1.8). In what follows we will make the assumption that there exists a time T such that for a wave map u with data (u_0, u_1) as in (2.1.7), the estimates in (2.1.8) hold on the interval $[0, T)$. That is, we have

$$\|du\|_{L_t^\infty([0,T); \dot{H}_x^1)} + \|du\|_{L_t^2([0,T); L_x^8)} \lesssim \varepsilon_0 \tag{2.1.9}$$

We will use this assumption to prove the global-in-time estimates (2.1.8).

Remark 1. The local well-posedness theory for the high regularity Cauchy problem for (1.1.5) is standard. For example, with $(M, g) = (\mathbb{R}^4, g)$ for a smooth perturbation g as in (2.1.1)–(2.1.4), if we have data $(u_0, u_1) \in H^s \times H^{s-1}(M; TN)$ for say, $s > 4 = \frac{d}{2} + 2$, then the Cauchy problem for (1.1.5) is locally well-posed. This can be proved using H^s energy estimates and a contraction argument. The proof relies on the fact that $H^s(\mathbb{R}^d)$ is an algebra for $s > \frac{d}{2}$, and can be found for example in [68].

Remark 2. We have only addressed the case $d = 4$ case here because this is the only dimension where we have applicable Strichartz estimates. In dimension 3, the endpoint $L_t^2 L_x^\infty$ estimate is forbidden. In higher dimensions, $d \geq 4$, the initial data is assumed to be small in $\dot{H}^s \times \dot{H}^{s-1}$ with $s = \frac{d}{2}$, but the estimates in [57] only apply when lower order terms are present if we have $s = 2$ or $s = 1$, see [57, Corollary 5 and Theorem 6]. This leaves $d = 4$ as the only option, as here $\frac{d}{2} = 2$.

2.2 Uniqueness

We use the extrinsic formulation (1.1.5) of the wave maps system to prove uniqueness. The argument given for uniqueness in [69] adapts perfectly to our case and we reproduce it below for completeness.

Suppose that $(u, \dot{u})$ and $(v, \dot{v})$ are two solutions to (1.1.5) of class $H^2 \times H^1((\mathbb{R}^4, g); TN)$ such that

$$(u, \dot{u})|_{t=0} = (v, \dot{v})|_{t=0} \tag{2.2.1}$$

In addition, assume that

$$\|du\|_{L_t^2 L_x^8} < \infty, \quad \|dv\|_{L_t^2 L_x^8} < \infty \tag{2.2.2}$$

Set $w = u - v$. Then w satisfies

$$\Box_\eta w = -\eta^{\alpha\beta}[S(u) - S(v)](\partial_\alpha u, \partial_\beta u) - \eta^{\alpha\beta} S(v)(\partial_\alpha u + \partial_\alpha v, \partial_\beta w)$$

By considering the pairing $\left\langle \Box_\eta w, \dot{w} \right\rangle$ and integrating over M we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|dw\|_{L^2}^2 &= \int_{\mathbb{R}^4} \left\langle \eta^{\alpha\beta}[S(u)-S(v)](\partial_\alpha u, \partial_\beta u), \dot{w} \right\rangle \sqrt{|g|}\, dx \\ &\quad + \int_{\mathbb{R}^4} \left\langle \eta^{\alpha\beta} S(v)(\partial_\alpha u + \partial_\alpha v, \partial_\beta w), \dot{w} \right\rangle \sqrt{|g|}\, dx \\ &= I(t) + II(t)\end{aligned}$$

Using that S and all of its derivatives are bounded we have

$$\begin{aligned}|I(t)| &\lesssim \int_{\mathbb{R}^4} |du|^2\, |w|\, |dw|\ dx \\ &\lesssim \|du\|_{L^8}^2 \|w\|_{L^4} \|dw\|_{L^2} \\ &\lesssim \|du\|_{L^8}^2 \|dw\|_{L^2}^2\end{aligned}$$

with the last inequality following from the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$.

To estimate $II(t)$, we exploit the fact the $S(u)(\cdot,\cdot) \in (T_u N)^\perp$ which gives

$$\langle S(u)(\cdot,\cdot), u_t \rangle = \langle S(v)(\cdot,\cdot), v_t \rangle = 0.$$

This implies that we can rewrite

$$\begin{aligned}
\left|\left\langle \eta^{\alpha\beta} S(v)(\partial_\alpha u + \partial_\alpha v, \partial_\beta w), \dot{w}\right\rangle\right| &= \left|\left\langle \eta^{\alpha\beta} S(v)(\partial_\alpha u + \partial_\alpha v, \partial_\beta w), \dot{u}\right\rangle\right| \\
&= \left|\left\langle \eta^{\alpha\beta} [S(v) - S(u)](\partial_\alpha u + \partial_\alpha v, \partial_\beta w), \dot{u}\right\rangle\right| \\
&\leq \left|\left\langle \eta^{\alpha\beta} [S(v) - S(u)](\partial_\alpha u, \partial_\beta w), \dot{u}\right\rangle\right| \\
&\quad + \left|\left\langle \eta^{\alpha\beta} [S(v) - S(u)](\partial_\alpha v, \partial_\beta w), \dot{u}\right\rangle\right| \\
&\lesssim (|du|^2 + |dv|^2)\, |w|\, |dw|
\end{aligned}$$

Hence we have

$$|II(t)| \lesssim (\|du\|_{L^8}^2 + \|dv\|_{L^8}^2)\|w\|_{L^4}\|dw\|_{L^2} \lesssim (\|du\|_{L^8}^2 + \|dv\|_{L^8}^2)\|dw\|_{L^2}^2$$

Putting this together we have

$$\frac{1}{2}\frac{d}{dt}\|dw\|_{L^2}^2 \lesssim (\|du\|_{L^8}^2 + \|dv\|_{L^8}^2)\|dw\|_{L^2}^2$$

Integrating in t and applying Gronwall's inequality gives us the uniform estimate

$$\|dw\|_{L_t^\infty L_x^2}^2 \leq \|dw(0)\|_{L^2}^2 \cdot \exp(C(\|du\|_{L_t^2 L_x^8}^2 + \|dv\|_{L_t^2 L_x^8}^2))$$

which implies uniqueness since $dw(0) = 0$.

2.3 Coulomb Frame & Elliptic Estimates

We follow [69] by exploiting the gauge invariance of the wave maps problem and rephrasing the wave maps equation in terms of the Coulomb frame. As discussed in Section 2.1.1, we can, without loss of generality, assume that TN is parallelizable, and we choose a global orthonormal frame $\tilde{e} = \{\tilde{e_1}, \ldots, \tilde{e_n}\}$. If $u : (\tilde{M}, \eta) \to (N, h)$ is a smooth map, then we can pull back $\tilde{e}$ to an orthonormal frame $\bar{e} = \tilde{e} \circ u$ of u^*TN. Now, let $B : \mathbb{R} \times M \longrightarrow SO(n)$. With B we can rotate this frame over each point $z \in \mathbb{R} \times M$ and obtain a new frame $e = (e_1, \ldots, e_n)$, with e_a given by

$$e_a = B_a^b \bar{e}_b$$

Observe that we can express the u^*TN-valued 1-form du in this new frame by finding 1-forms $q^a = q^a_\alpha dx^\alpha$ where $q^a_\alpha = u^*h(\partial_\alpha u, e_a)$, and writing

$$du = q^a e_a \tag{2.3.1}$$

For this frame e we have the associated connection form A. $A = (A_b^a)$ is a matrix of 1-forms obtained in the following way. Given the frame e, we obtain for each $s \in \mathbb{R}$ a map

$$\begin{aligned} De_a : \Gamma(T(\{s\}{\times}M)) &\longrightarrow \Gamma(u^*TN) \\ X &\longmapsto D_X e_a \end{aligned}$$

where D is the pull back connection on u^*TN and where for a vector bundle $E \to M$, $\Gamma(E)$ denotes the space of smooth sections. Equivalently, we can view De_a as a section of $T^*M \otimes u^*TN$. We can express this map in terms of the connection form A which can be viewed as the matrix of 1-forms so that

$$De_a = A_a^b \otimes e_b$$
$$De_a(X) = D_X e_a = A_a^b(X) e_b$$

Observe that this is the same as viewing De_a as a $\binom{1}{1}$-tensor on $T^*M \otimes u^*TN \to M$ in the sense that $De_a : TM \times u^*T^*N \to \mathbb{R}$ is a bilinear map over $C^\infty(M)$. Then we have that

$$A_a^b = u^*h(A_a^c \otimes e_c, e_b)$$

where u^*h is the metric on u^*TN. In local coordinates, A_a^b is given by $A_{a,\alpha}^b dx^\alpha$ where the coefficients of A_a^b are defined by $A_{a,\alpha}^b = A_a^b(\partial_\alpha)$. Hence if X is given in local coordinates by $X = X^\alpha \partial_\alpha$ we have that $D_X e_a = X^\alpha A_{a,\alpha}^b e_b$.

One should also note that for a fixed coordinate α, the matrix $(A_{b,\alpha}^a)$ is antisymmetric. That is, $A_{b,\alpha}^a = -A_{a,\alpha}^b$. To see this, simply differentiate the orthogonality condition of our orthonormal frame, $h(e_a, e_b) = \delta_{ab}$. This gives

$$\begin{aligned} 0 &= D\left(h(e_a, e_b)\right) \\ &= h(De_a, e_b) + h(e_a, De_b) \\ &= A_a^b + A_b^a \end{aligned}$$

The curvature tensor, F, on u^*TN can be represented in term of the connection form A. Viewed as a 2-form, F is given by $F = dA + A \wedge A$. We can also represent F in terms of the curvature tensor on TN. In local coordinates, F is given by $F_{\alpha\beta} = R(u)(\partial_\alpha u, \partial_\beta u)$.

As in [32, Lemma 4.1.3], we choose our rotation B so that at for each $s \in \mathbb{R}$, $B(s, \cdot)$

minimizes the functional

$$
\begin{aligned}
\Lambda(B(s)) &= \int_M \sum_{a,b=1}^{n} g^{-1}(A_a^b(s), A_a^b(s))\, dvol_g \\
&= \int_M \sum_{a,b=1}^{n} g^{\alpha\beta} A_{a,\alpha}^b(s) A_{a,\beta}^b(s) \sqrt{|g|}\, dx
\end{aligned}
$$

This gives us a frame e that we call the Coulomb frame. The Euler-Lagrange equations for this minimization problem are given by

$$
\frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} g^{\alpha\beta} A_\beta) = 0 \tag{2.3.2}
$$

The above equation implies that $\delta A = 0$ since the exterior co-differential, δ, on 1-forms is given in local coordinates by

$$
-\delta A = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} g^{\alpha\beta} A_\beta) = 0 \tag{2.3.3}
$$

Since the Hodge Laplacian Δ on M is given by $\Delta = d\delta + \delta d$, (2.3.3) implies the following differential equation of 1-forms for A

$$
\Delta A = \delta d A
$$

Using the fact that the curvature form F satisfies $F = dA + A \wedge A$ we can rewrite the above equation for A as

$$
\Delta A = \delta (F - A \wedge A) \tag{2.3.4}
$$

In local coordinates we can write this in components as

$$(\Delta A)_\gamma = -[\nabla^\alpha (F - A \wedge A)]_{\alpha\gamma} \tag{2.3.5}$$

where $\nabla^\alpha = g^{\alpha\beta}\nabla_\beta$ and ∇ denotes the Levi-Civita connection on M.

Observe that (2.3.5) can be written as system of elliptic equations for the components of A in local coordinates on M. We record this fact in the following lemma:

Lemma 2.3.1. *The components of A satisfy the following system of elliptic equations*

$$\begin{aligned} g^{ij}\partial_i\partial_j A_\gamma - g^{ij}\Gamma^k_{ij}\partial_\gamma A_k + \partial_\gamma g^{ij}\partial_j A_i - \partial_\gamma(g^{ij}\Gamma^k_{ij})A_k & \\ = g^{ij}\partial_j\left(F_{i\gamma} - [A_i, A_\gamma]\right) & \end{aligned} \tag{2.3.6}$$

where the $\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}\right)$ denote the Christoffel symbols on M.

Proof. We first expand the left-hand side of (2.3.5)

$$\begin{aligned} (\Delta A)_\gamma &= (d\delta A)_\gamma + (\delta d A)_\gamma \\ &= -\partial_\gamma(g^{ij}(\nabla_j A)_i) - g^{ij}(\nabla_j dA)_{i\gamma} \\ &= -\left(\partial_\gamma g^{ij}\right)(\nabla_j A)_i - g^{ij}\partial_\gamma\left(\partial_j A_i - \Gamma^k_{ij}A_k\right) \\ &\quad - g^{ij}\left(\partial_j (dA)_{i\gamma} - \Gamma^k_{ij}(dA)_{k\gamma} - \Gamma^k_{\gamma j}(dA)_{ik}\right) \\ &= -g^{ij}\partial_i\partial_j A_\gamma - \left(\partial_\gamma g^{ij}\right)(\nabla_j A)_i + g^{ij}\partial_\gamma\left(\Gamma^k_{ij}A_k\right) \\ &\quad + g^{ij}\Gamma^k_{ij}(dA)_{k\gamma} + g^{ij}\Gamma^k_{\gamma j}(dA)_{ik} \end{aligned}$$

Similarly, we expand the right-hand side of (2.3.5)

$$-[\nabla^i(F - A \wedge A)]_{i\gamma} = -g^{ij}\partial_j \left(F_{i\gamma} - \left[A_i, A_\gamma\right]\right) + g^{ij}\Gamma^k_{ij}\left(F_{k\gamma} - \left[A_k, A_\gamma\right]\right)$$
$$+ g^{ij}\Gamma^k_{j\gamma}\left(F_{ik} - [A_i, A_k]\right)$$

Equating the left and right hand sides and recalling that $(dA)_{ij} = F_{ij} - [A_i, A_j]$ we have

$$g^{ij}\partial_i\partial_j A_\gamma + \left(\partial_\gamma g^{ij}\right)(\nabla_j A)_i - g^{ij}\partial_\gamma\left(\Gamma^k_{ij}A_k\right) = g^{ij}\partial_j\left(F_{i\gamma} - \left[A_i, A_\gamma\right]\right)$$

which is exactly (2.3.6). $\square$

2.3.1 Connection Form Estimates

With the metric g as in (2.1.1)–(2.1.3) and ε small enough, we can use the elliptic system (2.3.6) to establish a variety of estimates for the connection form A. In particular, we can prove the following proposition which will be essential when deriving a priori estimates for wave maps.

Proposition 2.3.2. *Let (N, h) be a n-dimensional manifold smoothly embedded in $\mathbb{R}^m$ with bounded geometry and a bounded parallelizable structure. Let $u : (\mathbb{R} \times \mathbb{R}^4, \eta) \to (N, h)$ be a smooth map with $\eta = diag(-1, g)$ and g as in (2.1.1)–(2.1.3). Moreover, assume the bootstrap hypothesis,*

$$\sup_{t\in[0,T)} \|du\|_{\dot{H}^1} \lesssim \varepsilon_0 \tag{2.3.7}$$

Then, for each $t \in \mathbb{R}$, there exists a unique frame $e = (e_1, \ldots, e_n)$ for u^*TN with the associated connection form, A, satisfying the uniform-in-time estimates

(i) $\|A\|_{L^4} \lesssim \|du\|_{H^1} \lesssim \varepsilon_0$

(ii) $\|A\|_{\dot{W}^{1,\frac{8}{3}}} \lesssim \|du\|_{L^8}\|du\|_{\dot{H}^1}$

(iii) $\|A\|_{\dot{W}^{2,\frac{8}{5}}} \lesssim \|du\|_{L^8}\|du\|_{\dot{H}^1}$

(iv) $\|A\|_{L^\infty} \lesssim \|du\|^2_{L^{8,2}(\mathbb{R}^4)}$

as long as ε_0 is small enough. Also, the frame e, and hence A, depend continuously on t. Above, $L^{8,2} = L^{8,2}(\mathbb{R}^4)$ denotes the Lorentz space.

The estimates are deduced via a perturbative method as the assumptions in (2.1.1)–(2.1.3) imply that the left hand side of (2.3.6) is a slight perturbation of the flat Laplacian on $\mathbb{R}^4$. To simplify notation, in what follows we consider an elliptic operator of the form

$$L := g^{ij}\partial_i\partial_j + b^j\partial_j + c \tag{2.3.8}$$

and the elliptic system

$$LA_\ell = g^{ij}\partial_j G_{i\ell} \tag{2.3.9}$$

where $G_{i\ell} := F_{i\ell} - [A_i, A_\ell]$, and b and c satisfy

$$\|b\|_{L^{4,1}(\mathbb{R}^4)} \lesssim \varepsilon \tag{2.3.10}$$

$$\|\partial b\|_{L^{2,1}(\mathbb{R}^4)} \lesssim \varepsilon \tag{2.3.11}$$

$$\|c\|_{L^{2,1}(\mathbb{R}^4)} \lesssim \varepsilon \tag{2.3.12}$$

Since $\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$, it is clear that the left-hand side of (2.3.6) is essentially of this form.

We begin by recalling some basic elliptic estimates. Let g_0 denote the Euclidean metric on $\mathbb{R}^4$ and let $L_0 := g_0^{ij}\partial_i\partial_j$ denote the flat Laplacian on $\mathbb{R}^4$. Then we have

$$\|A\|_{\dot{W}^{s+2,p}} \lesssim \|L_0 A\|_{\dot{W}^{s,p}} \tag{2.3.13}$$

for every $s \in \mathbb{R}$ and for every $1 < p < \infty$. With (2.3.13) we can prove the following elliptic estimates for the connection form A.

Lemma 2.3.3. *Let A be the connection form associated to the Coulomb frame e. Then, if ε is small enough, we have the following uniform-in-time estimates*

(i) $\|A\|_{\dot{W}^{1,p}} \lesssim \|[A,A]\|_{L^p} + \|F\|_{L^p}$ *if* $1 < p < 4$

(ii) $\|A\|_{\dot{W}^{2,p}} \lesssim \|[A,A]\|_{\dot{W}^{1,p}} + \|F\|_{\dot{W}^{1,p}}$ *if* $1 < p < 2$.

where F denotes the curvature tensor on u^*TN.

Proof. Let L_0 and L be defined as above and write $LA = L_0A + (L - L_0)A$. Hence,

$$\|LA\|_{\dot{W}^{s,p}} \geq \|L_0 A\|_{\dot{W}^{s,p}} - \|(L - L_0)A\|_{\dot{W}^{s,p}}$$

We can use (2.3.13) to obtain

$$\|A\|_{\dot{W}^{s+2,p}} \lesssim \|LA\|_{\dot{W}^{s,p}} + \|(L - L_0)A\|_{\dot{W}^{s,p}} \tag{2.3.14}$$

for every $s \in \mathbb{R}$ and for $1 < p < \infty$. To prove (i), set $s = -1$ above to get

$$\begin{aligned}\|A\|_{\dot{W}^{1,p}} &\lesssim \|LA\|_{\dot{W}^{-1,p}} + \|(L - L_0)A\|_{\dot{W}^{-1,p}} \\ &\lesssim \left\|g^{-1}\,\partial G\right\|_{\dot{W}^{-1,p}} + \|b\,\partial A\|_{\dot{W}^{-1,p}} \\ &\quad + \|cA\|_{\dot{W}^{-1,p}} + \left\|(g^{-1} - g_0^{-1})\partial^2 A\right\|_{\dot{W}^{-1,p}}\end{aligned}$$

We claim that

$$\left\| g^{-1}\,\partial G \right\|_{\dot{W}^{-1,p}} \lesssim \|G\|_{L^p}$$

This follows from the dual estimate

$$\left\| g^{-1} f \right\|_{\dot{W}^{1,p'}} \lesssim \|f\|_{\dot{W}^{1,p'}} \tag{2.3.15}$$

To prove (2.3.15) observe that we have

$$\begin{aligned}
\left\| g^{-1} f \right\|_{\dot{W}^{1,p'}} &\lesssim \left\| \partial(g^{-1} f) \right\|_{L^{p'}} \\
&\lesssim \left\| (\partial g^{-1}) f \right\|_{L^{p'}} + \left\| g^{-1}(\partial f) \right\|_{L^{p'}} \\
&\lesssim \left\| \partial g^{-1} \right\|_{L^4} \|f\|_{L^r} + \left\| g^{-1} \right\|_{L^\infty} \|\partial f\|_{L^{p'}} \\
&\lesssim \|f\|_{\dot{W}^{1,p'}}
\end{aligned}$$

where the last inequality follows from (2.1.2) and the Sobolev embedding $\dot{W}^{1,p'} \hookrightarrow L^r$ since we have $\frac{1}{r} = \frac{1}{p'} - \frac{1}{4}$. Next, we assert that

$$\left\| (g^{-1} - g_0^{-1}) \partial^2 A \right\|_{\dot{W}^{-1,p}} \lesssim \varepsilon \left\| \partial^2 A \right\|_{\dot{W}^{-1,p}} \lesssim \varepsilon \|A\|_{\dot{W}^{1,p}}$$

Again, this follows from a duality argument. Observe that

$$\begin{aligned}
\|(g^{-1} - g_0^{-1}) f\|_{\dot{W}^{1,p'}} &\lesssim \|\partial(g^{-1} - g_0^{-1}) f\|_{\dot{W}^{1,p'}} + \|(g^{-1} - g_0^{-1}) \partial f\|_{\dot{W}^{1,p'}} \\
&\lesssim \|\partial g^{-1}\|_{L^4} \|f\|_{L^r} + \|(g^{-1} - g_0^{-1})\|_{L^\infty} \|\partial f\|_{L^{p'}} \\
&\lesssim \varepsilon \|f\|_{\dot{W}^{1,p'}}
\end{aligned}$$

where the last inequality is again due to (2.1.1), (2.1.2), and Sobolev embedding since $\frac{1}{r} =$

$\frac{1}{p'} - \frac{1}{4}$. To estimate $\|b\,\partial A\|_{\dot{W}^{-1,p}}$, we use Sobolev embedding, Hölder's inequality and (2.3.10). Indeed,

$$
\begin{aligned}
\|b\,\partial A\|_{\dot{W}^{-1,p}} &\lesssim \|b\,\partial A\|_{L^s} \\
&\lesssim \|b\|_{L^4}\|\partial A\|_{L^p} \\
&\lesssim \varepsilon\,\|A\|_{\dot{W}^{1,p}}
\end{aligned}
$$

where $\frac{1}{p} = \frac{1}{s} - \frac{1}{4}$. Finally, we show that

$$\|cA\|_{\dot{W}^{-1,p}} \lesssim \varepsilon\,\|A\|_{\dot{W}^{1,p}}$$

To see this, we again use Sobolev embedding and (2.3.12) to obtain

$$
\begin{aligned}
\|c\,A\|_{\dot{W}^{-1,p}} &\lesssim \|c\,A\|_{L^s} \\
&\lesssim \|c\|_{L^2}\|A\|_{L^r} \\
&\lesssim \varepsilon\|A\|_{\dot{W}^{1,p}}
\end{aligned}
$$

with $\frac{1}{p} = \frac{1}{s} - \frac{1}{4}$, $\frac{1}{s} = \frac{1}{2} + \frac{1}{r}$, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{4}$. Putting this all together we are able to conclude that

$$\|A\|_{\dot{W}^{1,p}} \lesssim \|G\|_{L^p} + \varepsilon\,\|A\|_{\dot{W}^{1,p}}$$

For ε small enough, this implies (i), since $G = F - A \wedge A$.

To prove (ii) we set $s = 0$ in (2.3.14), and use (2.1.1), (2.3.10), (2.3.12), and Sobolev embedding to obtain

$$
\begin{aligned}
\|A\|_{\dot W^{2,p}} &\lesssim \left\|g^{-1}\,\partial G\right\|_{L^p} + \|b\,\partial A\|_{L^p} + \|cA\|_{L^p} + \left\|(g^{-1}-g_0^{-1})\partial^2 A\right\|_{L^p} \\
&\lesssim \left\|g^{-1}\right\|_{L^\infty}\|\partial G\|_{L^p} + \|b\|_{L^4}\|\partial A\|_{L^s} + \|c\|_{L^2}\|A\|_{L^r} \\
&\quad + \left\|g^{-1}-g_0^{-1}\right\|_{L^\infty}\|\partial^2 A\|_{L^p} \\
&\lesssim \|G\|_{\dot W^{1,p}} + \varepsilon\|A\|_{\dot W^{2,p}}
\end{aligned}
$$

where $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{1}{4}$ and $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{2}{4}$. This proves (ii) as long as ε is small enough. $\square$

With the elliptic estimates in Lemma 2.3.3 we can prove Proposition 2.3.2 (i), (ii) and (iii).

Proof of Proposition 2.3.2 (i). This will follow from Lemma 2.3.3 (i) with $p = 2$, a contraction argument at one fixed time, and then a bootstrap argument to conclude the uniform-in-time estimates. We note that this argument also proves the existence of a unique Coulomb frame e with the associated connection form A having small L^4 norm.

To carry out the contraction argument we fix a time t_0 and we set X to be the space

$$X := \{A \in \dot H^1 \cap L^4\}$$

with the norm

$$\|A\|_X := \|A\|_{L^4} + \|A\|_{\dot H^1}$$

Of course by Sobolev embedding we have $\|A\|_X \lesssim \|A\|_{\dot H^1}$. We set X_{ε_0} to be

$$X_{\varepsilon_0} := \{A \in X : \|A\|_X \le \varepsilon_0\}$$

Define a map Φ that associates to each $\tilde{A} \in X_{\varepsilon_0}$ the solution A to the linear elliptic problem

$$LA_\ell = g^{ij}\partial_i(F_{j\ell} - [\tilde{A}_j, \tilde{A}_\ell]) \tag{2.3.16}$$

The existence of such a solution follows easily by the method of continuity, the key estimate here being

$$\|A\|_{\dot{H}^1} \lesssim \|LA\|_{\dot{H}^{-1}}$$

which was obtained in the course of proving Lemma 2.3.3 (i) with $p = 2$. We will show that if ε_0 and $\|du\|_{\dot{H}^1}$ are small enough, then $\Phi : X_{\varepsilon_0} \to X_{\varepsilon_0}$ and that Φ is a contraction mapping on this space. To see that $\Phi : X_{\varepsilon_0} \to X_{\varepsilon_0}$ we use Sobolev embedding and Lemma 2.3.3 (i) to obtain

$$\|A\|_X \lesssim \|A\|_{\dot{H}^1} \lesssim \|[\tilde{A}, \tilde{A}]\|_{L^2} + \|F\|_{L^2}$$

Recall that we can write $F_{\alpha\beta} = R(u)(\partial_\alpha u, \partial_\beta u)$ where R is the Riemannian curvature tensor on N. Hence

$$\|A\|_X \lesssim \|\tilde{A}\|_{L^4}^2 + \|R\|_{L^\infty}\|du\|_{L^4}^2$$

$$\leq C_1\|\tilde{A}\|_X^2 + C_2\|du\|_{\dot{H}^1}^2 \leq \varepsilon_0$$

as long as ε_0 and $\|du\|_{\dot{H}^1}$ are small enough. Next we show that $\Phi : X_{\varepsilon_0} \to X_{\varepsilon_0}$ is a contraction mapping. Let $\tilde{A}^1, \tilde{A}^2 \in X_{\varepsilon_0}$ and let A^1, A^2 be the associated solutions to (2.3.16).

Then $A^1 - A^2$ is a solution to

$$L(A^1_\ell - A^2_\ell) = g^{ij}\partial_i([\tilde{A}^1_j, \tilde{A}^1_\ell] - [\tilde{A}^2_j, \tilde{A}^2_\ell])$$

and hence we have estimates

$$\begin{aligned} \|A^1 - A^2\|_X \lesssim \|A^1 - A^2\|_{\dot{H}^1} &\lesssim \|[\tilde{A}^1, \tilde{A}^1] - [\tilde{A}^2, \tilde{A}^2]\|_{L^2} \\ &\lesssim \|\tilde{A}^1 - \tilde{A}^2\|_{L^4}\|\tilde{A}^1\|_{L^4} + \|\tilde{A}^1 - \tilde{A}^2\|_{L^4}\|\tilde{A}^2\|_{L^4} \\ &\lesssim \varepsilon_0 \|\tilde{A}^1 - \tilde{A}^2\|_X \end{aligned}$$

which proves that Φ is a contraction. Hence Φ has a unique fixed point $A = A(t_0)$ which solves (2.3.9) such that

$$\|A(t_0)\|_{L^4} \lesssim \varepsilon_0$$

To obtain this estimate for all times t with a uniform constant we again use Lemma 2.3.3 (i) with $p = 2$ to obtain for any time

$$\begin{aligned} \|A\|_{L^4} \lesssim \|A\|_{\dot{H}^1} &\lesssim \|[A, A]\|_{L^2} + \|F\|_{L^2} \\ &\lesssim \|A\|^2_{L^4} + \|du\|^2_{\dot{H}^1} \end{aligned}$$

As long as $\|du\|_{\dot{H}^1}$ is small enough we can use a bootstrap argument, with $\|A(t_0)\| \lesssim \varepsilon_0$ as our base case, to absorb the $\|A\|^2_{L^4}$ term on the left hand side and obtain

$$\|A\|_{L^4} \lesssim \varepsilon_0$$

for all times t, as desired. $\square$

Proof of Propostion 2.3.2 (ii) and (iii). To prove (ii) we set $p = \frac{8}{3}$ in Lemma 2.3.3 (i), giving

$$\|A\|_{\dot{W}^{1,\frac{8}{3}}} \lesssim \|[A,A]\|_{L^{\frac{8}{3}}} + \|F\|_{L^{\frac{8}{3}}}$$

First we claim that $\|[A,A]\|_{L^{\frac{8}{3}}} \lesssim \varepsilon \|A\|_{\dot{W}^{1,\frac{8}{3}}}$ and this term can thus be absorbed on the left-hand side above. Indeed,

$$\|[A,A]\|_{L^{\frac{8}{3}}} \lesssim \|A\|_{L^4} \|\partial A\|_{L^8}$$

$$\lesssim \varepsilon_0 \|A\|_{\dot{W}^{1,\frac{8}{3}}}$$

where the last inequality follows from Sobolev embedding and the previous estimate

$$\|A\|_{L^4} \lesssim \varepsilon.$$

Next we recall that $F = R(u)(du, du)$ and hence we have

$$\|F\|_{L^{\frac{8}{3}}} \lesssim \|du\|_{L^8} \|du\|_{L^4} \lesssim \|du\|_{L^8} \|du\|_{\dot{H}^1}$$

Putting this together implies gives

$$\|A\|_{\dot{W}^{1,\frac{8}{3}}} \lesssim \|du\|_{L^8} \|du\|_{\dot{H}^1}$$

as long as ε is small enough.

To prove (iii) we proceed in a similar fashion. We set $p = \frac{8}{5}$ in Lemma (2.3.3) (ii). This gives

$$\|A\|_{\dot{W}^{2,\frac{8}{5}}} \lesssim \|[A,A]\|_{\dot{W}^{1,\frac{8}{5}}} + \|F\|_{\dot{W}^{1,\frac{8}{5}}}$$

First we observe that $\|[A,A]\|_{\dot{W}^{1,\frac{8}{5}}} \lesssim \varepsilon \|A\|_{\dot{W}^{2,\frac{8}{5}}}$ and this term can thus be absorbed on the left-hand side above. In fact,

$$\begin{aligned}
\|[A,A]\|_{\dot{W}^{1,\frac{8}{5}}} &\lesssim \|A\partial A\|_{L^{\frac{8}{5}}} \\
&\lesssim \|A\|_{L^4} \|\partial A\|_{L^{\frac{8}{3}}} \\
&\lesssim \varepsilon_0 \|A\|_{\dot{W}^{2,\frac{8}{5}}}
\end{aligned}$$

where the last inequality follows from Sobolev embedding and the previous estimate

$$\|A\|_{L^4} \lesssim \varepsilon.$$

Next observe that

$$\partial_\gamma F_{\alpha\beta} = (\partial R(u))(\partial_\gamma u, \partial_\alpha u, \partial_\beta u) + R(u)(\partial_\gamma \partial_\alpha u, \partial_\beta u) + R(u)(\partial_\alpha u, \partial_\gamma \partial_\beta u)$$

Hence by Sobolev embedding and the assumption that $\|du\|_{\dot{H}^1} \lesssim \varepsilon_0$,

$$\|\partial F\|_{L^{\frac{8}{5}}} \lesssim \|\partial R(u)\|_{L^\infty}\|du\|_{L^4}\|du\|_{L^4}\|du\|_{L^8} + \|R\|_{L^\infty}\|\partial du\|_{L^2}\|du\|_{L^8}$$

$$\lesssim \|du\|_{\dot{H}^1}\|du\|_{L^8}$$

Putting this all together we have for small enough ε_0 that

$$\|A\|_{\dot{W}^{2,\frac{8}{5}}} \lesssim \|du\|_{\dot{H}^1}\|du\|_{L^8} \lesssim \|du\|_{L^8}$$

establishing (iii). □

To prove the pointwise estimates for the connection form in Propostion 2.3.2 (iv), we will need a few facts about Lorentz Spaces, $L^{p,r}(\mathbb{R}^4)$, including Sobolev embedding for Lorentz spaces and the Calderon-Zygmund theorem for Lorentz spaces. These facts, along with a few others, are reviewed in the appendix, see Section 2.9.3.

Now, again let $L_0 = g_0^{ij}\partial_i\partial_j$ be the flat Laplacian on $\mathbb{R}^4$ and let $K = L_0^{-1}$ be convolution with $k(x) = \dfrac{c}{|x|^2}$, the fundamental solution for L_0 in $\mathbb{R}^4$. We can then write

$$A = KLA + K(L_0 - L)A$$

In order to prove Proposition 2.3.2 (iv), we will need the following preliminary estimates for ∂A.

Lemma 2.3.4. *Let A denote the connection form associated to the Coulomb frame as in Proposition 2.3.2. Then, the following estimates hold uniformly in time:*

$$\|\partial A\|_{L^{4,1}} \lesssim \|du\|^2_{L^{8,2}} + \varepsilon\|A\|_{L^\infty}$$

Proof. With K, L and L_0 as above, write

$$A = KLA + K(L_0 - L)A$$

Then

$$\begin{aligned} KLA &= k * g^{ij}\partial_i G_j \\ &= (\partial_i k) * g^{ij} G_j - k * (\partial_i g^{ij}) G_j \end{aligned}$$

Then, formally, we have

$$\partial_\alpha(KLA) = (\partial_\alpha \partial_i k) * g^{ij} G_j - (\partial_\alpha k) * (\partial_i g^{ij}) G_j \tag{2.3.17}$$

Since, $\partial_\alpha \partial_i k$ is a Calderon-Zygmund kernel, we can use the Calderon-Zygmund theorem for Lorentz spaces, see Theorem 2.9.5 below, and Hölder's inequality for Lorentz spaces, see Lemma 2.9.3 (i) below, to obtain

$$\begin{aligned} \|(\partial_\alpha \partial_i k) * g^{ij} G_j\|_{L^{4,1}} &\lesssim \|g^{ij} G_j\|_{L^{4,1}} \\ &\lesssim \|[A, A]\|_{L^{4,1}} + \|F\|_{L^{4,1}} \\ &\lesssim \|A\|^2_{L^{8,2}} + \|du\|^2_{L^{8,2}} \end{aligned}$$

Using the fact that $L^{p,r} \subset L^{p,s}$ for $r < s$, Sobolev embedding for Lorentz spaces, see Lemma 2.9.4 below, and Proposition 2.3.2 (iii), we have

$$\|A\|_{L^{8,2}} \lesssim \|A\|_{L^{8,\frac{8}{5}}} \lesssim \|A\|_{\dot{W}^{2,\frac{8}{5}}} \lesssim \|du\|_{L^8} \|du\|_{\dot{H}^1}$$

Then using the bootstrap assumption that $\|du\|_{\dot{H}^1} \ll 1$ we can conclude that

$$\|A\|_{L^{8,2}} \lesssim \|du\|_{L^8} \lesssim \|du\|_{L^{8,2}} \tag{2.3.18}$$

Inserting this estimate above we can conclude that

$$\|(\partial_\alpha \partial_i k) * g^{ij} G_j\|_{L^{4,1}} \lesssim \|du\|^2_{L^{8,2}}$$

Next, we can use Young's inequality for Lorentz spaces, see Lemma 2.9.3 (ii) below, to show

$$\begin{aligned}
\|(\partial_\alpha k) * (\partial_i g^{ij}) G_j\|_{L^{4,1}} &\lesssim \|\partial_\alpha k\|_{L^{\frac{4}{3},\infty}} \|\partial_i g^{ij} G_j\|_{L^{2,1}} \\
&\lesssim \|\partial_i g^{ij}\|_{L^{4,\infty}} \|G_j\|_{L^{4,1}} \\
&\lesssim \|A^2\|_{L^{4,1}} + \|F\|_{L^{4,1}} \\
&\lesssim \|du\|^2_{L^{8,2}}
\end{aligned}$$

where above we have used (2.1.2). Now, to deal with the error term $K(L_0 - L)A$, write $(L_0 - L)A = \varepsilon^{ij}\partial_i \partial_j A - b^j \partial_j A - cA$ where $\varepsilon^{ij}(x) = g_0^{ij}(x) - g^{ij}(x)$. Then we have

$$\begin{aligned}
K(L_0 - L)A &= k * \varepsilon^{ij} \partial_i \partial_j A - k * b^j \partial_j A - k * cA \\
&= (\partial_i k) * \varepsilon^{ij} \partial_j A - k * (\partial_i \varepsilon^{ij}) \partial_j A - k * b^j \partial_j A - k * cA
\end{aligned}$$

Hence, formally we have

$$\begin{aligned}
\partial_\alpha (K(L_0 - L)A) = (\partial_\alpha \partial_i k) * \varepsilon^{ij} \partial_j A - (\partial_\alpha k) * (\partial_i \varepsilon^{ij}) \partial_j A \\
- (\partial_\alpha k) * b^j \partial_j A - (\partial_\alpha k) * cA
\end{aligned} \tag{2.3.19}$$

And as before, we use the Calderon-Zygmund theorem on the first term on the right-hand side above to get

$$\|(\partial_\alpha\partial_i k) * \varepsilon^{ij}\partial_j A\|_{L^{4,1}} \lesssim \sum_i \|\varepsilon^{ij}\partial_j A\|_{L^{4,1}} \lesssim \varepsilon\|\partial A\|_{L^{4,1}}$$

We estimate the other three terms on the right-hand side of (2.3.19) using Young's inequality for Lorentz spaces as follows

$$\begin{aligned}\|(\partial_\alpha k) * (\partial_i\varepsilon^{ij})\partial_j A\|_{L^{4,1}} &\lesssim \|\partial_\alpha k\|_{L^{\frac{4}{3},\infty}}\|(\partial_i\varepsilon^{ij})\partial_j A\|_{L^{2,1}} \\ &\lesssim \|\partial_i\varepsilon^{ij}\|_{L^{4,\infty}}\|\partial_j A\|_{L^{4,1}} \\ &\lesssim \varepsilon\|\partial A\|_{L^{4,1}}\end{aligned}$$

the last inequality following from the fact that $\partial_i\varepsilon^{ij} = \partial_i g^{ij} \in L^{4,\infty}$. We also have

$$\begin{aligned}\|\partial_\alpha k * cA\|_{L^{4,1}} &\lesssim \|\partial_\alpha k\|_{L^{\frac{4}{3},\infty}}\|cA\|_{L^{2,1}} \\ &\lesssim \|c\|_{L^{2,1}}\|A\|_{L^\infty} \\ &\lesssim \varepsilon\|A\|_{L^\infty}\end{aligned}$$

Above we have used the fact that

$$|\partial_\alpha k| \sim \frac{1}{|x|^3} \in L^{\frac{4}{3},\infty}(\mathbb{R}^4), \quad \|c\|_{L^{2,1}} \lesssim \varepsilon,$$

see Lemma 2.9.2, (*iii*) and (2.3.12). And lastly,

$$\begin{aligned}\|(\partial_\alpha k) * b^j\partial_j A\|_{L^{4,1}} &\lesssim \|\partial_\alpha k\|_{L^{\frac{4}{3},\infty}}\|b^j\partial_j A\|_{L^{2,1}} \\ &\lesssim \|b\|_{L^4\infty}\|\partial A\|_{L^{4,1}} \\ &\lesssim \varepsilon\|\partial A\|_{L^{4,1}}\end{aligned}$$

which follows by (2.3.10). Putting this all together establishes that

$$\|\partial A\|_{L^{4,1}} \lesssim \|du\|_{L^{8,2}}^2 + \varepsilon\|A\|_{L^\infty} + \varepsilon\|\partial A\|_{L^{4,1}}$$

which, for small ε, implies that

$$\|\partial A\|_{L^{4,1}} \lesssim \|du\|_{L^{8,2}}^2 + \varepsilon\|A\|_{L^\infty}$$

as desired. $\square$

Now we are able to prove Proposition (2.3.2) (iv).

Proof of Proposition 2.3.2 (iv). Since $A = KLA + K(L_0 - L)A$, it suffices to show that for every t the following two estimates hold:

$$\|KLA\|_{L^\infty} \lesssim \|du\|_{L^{8,2}}^2 \tag{2.3.20}$$

$$\|K(L_0 - L)A\|_{L^\infty} \lesssim \|du\|_{L^{8,2}}^2 + \varepsilon\|A\|_{L^\infty} \tag{2.3.21}$$

Observe that we can write

$$\begin{aligned} KLA &= k * g^{ij}\partial_i G_j \\ &= (\partial_i k) * g^{ij} G_j - k * (\partial_i g^{ij}) G_j \end{aligned}$$

By Lemma 2.9.3 (iii), we have that

$$\begin{aligned}
\|(\partial_i k) * g^{ij} G_j\|_{L^\infty} &\lesssim \|\partial_i k\|_{L^{\frac{4}{3},\infty}} \|g^{ij} G_j\|_{L^{4,1}} \\
&\lesssim \|[A, A]\|_{L^{4,1}} + \|F\|_{L^{4,1}} \\
&\lesssim \|A\|^2_{L^{8,2}} + \|F\|^2_{L^{4,1}} \\
&\lesssim \|du\|^2_{L^{8,2}}
\end{aligned}$$

where we have used (2.3.18) in the last inequality. Similarly,

$$\begin{aligned}
\|k * (\partial_i g^{ij}) G_j\|_{L^\infty} &\lesssim \|k\|_{L^{2,\infty}} \|\partial_i g^{ij} G_j\|_{L^{2,1}} \\
&\lesssim \|\partial_i g^{ij}\|_{L^{4,\infty}} \|G_j\|_{L^{4,1}} \\
&\lesssim \|G\|_{L^{4,1}} \\
&\lesssim \|du\|_{L^{8,2}}
\end{aligned}$$

This proves (2.3.20). To establish the error estimate (2.3.21) we again write

$$\begin{aligned}
K(L_0 - L)A &= k * \varepsilon^{ij} \partial_i \partial_j A - k * b^j \partial_j A - k * cA \\
&= \partial_i k * \varepsilon^{ij} \partial_j A - k * (\partial_i \varepsilon^{ij}) \partial_j A - (\partial_j k) * b^j A + k * (\partial_j b^j) A \\
&\quad - k * cA \\
&= \partial_i k * \varepsilon^{ij} \partial_j A - \partial_j k * (\partial_i \varepsilon^{ij}) A + k * (\partial_j \partial_i \varepsilon^{ij}) A - (\partial_j k) * b^j A \\
&\quad + k * (\partial_j b^j) A - k * cA
\end{aligned} \tag{2.3.22}$$

where as before $\varepsilon^{ij} = g_0^{ij} - g^{ij}$. Now, we can use Lemma 2.3.4 to control the first term on the right above

$$\begin{aligned}
\|\partial_i k * \varepsilon^{ij}\partial_j A\|_{L^\infty} &\lesssim \|\partial_i k\|_{L^{\frac{4}{3},\infty}} \|\varepsilon^{ij}\partial_j A\|_{L^{4,1}} \\
&\lesssim \varepsilon\|\partial A\|_{L^{4,1}} \\
&\lesssim \|du\|_{L^{8,2}}^2 + \varepsilon\|A\|_{L^\infty}
\end{aligned}$$

The other terms in (2.3.22) are estimated as follows:

$$\begin{aligned}
\|\partial_j k * (\partial_i \varepsilon^{ij})A\|_{L^\infty} &\lesssim \|\partial_j k\|_{L^{\frac{4}{3},\infty}} \|(\partial_i \varepsilon^{ij})A\|_{L^{4,1}} \\
&\lesssim \sum_j \|\partial_i g^{ij}\|_{L^{4,1}} \|A\|_{L^\infty} \\
&\lesssim \varepsilon\|A\|_{L^\infty}
\end{aligned}$$

We also have

$$\begin{aligned}
\|k * (\partial_i \partial_j \varepsilon^{ij})\|_{L^\infty} &\lesssim \|k\|_{L^{2,\infty}} \|(\partial_j \partial_i \varepsilon^{ij})A\|_{L^{2,1}} \\
&\lesssim \|(\partial_j \partial_i \varepsilon^{ij})\|_{L^{2,1}} \|A\|_{L^\infty} \\
&\lesssim \varepsilon\|A\|_{L^\infty}
\end{aligned}$$

The remaining terms are handled exactly in the same manner as these last two, using (2.3.10), (2.3.11) and (2.3.12) as needed. This proves (2.3.21). Finally, putting everything together, we have

$$\|A\|_{L^\infty} \lesssim \|KLA\|_{L^\infty} + \|K(L_0 - L)A\|_{L^\infty} \lesssim \|du\|_{L^{8,2}}^2 + \varepsilon\|A\|_{L^\infty}$$

which, for ε small enough, gives

$$\|A\|_{L^\infty} \lesssim \|du\|_{L^{8,2}}^2$$

as claimed. $\square$

2.3.2 Equivalence of Norms

In this section we again set $(M, g) = (\mathbb{R}^4, g)$ with g as in (2.1.1)-(2.1.4). Now that we have settled Proposition 2.3.2, we can show that in the case that e is the Coulomb frame, the extrinsic $\dot{H}_e^s$ norms of du are equivalent to the intrinsic $\dot{H}_i^s$ norms of $q = q^a e_a$ where q is defined, as in (2.3.1), by

$$du = q^a e_a$$

In the appendix, Section 2.9.1, we show using (2.1.1)-(2.1.4), that $\dot{H}_e^s((M, g); N)$ is equivalent to $\dot{H}_e^s((\mathbb{R}^4, g_0); N)$ and that $\dot{H}_i^s((M, g); N)$ is equivalent to $\dot{H}_i^s((\mathbb{R}^4, g_0); N)$. Therefore, it suffices to ignore the perturbed metric g on $\mathbb{R}^4$ and show that

$$\|du\|_{\dot{H}_e^s((\mathbb{R}^4, g_0); N)} \simeq \|q\|_{\dot{H}_c^s((\mathbb{R}^4, g_0); N)} \tag{2.3.23}$$

This will follow from Proposition 2.3.2. We proceed exactly as in [69, Section 4.3]. We reproduce their argument here. For each t, since e is an orthonormal frame, we have

$$|du|^2 = |q|^2 = \sum_{\alpha=0}^{4} |q_\alpha|^2$$

This implies that for $1 \leq p \leq \infty$ that the L^p norm of du is well defined and independent of the choice of frame and coincides with the "extrinsic" L^p norm of du. However, this "gauge" independence is in general lost when we consider norms of higher derivatives of du as the connection form A appears when relating the intrinsic and extrinsic representations, and A, in general, cannot be controlled. In the case of the Coulomb frame, we can use the smallness provided by Proposition 2.3.2 to prove the desired equivalence of Sobolev norms. To see this, let ψ be a section of u^*TN whose components in terms of the Coulomb frame e are given by

$$\psi = Q^a e_a = Qe \tag{2.3.24}$$

By the previous discussion we have $\|\psi\|_{L^2} = \|Q\|_{L^2}$. Recall that we can represent covariant derivatives of ψ in terms of the extrinsic partial derivatives of ψ and the second fundamental form by

$$\partial_k \psi = D_k \psi + S(u)(\partial_k u, \psi) \tag{2.3.25}$$

Using the representation (2.3.24) we also have

$$D_k \psi = (\partial_k Q + AQ)e \tag{2.3.26}$$

Combining (2.3.25) and (2.3.26), we obtain

$$\partial_k \psi = \partial_k Qe + AQe + B(u)(\partial_k u, Qe)$$

We can then use Proposition 2.3.2 (i), Sobolev embedding and the boundedness of the second fundamental form to obtain

$$\begin{aligned}\big|\|\partial\psi\|_{L^2} - \|\partial Q\|_{L^2}\big| &\lesssim \|AQ\|_{L^2} + \|duQ\|_{L^2} \\ &\lesssim (\|A\|_{L^4} + \|du\|_{L^4})\|Q\|_{L^4} \\ &\lesssim \varepsilon\|\partial Q\|_{L^2}\end{aligned} \tag{2.3.27}$$

This proves equivalence of the H^1 norms of Q and ψ. Interpolation then provides equivalence for the H^s norms for all $0 \leq s \leq 1$. To conclude the equivalence of all the H^s norms of q and du, we apply the above argument to $\psi = \nabla^\ell du$ for all $\ell \in \mathbb{N}$.

Note that a similar argument also proves the equivalence of the H^s norms of ψ if we instead used covariant derivatives on u^*TN. That is, we can also show that

$$\|DQ\|_{L^2} \simeq \|\partial Q\|_{L^2} = \|Q\|_{\dot{H}^1_i(\mathbb{R}^4;N)} \simeq \|Q\|_{\dot{H}^1_i(M;N)} \tag{2.3.28}$$

We will use (2.3.28) in Section 2.6 when we prove that higher regularity of wave maps is preserved by the evolution.

2.4 Wave Equation for du

In this section show that for any Riemannian manifold (M, g), if $u : \mathbb{R} \times M \to N$ is a smooth wave map, then we can derive wave equations of 1-forms for du. The wave equations of 1-forms imply a system of variable coefficient wave equations for the components of du. We emphasize that the content of this section holds for any Riemannian manifold (M, g) and not just the special case $(M, g) = (\mathbb{R}^4, g)$ with g as in (2.1.1)–(2.1.4).

We begin by expressing $du \in \Gamma(T^*\tilde{M} \otimes u^*TN)$ in terms of the Coulomb frame e as in

Proposition 2.3.2, by finding u^*TN-valued one-forms $q = q_\alpha dx^\alpha$ so that

$$du = q^a e_a \tag{2.4.1}$$

Here $q^a_\alpha = u^*h\,(\partial_\alpha u, e_a)$. Assuming that u is a wave map, we derive a wave equation of 1-forms for q. In what follows we let $\Box = d\delta + \delta d$ denote the Hodge Laplacian on p-forms over $\tilde{M} = \mathbb{R} \times M$, where d is the exterior derivative on $\tilde{M}$ and δ is the adjoint to d.

Lemma 2.4.1. *Let $u : (\tilde{M}, \eta) \to (N, h)$ be a smooth wave map. And let $q = du$ be the representation of du in the Coulomb frame e as in* (2.4.1)*. Then we have $\delta\, q^c = A^c_{a,\alpha}\, \eta^{\alpha\beta}\, q^a_\beta$.*

Proof. This follows from the fact that u is a wave map. We have that u is wave map if and only if

$$\frac{1}{\sqrt{|\eta|}} D_\alpha \left(\sqrt{|\eta|}\, \eta^{\alpha\beta} \partial_\beta u\right) = 0 \quad \iff \quad \frac{1}{\sqrt{|\eta|}} D_\alpha \left(\sqrt{|\eta|}\, \eta^{\alpha\beta} q^a_\beta e_a\right) = 0$$

Hence, we have

$$\begin{aligned} 0 =& \frac{1}{\sqrt{|\eta|}} D_\alpha \left(\sqrt{|\eta|}\, \eta^{\alpha\beta} q^a_\beta e_a\right) \\ =& \frac{1}{\sqrt{|\eta|}} \partial_\alpha \left(\sqrt{|\eta|}\, \eta^{\alpha\beta} q^c_\beta\right) e_c + \eta^{\alpha\beta} q^a_\beta A^c_{a,\alpha} e_c \\ =& -\delta q^c e_c + A^c_{a,\alpha} \eta^{\alpha\beta}\, q^a_\beta\, e_c \end{aligned}$$

Therefore,

$$\delta\, q^c = A^c_{a,\alpha}\, \eta^{\alpha\beta}\, q^a_\beta$$

as desired. $\square$

Lemma 2.4.2. *Let $u : (\tilde{M}, \eta) \to (N, h)$ be a smooth map and let $q = du$ be the representation of du in the Coulomb frame as in* (2.4.1). *Then we have $dq^c = -A^c_b \wedge q^b$.*

Proof. First we claim that $D_\alpha(\partial_\beta u) - D_\beta(\partial_\alpha u) = 0$. To see this recall that

$$D_\alpha(\partial_\beta u)^k = \partial_\alpha \partial_\beta u^k + \Gamma^k_{ij}(u)\partial_\alpha u^j \partial_\beta u^i$$

Then the claim follows from the fact that $\partial_\alpha \partial_\beta u^k = \partial_\beta \partial_\alpha u^k$ and the fact that $\Gamma^k_{ij} = \Gamma^k_{ji}$. The above implies that

$$D_\alpha(q^a_\beta\, e_a) - D_\beta(q^a_\alpha\, e_a) = 0$$

Now, recalling that the A is the connection form for the frame e we have that

$$\begin{aligned} 0 &= D_\alpha(q^b_\beta\, e_b) - D_\beta(q^b_\alpha\, e_b) \\ &= \left(\partial_\alpha q^c_\beta + A^c_{b,\alpha}\, q^b_\beta - \partial_\beta q^c_\alpha - A^c_{b,\beta}\, q^b_\alpha\right) e_c \\ &= \left(\partial_\alpha q^c_\beta - \partial_\beta q^c_\alpha + A^c_{b,\alpha}\, q^b_\beta - A^c_{b,\beta}\, q^b_\alpha\right) e_c \\ &= \left((dq^c)_{\alpha\beta} - (A^c_b \wedge q^b)_{\beta\alpha}\right) e_c \end{aligned}$$

and the lemma follows. ∎

Lemma 2.4.2 shows that in local coordinates on $\tilde{M}$ we have that $(dq^c)_{\alpha\beta}\, dx^\alpha \wedge dx^\beta = (A^c_b \wedge q^b)_{\beta\alpha}\, dx^\alpha \wedge dx^\beta$. We can abbreviate this by writing $dq = -A \wedge q$. Hence, by Lemma 2.4.1 and Lemma 2.4.2 we obtain the following equation for q

$$\Box q = d(\eta^{\alpha\beta} A_\alpha q_\beta) + \delta(-A \wedge q) \tag{2.4.2}$$

This is a system of wave equations for the u^*TN valued 1-form q. In coordinates we can express the operator δ on a 2-form ω in terms of Levi-Civita connection, ∇, on $\mathbb{R} \times M$ as follows

$$(\delta\omega)_\beta = -(\nabla^\alpha \omega)_{\alpha\beta} \tag{2.4.3}$$

where $\nabla^\alpha = \eta^{\alpha\beta}\nabla_\beta$. Hence, in components, the equations for q become

$$(\Box q)_\gamma = \partial_\gamma(\eta^{\alpha\beta} A_\alpha q_\beta) + \nabla^\alpha (A \wedge q)_{\alpha\gamma} \tag{2.4.4}$$

By expanding the right-hand side of (2.4.4) we obtain the following equation for q.

Proposition 2.4.3. *Let $u : (\tilde{M}, \eta) \to (N, h)$ be a smooth wave map. Let $q = du$ be the representation of du in the Coulomb frame, e as in* (2.4.1)*. Then q satisfies the following wave equation of* 1*-forms, written in components as*

$$(\Box q)_\gamma = F_{\gamma\alpha}\, q^\alpha + A_\alpha\, A^\alpha\, q_\gamma + (\nabla^\alpha A)_\alpha\, q_\gamma + 2A^\alpha\, (\nabla_\alpha q)_\gamma \tag{2.4.5}$$

where F is the curvature tensor on u^*TN.

Remark 3. Proposition 2.4.3 essentially amounts to differentiating the wave map equation (1.1.1) and then expressing the result in terms of the Coulomb frame. We emphasize that in order to obtain (2.4.5) we must begin with a wave map u.

Proof. We begin by expanding the right-hand side of (2.4.4)

$$(\Box q)_\gamma = \partial_\gamma(\eta^{\alpha\beta}A_\alpha q_\beta) + \nabla^\alpha(A\wedge q)_{\alpha\gamma}$$

$$= (\partial_\gamma\eta^{\alpha\beta})A_\alpha q_\beta + \eta^{\alpha\beta}(\partial_\gamma A_\alpha)q_\beta + \eta^{\alpha\beta}A_\alpha(\partial_\gamma q_\beta) + \eta^{\alpha\beta}\left(\nabla_\beta A\wedge q\right)_{\alpha\gamma}$$
$$+ \eta^{\alpha\beta}\left(A\wedge\nabla_\beta q\right)_{\alpha\gamma}$$

$$= (\partial_\gamma\eta^{\alpha\beta})A_\alpha q_\beta + \eta^{\alpha\beta}(\partial_\gamma A_\alpha)q_\beta + \eta^{\alpha\beta}A_\alpha(\partial_\gamma q_\beta) + \eta^{\alpha\beta}(\nabla_\beta A)_\alpha\, q_\gamma$$
$$- \eta^{\alpha\beta}(\nabla_\beta A)_\gamma\, q_\alpha + \eta^{\alpha\beta}A_\alpha(\nabla_\beta q)_\gamma - \eta^{\alpha\beta}A_\gamma(\nabla_\beta q)_\alpha$$

Now, observe that

$$\eta^{\alpha\beta}(\partial_\gamma A_\alpha)q_\beta = \partial_\alpha A_\gamma q^\alpha + F_{\gamma\alpha}q^\alpha - A_\gamma A_\alpha q^\alpha + A_\alpha A_\gamma q^\alpha$$

and by Lemma 2.4.1 we have

$$-\eta^{\alpha\beta}A_\gamma(\nabla_\beta q)_\alpha = -A_\gamma(\nabla^\alpha q)_\alpha = A_\gamma\delta q = A_\gamma A_\alpha q^\alpha$$

Also, Lemma 2.4.2 implies

$$\eta^{\alpha\beta}A_\alpha(\partial_\gamma q_\beta) = \eta^{\alpha\beta}A_\alpha(dq)_{\gamma\beta} + \eta^{\alpha\beta}A_\alpha\partial_\beta q_\gamma$$

$$= \eta^{\alpha\beta}A_\alpha(A\wedge q)_{\beta\gamma} + \eta^{\alpha\beta}A_\alpha\partial_\beta q_\gamma$$

$$= A_\alpha A^\alpha q_\gamma - A_\alpha A_\gamma q^\alpha + \eta^{\alpha\beta}A_\alpha\partial_\beta q_\gamma$$

Hence,

$$
\begin{aligned}
(\Box q)_\gamma = F_{\gamma\alpha} q^\alpha + A_\alpha A^\alpha q_\gamma + (\nabla^\alpha A)_\alpha \, q_\gamma + A^\alpha (\nabla_\alpha q)_\gamma + \eta^{\alpha\beta} A_\alpha \partial_\beta q_\gamma \\
+ (\partial_\gamma \eta^{\alpha\beta}) A_\alpha q_\beta + \eta^{\alpha\beta} \partial_\beta A_\gamma q_\alpha - \eta^{\alpha\beta} (\nabla_\beta A)_\gamma \, q_\alpha
\end{aligned}
$$

Next observe that

$$
\eta^{\alpha\beta} A_\alpha \partial_\beta q_\gamma = \eta^{\alpha\beta} A_\alpha (\nabla_\beta q)_\gamma + \eta^{\alpha\beta} A_\alpha \Gamma^\sigma_{\beta\gamma} q_\sigma
$$

and

$$
\eta^{\alpha\beta} \partial_\beta A_\gamma q_\alpha - \eta^{\alpha\beta} (\nabla_\beta A)_\gamma q_\alpha = \eta^{\alpha\beta} \Gamma^\sigma_{\beta\gamma} A_\sigma q_\alpha
$$

Therefore,

$$
\begin{aligned}
(\Box q)_\gamma = F_{\gamma\alpha} q^\alpha + A_\alpha A^\alpha q_\gamma + (\nabla^\alpha A)_\alpha \, q_\gamma + 2A^\alpha (\nabla_\alpha q)_\gamma \\
+ (\partial_\gamma \eta^{\alpha\beta}) A_\alpha q_\beta + \eta^{\alpha\beta} A_\alpha \Gamma^\sigma_{\beta\gamma} q_\sigma + \eta^{\alpha\beta} \Gamma^\sigma_{\beta\gamma} A_\sigma q_\alpha
\end{aligned}
$$

Finally, we claim that

$$
(\partial_\gamma \eta^{\alpha\beta}) A_\alpha q_\beta + \eta^{\alpha\beta} A_\alpha \Gamma^\sigma_{\beta\gamma} q_\sigma + \eta^{\alpha\beta} \Gamma^\sigma_{\beta\gamma} A_\sigma q_\alpha = 0 \tag{2.4.6}
$$

This follows from the fact that $\Gamma^\sigma_{\beta\gamma} = \frac{1}{2} \eta^{\sigma\delta} (\partial_\beta \eta_{\gamma\delta} + \partial_\gamma \eta_{\beta\delta} - \partial_\delta \eta_{\beta\gamma})$ and that $\partial_\gamma \eta^{\alpha\beta} = -\eta^{\alpha\delta} (\partial_\gamma \eta_{\delta\sigma}) \eta^{\sigma\beta}$, the latter statement being the general fact that for an invertible matrix

$G(x)$ we have $\partial_i G^{-1} = -G^{-1}\partial_i G G^{-1}$. To show (2.4.6), we write

$$
\begin{aligned}
(\partial_\gamma \eta^{\alpha\beta}) A_\alpha q_\beta + \eta^{\alpha\beta} A_\alpha \Gamma^\sigma_{\beta\gamma} q_\sigma + \eta^{\alpha\beta}\Gamma^\sigma_{\beta\gamma} A_\sigma q_\alpha \\
&= \left(\partial_\gamma \eta^{\alpha\beta} + \eta^{\alpha\sigma}\Gamma^\beta_{\sigma\gamma} + \eta^{\sigma\beta}\Gamma^\alpha_{\sigma\gamma}\right) A_\alpha q_\beta \\
&= \left(-\eta^{\alpha\delta}\eta^{\sigma\beta}\partial_\gamma \eta_{\delta\sigma}\right) A_{\alpha\beta} \\
&\quad + \left(\frac{1}{2}\eta^{\alpha\sigma}\eta^{\beta\delta}(\partial_\sigma g_{\delta\gamma} + \partial_\gamma g_{\sigma\delta} - \partial_\delta g_{\sigma\gamma})\right) A_\alpha q_\beta \qquad (2.4.7) \\
&\quad + \left(\frac{1}{2}\eta^{\sigma\beta}\eta^{\alpha\delta}(\partial_\sigma g_{\delta\gamma} + \partial_\gamma g_{\sigma\delta} - \partial_\delta g_{\sigma\gamma})\right) A_\alpha q_\beta \\
&= 0
\end{aligned}
$$

where the last line follows by swapping σ and δ in line (2.4.7) above. Therefore,

$$(\Box q)_\gamma = F_{\gamma\alpha} q^\alpha + A_\alpha A^\alpha q_\gamma + (\nabla^\alpha A)_\alpha\, q_\gamma + 2A^\alpha (\nabla_\alpha q)_\gamma$$

as claimed. $\Box$

Next, we examine the left hand side of (2.4.2). We claim that for a 1-form q, we can write $\Box q = \ddot{q} + \Delta q$, where Δ denotes the Hodge Laplacian on the Riemannian manifold M and $\ddot{q}$ is the 1-form given in local coordinates by $\ddot{q}(t,x) = \ddot{q}_\alpha(t,x)\, dx^\alpha$.

Lemma 2.4.4. *We can express $\Box$ in local coordinates on $\mathbb{R}\times M$ by*

$$(\Box q)_\gamma = \ddot{q}_\gamma + (\Delta q)_\gamma \qquad (2.4.8)$$

where here Δ is the Hodge Laplacian on 1-forms over M.

Remark 4. Despite the appearance of the + sign in expression (2.4.8), the expression $\Box =$

$\partial_t^2 + \Delta$ is, in fact, a hyperbolic operator as we will see in Proposition 2.4.6. The sign in (2.4.8) is simply due to our sign convention for the Hodge Laplacian Δ. Our convention is such that for a 0-form f, $\Delta f = -\Delta_g f$ where $\Delta_g = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j)$ is the Laplace-Beltami operator on M.

Proof. Let $\eta = \mathrm{diag}(-1, g)$ denote the metric on $\tilde{M} = \mathbb{R} \times M$. In the following argument, $0 \le \alpha, \beta, \gamma \le d$ will be indices denoting coordinates on $\mathbb{R} \times M$ and $1 \le i, j \le d$ will be indices denoting coordinates on M. Also we denote by d_M, (resp. δ_M), the exterior differential, (resp. co-differential), on M. It follows that

$$
\begin{aligned}
(\Box q)_\gamma &= (d\delta q)_\gamma + (\delta d q)_\gamma \\
&= -\partial_\gamma \left(\eta^{\alpha\beta}(\nabla_\beta q)_\alpha\right) - \eta^{\alpha\beta}(\nabla_\beta dq)_{\alpha\gamma} \\
&= -\partial_\gamma \left(-(\nabla_0 q)_0 + g^{ij}(\nabla_j q)_i\right) + (\nabla_0 dq)_{0\gamma} - g^{ij}(\nabla_j dq)_{i\gamma} \\
&= \partial_\gamma\partial_0 q_0 - \partial_\gamma \left(g^{ij}(\nabla_j q)_i\right) + \partial_0\partial_0 q_\gamma - \partial_\gamma\partial_0 q_0 \ - g^{ij}(\nabla_j dq)_{i\gamma} \\
&= \ddot{q}_\gamma + (d_M\delta_M q)_\gamma + (\delta_M d_M q)_\gamma \\
&= \ddot{q}_\gamma + (\Delta q)_\gamma
\end{aligned}
$$

Above we have used the fact that the Christoffel symbols $\Gamma^\delta_{\alpha\beta} = 0$ if either α, β, or δ are equal to 0. ☐

We can derive a coordinate representation for the Hodge Laplacian Δ on 1-forms in terms of the Laplace-Beltrami operator, Δ_g, on functions plus lower order terms.

Lemma 2.4.5. *The Hodge Laplacian on* 1*-forms* q *can be written in coordinates as*

$$(\Delta q)_\gamma = -\Delta_g q_\gamma + 2g^{ij}\Gamma^k_{j\gamma}\partial_i q_k + \partial_\gamma(g^{ij}\Gamma^k_{ij})q_k \qquad (2.4.9)$$

Proof. Here we will let d and δ denote the exterior differential and exterior co-differential on

M. Then,

$$\begin{aligned}
(\Delta q)_\gamma &= (d\delta q)_\gamma + (\delta d q)_\gamma \\
&= -\partial_\gamma \left(g^{ij} (\nabla_j q)_i \right) - g^{ij} \left(\nabla_j dq \right)_{i\gamma} \\
&= -(\partial_\gamma g^{ij})(\nabla_j q)_i - g^{ij} \partial_\gamma (\partial_j q_i - \Gamma^k_{ji} q_k) \\
&\quad - g^{ij} \left(\partial_j (dq)_{i\gamma} - \Gamma^k_{ji} (dq)_{k\gamma} - \Gamma^k_{j\gamma} (dq)_{ik} \right) \\
&= -(\partial_\gamma g^{ij}) \partial_j q_i + \partial_\gamma (g^{ij} \Gamma^k_{ji}) q_k - g^{ij} \partial_j \partial_i q_\gamma + g^{ij} \Gamma^k_{ji} \partial_k q_\gamma \\
&\quad + g^{ij} \Gamma^k_{j\gamma} \partial_i q_k - g^{ij} \Gamma^k_{j\gamma} \partial_k q_i
\end{aligned}$$

Recalling that $\Delta_g q_\gamma = g^{ij} \partial_j \partial_i q_\gamma - g^{ij} \Gamma^k_{ji} \partial_k q_\gamma$ and that $\partial_\gamma g^{ij} = -g^{ik} \partial_\gamma g_{km} g^{mj}$ we have then that

$$(\Delta q)_\gamma = -\Delta_g q_\gamma + \partial_\gamma (g^{ij} \Gamma^k_{ji}) q_k + g^{ik} \partial_\gamma g_{km} g^{mj} \partial_j q_i + g^{ij} \Gamma^k_{j\gamma} \partial_i q_k - g^{ij} \Gamma^k_{j\gamma} \partial_k q_i$$

Finally observe that

$$g^{ik} \partial_\gamma g_{km} g^{mj} \partial_j q_i + g^{ij} \Gamma^k_{j\gamma} \partial_i q_k - g^{ij} \Gamma^k_{j\gamma} \partial_k q_i = 2 g^{ij} \Gamma^k_{j\gamma} \partial_i q_k$$

Therefore

$$(\Delta q)_\gamma = -\Delta_g q_\gamma + 2 g^{ij} \Gamma^k_{j\gamma} \partial_i q_k + \partial_\gamma (g^{ij} \Gamma^k_{ij}) q_k$$

which is exactly (2.4.9). $\square$

Combining the results of the previous two lemmas with Proposition 2.4.3 gives us a system of nonlinear wave equations for the components of q. The following Proposition is the main result of this section and will be used to prove a priori estimates for the differential, du, of a wave map u.

Proposition 2.4.6. *Let $u : (\tilde{M}, \eta) \to (N, h)$ be a smooth wave map. Let $q = du$ be the representation of du in the Coulomb frame, e as in* (2.4.1)*. Then, the components of q satisfy the following system of variable coefficient wave equations:*

$$\begin{aligned} &\ddot{q}_\gamma - \Delta_g q_\gamma + 2g^{ij}\Gamma^k_{j\gamma}\partial_i q_k + \partial_\gamma(g^{ij}\Gamma^k_{ij})q_k \\ &\qquad = F_{\gamma\alpha}\, q^\alpha + A_\alpha\, A^\alpha\, q_\gamma + (\nabla^\alpha A)_\alpha\, q_\gamma + 2A^\alpha\, (\nabla_\alpha q)_\gamma \end{aligned} \tag{2.4.10}$$

Expanding the term $\Delta_g q_\gamma$, the left-hand side of the above system becomes

$$\ddot{q}_\gamma - g^{ij}\partial_i\partial_j q_\gamma + g^{ij}\Gamma^k_{ij}\partial_k q_\gamma + 2g^{ij}\Gamma^k_{j\gamma}\partial_i q_k + \partial_\gamma(g^{ij}\Gamma^k_{ij})q_k \tag{2.4.11}$$

2.5 A Priori Estimates

To derive a priori bounds for wave maps u we use the Strichartz estimates for variable coefficient wave equations proved in [57]. We require a Lorentz space refinement of the estimates in [57] obtained by a rephrasing in terms of Besov spaces and real interpolation. Equation (2.4.10), the decay assumptions on the metric g specified in (2.1.1)–(2.1.3), and [57, Theorems 4 and 6] imply the following estimates for q:

$$\|q\|_{L^2_t \dot{B}^{\frac{1}{6}}_{6,2}} + \|\partial q\|_{L^\infty_t L^2_x} \lesssim \|q[0]\|_{\dot{H}^1 \times L^2} + \|H\|_{L^1_t L^2_x} \tag{2.5.1}$$

where $H_\gamma := F_{\gamma\alpha}\, q^\alpha + A_\alpha\, A^\alpha\, q_\gamma + (\nabla^\alpha A)_\alpha\, q_\gamma + 2A^\alpha\, (\nabla_\alpha q)_\gamma$ is the nonlinearity in (2.4.10). There are a few things to note. The first is that we have extended the result in [57] to the case of a system of variable coefficient equations as q is the solution to such a system. However this extension is immediate as the methods in [57] allow us to treat the lower order terms in (2.4.10) perturbatively, and the principle part of our operator is diagonal. Hence the system of equations for q in (2.4.10) falls directly into the class of equations that are treated in [57] because of the assumptions in (2.1.1)–(2.1.3). The second observation is that a Besov norm appears on the left-hand side in (2.5.1). This refinement can be obtained by an easy modification of the proof of Lemma 19 in [57]. For completeness we carry out this refinement in Section 2.8.3.

To obtain a Lorentz space version of estimate (2.5.1) we use the Besov space embedding into Lorentz spaces, see Lemma 2.9.4, with $d = 4$, $s = \frac{1}{6}$, $q = 6$, $p = 8$ and $r = 2$ which gives

$$\dot{B}^{\frac{1}{6}}_{6,2}(\mathbb{R}^4) \hookrightarrow L^{8,2}(\mathbb{R}^4)$$

This, together with the estimate in (2.5.1), gives

$$\|q\|_{L^2_t L^{8,2}_x} + \|\partial q\|_{L^\infty_t L^2_x} \lesssim \|q[0]\|_{\dot{H}^1 \times L^2} + \|H\|_{L^1_t L^2_x} \tag{2.5.2}$$

We use Proposition 2.3.2, together with Sobolev embedding to estimate the various terms in H. In local coordinates on M, H is given by

$$\begin{aligned} H_\gamma = {} & \eta^{\gamma\alpha} F_{\gamma\beta}\, q_\alpha + \eta^{\alpha\beta} A_\alpha\, A_\beta\, q_\gamma + \eta^{\alpha\beta} (\partial_\beta A_\alpha)\, q_\gamma \\ & + 2\eta^{\alpha\beta} A_\beta \Gamma^\delta_{\alpha\gamma} q_\delta + 2\eta^{\alpha\beta} A_\beta\, (\partial_\alpha q_\gamma) \end{aligned} \tag{2.5.3}$$

Hence, at any time $t \in [0, T)$, (where T is chosen as in (2.3.7), for the sake of our bootstrap argument), we have

$$\begin{aligned}
\|H\|_{L^2_x} &\lesssim \|Fq\|_{L^2_x} + \|A^2 q\|_{L^2_x} + \|(\partial A)q\|_{L^2_x} + \|\Gamma A q\|_{L^2_x} + \|A\partial q\|_{L^2_x} \\
&\lesssim \|F\|_{L^4_x}\|q\|_{L^4_x} + \|A^2\|_{L^{\frac{8}{3}}_x}\|q\|_{L^8_x} + \|\partial A\|_{L^{\frac{8}{3}}_x}\|q\|_{L^8_x} \\
&\quad + \|\Gamma\|_{L^4_x}\|A\|_{L^8_x}\|q\|_{L^8_x} + \|A\|_{L^\infty_x}\|\partial q\|_{L^2_x} \\
&\lesssim \|q\|^2_{L^8_x}\|\partial q\|_{L^2_x} + \|A\|_{\dot{W}^{1,\frac{8}{3}}_x}\|q\|_{L^8_x} + \|q\|^2_{L^{8,2}_x}\|\partial q\|_{L^2_x} \\
&\lesssim \|q\|^2_{L^{8,2}_x}\|\partial q\|_{L^2_x}
\end{aligned} \tag{2.5.4}$$

where in the third inequality above we have used Proposition 2.3.2 and Sobolev embedding to show that $\|A^2\|_{L^{\frac{8}{3}}} \lesssim \|A\|_{L^4}\|A\|_{L^8} \lesssim \|A\|_{\dot{W}^{1,\frac{8}{3}}}$. This implies that we have the estimate

$$\begin{aligned}
\|q\|_{L^2_t([0,T);L^{8,2}_x)} + \|\partial q\|_{L^\infty_t([0,T);L^2_x)} &\lesssim \|q[0]\|_{\dot{H}^1\times L^2} \\
&\quad + \|q\|^2_{L^2_t([0,T);L^{(8,2)}_x)}\|\partial q\|_{L^\infty_t([0,T);L^2_x)} \\
&\lesssim \|q[0]\|_{\dot{H}^1\times L^2} \\
&\quad + \left(\|q\|_{L^2_t([0,T);L^{(8,2)}_x)} + \|\partial q\|_{L^\infty_t([0,T);L^2_x)}\right)^3
\end{aligned} \tag{2.5.5}$$

By the equivalence of the extrinsic and intrinsic norms of $du = q$, see Section 2.3.2, we can show

$$\|q[0]\|_{\dot{H}^1\times L^2} \lesssim \|du_0\|_{\dot{H}^1} + \|u_1\|_{\dot{H}^1} \lesssim \varepsilon_0 \tag{2.5.6}$$

Hence as long as ε_0 is sufficiently small, we can use a bootstrap/continuity-trapping argument to absorb the cubic term,

$$\left(\|q\|_{L^2([0,T);L^{(8,2)})} + \|\partial q\|_{L^\infty([0,T);L^2)}\right)^3$$

on the left-hand side in (2.5.5) and obtain the global in time estimate

$$\|q\|_{L^2L^{8,2}} + \|\partial q\|_{L^\infty L^2} \lesssim \|du_0\|_{\dot{H}^1} + \|u_1\|_{\dot{H}^1} \tag{2.5.7}$$

Using again the equivalence of the relevant extrinsic norms of du and intrinsic norms of q, see Section 2.3.2, and recalling that $\|du\|_{L^2L^8} \leq \|du\|_{L^2L^{8,2}}$, we obtain the desired global a priori bounds which we record in the following proposition:

Proposition 2.5.1. *Let* $(\tilde{M}, \eta) = (\mathbb{R} \times \mathbb{R}^4, \eta)$*, where* $\eta = \mathrm{diag}(-1, g)$*, and* g *satisfies the conditions* (2.1.1)–(2.1.3)*. Let* $u : (\tilde{M}, \eta) \to (N, h)$ *be a smooth wave map with initial data* (u_0, u_1) *satisfying* (2.1.7)*. Then* du *satisfies the following global, a priori estimates*

$$\|du\|_{L^2_t L^8_x} + \|du\|_{L^\infty_t \dot{H}^1_x} \lesssim \|du_0\|_{\dot{H}^1} + \|u_1\|_{\dot{H}^1} \lesssim \varepsilon_0 \tag{2.5.8}$$

2.6 Higher Regularity

In this section we show that higher regularity of the data is preserved. In particular, we show that if we begin with initial data, $(u_0, u_1) \in H^s \times H^{s-1}((\mathbb{R}^4, g), TN)$ for any $s \geq 2$, such that (2.1.7) holds, then the $H^s \times H^{s-1}$ norm of the solution, $(u(t), \dot{u}(t))$, to (1.1.1), is finite for any time t. This will allow us to immediately deduce global existence of wave maps with data $(u_0, u_1) \in H^s \times H^{s-1}$ satisfying (2.1.7) for $s \geq 5$, as any local solution to the Cauchy problem can then be extended past any finite time, T, using the high regularity local theory with data $(u(T), \dot{u}(T))$, which is finite in $H^s \times H^{s-1}$ due to the results in this section. We note that the a priori estimates, (2.5.8), and in particular the global control of $\|du\|_{L^2_t L^8_x}$, will play a key role in the argument. We formulate the main result of this section in the following proposition:

Proposition 2.6.1. *Let* $(\tilde{M}, \eta) = (\mathbb{R} \times \mathbb{R}^4, \eta)$*, where* $\eta = \mathrm{diag}(-1, g)$*, and* g *satisfies the conditions* (2.1.1)–(2.1.3)*. Let* $u : (\tilde{M}, \eta) \to (N, h)$ *be a solution to* (1.1.1) *with initial* *data* (u_0, u_1) *that is small in the sense of* (2.1.7). *Suppose in addition that* $(u_0, u_1) \in H^s \times H^{s-1}((\mathbb{R}^4, g), TN)$ *with* $s \geq 2$. *Then for any time* T, *the* $H^s \times H^{s-1}((\mathbb{R}^4, g), N)$ *norm of the solution* $(u(T), \dot{u}(T))$ *is finite. In particular,*

$$\sup_{0 \leq t \leq T} \|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq C_T \|u[0]\|_{H^s \times H^{s-1}} \tag{2.6.1}$$

where the constant, C_T, *depends on* T *and* ε_0.

To prove Proposition 2.6.1, we begin by differentiating (1.1.1) covariantly. Let $1 \leq \gamma \leq 4$ be a space index and let $q = du$ be the representation of du in the Coulomb frame. Then, recalling that $D_\alpha D_\beta - D_\beta D_\alpha = F_{\alpha\beta}$, we have

$$\begin{aligned}
0 =& D_\gamma \left(\frac{1}{\sqrt{|\eta|}} D_\alpha(\sqrt{|\eta|}\eta^{\alpha\beta} q_\beta) \right) \\
=& - D_\gamma \left(D_t q_t\right) + D_\gamma \left(\frac{1}{\sqrt{|g|}} D_\alpha(\sqrt{|g|} g^{\alpha\beta} q_\beta) \right) \\
=& - D_t D_t q_\gamma + \frac{1}{\sqrt{|g|}} D_\alpha(\sqrt{|g|} g^{\alpha\beta} D_\gamma q_\beta) \\
&+ F_{\gamma\alpha}\eta^{\alpha\beta} q_\beta + \partial_\gamma(g^{\alpha\beta}) D_\alpha q_\beta + \partial_\gamma \left(\frac{1}{\sqrt{|g|}} \partial_\alpha(\sqrt{|g|} g^{\alpha\beta}) \right) q_\beta
\end{aligned}$$

This implies that q satisfies the equation

$$\begin{aligned}
D_t D_t q_\gamma - \frac{1}{\sqrt{|g|}} &D_\alpha(\sqrt{|g|} g^{\alpha\beta} D_\beta q_\gamma) \\
&= F_{\gamma\alpha}\eta^{\alpha\beta} q_\beta + \partial_\gamma(g^{\alpha\beta}) D_\alpha q_\beta - \partial_\gamma \left(g^{\alpha\rho}\Gamma^\beta_{\alpha\rho} \right) q_\beta
\end{aligned} \tag{2.6.2}$$

Pairing this equation with $g^{\gamma\delta} D_t q_\delta$ as sections of $u^*TN \to M$ and integrating over M gives

$$\int_M \left\langle D_t D_t q_\gamma - \frac{1}{\sqrt{|g|}} D_\alpha(\sqrt{|g|} g^{\alpha\beta} D_\gamma q_\beta),\, g^{\gamma\delta} D_t q_\delta \right\rangle \sqrt{|g|}\, dx$$
$$= \int_M \left\langle F_{\gamma\alpha} \eta^{\alpha\beta} q_\beta,\, g^{\gamma\delta} D_t q_\delta \right\rangle \sqrt{|g|}\, dx + \int_M \left\langle \partial_\gamma (g^{\alpha\beta}) D_\alpha q_\beta,\, g^{\gamma\delta} D_t q_\delta \right\rangle \sqrt{|g|}\, dx$$
$$- \int_M \left\langle \partial_\gamma \left(g^{\alpha\rho} \Gamma^\beta_{\alpha\rho} \right) q_\beta,\, g^{\gamma\delta} D_t q_\delta \right\rangle \sqrt{|g|}\, dx$$

Integrating the second term on the left by parts gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Dq\|^2_{L^2} = &- \int_M g^{\alpha\beta} \partial_\alpha (g^{\gamma\delta}) \left\langle D_\gamma q_\beta,\, D_t q_\delta \right\rangle \sqrt{|g|}\, dx \\
&+ \int_M g^{\alpha\beta} g^{\gamma\delta} \left\langle D_\gamma q_\beta,\, F_{\alpha t} q_\delta \right\rangle \sqrt{|g|}\, dx \\
&+ \int_M \eta^{\alpha\beta} g^{\gamma\delta} \left\langle F_{\gamma\alpha} q_\beta,\, D_t q_\delta \right\rangle \sqrt{|g|}\, dx \\
&+ \int_M \partial_\gamma (g^{\alpha\beta}) g^{\gamma\delta} \left\langle D_\alpha q_\beta,\, D_t q_\delta \right\rangle \sqrt{|g|}\, dx \\
&- \int_M \partial_\gamma \left(g^{\alpha\rho} \Gamma^\beta_{\alpha\rho} \right) g^{\gamma\delta} \left\langle q_\beta,\, D_t q_\delta \right\rangle \sqrt{|g|}\, dx
\end{aligned}$$

where we define

$$\begin{aligned}
\|Dq\|^2_{L^2} := &\int_M g^{\gamma\delta} \left\langle D_\gamma q_t,\, D_\delta q_t \right\rangle \sqrt{|g|}\, dx \\
&+ \int_M g^{\alpha\beta} g^{\gamma\delta} \left\langle D_\gamma q_\beta,\, D_\alpha q_\delta \right\rangle \sqrt{|g|}\, dx
\end{aligned} \tag{2.6.3}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|Dq\|_{L^2}^2 \lesssim \|\partial g\|_{L^\infty}\|Dq\|_{L^2}^2 + \|F\|_{L^4}\|q\|_{L^4}\|Dq\|_{L^2} + \|\partial^2 g\|_{L^4}\|q\|_{L^4}\|Dq\|_{L^2} \tag{2.6.4}$$

$$\lesssim \|Dq\|_{L^2}^2\|q\|_{L^8}^2 + \|Dq\|_{L^2}^2$$

Integrating in time gives

$$\|Dq(t)\|_{L^2}^2 \le \|Dq(0)\|_{L^2}^2 + C\int_0^t \|Dq(s)\|_{L^2}^2(\|q(s)\|_{L^8}^2 + 1)\, ds$$

Hence by Gronwall's inequality we have

$$\begin{aligned}\|Dq(t)\|_{L^2}^2 &\le \|Dq(0)\|_{L^2}^2 \exp\left(C\int_0^t (\|q(s)\|_{L^8}^2 + 1)\, ds\right)\\ &\le \|Dq(0)\|_{L^2}^2 \exp\left(C(\|q\|_{L^2L^8}^2 + t)\right)\\ &\le \|Dq(0)\|_{L^2}^2 \exp\left(C(\varepsilon_0 + t)\right)\end{aligned} \tag{2.6.5}$$

The last inequality follows from the global a priori bounds, (2.5.8), proved in the previous section.

As explained in Section 2.3.2, see (2.3.27) and (2.3.28), the inequality in (2.6.5) is equivalent to a bound on $(u, \dot{u})$ in $\dot{H}^2(M;N) \times \dot{H}^1(M;N)$. We thus obtain a bound on $(u, \dot{u})$ in $H^2(M;N) \times H^1(M;N)$ by combining the above with the conservation of energy and the simple L^2 estimates obtained by the fundamental theorem of calculus and Minkowksi's inequality

$$\|u(t)\|_{L^2} \le \|u(0)\|_{L^2} + t\|\partial_t u(t)\|_{L^2} \tag{2.6.6}$$

Remark 5. Of course, we already have proved an even stronger result than (2.6.5) in the previous section where we showed that, in fact, $\|q(t)\|_{\dot{H}^1} \lesssim \varepsilon_0$ for any time t where the wave map u is defined. We have gone through the trouble in proving (2.6.5) here in order to establish the technique required to prove bounds on higher derivatives of q below.

To obtain bounds in $H^3(M;N) \times H^2(M;N)$ and in $H^4(M;N) \times H^3(M;N)$ we proceed in exactly the same manner as above, differentiating (2.6.2) two more times and obtaining wave equations for $D_\kappa q_\delta$ and for $D_\mu D_\kappa q_\delta$. Roughly, these are of the form

$$D_t D_t D_\kappa q_\gamma - \frac{1}{\sqrt{|g|}} D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} D_\beta D_\kappa q_\gamma \right) = D_\kappa (\eta^{\alpha\beta} F_{\gamma\alpha} q_\beta) + \text{lower order terms} \tag{2.6.7}$$

and

$$D_t D_t D_\mu D_\kappa q_\gamma - \frac{1}{\sqrt{|g|}} D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} D_\beta D_\mu D_\kappa q_\gamma \right) = = D_\mu D_\kappa (\eta^{\alpha\beta} F_{\gamma\alpha} q_\beta) + \text{lower order terms} \tag{2.6.8}$$

Proceeding as above, we pair (2.6.7) with $g^{\kappa\iota} g^{\gamma\delta} D_t D_\iota q_\delta$, and we pair (2.6.8) with

$$g^{\mu\nu} g^{\kappa\iota} g^{\gamma\delta} D_t D_\nu D_\iota q_\delta$$

and integrate over M to obtain for $\ell = 1, 2$

$$\frac{1}{2} \frac{d}{dt} \|D^{\ell+1} q\|_{L^2}^2 \lesssim \sum_{k=1}^{\ell+1} \|D^k q\|_{L^2}^2 + \|D^\ell (\eta F q)\|_{L^2} \|D^{\ell+1} q\|_{L^2} \tag{2.6.9}$$

We claim that (2.6.9) implies the estimate

$$\frac{1}{2}\frac{d}{dt}\|D^{\ell+1}q\|_{L^2}^2 \lesssim \sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}^2 + \left(\sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}^2\right)\|q\|_{L^8}^2 \tag{2.6.10}$$

In order to deduce (2.6.10) from (2.6.9), we need the following lemma:

Lemma 2.6.2. *For any time t and for $\ell = 1, 2$ we have*

$$\|D^\ell(\eta F q)\|_{L^2} \lesssim \sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}\|q\|_{L^8}^2 \tag{2.6.11}$$

Proof. In what follows we will freely use the equivalence of norms explained in Section 2.3.2. For $\ell = 1$, we have $\partial(\eta F q) = \partial\eta F q + \eta\partial F q + \eta F \partial q$. Schematically, recall that we have $F = R(u)(q, q)$ and hence $\partial F = (\partial R(u))(q, q, q) + 2R(u)(\partial q, q)$. Hence we have

$$\begin{aligned} \|\partial(\eta F q)\|_{L^2} &\lesssim \|\partial\eta\|_{L^4}\|q^3\|_{L^4} + \|\partial R(u)\|_{L^\infty}\|q\|_{L^4}\|q^3\|_{L^4} \\ &\quad + \|R(u)\|_{L^\infty}\|\partial q\|_{L^4}\|q^2\|_{L^4} \\ &\lesssim \|q^3\|_{L^4} + \|q\|_{\dot{H}^1}\|q^3\|_{L^4} + \|D^2 q\|_{L^2}\|q\|_{L^8}^2 \end{aligned} \tag{2.6.12}$$

Finally we claim that $\|q^3\|_{L^4} \lesssim \|D^2 q\|_{L^2}\|q\|_{L^8}^2$. This follows from the multiplicative Sobolev inequality, see [26, pg. 24]. Indeed,

$$\|q^3\|_{L^4} \lesssim \prod_{i=1}^{3}\|q\|_{L^{p_i}} \lesssim \prod_{i=1}^{3}\|D^2 q\|_{L^2}^{1-\theta_i}\|q\|_{L^8}^{\theta_i} \lesssim \|D^2 q\|_{L^2}\|q\|_{L^8}^2 \tag{2.6.13}$$

as long as we set $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{4}$, $\frac{1}{p_i} = \frac{\theta_i}{8}$ and $\theta_1 + \theta_2 + \theta_3 = 2$. For example, we can set $p_i = 12$ and $\theta_i = \frac{2}{3}$ for $i = 1, 2, 3$.

For $\ell = 2$ we have

$$\partial^2(\eta F q) = \partial^2\eta F q + \eta\partial^2 F q + \eta F\partial^2 q + 2\partial\eta\partial F q + 2\partial\eta F\partial q + 2\eta\partial F\partial q$$

And we have

$$\partial^2 F = (\partial^2 R(u))(q,q,q,q) + 5(\partial R(u))(\partial q,q,q) + 2R(u)(\partial^2 q,q) + 2R(u)(\partial q,\partial q)$$

Hence,

$$\|\partial^2(\eta F q)\|_L^2 \lesssim \|\partial^2\eta F q\|_{L^2} + \|\partial\eta\partial F q\|_{L^2} + \|\partial\eta F\partial q\|_{L^2} \tag{2.6.14}$$

$$+ \|\eta F\partial^2 q\|_{L^2} + \|\eta\partial F\partial q\|_{L^2} + \|\eta\partial^2 F q\|_{L^2} \tag{2.6.15}$$

The first three terms on the right-hand side of (2.6.14) all have derivatives hitting η and can be controlled as follows

$$\begin{aligned}
\|\partial^2\eta F q\|_{L^2} + \|\partial\eta\partial F q\|_{L^2} + \|\partial\eta F\partial q\|_{L^2} &\lesssim \|\partial^2\eta\|_{L^4}\|Fq\|_{L^4} + \|\partial\eta\|_{L^\infty}\|\partial F q\|_{L^2} \\
&\quad + \|\partial\eta\|_{L^\infty}\|F\partial q\|_{L^2} \\
&\lesssim \|q^3\|_{L^4} + \|\partial R(u)\|_{L^\infty}\|q\|_{L^4}\|q^3\|_{L^4} \\
&\quad + \|R(u)\|_{L^\infty}\|\partial q\|_{L^4}\|q^2\|_{L^4} \\
&\quad + \|\partial q\|_{L^4}\|q^2\|_{L^4} \\
&\lesssim \|D^2 q\|_{L^2}\|q\|_{L^8}^2
\end{aligned}$$

where the last line is deduced via the same argument as in (2.6.12) and (2.6.13). To estimate the first term in (2.6.15) we observe that

$$\|\eta F\partial^2 q\|_{L^2} \lesssim \|F\|_{L^4}\|\partial^2 q\|_{L^4} \lesssim \|D^3 q\|_{L^2}\|q\|_{L^8}^2$$

For the last two terms in (2.6.15) we have

$$
\begin{aligned}
\|\eta\partial F\partial q\|_{L^2} + \|\eta\partial^2 Fq\|_{L^2} \lesssim\ & \|(\partial R)(u)\|_{L^\infty}\|q^3\partial q\|_{L^2} \\
& + \|R(u)\|_{L^\infty}\|q(\partial q)^2\|_{L^2} \\
& + \|R(u)\|_{L^\infty}\|q^2\partial^2 q\|_{L^2} \\
& + \|(\partial^2 R)(u)\|_{L^\infty}\|q^5\|_{L^2}
\end{aligned}
\tag{2.6.16}
$$

$$
\begin{aligned}
\lesssim\ & \|q\|_{L^4}\|q\|_{L^{16}}^2\|\partial q\|_{L^8} + \|q\|_{L^8}\|\partial q\|_{L^{\frac{16}{3}}}^2 \\
& + \|q\|_{L^8}^2\|\partial^2 q\|_{L^4} + \|q\|_{L^8}^2\|q\|_{L^{12}}^3
\end{aligned}
$$

The multiplicative Sobolev inequality then implies

$$
\begin{aligned}
\|q\|_{16} &\lesssim \|D^3 q\|_{L^2}^{\frac{1}{3}}\|q\|_{L^8}^{\frac{2}{3}} \\
\|\partial q\|_{L^8} &\lesssim \|D^3 q\|_{L^2}^{\frac{1}{3}}\|q\|_{L^8}^{\frac{2}{3}} \\
\|\partial q\|_{L^{\frac{16}{3}}} &\lesssim \|D^3 q\|_{L^2}^{\frac{1}{2}}\|q\|_{L^8}^{\frac{1}{2}} \\
\|q\|_{L^{12}} &\lesssim \|D^3 q\|_{L^2}^{\frac{1}{3}}\|q\|_{L^4}^{\frac{2}{3}}
\end{aligned}
$$

Plugging these into (2.6.17), and using Sobolev embedding followed by the a priori bounds $\|q\|_{L^\infty\dot{H}^1} \lesssim \varepsilon_0$, we get

$$
\begin{aligned}
\|\eta\partial F\partial q\|_{L^2} + \|\eta\partial^2 Fq\|_{L^2} &\lesssim \|q\|_{\dot{H}^1}\|D^3 q\|_{L^2}\|q\|_{L^8}^2 \\
&\lesssim \|D^3 q\|_{L^2}\|q\|_{L^8}^2
\end{aligned}
$$

Putting this all together we conclude

$$
\|\partial^2(\eta Fq)\|_{L^2} \lesssim \left(\|D^2 q\|_{L^2} + \|D^3 q\|_{L^2}\right)\|q\|_{L^8}^2
$$

as desired. $\square$

Now, inserting the conclusion in Lemma 2.6.2 into (2.6.9) we have for $\ell = 1, 2$

$$\frac{1}{2}\frac{d}{dt}\|D^{\ell+1}q\|_{L^2}^2 \lesssim \left(\sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}^2\right)\left(\|q\|_{L^8}^2 + 1\right) \tag{2.6.17}$$

Together with (2.6.4) this implies for $\ell = 1, 2$ that

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}^2 \lesssim \left(\sum_{k=1}^{\ell+1}\|D^k q\|_{L^2}^2\right)\left(\|q\|_{L^8}^2 + 1\right) \tag{2.6.18}$$

Integrating in time, applying Gronwall's inequality and using the a priori estimates

$$\|q\|_{L^2L^8} \lesssim \ \varepsilon_0,$$

gives

$$\sum_{k=1}^{\ell+1}\|D^k q(t)\|_{L^2}^2 \leq \left(\sum_{k=1}^{\ell+1}\|D^k q(0)\|_{L^2}^2\right)\exp\left(C(\varepsilon_0 + t)\right) \tag{2.6.19}$$

for $\ell = 1, 2$. This implies that the $H^3(M;N) \times H^2(M;N)$ (resp. $H^4(M;N) \times H^3(M;N)$) norm of the solution $(u, \dot{u})$ remains finite for all time assuming the data (u_0, u_1) is bounded in $H^3(M;N) \times H^2(M;N)$, (resp. $H^4(M;N) \times H^3(M;N)$).

To deal with higher derivatives, $s \geq 5$, we note that (2.6.19) implies that

$$q(t) \in H^3 \hookrightarrow L_x^\infty,$$

and hence we can bootstrap the preceding argument, in particular Lemma 2.6.2, to all higher derivatives. For the global existence proof to come in the next section, we only need to do the case $s = 5$ as we have a local well-posedness theory for (1.1.2) at this regularity, see for example [68, Chapter 5].

2.7 Existence & Proof of Theorem 2.1.1

In this section, we the complete the proof of Theorem 2.1.1. We begin by establishing the existence statement in Theorem 2.1.1. The argument here follows exactly as in [69]. As explained in Section 2.9.2, we can find a sequence of smooth data $(u_0^k, u_1^k) \in C^\infty \times C^\infty(M; TN)$ such that $(u_0^k, u_1^k) \to (u_0, u_1)$ in $H^2 \times H^1(M; TN)$ as $k \to \infty$. Using the high regularity, local well-posedness theory, we can find smooth local solutions u^k to the Cauchy problem (1.1.1) with data (u_0^k, u_1^k) satisfying

$$\|u_0^k\|_{\dot{H}^2} + \|u_1^k\|_{\dot{H}^1} < \varepsilon_0 \tag{2.7.1}$$

for large enough k. Then, by the a priori bounds in Proposition 2.5.1 and the regularity results in Proposition 2.6.1, these local solutions u^k can be extended as smooth solutions of (1.1.1) for all time satisfying the uniform in k, global-in-time estimates

$$\|du^k\|_{L^\infty \dot{H}^1} + \|du^k\|_{L^2 L^8} \lesssim \|u_0^k\|_{\dot{H}^2} + \|u_1^k\|_{\dot{H}^1} \lesssim \varepsilon_0 \tag{2.7.2}$$

for large enough k. To see this, suppose that the smooth local solution u^k exists on the time interval $[0, T)$. Then, by Proposition 2.5.1 and Proposition 2.6.1 we have, say, that the $H^5 \times H^4$ norm of $(u^k(T), \dot{u}^k(T))$ is finite. Hence, we can apply the high regularity local well-posedness theory again to the Cauchy problem with data $(u^k(T), \dot{u}^k(T))$ obtaining a positive time of existence, T_1. By the uniqueness theory, this solution agrees with u^k, thereby extending u^k to the interval $[0, T + T_1)$. This implies that u^k is, in fact a global solution, as it can always be extended.

Now, by (2.7.2), we can find a subsequence, u^k such that $u^k \rightharpoonup u$ weakly in H^2_{loc}. We also have

$$\|du\|_{L^\infty \dot{H}^1} + \|du\|_{L^2 L^8} \lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} \lesssim \varepsilon_0 \tag{2.7.3}$$

By Rellich's theorem, we can find a further subsequence so that $du^k \to du$ pointwise almost everywhere. It follows that u is a global solution to (1.1.1) with data (u_0, u_1). We have now completed the proof of Theorem 2.1.1. We summarize the entire proof below.

Proof of Theorem 2.1.1. In Proposition 2.5.1 we established the global a priori bounds (2.1.8) for smooth wave maps $(u, \dot{u})$ with initial data (u_0, u_1) that satisfies (2.1.7). Now, given data $(u_0, u_1) \in H^2 \times H^1((\mathbb{R}^4, g), TN)$ satisfying (2.1.7) the above argument concludes the existence of a global wave map $(u, \dot{u}) \in C^0(\mathbb{R}; H^2 \times H^1)$. Proposition 2.5.1, and in particular the global control of the $L^2_t L^8_x$ norm of du allowed us to deduce the global regularity result, Proposition 2.6.1, which not only drives the existence proof above, but also shows that higher regularity of the data is preserved. Finally, the global control of the $L^2_t L^8_x$ norm of du validates the uniqueness proof in Section 2.2. $\square$

2.8 Linear Dispersive Estimates for Wave Equations on a Curved Background

In this section we outline the linear dispersive estimates for variable coefficient wave equations established by Metcalfe and Tataru in [57]. We review a portion of the argument in [57] with the necessary extensions needed to prove (2.5.1). It is suggested that the reader refer to [57] when reading this section as here we detail only the parts where changes have been made to suit our needs. In order to facilitate this joint reading we will try to use as much of the same notation as possible. We begin with a brief summary.

We say that (ρ, p, q) is a Strichartz pair if $2 \le p \le \infty$, $2 \le q < \infty$, and if (ρ, p, q) satisfies the following two conditions

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \rho \tag{2.8.1}$$

$$\frac{1}{p} + \frac{d-1}{2q} \leq \frac{d-1}{4} \tag{2.8.2}$$

with the exception of the forbidden endpoint $(1, 2, \infty)$, if $d = 3$.

In [57], Metcalfe and Tataru prove global Strichartz estimates for variable coefficient wave equations of the form

$$Pv = f \tag{2.8.3}$$
$$v[0] = (v_0, v_1)$$

where P is the second order hyperbolic operator,

$$P(t, x, \partial) = -\partial_t^2 + \partial_\alpha(a^{\alpha\beta}(x)\partial_\beta) + b^\alpha(x)\partial_\alpha + c(x) \tag{2.8.4}$$

In fact, in [57] time-dependent coefficients are considered as well, but we will restrict our attention to the time-independent case for our purposes. Here we assume that the matrix a is positive definite and the coefficients a, b, c satisfy the weak asymptotic flatness conditions

$$\sum_{j\in\mathbb{Z}} \sup_{|x|\sim 2^j} |x|^2 \left|\partial^2 a(x)\right| + |x|\,|\partial a(x)| + |a(x) - g_0| \leq \tilde{\varepsilon} \tag{2.8.5}$$

where g_0 denotes the diagonal matrix $\mathrm{diag}(1, \ldots, 1)$. And

$$\sum_{j\in\mathbb{Z}} \sup_{|x|\sim 2^j} |x|^2\,|\partial b(x)| + |x|\,|b(x)| \leq \tilde{\varepsilon} \tag{2.8.6}$$

$$\sum_{j\in\mathbb{Z}} \sup_{|x|\sim 2^j} |x|^4\,|c(x)|^2 \leq \tilde{\varepsilon} \tag{2.8.7}$$

Given the assumptions in (2.1.1)–(2.1.3), it is clear that our wave equation for q in (2.4.10) is of this form. Metcalfe and Tataru introduce the following function spaces in order to deduce localized energy estimates and control error terms later on.

Let S_k denote the kth Littlewood Paley projection. Set $A_j = \mathbb{R} \times \{|x| \simeq 2^j\}$ and $A_{<j} = \mathbb{R} \times \{|x| \lesssim 2^j\}$. For a function v of frequency 2^k define the norm

$$\|v\|_{X_k} := 2^{\frac{k}{2}} \|v\|_{L^2_{t,x}(A_{<-k})} + \sup_{j \geq -k} \| \, |x|^{-\frac{1}{2}} \, v\|_{L^2_{t,x}(A_j)} \tag{2.8.8}$$

With this we can define the global norm

$$\|v\|^2_{X^s} := \sum_{k \in \mathbb{Z}} 2^{2sk} \|S_k v\|^2_{X_k} \tag{2.8.9}$$

for $-\frac{d+1}{2} < s < \frac{d+1}{2}$. The space X^s is defined to be the completion of all Schwartz functions with respect to the X^s norm defined above. For the dual space $Y^s = (X^{-s})'$ we have the norm

$$\|f\|^2_{Y^s} = \sum_{k \in \mathbb{Z}} 2^{2sk} \|S_k f\|^2_{X'_k} \tag{2.8.10}$$

for $-\frac{d+1}{2} < s < \frac{d+1}{2}$. We refer the reader to [57] for details regarding the structure of these spaces.

With this setup, Metcalfe and Tataru are able to prove the following results:

1. Establish $\dot{H}^s$ localized energy estimates for the operator P, see [57, Definition 2, Theorem 4 and Corollary 5].

2. Construct a global-in-time parametrix, K, for the operator P, and prove Strichartz estimates for this parametrix. Error terms are controlled in the localized energy spaces, see [57, Propositions 15-17 and Lemmas 19-21].

3. Combine the localized energy estimates with the Strichartz and error estimates for the parametrix to prove global Strichartz estimates for solutions to (2.8.3), see [57, Theorem 6].

To prove these results, Metcalfe and Tataru are able to make a number of simplifications that allow them to treat the lower order terms in P as small perturbations and work instead with only the principal part of P, denoted by $P_a = -\partial_t^2 + \partial_\alpha a^{\alpha\beta}\partial_\beta$.

Let χ_j be smooth spatial Littlewood-Paley multipliers, i.e.

$$1 = \sum_{j\in\mathbb{Z}} \chi_j(x), \qquad \mathrm{supp}(\chi_j) \subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}$$

And set

$$\chi_{<j}(x) := \sum_{k<j} \chi_k(x), \qquad S_j := \sum_{k<j} S_k$$

We then define frequency localized coefficients

$$a_{(k)}^{\alpha\beta} := g_0^{\alpha\beta} + \sum_{\ell<k-4} (S_{<\ell}\chi_{<k-2\ell})S_\ell a^{\alpha\beta} \tag{2.8.11}$$

corresponding frequency localized operators

$$P_{(k)} := -\partial_t^2 + \partial_\alpha(a_{(k)}^{\alpha\beta}\partial_\beta) \tag{2.8.12}$$

used on functions of frequency k, and the global operators

$$\tilde{P} := \sum_{k\in\mathbb{Z}} P_{(k)}S_k \tag{2.8.13}$$

In order to prove (2.5.1), we only need to make a small alteration to the proof of the

Strichartz estimates for the parametrix, K. At first, the parametrix construction occurs on the level of the frequency localized operator, P_0, see [57, Propositions 15-17]. In particular, they prove the following result.

Proposition 2.8.1 ([57], Proposition 17). *Assume that $\tilde{\varepsilon}$ is sufficiently small, and assume that f is localized at frequency 0. Then there is a parametrix K_0 for $P_{(0)}$ which has the following properties:*

(i) (regularity) For any Strichartz pairs (p_1, q_1) respectively (p_2, q_2) with $q_1 \leq q_2$, we have

$$\|\partial K_0 f\|_{L_t^{p_1} L_x^{q_1} \cap X_0} \lesssim \|f\|_{L_t^{p_2'} L_x^{q_2'}} \tag{2.8.14}$$

(ii) (error estimate) For any Strichartz pair (p, q) we have

$$\|(P_{(0)} K_0 - 1) f\|_{X_0'} \lesssim \|f\|_{L_t^{p'} L_x^{q'}} \tag{2.8.15}$$

The next step is to move from these frequency localized parametrices to a construction of a parametrix for P_a, and this is where we make a slight alteration. To begin, Metcalfe and Tataru prove that the operator $P_{(0)}$ in Proposition 2.8.1 can be replaced with $\tilde{P}$, see [57, Lemma 10], on functions localized at frequency 0. To construct parametrices, K_j, at any frequency j, for $\tilde{P}$ we rescale, setting

$$K_j f(t, x) = 2^{-2j} K_0(f_{2^{-j}})(2^j t, 2^j x) \tag{2.8.16}$$

where $f_{2^{-j}}(t, x) = f(2^{-j} t, 2^{-j} x)$. Next, set

$$K := \sum_{j \in \mathbb{Z}} K_j S_j \tag{2.8.17}$$

With these definitions it is straightforward to prove the following lemma, which is our altered version of [57, Lemma 19]. Recall that the homogeneous Besov norm of a function φ is given by

$$\|\varphi\|_{\dot{B}^s_{p,q}} = \left(\sum_{j \in Z} (2^{sj} \|S_j \varphi\|_{L^p})^q \right)^{\frac{1}{q}}$$

Then we have

Lemma 2.8.2 (Besov space version of Lemma 19 in [57])**.** *The parametrix K has the following properties:*

(i) (regularity) For any Strichartz pairs (ρ_1, p_1, q_1), respectively (ρ_2, p_2, q_2) with $q_1 \leq q_2$ we have

$$\|\partial K f\|_{L_t^{p_1} \dot{B}^{s-\rho_1}_{q_1,2} \cap X^s} \lesssim \|f\|_{|\partial_x|^{-s-\rho_2} L_t^{p_2'} L_x^{q_2'}} \tag{2.8.18}$$

(ii) (error) For any Strichartz pair (ρ, p, q), we have

$$\|(P_a K - 1) f\|_{Y^s} \lesssim \|f\|_{|\partial_x|^{-s-\rho} L_t^{p'} L_x^{q'}} \tag{2.8.19}$$

Proof. We begin by extending the results of Lemma 2.8.1 to the parametrices K_j. Observe that $\partial K_j f(t,x) = 2^{-j} \partial K_0(f_{2^{-j}})(2^j t, 2^j x)$. Hence, for a function f localized at frequency j, we have

$$\begin{aligned}
\|\partial K_j f\|_{L_t^{p_1} L_x^{q_1}} &= 2^{-j} 2^{j(-\frac{1}{p_1} - \frac{d}{q_1})} \|\partial K_0(f_{2^{-j}})\|_{L_t^{p_1} L_x^{q_1}} \\
&\lesssim 2^{j(-1-\frac{1}{p_1} - \frac{d}{q_1})} \|f_{2^{-j}}\|_{L_t^{p_2'} L_x^{q_2'}} \\
&= 2^{j(-1-\frac{1}{p_1} - \frac{d}{q_1})} 2^{j(\frac{1}{p_2'} + \frac{d}{q_2'})} \|f\|_{L_t^{p_2'} L_x^{q_2'}}
\end{aligned}$$

Therefore, by (2.8.1) we obtain

$$2^{j(\frac{d}{2}+1-\rho_1)}\|\partial K_j f\|_{L_t^{p_1}L_x^{q_1}} \lesssim 2^{j(\frac{d}{2}+1+\rho_2)}\|f\|_{L_t^{p_2'}L_x^{q_2'}} \tag{2.8.20}$$

for functions f localized at frequency j.

We also need to estimate $\|\partial K_j f\|_{X_j}$. Let f again be localized at frequency j. Observe that

$$\begin{aligned} 2^{\frac{j}{2}}\|\partial K_j f\|_{L^2_{t,x}(A_{<-j})} &= 2^{\frac{j}{2}}\left(\int_{|x|\leq 2^{-j}} \left|\partial K_j f(t,x)\right|^2 \, dx\, dt\right)^{\frac{1}{2}} \\ &= 2^{j(-1-\frac{d}{2})}\|\partial K_0 f_{2^{-j}}\|_{L^2_{t,x}(A_{<0})} \end{aligned}$$

Therefore we can apply Proposition 2.8.1 to deduce

$$\begin{aligned} 2^{j(\frac{d}{2}+1)}2^{\frac{j}{2}}\|\partial K_j f\|_{L^2_{t,x}(A_{<-j})} &\lesssim \|f_{2^{-j}}\|_{L_t^{p_2'}L_x^{q_2'}} \\ &= 2^{j(\frac{1}{p_2'}+\frac{d}{q_2'})}\|f\|_{L_t^{p_2'}L_x^{q_2'}} \\ &= 2^{j(\frac{d}{2}+1+\rho_2)}\|f\|_{L_t^{p_2'}L_x^{q_2'}} \end{aligned}$$

Similarly one can show for any $k \geq -j$ that

$$2^{j(\frac{d}{2}+1)}\left\||x|^{-\frac{1}{2}}\,\partial K_j f\right\|_{L^2_{t,x}(A_k)} \lesssim 2^{j(\frac{d}{2}+1+\rho_2)}\|f\|_{L_t^{p_2'}L_x^{q_2'}}$$

Hence,

$$\|\partial K_j f\|_{X_j} \lesssim 2^{j\rho_2}\|f\|_{L_t^{p_2'}L_x^{q_2'}}$$

Putting this all together we obtain the frequency j version of Proposition 2.8.1 (i):

$$2^{-j\rho_1}\|\partial K_j f\|_{L_t^{p_1}L_x^{q_1}} + \|\partial K_j f\|_{X_j} \lesssim 2^{j\rho_2}\|f\|_{L_t^{p_2'}L_x^{q_2'}} \tag{2.8.21}$$

The next step is to use the Littlewood-Paley theorem to sum up these frequency localized pieces. As a preliminary step we observe that (2.8.21) implies that for each s

$$2^{2j(s-\rho_1)}\|\partial K_j f\|^2_{L_t^{p_1}L_x^{q_1}} + 2^{2js}\|\partial K_j f\|^2_{X_j} \lesssim 2^{2j(s+\rho_2)}\|f\|^2_{L_t^{p_2'}L_x^{q_2'}} \tag{2.8.22}$$

Then, we have

$$\|\partial K f\|_{L_t^{p_1}\dot{B}^{s-\rho_1}_{q_1,2}} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{2j(s-\rho_1)} \left\| S_j \partial \sum_{\ell\in\mathbb{Z}} K_\ell S_\ell f \right\|^2_{L_x^{q_1}} \right)^{\frac{1}{2}} \right\|_{L_t^{p_1}} \tag{2.8.23}$$

$$\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} 2^{2j(s-\rho_1)} \left\| \partial K_j S_j f \right\|^2_{L_x^{q_1}} \right)^{\frac{1}{2}} \right\|_{L_t^{p_1}} \tag{2.8.24}$$

$$\lesssim \left(\sum_{j} 2^{2j(s-\rho_1)} \left\| \partial K_j S_j f \right\|^2_{L_t^{p_1}L_x^{q_1}} \right)^{\frac{1}{2}} \tag{2.8.25}$$

$$\lesssim \left(\sum_{j} 2^{2j(s+\rho_2)} \left\| S_j f \right\|^2_{L_t^{p_2'}L_x^{q_2'}} \right)^{\frac{1}{2}} \tag{2.8.26}$$

$$\lesssim \left\| \left(\sum_{j} 2^{2j(s+\rho_2)} \left| S_j f \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^{p_2'}L_x^{q_2'}} \tag{2.8.27}$$

$$\lesssim \|f\|_{|\partial_x|^{-s-\rho_2}L_t^{p_2'}L_x^{q_2'}} \tag{2.8.28}$$

Above, (2.8.25), (2.8.26), (2.8.27) and (2.8.28) follow, respectively, from Minkowski's inequality, estimate (2.8.22), the dual estimate to Minkowski, and the Littlewood-Paley theorem.

Finally, we have

$$\|\partial K f\|_{X^s} = \left(\sum_{j\in\mathbb{Z}} 2^{2js} \left\| S_j \partial \sum_{\ell\in\mathbb{Z}} K_\ell S_\ell f \right\|_{X_j}^2 \right)^{\frac{1}{2}} \tag{2.8.29}$$

$$\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{2js} \left\| \partial K_j S_j f \right\|_{X_j}^2 \right)^{\frac{1}{2}} \tag{2.8.30}$$

$$\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{2j(s+\rho_2)} \left\| S_j f \right\|_{L_t^{p_2'} L_x^{q_2'}}^2 \right)^{\frac{1}{2}} \tag{2.8.31}$$

$$\lesssim \|f\|_{|\partial_x|^{-s-\rho_2} L_t^{p'} L_x^{q'}} \tag{2.8.32}$$

where (2.8.32) follows from the dual to Minkowski's inequality and the Littlewood-Paley theorem. The proof of (2.8.19) follows exactly as in [57]. □

We can carry out the rest of the argument exactly as in [57] except with Lemma 2.8.2 in place of [57, Lemma 19], to obtain the following Besov space version of [57, Theorem 6].

Theorem 2.8.3 (Besov space version of Theorem 6 in [57]). *Let $d \geq 4$. Assume that the coefficients $a^\alpha\beta$, b^α, c satisfy* (2.8.5), (2.8.6), *and* (2.8.7) *with $\tilde{\varepsilon}$ sufficiently small. Let (ρ_1, p_1, q_1) and (ρ_2, p_3, q_2) be two Strichartz pairs and assume further that $s = 0$ or $s = -1$. Then the solution v to* (2.8.3) *satisfies*

$$\|\partial v\|_{L_t^{p_1} \dot{B}^{s-\rho_1}_{q_1,2}} + \|\partial v\|_{X^s} \lesssim \|v[0]\|_{\dot{H}^{s+1}\times \dot{H}^s} + \|f\|_{|\partial_x|^{-s-\rho_2} L_t^{p_2'} L_x^{q_2'} + Y^s} \tag{2.8.33}$$

To obtain (2.5.1) we set $s = 0$, $\rho_1 = \frac{5}{6}$, $p = 2$, $q = 6$, $\rho_2 = 0$, $p_2 = 1$ and $q_2 = 2$ in (2.8.33) giving

$$\|v\|_{L^2 \dot{B}^{\frac{1}{6}}_{6,2}} \lesssim \|v[0]\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1 L^2} \tag{2.8.34}$$

We combine this which the energy estimates which correspond to $s = 0$, $\rho_1 = 0$, $p = \infty$, $q = 2$, $\rho_2 = 0$, $p_2 = 1$, $q_2 = 2$ and $d = 4$ in (2.8.33) giving

$$\|v\|_{L^2_t(\mathbb{R};(\dot{B}^{\frac{1}{6}}_{6,2}(\mathbb{R}^4))} + \|\partial v\|_{L^\infty_t(\mathbb{R};(L^2_x(\mathbb{R}^4))} \lesssim \\ \|v[0]\|_{\dot{H}^1\times L^2(\mathbb{R}^4)} + \|f\|_{L^1_t(\mathbb{R};(L^2_x(\mathbb{R}^4))} \quad (2.8.35)$$

which is exactly (2.5.1).

2.9 Appendix

2.9.1 Sobolev Spaces

We have interchangeably used two different definitions of Sobolev spaces throughout this chapter. The difference in the definitions arises from the different ways that we can view maps $f : M \to N$ and their differentials $df : TM \to u^*TN$. On one hand, we can take the extrinsic viewpoint, where we consider the isometric embedding of $N \hookrightarrow \mathbb{R}^m$ and view TN as a subspace of $\mathbb{R}^m$. Here we view f as a map $M \to \mathbb{R}^m$ with values in N and $df : TM \to \mathbb{R}^m$ with values in TN. On the other hand, we can view things intrinsically, and exploit the parallelizable structure on TN. We outline these different approaches below, and show that if we take the Coulomb frame on u^*TN these approaches are equivalent for our purposes. Furthermore, we show that if $(M, g) = (\mathbb{R}^4, g)$ with g as in (2.1.1)–(2.1.4) then all of the following spaces are equivalent to those that would arise if we had set M to be $\mathbb{R}^4$ with the Euclidean metric.

Extrinsic Approach

Taking the extrinsic point of view, we consider maps $f : (M, g) \to (\mathbb{R}^m, \langle\cdot,\cdot\rangle)$. Hence, we can write $f = (f^1, \ldots, f^m)$ with the differential $df = (df^1, \ldots, df^m)$. For such f and for

$1 < p < \infty$, we define the norm

$$
\begin{aligned}
\|f\|_{W_e^{k,p}} &= \sum_{\ell=0}^{k} \left(\sum_{a=1}^{m} \int_M \left| \nabla^\ell f^a \right|_g^p \mathrm{dvol}_g \right)^{\frac{1}{p}} \\
&= \sum_{\ell=0}^{k} \left(\sum_{a=1}^{m} \int_M \left(g^{i_1 j_1} \cdots g^{i_\ell j_\ell} (\nabla^\ell f^a)_{i_1,\dots,i_\ell} (\nabla^\ell f^a)_{j_1 \dots, j_\ell} \right)^{\frac{p}{2}} \sqrt{|g|}\, dx \right)^{\frac{1}{p}}
\end{aligned}
\tag{2.9.1}
$$

where ∇^ℓ denotes the ℓth covariant derivative on M with the convention that $\nabla^0 f^a = f^a$, see [31] . For example, the components in local coordinates of ∇f^a are given by $(\nabla f^a)_i = (df)_i = \partial_i f$ while the components in local coordinates for $\nabla^2 f^a$ are given by

$$
(\nabla^2 f^a)_{ij} = \partial_{ij} f^a - \Gamma_{ij}^k f_k^a
$$

We define $W_e^{k,p}(M, \mathbb{R}^m)$ to be the completion of $\{f \in C^\infty(M; \mathbb{R}^m) : \|f\|_{W_e^{k,p}} < \infty\}$ with respect to the above norm, (the subscript e here stands for extrinsic). We then define $W_e^{k,p}(M, N)$ to be the space of functions $\{f \in W_e^{k,p}(M, \mathbb{R}^m) : \ f(x) \ \in \ N, \text{a.e.}\}$. The homogeneous Sobolev spaces $\dot{W}_e^{k,p}(M; N)$ are defined similarly.

Remark 6. The one drawback with this definition is that $C^\infty(M; N)$ may not be dense in $W^{1,p}(M; N)$ for $p < \dim M$, for a generic compact manifold N. For example, in [66], Schoen and Uhlenbeck show that $f(x) = \frac{x}{|x|} \in H^1(B^3; S^2)$ cannot be approximated by C^∞ maps from $B^3 \to S^2$ in $H^1(B^3; S^2)$, see [54] for a proof. This poses a potential difficulty for us as we required the density of $C^\infty(M, TN)$ in $H^1(M; TN)$ in order to approximate the data $(u_0, u_1) \in H^2 \times H^1(M; TN)$ by smooth functions in our existence argument. Thankfully, this difficulty can be avoided using the equivalence of the extrinsic and intrinsic definitions of Sobolev spaces which will be argued below.

With $(M, g) = (\mathbb{R}^4, g)$, with g as in (2.1.1)–(2.1.4), and ε small enough, we can show that these "covariant" Sobolev Spaces $W_e^{k,p}((\mathbb{R}^4, g); N)$ are equivalent to the "flat" Sobolev spaces $W_e^{k,p}((\mathbb{R}^4, \langle\cdot,\cdot\rangle); N)$.

Lemma 2.9.1. *Let* $(M,g) = (\mathbb{R}^4, g)$ *with* g *as in* (2.1.1)–(2.1.4) *and let* $1 < p < \infty$. *Then* $\dot{W}^{k,p}((\mathbb{R}^4, g))$ *is equivalent to* $\dot{W}^{k,p}(\mathbb{R}^4, g_0)$ *where* g_0 *is the Euclidean metric on* $\mathbb{R}^4$. *In particular, if* $f : \mathbb{R}^4 \to \mathbb{R}^m$ *then for every* $k \in \mathbb{N}$ *we have*

$$\|\partial^k f\|_{L^p(\mathbb{R}^4)} \simeq \|\nabla^k f\|_{L^p(\mathbb{R}^4, g)} \tag{2.9.2}$$

Proof. As the above norms are defined component-wise for $f = (f^1, \ldots, f^m)$, it enough to prove the statement for functions $f : (M,g) \to \mathbb{R}$ instead of for maps $f : (M,g) \to \mathbb{R}^m$ with values in N. We also will only prove this lemma in detail for a few easy cases, namely for $k = 0, 1$ and for $k = 2, p = 2$. These, in fact, include all the cases that we need. The other cases follow by similar arguments.

By (2.1.1) it is clear that $\sqrt{|g(x)|}$ is a bounded function on $\mathbb{R}^4$. Hence, for every k we have

$$\int_{\mathbb{R}^4} \left|\nabla^k f\right|_g^p \sqrt{|g|}\, dx \simeq \int_{\mathbb{R}^4} \left|\nabla^k f\right|_g^p dx$$

This proves the lemma for $k = 0$. In local coordinates we have, for $k = 1$, that $(\nabla f)_i := (df)_i = \partial_i f$ and $|\partial f|_g^2 = g^{ij}\partial_i f \partial_j f$. Letting g_0 denote the Euclidean metric we have, for p even, that

$$\begin{aligned}
|\nabla f|_g^p - |\partial f|^p &= (g^{ij}\partial_i f \partial_j f)^{\frac{p}{2}} - (g_0^{ij}\partial_i f \partial_j f)^{\frac{p}{2}} \\
&= (g^{ij} - g_0^{ij})(\partial_i f \partial_j f) \sum_{\ell=1}^{\frac{p}{2}} (g^{ab}\partial_a f \partial_b f)^{\frac{p}{2}-\ell} (g_0^{cd}\partial_c f \partial_d f)^{\ell-1}
\end{aligned}$$

Hence by (2.1.1) we have

$$\left| \|\nabla f\|_{L^p(\mathbb{R}^4,g)}^p - \|\partial f\|_{L^p(\mathbb{R}^4)}^p \right| \lesssim \varepsilon \int_{\mathbb{R}^4} |\partial f|^p \, dx$$

For p odd we interpolate. This proves the case $k = 1$. For $k = 2, p = 2$ we have in local coordinates that

$$(\nabla^2 f)_{ij} = \partial_{ij} f - \Gamma_{ij}^{\ell} \partial_{\ell} f$$

where here Γ_{ij}^{l} are the Christoffel symbols for $(\mathbb{R}^4, g)$. We also have

$$\left|\nabla^2 f\right|^2 = g^{ik} g^{j\ell} (\nabla^2 f)_{ij} (\nabla^2 f)_{k\ell}$$

Hence, using (2.1.1)-(2.1.2) and the Sobolev embedding we have

$$\begin{aligned}
&\left| \|\nabla^2 f\|_{L^2(\mathbb{R}^4,g)}^2 - \|\partial^2 f\|_{L^2(\mathbb{R}^4)}^2 \right| \\
&\simeq \left| \int_{\mathbb{R}^4} g^{ik} g^{j\ell} (\partial_{ij} f - \Gamma_{ij}^a \partial_a f)(\partial_{k\ell} f - \Gamma_{k\ell}^b \partial_b f) \; - g_0^{ik} g_0^{j\ell} (\partial_{ij} f)(\partial_{k\ell} f) \, dx \right| \\
&\lesssim \left| \int_{\mathbb{R}^4} (g^{ik} g^{j\ell} - g_0^{ik} g_0^{j\ell})(\partial_{ij} f)(\partial_{k\ell} f) \, dx \right| + 2 \left| \int_{\mathbb{R}^4} g^{ik} g^{j\ell} \, \partial_{ij} f \, \Gamma_{k\ell}^a \, \partial_a f \, dx \right| \\
&\quad + \left| \int_{\mathbb{R}^4} g^{ik} g^{j\ell} \, \Gamma_{ij}^a \, \partial_a f \, \Gamma_{k\ell}^b \, \partial_b f \, dx \right| \\
&\lesssim \varepsilon^2 \|\partial^2 f\|_{L^2(\mathbb{R}^4)}^2 + \|\partial^2 f\|_{L^2(\mathbb{R}^4)} \|\Gamma\|_{L^4(\mathbb{R}^4)} \|\partial f\|_{L^4(\mathbb{R}^4)} + \|\partial f\|_{L^4(\mathbb{R}^4)}^2 \|\Gamma\|_{L^4(\mathbb{R}^4)}^2
\end{aligned}$$

Now, recall that $\Gamma_{ij}^a = \frac{1}{2} g^{ab} (\partial_i g_{bj} + \partial_j g_{ib} - \partial_b g_{ij})$. Hence by (2.1.2), we have $\|\Gamma\|_{L^4(\mathbb{R}^4)} \lesssim \varepsilon$. Using the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ and the above inequalities we have

$$\left| \|\nabla^2 f\|_{L^2(\mathbb{R}^4,g)}^2 - \|\partial^2 f\|_{L^2(\mathbb{R}^4)}^2 \right| \lesssim \varepsilon \|\partial^2 f\|_{L^2(\mathbb{R}^4)}^2$$

proving (2.9.2) in the case $k = 2$, $p = 2$. $\square$

Intrinsic Approach

Next, we use the parallelizable structure on TN to define "intrinsic" Sobolev spaces for maps $\psi : TM \to u^*TN$.

Let $\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_n)$ be a global orthonormal frame on TN and let $\bar{e} = (\bar{e}_1, \dots, \bar{e}^n)$ be the induced orthonormal frame on u^*TN obtained via pullback. Now, let $\psi : TM \to u^*TN$ be a smooth map, i.e., ψ is a u^*TN valued 1-form on M. Then ψ can be written in terms of the orthonormal frame $\bar{e}$ on u^*TN. The components of ψ in the frame $\bar{e}$ are then given by $\psi^a = \langle \psi, \bar{e}_a \rangle_{u^*h}$ and each of these can be viewed as a 1-form on M, i.e., a section of T^*M, and can be written in local coordinates as $\psi^a = \psi^a_\alpha dx^\alpha$.

One way to define the Sobolev norms of ψ is to ignore the covariant structure on u^*TN and say that $\psi \in \dot{W}^{k,p}_i(M;N)$, (the index i here stands for intrinsic), if all of the components, ψ^a, are in $\dot{W}^{k,p}(M;\mathbb{R})$. And we define

$$\begin{aligned}\|\psi\|^p_{\dot{W}^{k,p}_i(M;N)} &:= \sum_{a=1}^n \|\psi^a\|^p_{\dot{W}^{k,p}(M)} = \sum_{a=1}^n \int_M \left|\nabla^k \psi^a\right|^p_g \mathrm{dvol}_g \\ &= \sum_{a=1}^n \int_M \left(g^{i_1 j_1} \cdots g^{i_{k+1}, j_{k+1}} (\nabla^k \psi)^a_{i_1,\dots,i_{k+1}} (\nabla^k \psi)^a_{j_1,\dots,j_{k+1}}\right)^{\frac{p}{2}} \sqrt{|g|}\, dx\end{aligned} \tag{2.9.3}$$

where ∇^k denotes the kth covariant derivative on M. By the same argument as above, we can show that in our case, with $(M,g) = (\mathbb{R}^4, g)$ and g as in (2.1.1)–(2.1.4), these spaces are equivalent to the case where we have the Euclidean metric on $\mathbb{R}^4$, that is, there exist constants c, C such that

$$\|\partial^k \psi^a\|_{L^p(\mathbb{R}^4)} \simeq \|\nabla^k \psi^a\|_{L^p(M)} \tag{2.9.4}$$

The one glaring issue here, is that this construction will depend, in general, on the choice of frame $\bar{e}$. We can avoid this confusion though in the case where the frame e is the Coulomb frame as in this case the intrinsic norms are equivalent to their extrinsic counterparts in the cases we will be interested in. This issue was addressed in Section 2.3.2.

2.9.2 Density of $C^\infty \times C^\infty(M;TN)$ in $H^2 \times H^1(M;TN)$

We set $(M,g) = (\mathbb{R}^4, g)$ with g as in (2.1.1)-(2.1.4). In the existence argument for wave maps we claimed the existence of a sequence of smooth data $(u_0^k, u_1^k) \to (u_0, u_1)$ in $H^2 \times H^1(M;TN)$. Here we show that such a sequence does, in fact, exist. That is, we show that $C^\infty \times C^\infty(M;TN)$ is dense in in $H^2 \times H^1(M;TN)$.

First, observe that $C^\infty(M;N)$ is dense in $H_e^2(M;N)$, see [7, Lemma A.12]. Hence we can find a sequence of smooth maps u_0^k such that $u_0^k \to u_0$ in $H^2(M;N)$.

Finding a sequence of smooth maps $u_1^k : M \to TN$ such that $u_1^k(x) \in T_{u_0^k(x)}N$ approximating u_1 in $H^1(M;TN)$ is not as straightforward as we do not know a priori that $C^\infty(M;TN)$ is dense in $H_e^1(M;TN)$. However, we can use the equivalence of the norms $H_e^1(M;TN)$ and $H_i^1(M;TN)$ proved in the previous section to get around this issue.

Let e denote the Coulomb frame on u_0^*TN. Since u_1 is a section of u_0^*TN, we can find one-forms q_1^a over M so that $u_1 = q_1^a e_a$. By the equivalence of the norms $H_e^1(M;TN)$ and $H_i^1(M;TN)$, we see that $u_1 \in H_e^1(M;TN)$ if and only if $q_1^a \in H^1(TM;\mathbb{R})$. Since C^∞ is dense in $H^1(TM;\mathbb{R}) \simeq H^1(\mathbb{R}^4;\mathbb{R})$ we can find smooth $(q_1^a)^k$ such that $(q_1^a)^k \to q_1^a$ in $H^1(TM;\mathbb{R})$. Now, for each smooth map $u_0^k : M \to N$ we can find the associated Coulomb frame $e^k = (e_1^k, \ldots, e_n^k)$. We then define smooth sections $u_1^k := (q_1^a)^k e_a^k$ and by the equivalence of norms explained in Section 2.3.2 we have $u_1^k \to u_1$ in $H_e^1(M;TN)$ as desired.

2.9.3 Lorentz Spaces

To prove the pointwise estimates for the connection form A associated to the Coulomb gauge we need a few general facts about Lorentz spaces. We review these facts below. $L^{p,r}(\mathbb{R}^d)$

functions are measured with the norm

$$\|f\|_{L^{p,r}} = \left(\int_0^\infty t^{\frac{r}{p}} f^*(t)^r \, \frac{dt}{t} \right)^{\frac{1}{r}}$$

for $0 < r < \infty$. If $r = \infty$, then

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t)$$

where above we have

$$f^*(t) = \inf\{\alpha : d_f(\alpha) \leq t\}$$

$$d_f(\alpha) = \text{meas}\{x : |f(x)| > \alpha\}$$

A consequence of real interpolation theory is that Lorentz spaces can also be characterized as the interpolation spaces given by

$$L^{p,r}(\mathbb{R}^d) = (L^{p_0}, L^{p_1})_{\theta,r} \tag{2.9.5}$$

where $1 \leq p_0 < p_1 \leq \infty$, $p_0 < r \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We refer the reader to [4, Chapter 5.2] for more details.

Note that the $L^{p,\infty}$ norm is the same as the weak-L^p norm. Below we record some general properties of Lorentz spaces that were needed in the proof of Proposition 2.3.2. We refer the reader to [29], [4], [61], and [80] for more details.

Lemma 2.9.2. *Suppose that* $0 < p \leq \infty$ *and* $0 < r < s \leq \infty$. *Then*

(i) $L^{p,p} = L^p$

(ii) *If* $r < s$ *then* $L^{p,r} \subset L^{p,s}$

(iii) *If* $h : \mathbb{R}^d \to \mathbb{R}$ *is defined by* $h(x) = \dfrac{1}{|x|^\alpha}$*, then* $h \in L^{\frac{d}{\alpha},\infty}$.

The proof of Lemma 2.9.2 follows easily from the definitions and can be found for example in [29, Chapter 1.4.2]. We also needed the Lorentz space versions of Hölder's inequality and Young's inequality and the following duality statement.

Lemma 2.9.3. *Suppose that* $f \in L^{p_1,r_1}$ *and* $g \in L^{p_2,r_2}$ *where* $1 \leq p_1, p_2 < \infty$ *and* $1 \leq r_1, r_2 \leq \infty$*. Then,*

(i) $\|fg\|_{L^{p,r}} \lesssim \|f\|_{L^{p_1,r_1}} \|g\|_{L^{p_2,r_2}}$ *if* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$

(ii) $\|f * g\|_{L^{p,r}} \lesssim \|f\|_{L^{p_1,r_1}} \|g\|_{L^{p_2,r_2}}$ *if* $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$

(iii) $(L^{p,r})' = L^{p_1,r_1}$ *for* $1 < p < \infty$, $1 < r < \infty$ *and* $(L^{p,1})' = L^{p_1,\infty}$ *for* $1 < p < \infty$, *where* $\frac{1}{p} + \frac{1}{p_1} = 1$ *and* $\frac{1}{r} + \frac{1}{r_1} = 1$

To prove (i) above observe that $(fg)^*(t) \leq f^*(\frac{t}{2}) g^*(\frac{t}{2})$, see [29, Proposition 1.4.5]. Then apply Hölder's inequality. We refer the reader to [61] for the proof of (ii) above. And (iii) is proved in [29, Theorem 1.4.17].

We also require Sobolev embedding theorems for Lorentz spaces which can be obtained via real interpolation. A detailed proof can be found in [80, Chapter 32].

Lemma 2.9.4 (Sobolev embedding for Lorentz spaces)**.** *If* $0 < s < \dfrac{d}{q}$ *and* $\dfrac{1}{p} = \dfrac{1}{q} - \dfrac{s}{d}$ *then* $\dot{W}^{s,q}(\mathbb{R}^d) \hookrightarrow L^{p,q}(\mathbb{R}^d)$ *and* $\dot{B}^s_{q,r}(\mathbb{R}^d) \hookrightarrow L^{p,r}(\mathbb{R}^d)$.

To give an idea of why Lemma 2.9.4 is true, we demonstrate a special case, namely that

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^{p,2}(\mathbb{R}^d) \tag{2.9.6}$$

for $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$. Observe that this is a strengthening of the standard Sobolev inequality which says that $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$ since $L^{p,2}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$. The proof of (2.9.6) relies on Plancherel's theorem and real interpolation. Let $\mathcal{F}$ denote the Fourier transform. Let $f \in \dot{H}^s(\mathbb{R}^d)$, which means that $|\xi|^s \mathcal{F}f \in L^2(\mathbb{R}^d)$. Also note that if $0 < s < \frac{d}{2}$ then

$$|\xi|^{-s} \in L^{\frac{d}{s},\infty}(\mathbb{R}^d)$$

Hence, by Hölder's inequality for Lorentz spaces

$$\|\mathcal{F}f\|_{L^{\gamma,2}} = \| \, |\xi|^s \mathcal{F}f \, |\xi|^{-s} \, \|_{L^{\gamma,2}} \lesssim \| \, |\xi|^s \mathcal{F}f\|_{L^{2,2}} \| \, |\xi|^{-s} \, \|_{L^{\frac{d}{s},\infty}} < \infty$$

for $\frac{1}{\gamma} = \frac{1}{2} + \frac{s}{d}$. Now recall that $\mathcal{F}^{-1} : L^1 \to L^\infty$ and $\mathcal{F}^{-1} : L^2 \to L^2$. Therefore, by real interpolation

$$\mathcal{F}^{-1} : (L^1, L^2)_{\theta,2} \to (L^\infty, L^2)_{\theta,2}$$

which, by (2.9.5) is exactly the statement that

$$\mathcal{F}^{-1} : L^{\alpha,2}(\mathbb{R}^d) \to L^{\beta,2}(\mathbb{R}^d)$$

where $\frac{1}{\alpha} = 1 - \frac{\theta}{2}$ and $\frac{1}{\beta} = \frac{\theta}{2}$ and we notice that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Hence, with $\frac{1}{\gamma} = \frac{1}{2} + \frac{s}{d}$ we have that $\mathcal{F}f \in L^{\gamma,2}(\mathbb{R}^d)$ which implies that $f \in L^{\gamma',2}(\mathbb{R}^d)$ where $\frac{1}{\gamma'} = \frac{1}{2} - \frac{s}{d}$ which is exactly (2.9.6).

The L^p and Besov space versions of this statement are slightly more complicated to prove as they require additional facts from real interpolation theory and we refer the reader to [80] for a detailed proof.

We also need the following version of the Calderon-Zygmund theorem for Lorentz spaces.

Theorem 2.9.5 (Calderon-Zygmund theorem for Lorentz spaces)**.** *Let T be a Calderon-*

Zygmund operator. Then $T : L^{p,r} \to L^{p,r}$ *for* $1 < p < \infty$ *and* $1 \le r \le \infty$*,*

$$\|Tf\|_{L^{p,r}} \lesssim \|f\|_{L^{p,r}}$$

where the constant above does not depend on r*.*

This extension of the Calderon-Zygmund theorem is an easy consequence of the L^p version given the following interpolation theorem of Calderon, see [4, Theorem 5.3.4].

Theorem 2.9.6 (Calderon's interpolation theorem)**.** *Let* T *be a linear operator and suppose that*

$$T : L^{p_1,\rho} \to L^{q_1,\infty}$$
$$T : L^{p_2,\rho} \to L^{q_2,\infty}$$

where $\rho > 0$*. Then,*

$$T : L^{p,r} \to L^{q,s}$$

as long as $0 < r \le s \le \infty$*,* $p_1 \neq p_2$*,* $q_1 \neq q_2$*,* $\dfrac{1}{p} = \dfrac{(1-\theta)}{p_1} + \dfrac{\theta}{p_2}$*, and* $\dfrac{1}{q} = \dfrac{(1-\theta)}{q_1} + \dfrac{\theta}{q_2}$ *for* $\theta \in (0,1)$*.*

Proof of Theorem 2.9.5. Let T be a Calderon-Zygmund operator. To prove that $T : L^{p,q} \to L^{p,q}$, find p_1, p_2, θ so that $1 < p_1 < p < p_2 < \infty$ and $\dfrac{1}{p} = \dfrac{(1-\theta)}{p_1} + \dfrac{\theta}{p_2}$. Then we have $T : L^{p_1,p_1} \to L^{p_1,\infty}$ and $T : L^{p_2,p_1} \to L^{p_2,\infty}$ since

$$\|Tf\|_{L^{p_1,\infty}} \lesssim \|Tf\|_{L^{p_1,p_1}} = \|Tf\|_{L^{p_1}} \lesssim \|f\|_{L^{p_1}} = \|f\|_{L^{p_1,p_1}}$$
$$\|Tf\|_{L^{p_2,\infty}} \lesssim \|Tf\|_{L^{p_2,p_2}} = \|Tf\|_{L^{p_2}} \lesssim \|f\|_{L^{p_2}} = \|f\|_{L^{p_2,p_2}} \lesssim \|f\|_{L^{p_2,p_1}}$$

Therefore, by Theorem 2.9.6, we have $T : L^{p,q} \to L^{p,q}$ for every $q > 0$. $\square$

CHAPTER 3

3D WAVE MAPS EXTERIOR TO A BALL

3.1 Introduction

In this chapter, we begin our investigation of equivariant wave maps from $3+1$-dimensional Minkowski space exterior to a ball and with S^3 as target. To be specific, let $B \subset \mathbb{R}^3$ be the unit ball in $\mathbb{R}^3$. We consider wave maps $U : \mathbb{R} \times (\mathbb{R}^3 \setminus B) \to S^3$ with a Dirichlet condition on ∂B, i.e., $U(\partial B) = \{N\}$ where N is a fixed point on S^3. In the usual equivariant formulation of this equation, where ψ is the azimuth angle measured from the north pole, the equation for the ℓ-equivariant wave map from $\mathbb{R}^{3+1} \to S^3$ reduces to

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \ell(\ell+1)\frac{\sin(2\psi)}{2r^2} = 0 \tag{3.1.1}$$

We restrict to $\ell = 1$ and $r \geq 1$ with Dirichlet boundary condition $\psi(1,t) = 0$ for all $t \geq 0$. In other words, we are considering the Cauchy problem

$$\begin{aligned}
&\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1, \\
&\psi(1,t) = 0, \quad \forall\, t \geq 0, \\
&\vec{\psi}(0) = (\psi_0, \psi_1)
\end{aligned} \tag{3.1.2}$$

The conserved energy is

$$\mathcal{E}(\psi, \psi_t) = \int_1^\infty \frac{1}{2}\big(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\big) r^2\, dr \tag{3.1.3}$$

Any $\psi(r,t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(\infty, t) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed.

The natural space to place the solution into for $n = 0$ is the *energy space* $\mathcal{H} := (\dot{H}_0^1 \times$

$L^2)((1,\infty))$ with norm

$$\|(\psi,\dot{\psi})\|_{\mathcal{H}}^2 := \int_1^\infty (\psi_r^2(r) + \dot{\psi}^2(r))\, r^2\, dr \tag{3.1.4}$$

Here $\dot{H}_0^1((1,\infty))$ is the completion of the smooth functions on $(1,\infty)$ with compact support under the first norm on the right-hand side of (3.1.4).

The exterior equation (3.1.2) was proposed by Bizon, Chmaj, and Maliborski [5] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [2] as an easier alternative to the Skyrmion equation. Moreover, [2] stresses the analogy with the damped pendulum which plays an important role in our analysis. Numerical simulations described in [5] indicate that in each equivariance class and topological class given by the boundary value $n\pi$ at $r=\infty$ *every solution* scatters to the unique harmonic map that lies in this class. In this chapter we verify this conjecture for $\ell = 1, n = 0$. These solutions start at the north-pole and eventually return there. For $n \geq 1$ we obtain a perturbative result in this chapter. In the next chapter we prove the full conjecture for the higher topological classes, $n \geq 1$.

Theorem 3.1.1. *Consider the topological class defined by equivariance $\ell = 1$ and degree $n = 0$. Then for any smooth energy data in that class there exists a unique global and smooth evolution to* (3.1.2) *which scatters to zero in the sense that the energy of the wave map on an arbitrary but fixed compact region vanishes as $t \to \infty$.*

The scattering property can also be phrased in the following fashion: one has

$$(\psi,\psi_t)(t) = (\varphi,\varphi_t)(t) + o_{\mathcal{H}}(1) \quad t\to\infty \tag{3.1.5}$$

where $(\varphi, \varphi_t) \in \mathcal{H}$ solves the linearized version of (3.1.2), i.e.,

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{2\varphi}{r^2} = 0, \; r \geq 1, \; \varphi(1,t) = 0 \tag{3.1.6}$$

We prove Theorem 3.1.1 by means of the Kenig-Merle method [36], [37]. The most novel aspect of our implementation of this method lies with the rigidity argument. Indeed, in order to prove Theorem 3.1.1 *without any upper bound on the energy* we demonstrate that the natural virial functional is globally coercive on $\mathcal{H}$. This requires a detailed variational argument, the most delicate part of which consists of a phase-space analysis of the Euler-Lagrange equation.

The advantage of this model lies with the fact that removing the unit ball eliminates the scaling symmetry and also renders the equation subcritical relative to the energy. Both of these features are in stark contrast to the same equation on $3+1$-dimensional Minkowski space, which is known to be super-critical and to develop singularities in finite time, see Shatah [67] and also Shatah, Struwe [68].

Another striking feature of this model, which fails for the $2+1$-dimensional analogue, lies with the fact that it admits infinitely many stationary solutions $Q_n(r)$ which satisfy $Q_n(1) = 0$ and $\lim_{r\to\infty} Q_n(r) = n\pi$, for each $n \geq 1$. These solutions have minimal energy in the class of all functions of finite energy which satisfy the $n\pi$ boundary condition at $r = \infty$, and they are the unique stationary solutions in that class. We denote the latter class by $\mathcal{H}_n$.

Theorem 3.1.2. *For any $n \geq 1$ there exists $\varepsilon > 0$ small with the property that for any smooth data $(\psi_0, \psi_1) \in \mathcal{H}_n$ such that*

$$\|(\psi_0, \psi_1) - (Q_n, 0)\|_{\mathcal{H}} < \varepsilon$$

the solution to (3.1.1) *with data (ψ_0, ψ_1) exists globally, is smooth, and scatters to $(Q_n, 0)$ as $t \to \infty$.*

The same result applies as well to higher equivariance classes $\ell \geq 2$, after some fairly obvious modifications of the arguments in Section 3.5. However, for the sake of simplicity we restrict ourselves to $\ell = 1$. Scattering here means that on compact regions in space one has $(\psi, \psi_t)(t) - (Q_n, 0) \to (0, 0)$ in the energy topology, or alternatively

$$(\psi, \psi_t)(t) = (Q_n, 0) + (\varphi, \varphi_t)(t) + o_{\mathcal{H}}(1) \quad t \to \infty \tag{3.1.7}$$

where φ solves (3.1.6). Bizoń, Chmaj, and Maliborski [5] conducted numerical experiments which suggest that Theorem 3.1.2 holds with $\varepsilon = \infty$. We prove this much stronger theorem in the next chapter, completing the soliton resolution theory for this model. This requires a completely novel approach however as the methods of this chapter do not seem to extend to the cases $n \geq 1$. The main difficulty with this approach to a rigidity theory lies with the coercivity of the virial functional centered at the harmonic maps Q_n. Indeed, in Section 3.4, we establish the global coercivity of the virial functional centered at zero. This hinges crucially on the fact that the Euler-Lagrange equation of the associated variational problem can be transformed into an autonomous system in the plane which we analyze by a rigorous study of the phase portrait. For the nonzero Q_n we lose this reduction to an autonomous system, making any rigorous statement about the Euler-Lagrange equation associated to the virial functional centered at Q_n very difficult. Furthermore, no explicit expression is known for the Q_n which makes even the perturbative analysis — in and of itself useless for the Kenig-Merle method — of this virial functional very non-obvious. Therefore the case $n \geq 1$ requires a different strategy from the one we employ here.

3.2 Basic well-posedness and scattering

One has the following version of Hardy's inequality in $\dot{H}^1(1,\infty)$:

$$\int_1^\infty \psi^2(r)\,dr \leq 4\int_1^\infty \psi_r^2(r)r^2\,dr \tag{3.2.1}$$

proved by integration by parts:

$$\int_1^\infty \psi^2(r)\,dr + \psi^2(1) = -2\int_1^\infty r\psi_r(r)\psi(r)\,dr \tag{3.2.2}$$

and an application of Cauchy-Schwarz. This shows in particular that $\mathcal{E}(\vec{\psi}) \simeq \|\vec{\psi}\|_{\mathcal{H}}^2$ where $\vec{\psi} = (\psi, \dot{\psi})$. Another useful fact is the Strauss estimate:

$$|\psi(r)| \leq 2r^{-\frac{1}{2}}\|\psi\|_{\dot{H}^1(1,\infty)} \quad \forall r \geq 1 \tag{3.2.3}$$

which in particular implies that $\|\psi\|_\infty \leq 2\|\psi\|_{\dot{H}^1}$. Since the nonlinearity in (3.1.2) is globally Lipschitz due to $r \geq 1$, energy estimates immediately imply the following global well-posedness result. In what follows, $\mathbb{R}^d_* := \mathbb{R}^d \setminus B$ where B is the unit ball at the origin.

Proposition 3.2.1. *For any* $(\psi_0, \psi_1) \in \mathcal{H}$ *the Cauchy problem* (3.1.2) *has a unique global solution*

$$\psi \in C([0,\infty); \dot{H}^1_0(1,\infty)),\ \psi_t \in C([0,\infty), L^2(1,\infty)) \tag{3.2.4}$$

in the Duhamel sense which depends continuously on the data. Moreover, $\mathcal{E}(\vec{\psi}(t)) = constant$ *and we have persistence of regularity.*

Proof. Just write the equation in Duhamel form and apply the standard energy estimate to obtain local well-posedness. To be more precise, we write

$$\begin{aligned}\vec{\psi}(t) &= S_0(t)\vec{\psi}(0) + \int_0^t S_0(t-s)(0, N(\psi))(s)\,ds, \\ N(\psi)(t,r) &:= -\frac{\sin(2\psi(t,r))}{r^2}\end{aligned} \tag{3.2.5}$$

where $S_0(t)$ is the linear evolution of the wave equation in $\mathbb{R}_t \times \mathbb{R}^3_*$, with a Dirichlet condition at $r = 1$ (everything can be taken to be radial, of course). By the conservation of energy one has

$$\|S_0(t)\vec{\psi}(0)\|_{\mathcal{H}} = \|\vec{\psi}(0)\|_{\mathcal{H}} \tag{3.2.6}$$

whence

$$\begin{aligned}\|\vec{\psi}(t)\|_{\mathcal{H}} &\lesssim \|\vec{\psi}(0)\|_{\mathcal{H}} + \int_0^t \|\psi(s)\|_2\,ds \\ &\lesssim \|\vec{\psi}(0)\|_{\mathcal{H}} + t \sup_{0<s<t} \|\psi(s)\|_2\end{aligned} \tag{3.2.7}$$

So we can set up a contraction in the space $L_t^\infty(I;\mathcal{H})$ where $I = [0,T)$ and T is small depending only on the size of $\|\vec{\psi}(0)\|_{\mathcal{H}}$. The global statement therefore follows by energy conservation. □

As in [68] we refer to these energy Duhamel solutions as *strong solutions*. For the scattering problem the formulation (3.1.2) is less convenient due to the linear term in the nonlinearity:

$$\frac{\sin(2\psi)}{r^2} = \frac{2\psi}{r^2} + \frac{\sin(2\psi) - 2\psi}{r^2} = \frac{2\psi}{r^2} + \frac{O(\psi^3)}{r^2} \tag{3.2.8}$$

The presence of the strong repulsive potential $\frac{2}{r^2}$ indicates that the linearized operator of (3.1.2) has more dispersion than the three-dimensional wave equation. In fact, it has the same dispersion as the five-dimensional wave equation as the following standard reduction shows.

We set $\psi = ru$ which leads to the equation

$$u_{tt} - u_{rr} - \frac{4}{r}u_r + \frac{\sin(2ru) - 2ru}{r^3} = 0, \quad r \geq 1,\ u(1,t) = 0 \tag{3.2.9}$$

The nonlinearity is of the form $N(u,r) := u^3\, Z(ru)$ where Z is a smooth function, and the linear part is the d'Alembertian in $\mathbb{R}_t \times \mathbb{R}^5_*$.

To relate strong solutions of (3.1.2) with those of (3.2.9) we first note that

$$\int_1^\infty \psi_r^2(r) r^2\, dr \simeq \int_1^\infty u_r^2(r) r^4\, dr \tag{3.2.10}$$

via Hardy's inequality and the relations

$$\psi_r = ru_r + u = ru_r + \frac{\psi}{r}$$

Therefore, the map $\mathcal{H} \ni \vec{\psi} \to \frac{1}{r}\vec{\psi} =: \vec{u} \in \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$ is an isomorphism and in what follows we will use the notation $\mathcal{H}$ for both spaces without further comment. Second, there is the following Strauss estimate in $\mathbb{R}^5_*$:

$$|u(r)| \lesssim r^{-\frac{3}{2}} \|u\|_{\dot{H}^1} \tag{3.2.11}$$

Proposition 3.2.2. *The exterior Cauchy problem for* (3.2.9) *is globally well-posed in* $\dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$. *Moreover, a solution* u *scatters as* $t \to \infty$ *to a free wave, i.e., a solution* $\vec{v} \in \mathcal{H}$ *of*

$$\Box v = 0,\ r \geq 1,\ v(1,t) = 0,\ \forall t \geq 0 \tag{3.2.12}$$

if and only if $\|u\|_S < \infty$ *where* $S = L^3_t([0,\infty); L^6_x(\mathbb{R}^5_*))$. *In particular, there exists a constant*

$\delta > 0$ *small so that if* $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, *then* u *scatters to free waves as* $t \to \pm\infty$.

Proof. By the global Strichartz[1] estimates of Smith-Sogge [72] for the free wave equation outside a convex obstacle every energy solution of (3.2.12) satisfies

$$\|v\|_{L_t^3(\mathbb{R};\dot{W}_x^{\frac{1}{2},3}(\mathbb{R}_*^5))} \lesssim \|\vec{v}(0)\|_{\mathcal{H}} \tag{3.2.13}$$

We claim the embedding $\dot{W}_x^{\frac{1}{2},3} \hookrightarrow L_x^6$ for radial functions in $r \geq 1$ in $\mathbb{R}_*^5$. Indeed, one checks via the fundamental theorem of calculus that $\dot{W}_x^{1,3} \hookrightarrow L_x^\infty$. More precisely,

$$|f(r)| \leq r^{-\frac{2}{3}} \|f\|_{\dot{W}_x^{1,3}} \tag{3.2.14}$$

Interpolating this with the embedding $L^3 \hookrightarrow L^3$ we obtain the claim. From (3.2.13) we infer the weaker Strichartz estimate

$$\|v\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}_*^5))} \lesssim \|\vec{v}(0)\|_{\mathcal{H}} \tag{3.2.15}$$

which suffices for our purposes. Indeed, applying it to the equation

$$\Box u = u^3 Z(ru) = N(u),\ r \geq 1$$

and estimating the inhomogeneous term in $L_t^1 L_x^2$, implies for any time interval $I \ni 0$

$$\|u\|_{L_t^3(I;L_x^6)} + \|\vec{u}\|_{L_t^\infty;\mathcal{H}} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|u\|^3_{L_t^3(I;L_x^6)} \tag{3.2.16}$$

1. Due to the radial assumption and the simple geometry, one does not need to resort to the sophisticated construction in [72]. Indeed, grazing and gliding rays cannot occur in this setting which is the main difficulty in the general case and which is addressed by means of the Melrose-Taylor parametrix in [72]. For the radial problem outside the ball one can instead rely on an elementary and explicit parametrix.

By the usual continuity argument (expanding I) this implies

$$\|\vec{u}(0)\|_{\mathcal{H}} < \delta \Longrightarrow \|u\|_S \lesssim \delta$$

Moreover, the scattering is also standard. Indeed, denoting the free propagator in $\mathbb{R}^5_*$ with a Dirichlet boundary condition again by $S_0(t)$, we seek $\vec{v}(0) \in \mathcal{H}$ such that

$$\vec{u}(t) = S_0(t)\vec{v}(0) + o_{\mathcal{H}}(1)$$

as $t \to \infty$. In view of the Duhamel representation of $\vec{u}$ and using the group property and unitarity of S_0 this is tantamount to

$$\vec{v}(0) = \vec{u}(0) + \int_0^\infty S_0(-s)(0, N(u(s)))\, ds \tag{3.2.17}$$

The integral on the right-hand side is absolutely convergent in $\mathcal{H}$ provided $\|u\|_S < \infty$. The necessity of the latter condition follows from the fact that free waves satisfy it, whence by the small data theory (applied to large times) it carries over to any nonlinear wave that scatters. $\square$

We remark that in the ψ formulation, the scattering of Proposition 3.2.2 means precisely (3.1.5), (3.1.6).

To prove Theorem 3.1.1 we therefore need to show that every energy solution ψ of (3.1.2) has the property that in the u-formulation $\|u\|_S < \infty$. This will be done by means of the Kenig-Merle concentration-compactness approach [36], [37].

3.3 Concentration Compactness

In this section, we prove the following result.

Proposition 3.3.1. *Suppose that Theorem 3.1.1 fails. Then there exists a nonzero energy solution to* (3.1.2) *(referred to as critical element)* $\vec{\psi}(t)$ *for* $t \geq 0$ *with the property that the trajectory*

$$\mathcal{K}_+ := \{\vec{\psi}(t) \mid t \geq 0\}$$

is precompact in $\mathcal{H}$.

In the following section we then lead this to a contradiction via a virial-type rigidity argument. To prove Proposition 3.3.1 we may work in the u-formulation of equation (3.2.9) since the map $u = r^{-1}\psi$ is an isomorphism between $\mathcal{H}$ in $\mathbb{R}^5_*$ and $\mathbb{R}^3_*$, respectively.

To proceed, we need the following version of the Bahouri-Gérard decomposition [1]. As before, "free" waves refer to solutions of (3.2.12). The following two lemmas are standard, see in particular Chapter 2 of the book [59].

Lemma 3.3.2. *Let* $\{u_n\}$ *be a sequence of free radial waves bounded in* $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$. *Then after replacing it by a subsequence, there exist a sequence of free solutions* v^j *bounded in* $\mathcal{H}$, *and sequences of times* $t^j_n \in \mathbb{R}$ *such that for* γ^k_n *defined by*

$$u_n(t) = \sum_{1\leq j<k} v^j(t + t^j_n) + \gamma^k_n(t) \tag{3.3.1}$$

we have for any $j < k$, $\vec{\gamma}^k_n(-t^j_n) \rightharpoonup 0$ *weakly in* $\mathcal{H}$ *as* $n \to \infty$, *as well as*

$$\lim_{n\to\infty} |t^j_n - t^k_n| = \infty \tag{3.3.2}$$

and the errors γ^k_n *vanish asymptotically in the sense that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma^k_n\|_{(L^\infty_t L^p_x \cap L^3_t L^6_x)(\mathbb{R}\times\mathbb{R}^5_*)} = 0 \quad \forall\ \frac{10}{3} < p < \infty \tag{3.3.3}$$

Finally, one has orthogonality of the free energy

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1\le j<k} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{\gamma}_n^k\|_{\mathcal{H}}^2 + o(1) \tag{3.3.4}$$

as $n \to \infty$.

Proof. Recall the Sobolev embeddings $\dot{H}_0^1(\mathbb{R}_*^5) \hookrightarrow L^{\frac{10}{3}} \cap L^\infty(\mathbb{R}_*^5)$ for radial functions. Moreover, for any $p \in (\frac{10}{3}, \infty)$ the embedding is compact. Since γ_n^k is bounded in $\dot{H}_0^1$, interpolation with these, as well as the Strichartz estimates from [72] implies that it suffices to bound the remainder in $L_t^\infty L_x^p$ for any fixed $p \in (\frac{10}{3}, \infty)$. Fix such a p. Let $\gamma_n^0 := u_n$ and $k = 0$. If

$$\nu^k := \limsup_{n\to\infty} \|\gamma_n^k\|_{L_t^\infty L_x^p} = 0,$$

then we are done by putting $\gamma_n^\ell = \gamma_n^k$ for all $\ell > k$. Otherwise, there exists a sequence $t_n^k \in \mathbb{R}$ such that $\|\gamma_n^k(-t_n^k)\|_{L_x^p} \ge \nu^k/2$ for large n. Since $\vec{\gamma}_n^k(-t_n^k) \in \mathcal{H}$ is bounded, after extracting a subsequence it converges weakly in $\mathcal{H}$, and $\gamma_n^k(-t_n^k)$ converges strongly in $L_x^p(\mathbb{R}_*^5)$. Let v^k be the free wave given by the limit

$$\lim_{n\to\infty} \vec{\gamma}_n^k(-t_n^k) = \vec{v}^k(0)$$

By Sobolev $\|v^k(0)\|_{\dot{H}_0^1(\mathbb{R}_*^5)} \gtrsim \nu^k$. We repeat the same procedure inductively in $k \ge 1$. As before, let $S_0(t)$ denote the free exterior propagator in $\mathcal{H}$. If $t_n^j - t_n^k \to c \in \mathbb{R}$ for some $j < k$, then

$$\vec{\gamma}_n^k(-t_n^k) = S_0(t_n^j - t_n^k)\vec{\gamma}_n^k(-t_n^j) \to 0,$$

weakly in $\mathcal{H}$. To see this, it suffices to show that

$$\langle \vec{\gamma}_n^k(-t_n^k) \mid \vec{\phi} \rangle \to 0 \qquad n \to \infty$$

for any Schwartz function $\vec{\phi}$. But one has

$$\langle \vec{\gamma}_n^k(-t_n^k) \mid \vec{\phi} \rangle = \langle \vec{\gamma}_n^k(-t_n^j) \mid S_0(t_n^k - t_n^j)\vec{\phi} \rangle \to 0$$

since $S_0(t_n^k - t_n^j)\vec{\phi} \to S_0(-c)\vec{\phi}$ strongly in L^2. Hence $|t_n^j - t_n^k| \to \infty$ as long as $\vec{v}^k \neq 0$. Then for all $j \leq k$,

$$\vec{\gamma}_n^{k+1}(-t_n^j) = \vec{\gamma}_n^k(-t_n^j) - \vec{v}^k(t_n^k - t_n^j) \rightharpoonup 0$$

weakly in $\mathcal{H}$. Indeed, if $j < k$ then this follows from the inductive assumption, whereas for $j = k$ it follows by construction.

To prove (3.3.4), expand (without loss of generality at $t = 0$)

$$\|\vec{u}_n(0)\|_{\mathcal{H}}^2 = \Big\| \sum_{1 \leq j < k} \vec{v}^j(t_n^j) + \vec{\gamma}_n^k(0) \Big\|_{\mathcal{H}}^2$$

The cross terms are all $o(1)$ as $n \to \infty$: for $k > j \neq \ell$, and with the scalar product in $\mathcal{H}$,

$$\begin{aligned} \langle \vec{v}^j(t_n^j) \mid \vec{v}^\ell(t_n^\ell) \rangle &= \langle \vec{v}^j(0) \mid S_0(t_n^\ell - t_n^j)\vec{v}^\ell(0) \rangle \to 0 \\ \langle \vec{v}^j(t_n^j) \mid \vec{\gamma}_n^k(0) \rangle &= \langle \vec{v}^j(0) \mid \vec{\gamma}_n^k(-t_n^j) \rangle \to 0 \end{aligned} \tag{3.3.5}$$

The first line of (3.3.5) vanishes as $n \to \infty$ due to $\|S_0(t_n^\ell - t_n^j)\vec{\phi}\|_\infty \to 0$ for any Schwartz function $\vec{\phi}$ since $|t_n^\ell - t_n^j| \to \infty$, by the pointwise decay of free waves with Schwartz data; as usual this suffices since we can approximate $\vec{v}^j(0), \vec{v}^\ell(0)$ by Schwartz functions. The second line vanishes by $\vec{\gamma}_n^k(-t_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$.

Finally, one uses (3.3.4) to conclude that $\nu^j \to 0$:

$$\limsup_{n\to\infty} \|\vec{u}_n\|_{\mathcal{H}}^2 \geq \sum_{j<k} \|\vec{v}^j\|_{\mathcal{H}}^2 \gtrsim \sum_{j<k} (\nu^j)^2$$

uniformly in k. The final inequality follows from the radial Sobolev embedding (in other words, Sobolev embedding and compactness). Hence, $\limsup_{n\to\infty} \|\gamma_n^k\|_{L_t^\infty L_x^p} = \nu^k \to 0$, as $k \to \infty$. $\square$

Applying this decomposition to the nonlinear equation requires a perturbation lemma which we now formulate. All spatial norms are understood to be on $\mathbb{R}^5_*$. The exterior propagator $S_0(t)$ is as above.

Lemma 3.3.3. *There are continuous functions $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), $u, v \in C(I; \dot{H}_0^1) \cap C^1(I; L^2)$ radial functions satisfying for some $A > 0$*

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|\vec{v}\|_{L^\infty(I;\mathcal{H})} + \|v\|_{L_t^3(I;L_x^6)} \leq A$$
$$\|\mathrm{eq}(u)\|_{L_t^1(I;L_x^2)} + \|\mathrm{eq}(v)\|_{L_t^1(I;L_x^2)} + \|w_0\|_{L_t^3(I;L_x^6)} \leq \varepsilon \leq \varepsilon_0(A),$$

where $\mathrm{eq}(u) := \Box u + u^3 Z(ru)$ *in the sense of distributions, and* $\vec{w}_0(t) := S_0(t-t_0)(\vec{u}-\vec{v})(t_0)$ *with $t_0 \in I$ arbitrary but fixed. Then*

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L_t^\infty(I;\mathcal{H})} + \|u - v\|_{L_t^3(I;L_x^6)} \leq C_0(A)\varepsilon.$$

In particular, $\|u\|_{L_t^3(I;L_x^6)} < \infty$.

Proof. Let $X := L_t^3 L_x^6$ and

$$w := u - v, \quad e := \Box(u - v) + u^3 Z(ru) - v^3 Z(rv) = \mathrm{eq}(u) - \mathrm{eq}(v).$$

There is a partition of the right half of I as follows, where $\delta_0 > 0$ is a small absolute constant which will be determined below:

$$
\begin{aligned}
&t_0 < t_1 < \cdots < t_n \leq \infty, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_n), \\
&\|v\|_{X(I_j)} \leq \delta_0 \quad (j = 0, \ldots, n-1), \quad n \leq C(A, \delta_0).
\end{aligned}
$$

We omit the estimate on $I \cap (-\infty, t_0)$ since it is the same by symmetry. Let $\vec{w}_j(t) := S_0(t - t_j)\vec{w}(t_j)$ for all $0 \leq j < n$. Then

$$
\vec{w}(t) = \vec{w}_0(t) + \int_{t_0}^{t} S_0(t-s)(0, e - (v+w)^3 Z(r(v+w)) + v^3 Z(rv))(s)\, ds \tag{3.3.6}
$$

which implies that, for some absolute constant $C_1 \geq 1$,

$$
\begin{aligned}
\|w - w_0\|_{X(I_0)} &\lesssim \|(v+w)^3 Z(r(v+w)) - v^3 Z(rv) - e\|_{L_t^1 L_x^2(I_0)} \\
&\leq C_1(\delta_0^2 + \|w\|_{X(I_0)}^2)\|w\|_{X(I_0)} + C_1\varepsilon
\end{aligned} \tag{3.3.7}
$$

To estimate the differences involving the Z function we invoke its smoothness as well as the fact that by radiality, ru and rv are bounded pointwise in terms of the energy of u and v, respectively (which we assume to be bounded by A). Note that $\|w\|_{X(I_0)} < \infty$ provided I_0 is a finite interval. If I_0 is half-infinite, then we first need to replace it with an interval of the form $[t_0, N)$, and let $N \to \infty$ after performing the estimates which are uniform in N. Now assume that $C_1\delta_0^2 \leq \frac{1}{4}$ and fix δ_0 in this fashion. By means of the continuity method (which refers to using that the X-norm is continuous in the upper endpoint of I_0), (3.3.7) implies that $\|w\|_{X(I_0)} \leq 8C_1\varepsilon$. Furthermore, Duhamel's formula implies that

$$
\vec{w}_1(t) - \vec{w}_0(t) = \int_{t_0}^{t_1} S_0(t-s)(0, e - (v+w)^3 Z(r(v+w)) + v^3 Z(rv))(s)\, ds
$$

whence also

$$\|w_1 - w_0\|_{X(\mathbb{R})} \lesssim \int_{t_0}^{t_1} \|(e - (v+w)^3 Z(r(v+w)) + v^3 Z(rv))(s)\|_2 \, ds \tag{3.3.8}$$

which is estimated as in (3.3.7). We conclude that $\|w_1\|_{X(\mathbb{R})} \le 8C_1\varepsilon$. In a similar fashion one verifies that for all $0 \le j < n$

$$\begin{aligned} \|w - w_j\|_{X(I_j)} + \|w_{j+1} - w_j\|_{X(\mathbb{R})} &\lesssim \|e - (v+w)^3 Z(r(v+w)) + v^3 Z(rv)\|_{L^1_t L^2_x(I_j)} \\ &\le C_1(\delta_0^2 + \|w\|^2_{X(I_j)})\|w\|_{X(I_j)} + C_1\varepsilon \end{aligned} \tag{3.3.9}$$

where $C_1 \ge 1$ is as above. By induction in j one obtains that

$$\|w\|_{X(I_j)} + \|w_j\|_{X(\mathbb{R})} \le C(j)\,\varepsilon \quad \forall\, 1 \le j < n$$

This requires that $\varepsilon < \varepsilon_0(n)$ which can be done provided $\varepsilon_0(A)$ is chosen small enough. Repeating the estimate (3.3.9) once more, but with the energy piece $L^\infty_t \mathcal{H}$ included on the left-hand side, we can now bound the $S(I)$-norm on w. □

We can now apply standard arguments to prove the main result of this section. Without further mention, all functions are radial.

Proof of Proposition 3.3.1. Suppose that the theorem fails. Then there exists a bounded sequence $\vec{u}_n := (u_{0,n}, u_{1,n}) \in \mathcal{H}$ with

$$\|\vec{u}_n\|_{\mathcal{H}} \to E_* > 0, \quad \|u_n\|_S \to \infty$$

where u_n denotes the global evolution of $\vec{u}_n$ of (3.2.9). We may assume that E_* is minimal with this property. Applying Lemma 3.3.2 to the free evolutions of $\vec{u}_n(0)$ yields free waves

v^j and times t_n^j as in (3.3.1). Let U^j be the nonlinear profiles of (v^j, t_n^j), i.e., those energy solutions of (3.2.9) which satisfy

$$\lim_{t \to t_\infty^j} \|\vec{v}^j(t) - \vec{U}^j(t)\|_{\mathcal{H}} \to 0$$

where $\lim_{n\to\infty} t_n^j = t_\infty^j \in [-\infty, \infty]$. The U^j exist locally around $t = t_\infty^j$ by the local existence and scattering theory, see Proposition 3.2.2. Locally around $t = 0$ one has the following *nonlinear profile decomposition*

$$u_n(t) = \sum_{j<k} U^j(t + t_n^j) + \gamma_n^k(t) + \eta_n^k(t) \tag{3.3.10}$$

where $\|\vec{\eta}_n^k(0)\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Now suppose that either there are two non-vanishing v^j, say v^1, v^2, or that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|\vec{\gamma}_n^k\|_{\mathcal{H}} > 0 \tag{3.3.11}$$

Note that the left-hand side does not depend on time since γ_n^k is a free wave. By the minimality of E_* and the orthogonality of the energy (3.3.4) each U^j is a global solution and scatters with $\|U^j\|_{L_t^3 L_x^6} < \infty$.

We now apply Lemma 3.3.3 on $I = \mathbb{R}$ with $u = u_n$ and

$$v(t) = \sum_{j<k} U^j(t + t_n^j) \tag{3.3.12}$$

That $\|\mathrm{eq}(v)\|_{L_t^1 L_x^2}$ is small for large n follows from (3.3.2). To see this, note that with

$N(v) := v^3 Z(rv)$,

$$\begin{aligned}\mathrm{eq}(v) &= \Box v + v^3 Z(rv) \\ &= -\sum_{j<k} N(U^j(t+t_n^j)) + N\big(\sum_{j<k} U^j(t+t_n^j)\big)\end{aligned}$$

The difference on the right-hand side here only consists of terms which involve at least one pair of distinct j, j'. But then $\|\mathrm{eq}(v)\|_{L_t^1 L_x^2} \to 0$ as $n \to \infty$ by (3.3.2). In order to apply Lemma 3.3.3 it is essential that

$$\limsup_{n\to\infty} \big\| \sum_{j<k} U^j(t+t_n^j) \big\|_{L_t^3 L_x^6} \le A < \infty \tag{3.3.13}$$

uniformly in k, which follows from (3.3.2), (3.3.4), and Proposition 3.2.2. The point here is that the sum can be split into one over $1 \le j < j_0$ and another over $j_0 \le j < k$. This splitting is performed in terms of the energy, with j_0 being chosen such that for all $k > j_0$

$$\limsup_{n\to\infty} \sum_{j_0 \le j < k} \|\vec{U}^j(t_n^j)\|_{\mathcal{H}}^2 \le \varepsilon_0^2 \tag{3.3.14}$$

where ε_0 is fixed such that the small data result of Proposition 3.2.2 applies. Clearly, (3.3.14) follows from (3.3.4). Using (3.3.2) as well as the small data scattering theory one now obtains

$$\begin{aligned}\limsup_{n\to\infty} \Big\| \sum_{j_0 \le j < k} U^j(\cdot + t_n^j) \Big\|_{L_t^3 L_x^6}^3 &= \sum_{j_0 \le j < k} \big\| U^j(\cdot) \big\|_{L_t^3 L_x^6}^3 \\ &\le C \limsup_{n\to\infty} \Big(\sum_{j_0 \le j < k} \|\vec{U}^j(t_n^j)\|_{\mathcal{H}}^2 \Big)^{\frac{3}{2}}\end{aligned} \tag{3.3.15}$$

with an absolute constant C. This implies (3.3.13), uniformly in k.

Hence one can take k and n so large that Lemma 3.3.3 applies to (3.3.10) whence

$$\limsup_{n\to\infty} \|u_n\|_{L_t^3 L_x^6} < \infty$$

which is a contradiction. Thus, there can be only one nonvanishing v^j, say v^1, and moreover

$$\limsup_{n\to\infty} \|\vec{\gamma}_n^2\|_{\mathcal{H}} = 0 \tag{3.3.16}$$

Thus, $\|\vec{U}^1\|_{\mathcal{H}} = E_*$. By the preceding, necessarily

$$\|U^1\|_{L_t^3 L_x^6} = \infty \tag{3.3.17}$$

Therefore, $U^1 =: u_*$ is the desired critical element. Suppose that

$$\|u_*\|_{L_t^3([0,\infty);L_x^6)} = \infty \tag{3.3.18}$$

Then we claim that

$$\mathcal{K}_+ := \{\vec{u}_*(t) \mid t \geq 0\}$$

is precompact in $\mathcal{H}$. If not, then there exists $\delta > 0$ so that for some infinite sequence $t_n \to \infty$ one has

$$\|\vec{u}_*(t_n) - \vec{u}_*(t_m)\|_{\mathcal{H}} > \delta \quad \forall\, n > m \tag{3.3.19}$$

Applying Lemma 3.3.2 to $U^1(t_n)$ one concludes via the same argument as before based on the minimality of E_* and (3.3.17) that

$$\vec{u}_*(t_n) = \vec{V}(\tau_n) + \vec{\gamma}_n(0) \tag{3.3.20}$$

where $\vec{V}$, $\vec{\gamma}_n$ are free waves in $\mathcal{H}$, and τ_n is some sequence in $\mathbb{R}$. Moreover, $\|\vec{\gamma}_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. If $\tau_n \to \tau_\infty \in \mathbb{R}$, then (3.3.20) and (3.3.19) lead to a contradiction. If $\tau_n \to \infty$, then

$$\|V(\cdot + \tau_n)\|_{L_t^3([0,\infty);L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies via the local wellposedness theory that $\|u_*(\cdot + t_n)\|_{L_t^3([0,\infty);L_x^6)} < \infty$ for all large n, which is a contradiction to (3.3.18). If $\tau_n \to -\infty$, then

$$\|V(\cdot + \tau_n)\|_{L_t^3((-\infty,0];L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies that $\|u_*(\cdot + t_n)\|_{L_t^3((-\infty,0];L_x^6)} < C < \infty$ for all large n where C is some fixed constant. Passing to the limit yields a contradiction to (3.3.17) and (3.3.19) is seen to be false, concluding the proof of compactness of $\mathcal{K}_+$. $\square$

3.4 The rigidity argument

In this section we complete the proof of Theorem 3.1.1 by showing that a critical element as given by Proposition 3.3.1 does not exist. This is based on the virial identity exterior to the ball. The main novelty here lies with the fact that due to the radial assumption in $\mathbb{R}^3_*$ we are able to show that the nonlinear functional arising in this virial identity is *globally coercive* on the energy space. In contrast, for equivariant energy critical wave maps in the energy class, Côte, Kenig, Merle [17] needed an upper bound on the energy in order to apply the virial argument. In particular, we have the following proposition.

Proposition 3.4.1 (Rigidity Property)**.** *Let $(\psi_0, \psi_1) \in \mathcal{H}$, and denote by $\vec{\psi}(t)$ the associated* *global in time solution to* (3.1.2) *given by Proposition 3.2.1. Suppose that the trajectory*

$$\mathcal{K}_+ := \{\vec{\psi}(t) \mid t \geq 0\}$$

is precompact in $\mathcal{H}$. *Then* $\psi \equiv 0$.

The proof of Proposition 3.4.1 relies on the following two results related to the virial identity for solutions to (3.1.2). In what follows we let $\chi \in C_0^\infty(\mathbb{R})$ be an even function so that $\chi(r) = 1$ for $|r| \leq 1$, $\text{supp}(\chi) \in [-2, 2]$ and $\chi(r) \in [0, 1]$ for every $r \in \mathbb{R}$. Define $\chi_R(r) := \chi(R^{-1}r)$.

Lemma 3.4.2. *Let* $\vec{\psi}(t) \in \mathcal{H}$ *be a solution to* (3.1.2). *Then, for every* $T \in \mathbb{R}$ *we have*

$$\begin{aligned}
\left\langle \chi_R \dot{\psi} | r\psi_r \right\rangle \Big|_0^T &\leq \int_0^T \left\{ -\frac{3}{2} \int_1^\infty \dot{\psi}^2 \, r^2 \, dr + \frac{1}{2} \int_1^\infty \psi_r^2 \, r^2 \, dr \right\} dt && (3.4.1)\\
&\quad + \int_0^T \left\{ \int_1^\infty \sin^2(\psi) \, dr + O(\mathcal{E}_R^\infty(\vec{\psi})) \right\} dt\\
\left\langle \chi_R \dot{\psi} | \psi \right\rangle \Big|_0^T &= \int_0^T \left\{ \int_1^\infty \dot{\psi}^2 \, r^2 \, dr - \int_1^\infty \psi_r^2 \, r^2 \, dr - \int_1^\infty \psi \sin(2\psi) \, dr \right\} dt && (3.4.2)\\
&\quad + \int_0^T \left\{ O(\mathcal{E}_R^\infty(\vec{\psi})) + O\left(\int_R^\infty \psi^2 \, dr \right) \right\} dt
\end{aligned}$$

where here, the brackets $\langle \cdot | \cdot \rangle$ *refer to the* $L^2_{rad}(\mathbb{R}^3_*)$ *pairing* $\langle f | g \rangle := \int_1^\infty f(r) g(r) r^2 \, dr$ *and*

$$\mathcal{E}_R^\infty(\vec{\psi}) := \frac{1}{2} \int_R^\infty \left(\dot{\psi}^2 + \psi_r^2 + \frac{2 \sin^2(\psi)}{r^2} \right) r^2 \, dr \tag{3.4.3}$$

Proof. We first establish (3.4.1) for solutions $\vec{\psi}(t) \in C_0^\infty \times C_0^\infty(\mathbb{R}^3_*)$.

$$
\begin{aligned}
\frac{d}{dt}\left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle &= \left\langle \chi_R \ddot{\psi} \,|\, r\psi_r \right\rangle + \left\langle \chi_R \dot{\psi} \,|\, r\dot{\psi}_r \right\rangle \\
&= \left\langle \chi_R \left(\psi_{rr} + \frac{2}{r}\psi_r - \frac{\sin(2\psi)}{r^2} \right) \,|\, r\psi_r \right\rangle + \left\langle \chi_R \dot{\psi} \,|\, r\dot{\psi}_r \right\rangle \\
&= \frac{1}{2}\int_1^\infty \partial_r(\psi_r^2)(\chi_R r^3)\, dr + 2\int_1^\infty \chi_R\, \psi_r^2 r^2\, dr \\
&\quad - \int_1^\infty \partial_r(\sin^2(\psi))\chi_R r\, dr + \frac{1}{2}\int_1^\infty \partial_r(\dot{\psi}^2)\chi_R r^3\, dr
\end{aligned}
$$

Integrating by parts, the preceding line can be further simplified as follows:

$$
\begin{aligned}
&= -\frac{3}{2}\int_1^\infty \chi_R \dot{\psi}^2 r^2\, dr + \frac{1}{2}\int_1^\infty \chi_R \psi_r^2 r^2\, dr + \int_1^\infty \chi_R \sin^2(\psi)\, dr - \frac{1}{2}\psi_r^2(t,1) \\
&\quad + \frac{1}{2}\int_1^\infty \left(\psi_r^2 - \dot{\psi}^2 + \frac{2\sin^2(\psi)}{r^2} \right) r\, \chi_R'\, r^2\, dr
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{3}{2}\int_1^\infty \dot{\psi}^2 r^2\, dr + \frac{1}{2}\int_1^\infty \psi_r^2\, r^2\, dr + \int_1^\infty \sin^2(\psi)\, dr - \frac{1}{2}\psi_r^2(t,1) \\
&\quad - \int_1^\infty (1-\chi_R)\left(-\frac{3}{2}\dot{\psi}^2 r^2 + \frac{1}{2}\psi_r^2\, r^2 + \frac{\sin^2(\psi)}{r^2} \right) r^2\, dr \\
&\quad + \frac{1}{2}\int_1^\infty \left(\psi_r^2 - \dot{\psi}^2 + \frac{2\sin^2(\psi)}{r^2} \right) r\, \chi_R'\, r^2\, dr
\end{aligned}
$$

Next, observe that

$$
\left| \int_1^\infty (1-\chi_R)\left(-\frac{3}{2}\dot{\psi}^2 r^2 + \frac{1}{2}\psi_r^2\, r^2 + \frac{\sin^2(\psi)}{r^2} \right) r^2\, dr \right| \lesssim \mathcal{E}_R^\infty(\vec{\psi})
$$

And similarly, since $\mathrm{supp}(\chi'(R^{-1}\cdot)) \cap [1,\infty) \subset [R, 2R]$, we have

$$\begin{aligned}
&\left|\frac{1}{2}\int_1^\infty \left(\psi_r^2 - \dot{\psi}^2 + \frac{2\sin^2(\psi)}{r^2}\right) r\,\chi_R'\, r^2\, dr\right| \\
&\le \frac{1}{2}\int_1^\infty \left(\psi_r^2 + \dot{\psi}^2 + \frac{2\sin^2(\psi)}{r^2}\right) R^{-1} r\, \left|\chi'(R^{-1}r)\right| \, r^2\, dr \\
&\lesssim \mathcal{E}_R^\infty(\vec{\psi})
\end{aligned}$$

Putting this together, we obtain

$$\begin{aligned}
\frac{d}{dt}\left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle &= -\frac{3}{2}\int_1^\infty \dot{\psi}^2 r^2\, dr + \frac{1}{2}\int_1^\infty \psi_r^2\, r^2\, dr + \int_1^\infty \sin^2(\psi)\, dr \\
&\quad - \frac{1}{2}\psi_r^2(t,1) + O(\mathcal{E}_R^\infty(\vec{\psi})) \\
&\le -\frac{3}{2}\int_1^\infty \dot{\psi}^2 r^2\, dr + \frac{1}{2}\int_1^\infty \psi_r^2\, r^2\, dr \\
&\quad + \int_1^\infty \sin^2(\psi)\, dr + O(\mathcal{E}_R^\infty(\vec{\psi}))
\end{aligned}$$

By integrating the above inequality in time from 0 to T we obtain (3.4.1) for smooth solutions. Our well-posedness theory for (3.1.2) then allows us to extend (3.4.1) to all energy class solutions $\vec{\psi}(t) \in \mathcal{H}$ via an approximation argument.

We proceed in a similar fashion to prove (3.4.2). Thus, for smooth ψ we have by direct calculation,

$$\frac{d}{dt}\left\langle \chi_R\dot{\psi} \mid \psi \right\rangle = \left\langle \chi_R\ddot{\psi} \,|\, \psi \right\rangle + \left\langle \chi_R\dot{\psi} \,|\, \dot{\psi} \right\rangle$$

$$= \left\langle \chi_R \left(\psi_{rr} + \frac{2}{r}\psi_r - \frac{\sin(2\psi)}{r^2} \right) \,|\, \psi \right\rangle + \left\langle \chi_R\dot{\psi} \,|\, \dot{\psi} \right\rangle$$

$$= \left\langle \frac{\chi_R}{r^2}\partial_r \left(r^2\psi_r \right) \,|\, \psi \right\rangle - \left\langle \chi_R \frac{\sin(2\psi)}{r^2} \,|\, \psi \right\rangle + \left\langle \chi_R\dot{\psi} \,|\, \dot{\psi} \right\rangle$$

Integrating by parts, the above simplifies as follows:

$$\begin{aligned}
&= \int_1^\infty \chi_R\dot{\psi}^2 r^2\, dr - \int_1^\infty \chi_R\psi_r^2\, r^2\, dr - \int_1^\infty \chi_R\psi\sin(2\psi)\, dr \\
&\quad - \int_1^\infty \psi_r\psi\chi_R' r^2\, dr \\
&= \int_1^\infty \dot{\psi}^2 r^2\, dr - \int_1^\infty \psi_r^2\, r^2\, dr - \int_1^\infty \psi\sin(2\psi)\, dr \\
&\quad - \int_1^\infty (1-\chi_R)\left(\dot{\psi}^2 - \psi_r^2\right) r^2\, dr \\
&\quad + \int_1^\infty \left\{ (1-\chi_R)\psi\sin(2\psi) + \frac{1}{2}\psi^2\partial_r(\chi_R' r^2) \right\} dr
\end{aligned}$$

As before we have,

$$\left| - \int_1^\infty (1-\chi_R)\left(\dot{\psi}^2 - \psi_r^2\right) r^2\, dr \right| \le \mathcal{E}_R^\infty(\vec{\psi})$$

And, since $|\psi \sin(2\psi)| \leq 2\psi^2$, we can deduce that

$$\begin{aligned}
&\left| \int_1^\infty \left\{ (1-\chi_R)\psi \sin(2\psi) + \frac{1}{2}\psi^2 \partial_r(\chi_R' r^2) \right\} dr \right| \\
&\lesssim \int_1^\infty (1-\chi_R)\psi^2 \, dr + \int_1^\infty \psi^2 \left|\chi'(R^{-1}r)\right| R^{-1} r \, dr + \int_1^\infty \psi^2 \left|\chi''(R^{-1}r)\right| R^{-2} r^2 \, dr \\
&\lesssim \int_R^\infty \psi^2 \, dr
\end{aligned}$$

Therefore, we see thatf

$$\begin{aligned}
\frac{d}{dt} \left\langle \chi_R \dot{\psi} | \psi \right\rangle = &\int_1^\infty \dot{\psi}^2 r^2 \, dr - \int_1^\infty \psi_r^2 \, r^2 \, dr - \int_1^\infty \psi \sin(2\psi) \, dr \\
&+ O\left(\mathcal{E}_R^\infty(\vec{\psi})\right) + O\left(\int_R^\infty \psi^2 \, dr\right)
\end{aligned}$$

Integrating the above in time from 0 to T proves (3.4.2) for smooth solutions. Approximating energy solutions by smooth solutions concludes the proof. □

From (3.4.1) and (3.4.2) we construct a nonlinear functional, $\mathcal{L} : \mathcal{H} \to \mathbb{R}$, whose global coercivity on $\mathcal{H}$ is a key ingredient in the proof of Theorem 3.4.1. Using Lemma 3.4.1 we consider the following linear combination of (3.4.1) and (3.4.2):

$$\begin{aligned}
\left\langle \chi_R \dot{\psi} \,|\, r\psi_r + \frac{29}{20}\psi \right\rangle \Big|_0^T \leq &-\int_0^T \left[\int_1^\infty \left(\frac{1}{20}\dot{\psi}^2 + \frac{19}{20}\psi_r^2 \right) r^2 \, dr \right] dt \\
&+ \int_0^T \left[\int_1^\infty \left(\sin^2(\psi) - \frac{29}{20}\psi \sin(2\psi) \right) dr \right] dt \\
&+ \int_0^T \left[O\left(\mathcal{E}_R^\infty(\vec{\psi})\right) + O\left(\int_R^\infty \psi^2 \, dr \right) \right] dt
\end{aligned} \tag{3.4.4}$$

We define $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ as follows

$$\mathcal{L}(\vec{\psi}) := -\int_1^\infty \left(\frac{1}{20}\dot{\psi}^2 + \frac{19}{20}\psi_r^2 \right) r^2 \, dr + \int_1^\infty \left(\sin^2(\psi) - \frac{29}{20}\psi \sin(2\psi) \right) dr \tag{3.4.5}$$

Lemma 3.4.3. *Let $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ be defined as in* (3.4.5)*. Then for every $\vec{\psi} = (\psi(t), \dot{\psi}(t)) \in \mathcal{H}$*

we have

$$\mathcal{L}(\vec{\psi}) \leq -\frac{1}{20}\int_1^\infty \left(\dot{\psi}^2 + \psi_r^2\right) r^2\, dr \leq -\frac{1}{180}\mathcal{E}(\vec{\psi}) \tag{3.4.6}$$

We postpone the proof of Lemma 3.4.3, and first use it to prove Proposition 3.4.1.

Proof of Proposition 3.4.1. Suppose $\vec{\psi}(t) \in \mathcal{H}$ satisfies the conditions of Proposition 3.4.1, i.e., suppose that

$$K_+ := \{\vec{\psi}(t) \mid t \geq 0\}$$

is pre-compact in $\mathcal{H}$. Note that the pre-compactness of K_+ in $\mathcal{H}$ implies, by Hardy's inequality, that K_+ is also pre-compact in $L^2(\mathbb{R}^3_*, dr)$ where

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^3_*, dr)} := \int_1^\infty \psi(t)^2\, dr$$

Then, for every $\varepsilon > 0$ there exists $R(\varepsilon)$ such that for every $t \geq 0$ we have

$$\mathcal{E}^\infty_{R(\varepsilon)}(\vec{\psi}(t)) + \int_{R(\varepsilon)}^\infty \psi(t)^2\, dr < \varepsilon \tag{3.4.7}$$

Now, by (3.4.4) and Lemma 3.4.3, we have that for all T

$$\left\langle \chi_R \dot{\psi} \,|\, r\psi_r + \frac{29}{20}\psi \right\rangle \Big|_0^T \leq \int_0^T \left[\mathcal{L}(\vec{\psi}) + O\left(\mathcal{E}^\infty_R(\vec{\psi}(t)) + \int_R^\infty \psi(t)^2\, dr\right)\right] dt$$

$$\leq \int_0^T \left[-\frac{\mathcal{E}(\vec{\psi})}{180} + O\left(\mathcal{E}^\infty_R(\vec{\psi}(t)) + \int_R^\infty \psi(t)^2\, dr\right)\right] dt$$

Using (3.4.7), we fix R large enough so that

$$\sup_{t\geq 0} O\left(\mathcal{E}_R^\infty(\vec{\psi}(t)) + \int_R^\infty \psi(t)^2\, dr\right) < \frac{\mathcal{E}(\vec{\psi})}{360}$$

Therefore, we deduce that

$$\left\langle \chi_R\dot{\psi} \,|\, r\psi_r + \frac{29}{20}\psi\right\rangle\Big|_0^T \leq -\frac{1}{360}\mathcal{E}(\vec{\psi})T \tag{3.4.8}$$

for every $T > 0$. However, we can use Hardy's inequality and the conservation of energy to estimate the left hand side of the above inequality as follows,

$$\begin{aligned}\left|\left\langle \chi_R\dot{\psi} \,|\, r\psi_r + \frac{29}{20}\psi\right\rangle\right| &\leq \left|\int_1^\infty \chi_R\dot{\psi}\psi_r\, r^3\, dr\right| + C\left|\int_1^\infty \chi_R\dot{\psi}\psi\, r^2\, dr\right| \\ &\lesssim R\int_1^\infty (\dot{\psi}^2 + \psi_r^2 + \frac{\psi^2}{r^2})\, r^2\, dr \\ &\lesssim R\mathcal{E}(\vec{\psi})\end{aligned}$$

Combining the above with (3.4.8) we conclude that

$$T\frac{1}{360}\mathcal{E}(\vec{\psi}) \lesssim R\,\mathcal{E}(\vec{\psi})$$

for all $T > 0$, which, since $\mathcal{E}(\vec{\psi}) = \text{const}$, implies that $T \leq CR$. And this contradicts the fact that $\vec{\psi}$ exists globally in time. $\square$

We can now complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Suppose that Theorem 3.1.1 fails. Then Proposition 3.3.1 implies the existence of a critical element, i.e., a nonzero energy class solution $\vec{\psi}(t) \in \mathcal{H}$ to (3.1.2) such that the trajectory $K_+ = \{\vec{\psi}(t) | t \geq 0\}$ is pre-compact in $\mathcal{H}$. However, Proposition 3.4.1 implies that any such solution must be identically zero, which contradicts the fact that the critical element is nonzero. $\square$

3.4.1 Proof of Lemma 3.4.3

The remaining piece of the argument is the proof of Lemma 3.4.3. To begin we define $\Lambda : \dot{H}_0^1(1, \infty) \to \mathbb{R}$ by

$$\Lambda(\psi) := -\frac{9}{10} \int_1^\infty \psi_r^2 \, r^2 \, dr + \int_1^\infty \left(\sin^2(\psi) - \frac{29}{20} \psi \sin(2\psi) \right) \, dr \tag{3.4.9}$$

And we note that in order to prove Lemma 3.4.3, it suffices to show that

$$\Lambda(\psi) \leq 0 \quad \text{for every} \quad \psi \in \dot{H}_0^1(1, \infty) \tag{3.4.10}$$

Indeed, if (3.4.10) holds then

$$\mathcal{L}(\vec{\psi}) = -\frac{1}{20} \int_1^\infty \left(\dot{\psi}^2 + \psi_r^2 \right) r^2 \, dr + \Lambda(\psi)$$

$$\leq -\frac{1}{20} \int_1^\infty \left(\dot{\psi}^2 + \psi_r^2 \right) r^2 \, dr$$

which is exactly (3.4.6). For each $R > 1$, define

$$\mathcal{A}_R := \{ \psi \in \dot{H}_0^1(1, \infty) \mid \psi(r) = 0 \text{ for every } r \geq R \}$$

Observe that $\mathcal{A}_R = \dot{H}_0^1(1, R)$ where the subscript 0 indicates Dirichlet boundary conditions at both $r = 1$ and $r = R$. We start by deducing (3.4.10) on $\mathcal{A}_R$ for each $R > 1$.

Lemma 3.4.4. *For each $R > 1$ the restriction $\Lambda|_{\mathcal{A}_R} : \mathcal{A}_R \to \mathbb{R}$ satisfies $\Lambda(\psi) \leq 0$ for every*

$\psi \in \mathcal{A}_R$.

Assuming Lemma 3.4.4, we can extend (3.4.10) to all of $\dot{H}^1_0(1,\infty)$ via an approximation argument as follows. To simplify notation, set

$$\begin{aligned} F(\psi) &:= \sin^2(\psi) - \frac{29}{20}\psi \sin(2\psi) \\ N(\psi) &:= \int_1^\infty F(\psi(r))\, dr \\ E(\psi) &:= \frac{1}{2}\int_1^\infty \psi_r^2(r)\, r^2\, dr \end{aligned}$$

Then,

$$\Lambda(\psi) = -\frac{9}{5}E(\psi) + N(\psi)$$

Proof that Lemma 3.4.4 implies Lemma 3.4.3. We assume that Lemma 3.4.4 is true but (3.4.10) fails. Then there exists $\psi \in \dot{H}^1_0(1,\infty)$ such that

$$\Lambda(\psi) = \delta > 0 \tag{3.4.11}$$

For each $k \in \mathbb{N}$ define $\phi_k \in C_0^\infty(\mathbb{R})$ so that $\phi_k(r) = 1$ for $0 \le r \le k$, $\phi_k \equiv 0$ for $r \ge 2k$ and $\left|\phi_k'(r)\right| \lesssim \frac{1}{k}$. Then set $\psi_k := \phi_k \psi$. Note that for each k, $\psi_k \in \mathcal{A}_{2k}$ and that

$$E(\psi_k) \to E(\psi) \text{ as } k \to \infty$$

$$N(\psi_k) \to N(\psi) \text{ as } k \to \infty$$

Hence, by (3.4.11), there exists $k_0 \in \mathbb{N}$ such that

$$\Lambda(\psi_k) \ge \frac{\delta}{2} > 0$$

for $k \geq k_0$, and this contradicts Lemma 3.4.4. $\square$

Therefore, it remains to establish Lemma 3.4.4. In what follows we fix $R > 1$. The goal is to show via a variational argument that $\psi \equiv 0$ maximizes $\Lambda|_{\mathcal{A}_R}$. Since $\Lambda(0) = 0$, this would prove Lemma 3.4.4.

We claim that Λ defines a bounded functional on $\mathcal{A}_R$. To see this, observe that for every x, we have $|F(x)| \leq 2\,|x|$. Hence by the Strauss estimate, (3.2.3), and the fact that we are in $\mathcal{A}_R$, we have

$$N(\psi) \leq 2 \int_1^R |\psi(r)| \, dr \leq 8R\sqrt{E(\psi)}$$

Therefore,

$$\Lambda(\psi) \leq -\frac{9}{5}E(\psi) + 8R\sqrt{E(\psi)} \leq C(R) \tag{3.4.12}$$

Since Λ is bounded on $\mathcal{A}_R$ and $\Lambda(0) = 0$, we define $0 \leq \mu \leq C(R)$ by

$$\mu := \sup_{\psi \in \mathcal{A}_R} \Lambda(\psi)$$

Now, let $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{A}_R$ be a maximizing sequence, i.e., $\Lambda(\psi_n) \to \mu$ as $n \to \infty$. We claim that $E(\psi_n) \leq C$. If not, then there exists a subsequence, $\{\psi_{n_k}\}$ such that $E(\psi_{n_k}) \to \infty$. But then, by (3.4.12), we would have $\Lambda(\psi_{n_k}) \to -\infty$, which contradicts the fact that $\{\psi_n\}$ is maximizing and $\mu \geq 0$. Since $E(\psi_n) = \frac{1}{2}\|\psi_n\|_{\dot{H}^1}^2 \leq C$ we can extract a subsequence, still denoted by $\{\psi_n\}$, so that

$$\begin{aligned}
&\psi_n \rightharpoonup \psi_\infty \in \dot{H}_0^1 \\
&\psi_n \to \psi_\infty \in L^2_{\text{loc}} \\
&\psi_n \to \psi_\infty \text{ pointwise a.e. on } [1, R]
\end{aligned}$$

And, since $\mathcal{A}_R = \dot{H}^1_0(1,R)$, the boundary conditions are automatically satisfied and we have $\psi_\infty \in \mathcal{A}_R$. Next, we claim that ψ_∞ is in fact a maximizer, i.e., $\Lambda(\psi_\infty) = \mu$. On the one hand, since μ is the supremum, $\Lambda(\psi_\infty) \leq \mu$. To prove the other direction we remark that by the lower semi-continuity of weak limits we have that

$$\liminf_{n} E(\psi_n) \geq E(\psi_\infty)$$

Also, since $|F(\psi_n)| \leq 3\psi_n^2 \leq 6E(\psi_n) \leq C$, by the bounded convergence theorem, we see that

$$\lim_{n\to\infty} N(\psi_n) = N(\psi_\infty).$$

Putting this together we get

$$\begin{aligned}
\Lambda(\psi_\infty) - \mu &= \lim_{n\to\infty} (\Lambda(\psi_\infty) - \Lambda(\psi_n)) \\
&= \lim_{n\to\infty} \left(-\frac{9}{5}E(\psi_\infty) + \frac{9}{5}E(\psi_n) + N(\psi_\infty) - N(\psi_n) \right) \\
&\geq \frac{9}{5} \liminf_{n\to\infty} (-E(\psi_\infty) + E(\psi_n)) + \liminf_{n\to\infty} (N(\psi_\infty) - N(\psi_n)) \\
&\geq \liminf_{n\to\infty} (N(\psi_\infty) - N(\psi_n)) = 0
\end{aligned}$$

Hence $\Lambda(\psi_\infty) = \mu$ and so $\psi := \psi_\infty \in \mathcal{A}_R$ is our maximizer. Now, let $\eta \in C^\infty_0(1,R)$ and consider compact variations $\psi_\varepsilon := \psi + \varepsilon\eta$ of ψ. Since ψ is a maximizer for $\Lambda|_{\mathcal{A}_R}$, it follows that

$$0 = \frac{d}{d\varepsilon}\Lambda(\psi_\varepsilon)|_{\varepsilon=0} = -\frac{9}{5}\int_1^\infty \psi_r \eta_r \, r^2 \, dr + \int_1^\infty F'(\psi)\eta \, dr$$

$$= \int_1^\infty \left(\frac{9}{5} r^{-2} \partial_r (r^2 \psi_r) + \frac{F'(\psi)}{r^2} \right) \eta \, r^2 \, dr$$

This implies that ψ satisfies the following Euler-Lagrange equation

$$\psi_{rr} + \frac{2}{r}\psi_r = -\frac{5}{9}\frac{F'(\psi)}{r^2} \tag{3.4.13}$$
$$\psi(1) = 0,\ \psi(R) = 0$$

where the boundary conditions originate with the requirement that $\psi \in \mathcal{A}_R$. Setting $r = e^x$ and defining $\varphi(x) := \psi(e^x)$ we obtain the following autonomous differential equation for φ:

$$\varphi'' + \varphi' = f(\varphi) \tag{3.4.14}$$
$$\varphi(0) = 0,\ \varphi(\log(R)) = 0$$

where $f(\varphi) := -\frac{5}{9}F'(\varphi) = \frac{1}{4}\sin(2\varphi) + \frac{29}{18}\varphi\cos(2\varphi)$. We claim that $\varphi \equiv 0$ is the only solution to (3.4.14). Note that this implies Lemma 3.4.4 since then $\psi \equiv 0$ would be the unique maximizer for $\Lambda|_{\mathcal{A}_R}$ and $\Lambda(0) = 0$. We formulate the claim as a general lemma about the differential equation (3.4.14).

Lemma 3.4.5. *Let $f(x) := \frac{1}{4}\sin(2x) + \frac{29}{18}x\cos(2x)$. Suppose that $x(t)$ is a solution to*

$$\ddot{x} + \dot{x} = f(x) \tag{3.4.15}$$

and suppose that $x(0) = 0$ and that there exists a $T > 0$ such that $x(T) = 0$. Then $x \equiv 0$.

We note that the conclusion of Lemma 3.4.5 depends highly on the exact form the function f. In fact, the lemma fails if we replace f with $\frac{3}{2}f$. Such a change would amount to requiring a smaller fraction of $E(\psi)$ to dominate $N(\psi)$ in (3.4.10). This subtlety necessitates the careful analysis that is carried out in the proof.

The proof of Lemma 3.4.5 will consist of a detailed analysis of the phase portrait associated to (3.4.15). Letting $y(t) := \dot{x}(t)$, and setting

$$v(t) := (x(t), y(t))^{\mathrm{tr}}$$
$$N(x,y) := (y, -y + f(x))^{\mathrm{tr}}$$

we rewrite (3.4.15) as the following system

$$\dot{v} := \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + f(x) \end{pmatrix} =: N(v) \tag{3.4.16}$$

We can make a few immediate observations about the behavior of solutions to (3.4.16). First we note that since $|N(v)| \leq C\,|v|$, Gronwall's inequality implies that solutions are unique and exist globally in time. Let Φ_t denote the flow.

Next observe that equilibria of (3.4.16) are all *hyperbolic* (following the terminology of Wiggins [91]) and that they occur at the points $v_j := (x_j, 0)$, where x_j is a zero of f, i.e., $f(x_j) = 0$. To see this we linearize about the equilibrium v_j, which results in the the equation

$$\dot{\xi} = \nabla N(v_j)\xi \tag{3.4.17}$$

where

$$\nabla N(v_j) = \begin{pmatrix} 0 & 1 \\ f'(x_j) & -1 \end{pmatrix}$$

The eigenvalues of $\nabla N(v_j)$ are given by

$$\lambda_{\pm}(v_j) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4f'(x_j)} \tag{3.4.18}$$

To proceed, a more careful examination of the zeros of f is required. We can order the zeros x_j so that

$$\dots x_{-j} < \dots < x_{-1} < 0 =: x_0 < x_1 < \dots < x_j \dots$$

We note that since f is odd one has $x_{-j} = -x_j$ and it suffices to look at only those x_j such that $x_j \geq 0$. Indeed, all properties of the phase portrait on the right-half plane are identical to those on the left-half plane after a reflection about the origin.

First, observe that $x_0 := 0$ satisfies $f(x_0) = 0$ and $f'(x_0) = \frac{19}{9} > 2$. Hence, $\lambda_+(v_0) > -\frac{1}{2} + \frac{3}{2} = 1 > 0$ and $\lambda_-(v_0) < -\frac{1}{2}$. This means that (3.4.16) has a *saddle* at $v_0 = (0,0)$. Next, we see that due to the oscillatory nature of f and the fact that $f'(0) > 0$ we can deduce that $f'(x_j) > 0$ for j even, and $f'(x_j) < 0$ for j odd. It is also straightforward to show that $\left|f'(x_j)\right| > 1$ for every $j > 0$. These facts, together with (3.4.18) imply that

$$\begin{aligned}
&\mathrm{Re}\left(\lambda_\pm(v_j)\right) < 0 \text{ if } j \text{ is odd} \\
&\lambda_+(v_j) > 0, \text{ and } \lambda_-(v_j) < 0 \text{ if } j \text{ is even}
\end{aligned}$$

Hence (3.4.16) has *sinks* at each x_j for j even, and *saddles* at each x_j for j odd. Also we note that in a neighborhood $V_j \ni v_j$, the equilibira v_j, for j even, each have a 1-dimensional invariant stable manifold

$$W_j^s := \{v \in V_j \mid \Phi_t(v) \in V_j \;\forall\, t \geq 0,\; \Phi_t(v) \to v_j \text{ exponentially as } t \to +\infty\}$$

and a 1-dimensional invariant unstable manifold

$$W_j^u := \{v \in V_j \mid \Phi_t(v) \in V_j \;\forall\, t \leq 0,\; \Phi_t(v) \to v_j \text{ exponentially as } t \to -\infty\}$$

that are tangent to the respective invariant subspaces of the the linearized vector field corresponding to the right hand side of (3.4.17) at the point v_j. For j even, the stable invariant linear subspace at v_j is spanned by $\xi_-(v_j) = (1, \lambda_-(v_j))$ and the unstable invariant subspace is spanned by $\xi_+(v_j) = (1, \lambda_+(v_j))$. The equilibria v_j, for j odd, each have a two dimensional invariant stable manifold, (see, for example, [59], Chapt. 3.2).

Our goal is to demonstrate the impossibility of a trajectory $v(t)$ such that $v(0) = (0, y_0)$ and $v(T) = (0, y_T)$ with $y_0 \neq 0$ and $T \in \mathbb{R}$. By symmetry considerations we can restrict ourselves to the case $y_0 > 0$. We rule out such a trajectory by showing that solutions with data on the unstable invariant manifolds at the equilibria v_j, for j even, have the following properties:

Lemma 3.4.6. *Let $j = 2\ell$ be even. Denote by $v_j^+ = (x_j^+, y_j^+)$ the unique trajectory with data in W_j^u such that there exists a $\tau_1 > 0$ large enough so that $y_j^+(t) > 0$ for all $t < -\tau_1$. And denote by $v_j^- = (x_j^-, y_j^-)$ the unique trajectory in W_j^u such that there exists a $\tau_2 > 0$ large enough so that $y_j^-(t) < 0$ for all $t < -\tau_2$. Then, the following statements hold.*

(i) There exists $T_1 \in \mathbb{R}$ such that $v_j^+(T_1) = (p_j^+, 0)$ with $p_j^+ \in (x_{j+1}, x_{j+2})$.

(ii) There exists $T_2 \in \mathbb{R}$ such that $v_j^-(T_2) = (p_j^-, 0)$ with $p_j^- \in (x_{j-2}, x_{j-1})$.

We assume that T_1, T_2 are minimal with the stated properties.

The conclusion of Lemma 3.4.6 is depicted in Figure 3.1.

Proof that Lemma 3.4.6 implies Lemma 3.4.5. Suppose we start with data $v(0) = (0, y_0)$ with $y_0 > 0$. Then, since the right hand side of (3.4.16) is given by $(y, -y)^{\mathrm{tr}}$ on the line $\{x = 0\}$, the trajectory $v(t)$ enters the right-half plane in forward time. Note that $v(t)$ can never cross back into the left-half plane when $y(t) > 0$ since the line $\{x = 0, y > 0\}$ is repulsive with respect the forward trajectory of v. Hence, in order for there to be a time $T > 0$ such that $v(T) = (0, y(T))$ the trajectory must first cross into the lower-half plane. However, $v(t)$ must then either lie in the stable manifold W_j^s for some even j, or by

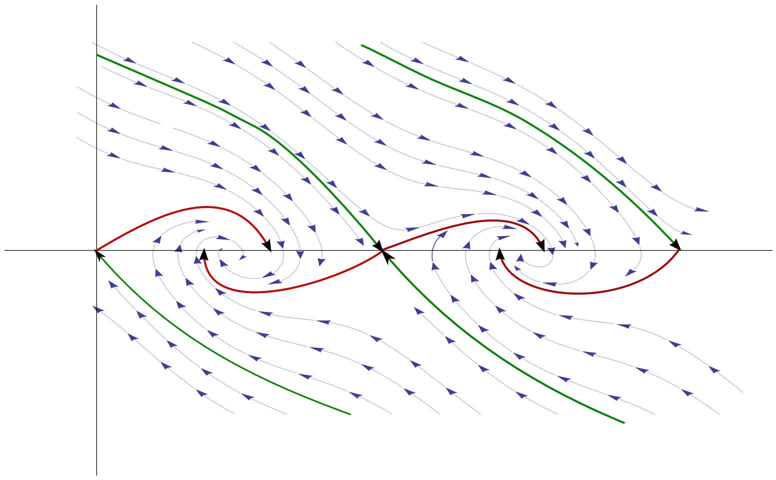

Figure 3.1: The figure above represents a slice of the phase portrait associated to (3.4.16). The red flow lines represent the unstable manifolds, W_j^u, associated to the v_j, and the green flow lines represent the stable manifolds, W_j^s, associated to the v_j.

Lemma 3.4.6 (i) it crosses the x-axis between x_k and x_{k+1} for some k odd. But then, if the latter occurs, by Lemma 3.4.6 (ii), the flow must cross back into the the upper-half plane again at some point strictly between x_{k-1} and x_k. If we track the trajectory further, (i) and (ii) will, in fact, force $v(t)$ into the sink at x_k, thus preventing it from ever reaching the y-axis. By the reflection symmetry of (3.4.16), the same logic works if we begin with data $v(0) = (0, y_0)$ with $y_0 < 0$. □

To simplify the picture we begin by dividing the phase plane into strips by defining $\Omega_{j/2+1} = [x_j, x_{j+2}] \times \mathbb{R}$ for $j \in 2\mathbb{Z}$. We first verify Lemma 3.4.6 in Ω_1 and in Ω_2 and then we will renormalize (3.4.16) in order to treat cases (i) and (ii) in Ω_ℓ for $\ell \geq 3$.

Proof of Lemma 3.4.6 on Ω_1 and Ω_2. The main tool in the proof of Lemma 3.4.6 in Ω_1 and Ω_2 will be the following identity which is obtained by multiplying equation (3.4.15) by $\dot{x}$ and integrating from $t = t_0$ to $t = t_1$.

$$\int_{t_0}^{t_1} \ddot{x}(s)\dot{x}(s)\, ds + \int_{t_0}^{t_1} \dot{x}(s)^2\, ds = \int_{t_0}^{t_1} f(x(s))\dot{x}(s)\, ds \tag{3.4.19}$$

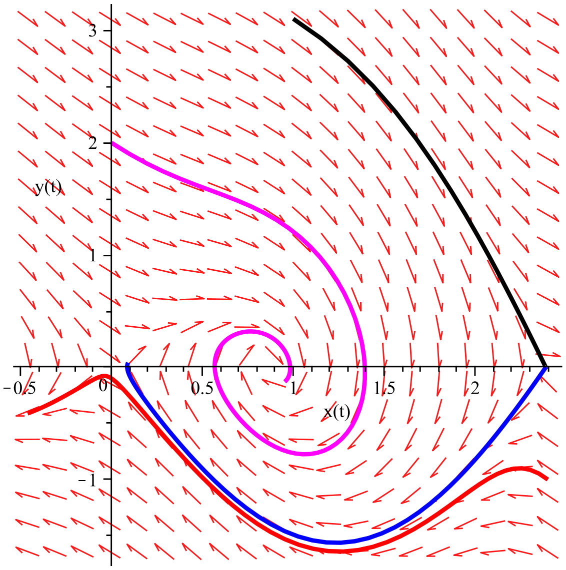

Figure 3.2: A schematic depiction of the flow in the first strip Ω_1.

Substituting $y = \dot{x}$ this becomes

$$\frac{1}{2}(y^2(t_1) - y^2(t_0)) + \int_{t_0}^{t_1} y^2(s)\,ds = F(x(t_1)) - F(x(t_0)) \tag{3.4.20}$$

where $F(x) := \frac{5}{18}\cos(2x) + \frac{29}{36}x\sin(2x)$ is a primitive for f.

We will also need to approximate the zeros $x_0, x_1, \ldots, x_4$. We can do this to any degree of precision, although a rather rough approximation will suffice. By inspection, the zero x_j is close to the point $\frac{2j-1}{4}\pi$ for $j \geq 1$. Indeed we have,

$$x_0 = 0,\ x_1 \approx 0.8733,\ x_2 \approx 2.3886,\ x_3 \approx 3.9466,\ x_4 \approx 5.51186 \tag{3.4.21}$$

First we show (i) on Ω_1. We would like to show that there exists $T \in (-\infty, \infty]$ and $p \in [x_1, x_2]$ so that $v_0^+(T) = (p, 0)$. In the process we will also show that $x_0^+(t) \leq x_{j+2}$ for all $t \in \mathbb{R}$.

Note that on the line $\{x = x_j\}$ in the phase plane the right-hand side of (3.4.16) is equal to $(y, -y)^{\mathrm{tr}}$. Hence, the trajectory $v_0^+(t)$ can never enter the left-half plane $\{x < 0\}$ by crossing the line $\{x = 0, y > 0\}$ as the vector field $(y, -y)^{\mathrm{tr}}$ is repulsive along this line in forward time. Also, since $|f(x)| \leq 3$ on $[0, x_2]$ the vector field $(y, -y+f(x))^{\mathrm{tr}}$ prevents $v_0^+(t)$ from ever crossing above the line segment $\{0 \leq x \leq x_2, y = 4\}$. Similarly, $v_0^+(t)$ can never cross from the upper into the lower-half plane through the line segment $\{0 < x < x_1, y = 0\}$, since $f(x) > 0$ on $(0, x_1)$ and thus the vector field $(0, f(x))^{\mathrm{tr}}$ repulses such a trajectory in forward time.

Therefore, the only remaining possibilities for the forward trajectory $v_0^+(t)$ are for Lemma 3.4.6 (i) to hold, or for one of the following two scenarios to occur: the trajectory crosses the line $\{x = x_2, y > 0\}$ in finite time, or it is heteroclinic connecting the saddles $(x_0, 0)$ and $(x_2, 0)$. Suppose that either of the latter two cases occurs. Then, there exists $T \in \mathbb{R} \cup \{\infty\}$ such that $v_0^+(T) = (x_2, y(T))$ with $y(T) \geq 0$. But then, letting $t_0 \to -\infty$ in (3.4.20) we would have

$$\frac{1}{2}y^2(T) + \int_{\infty}^{T} y^2(s)\, ds = F(x_2) - F(0) \approx -2.1799 < -2$$

which is a contradiction since the left hand side is strictly positive. This proves (i) for Ω_1. The proof of (i) for Ω_2 is identical. One first shows that the only possibilities for the trajectory $v_2^+(t)$ are for either (i) to hold, or for it to cross the line $\{x = x_4,\ y > 0\}$ in finite time, or to be to heteroclinic. And the latter two scenarios are impossible by (3.4.20) since then there would be a $T \in \mathbb{R} \cup \{\infty\}$ so that

$$\frac{1}{2}y^2(T) + \int_{\infty}^{T} y^2(s)\, ds = F(x_4) - F(x_2) \approx -2.52841 < -2$$

which contradicts the positivity of the left-hand-side above.

We will also use (3.4.20) to prove (ii), although we will not get by as easily as in the proof of (i), as we will need to estimate the size of the left hand side of (3.4.20) to obtain a contradiction. This will be achieved via the construction of a Lyapunov functional. Unfortunately, this is somewhat delicate as can been seen by means of the blue line in Figure 3.2 which is the unstable manifold W_2^u as computed by Maple. While it does visibly fall into the sink, it does so much less dramatically than W_0^u. For (*ii*), the relevant trajectory in Ω_1 is $v_2^-(t)$ which has data $v_2^-(-\infty) = (x_2, 0)$ and satisfies $y_2^-(t) < 0$ for $t \leq -\tau_2$. By symmetry, we can instead consider the trajectory $v_{-2}^+(t)$ in W_{-2}^u so that $y_{-2}^+(t) > 0$ for $t < -\tau$. This trajectory lies in Ω_{-1}.

Again one shows that either (*ii*) holds, or the forward trajectory $v_{-2}^+(t)$ reaches the line $\{x = 0, y \geq 0\}$ in finite or infinite positive time. In order to arrive at a contradiction, we assume that the latter occurs. That is, we assume that there exists $T \in \mathbb{R} \cup \{\infty\}$ such that $v_{-2}^+(T) = (0, y_{-2}^+(T))$ with $y_{-2}^+(T) \geq 0$. In this case we are able to use the attractive nature of the fixed point $(x_{-1}, 0)$ to construct a subset $\Sigma \subset \Omega_{-1}$ so that the flow $v_{-2}^+(t)$ cannot enter Σ. In other words, the boundary of Σ will be repulsive with respect to the forward trajectory of v_{-2}^+.

To construct Σ, we define three polynomials. First define p_1 as a function of x:

$$\begin{aligned}
p_1(x) := &-\frac{3}{1000} + \frac{110}{47}\left(x + \frac{43}{18}\right) - \frac{89}{222}\left(x + \frac{43}{18}\right)^2 - \frac{23}{42}\left(x + \frac{43}{18}\right)^3 \\
&+ \frac{7}{85}\left(x + \frac{43}{18}\right)^4 + \frac{8}{303}\left(x + \frac{43}{18}\right)^5 - \frac{1}{446}\left(x + \frac{43}{18}\right)^6 - \frac{1}{760}\left(x + \frac{43}{18}\right)^7 \\
&+ \frac{1}{4035}\left(x + \frac{43}{18}\right)^8 - \frac{1}{13999}\left(x + \frac{43}{18}\right)^9
\end{aligned}$$

Then, define p_2 and p_3 as functions of y as follows:

$$\begin{aligned} p_2(y) := &-\frac{6627}{638000} - \frac{17913}{29000}y - \frac{19}{75}\left(y-\frac{21}{22}\right)^2 - \frac{17}{80}\left(y-\frac{21}{22}\right)^3 \\ &- \frac{29}{106}\left(y-\frac{21}{22}\right)^4 - \frac{36}{115}\left(y-\frac{21}{22}\right)^5 - \frac{9}{20}\left(y-\frac{21}{22}\right)^6 - \frac{19}{31}\left(y-\frac{21}{22}\right)^7 \\ &- \frac{32}{35}\left(y-\frac{21}{22}\right)^8 - \frac{42}{31}\left(y-\frac{21}{22}\right)^9 \end{aligned}$$

and

$$\begin{aligned} p_3(y) := &-\frac{104159}{877500} - \frac{9383}{19500}y - \frac{18}{113}\left(y-\frac{3}{5}\right)^2 + \frac{2}{365}\left(y-\frac{3}{5}\right)^3 \\ &- \frac{38}{291}\left(y-\frac{3}{5}\right)^4 + \frac{3}{50}\left(y-\frac{3}{5}\right)^5 - \frac{21}{158}\left(y-\frac{3}{5}\right)^6 + \frac{6}{71}\left(y-\frac{3}{5}\right)^7 \\ &- \frac{2}{15}\left(y-\frac{3}{5}\right)^8 + \frac{7}{82}\left(y-\frac{3}{5}\right)^9 - \frac{31}{278}\left(y-\frac{3}{5}\right)^{10} + \frac{6}{121}\left(y-\frac{3}{5}\right)^{11} \end{aligned}$$

Finally, we set $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where,

$$\begin{aligned} \Sigma_1 &:= \left\{(x,y) \in \Omega_{-1} \mid -\frac{43}{18} + \frac{3}{1000} < x < -\frac{3}{5},\, 0 < y < p_1\left(-\frac{3}{5}\right)\right\} \\ \Sigma_2 &:= \left\{(x,y) \in \Omega_{-1} \mid -\frac{3}{5} < x < p_2(y),\, \frac{3}{5} < y < \frac{21}{22}\right\} \\ \Sigma_3 &:= \left\{(x,y) \in \Omega_{-1} \mid -\frac{3}{5} < x < p_3(y),\, 0 < y < \frac{3}{5}\right\} \end{aligned}$$

The region Σ is pictured in Figure 3.3. A few words are required in order to explain how one goes about constructing the region Σ, and in particular, about how one finds the functions p_k. To choose p_1, one begins by finding an approximate solution to (3.4.15) with data slightly to the right of x_{-2} via power series expansions. This approximate solution is then shifted downward by a small amount, here we take $\frac{3}{1000}$. As we will see below, this downward shift ensures that the resulting function forms a curve that is, at least initially, a Lyapunov functional in that it is repulsive with respect to the true trajectory emanating from x_{-2},

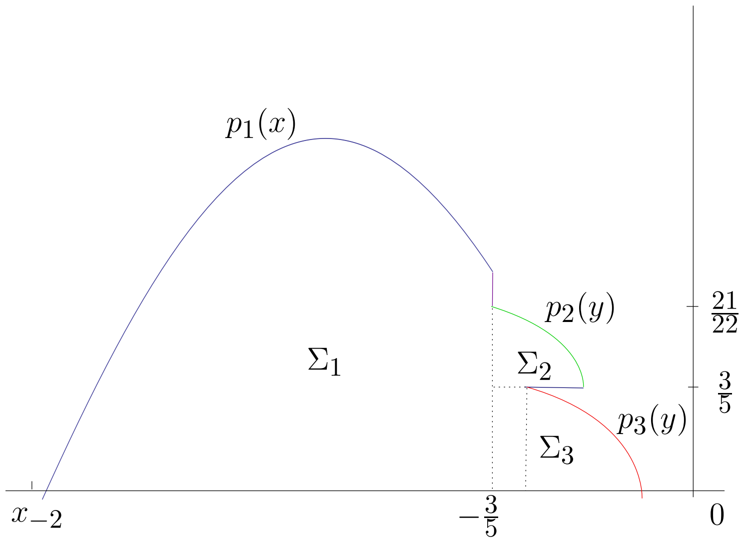

Figure 3.3: The region $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ pictured above has the property that $\partial\Sigma$ is repulsive with respect to the unstable manifold W^u_{-2}.

i.e., the unstable manifold W^u_{-2}. We then define p_1 by approximating the coefficients of the polynomial we found by rationals. We cease to use the graph of p_1 as the boundary of Σ when it ceases to possess the desired Lyapunov properties. We then define p_2 and p_3 in similar fashions making sure that all of the respective graphs are eventually joined together by curves that are also Lyapunov. In the case of the segment joining the graph of p_1 and p_2 this is achieved with a vertical line as depicted in Figure 3.3. For p_2 and p_3 the matching is done with a horizontal line.

We claim that the boundary of Σ is repulsive with respect to the trajectory $v^+_{-2}(t)$. To see this, it suffices to show that the outward normal ν on $\partial\Sigma \cap \{y > 0\}$ satisfies

$$\nu \cdot N \geq 0 \tag{3.4.22}$$

where $N := (y, -y + f(x))^{\text{tr}}$ is the vector field (3.4.16). There are five components to $\partial\Sigma \cap \{y > 0\}$. Three components are given by the graphs of $p_1, \ldots, p_3$, and we label these components $\partial\Sigma_1, \ldots, \partial\Sigma_3$. The other two components are given by the vertical segment, $\partial\Sigma_4$, connecting the point $(-\frac{3}{5}, \frac{21}{22})$ to $(-\frac{3}{5}, p_1(-\frac{3}{5}))$, and the horizontal segment, $\partial\Sigma_5$, connecting the point $(p_3(\frac{3}{5}), \frac{3}{5})$ to $(p_2(\frac{3}{5}), \frac{3}{5})$. We must check that (3.4.22) holds on each component.

On $\partial\Sigma_1$ the outward normal ν_1 is given by $\nu_1 = (-p_1'(x), 1)$. On $\partial\Sigma_2$, $\nu_2 = (1, -p_2'(y))$. Similarly, $\nu_3 = (1, -p_3(y))$. Finally, $\nu_4 = (1, 0)$ and $\nu_5 = (0, -1)$. And, it is elementary to check that indeed,

$$\begin{aligned}
&\nu_1 \cdot N = f(x) - p_1(x)(1 + p_1'(x)) > 0 \text{ for every } -\frac{43}{18} \leq x \leq -\frac{3}{5} \\
&\nu_2 \cdot N = y + p_2'(y)(y - f(p_2(y)) > 0 \text{ for every } \frac{3}{5} < y \leq \frac{21}{22} \\
&\nu_3 \cdot N = y + p_3'(y)(y - f(p_3(y)) > 0 \text{ for every } 0 \leq y \leq \frac{3}{5}
\end{aligned}$$

as well as

$$\begin{aligned}
&\nu_4 \cdot N = y > 0 \text{ for every } \frac{3}{5} \leq y \leq \frac{21}{22} \\
&\nu_5 \cdot N = \frac{3}{5} - f(x) > 0 \text{ for every } p_3\,(3/5) \leq x \leq p_2\,(3/5)
\end{aligned}$$

Now, by (3.4.20), we have that

$$\frac{1}{2}y^2(T) + \int_{-\infty}^{T} y^2(s)\,ds = F(0) - F(x_{-2}) \approx 2.1799 < 2.18 \tag{3.4.23}$$

However, we claim that

$$\int_{-\infty}^{T} y^2(s)\,ds > \text{Area}(\Sigma) > 2.18 \tag{3.4.24}$$

To prove (3.4.24), we first make the claim that under our current assumptions, the integral on the left-hand side of (3.4.24) is greater than the area of the region bounded by the trajectory $v_{-2}^{+}(t)$ and the lines $\{x \leq 0\}$ and $\{y = 0\}$. To see this recall that $v_{-2}^{+}(t)$ lies on the unstable manifold W^u_{-2} and hence locally we can either write $y^+_{-2}(t) = y(x(t))$ or $x^+_{-2}(t) = x(y(t))$. Assume that for $\tau_0 < t < \tau_1$ we can write $y = y(x)$. Then, $x(\tau_0) < x(\tau_1)$ and

$$\int_{\tau_0}^{\tau_1} y^2(s)\,ds = \int_{\tau_0}^{\tau_1} y(x(s))\dot{x}(s)\,ds = \int_{x(\tau_0)}^{x(\tau_1)} y(x)\,dx$$

which, since $y(t) \geq 0$, is, in fact, the area of the region bounded by the trajectory $v^+_{-2}(t)$, the line $\{y = 0\}$, and the lines $\{x = x(\tau_0)\}$ and $\{x = x(\tau_1)\}$.

Next suppose we can write $x = x(y)$ for $\tau_2 < t < \tau_3$ and that $y(\tau_2) > y(\tau_3)$. Since all vertical lines in Ω_{-1} have the property that they cannot be crossed by the flow from right to left in forward time we have that $x(y(\tau_2)) \leq x(y(\tau_3))$. Observe that if $x = x(y(t))$ then $\dot{x} = x'(y)\dot{y}$, and hence

$$\begin{aligned}
\int_{\tau_2}^{\tau_3} \dot{x}(s)^2\,ds &= \int_{\tau_2}^{\tau_3} y(s)x'(y(s))\dot{y}(s)\,ds = \int_{y(\tau_2)}^{y(\tau_3)} y\,x'(y)\,dy \\
&= \int_{y(\tau_3)}^{y(\tau_2)} x(y)\,dy + y(\tau_3)x(y(\tau_3)) - y(\tau_2)x(y(\tau_2))
\end{aligned}$$

but this can further be estimated from below by

$$\begin{aligned}
&\geq \int_{y(\tau_3)}^{y(\tau_2)} x(y)\,dy + (y(\tau_3) - y(\tau_2))x(y(\tau_2)) \\
&= \int_{y(\tau_3)}^{y(\tau_2)} \left[x(y) + x(y(\tau_2))\right]\,dy
\end{aligned}$$

where the last line is exactly the area of the region bounded by $v^+_{-2}(t)$, and the lines $\{x = x(\tau_2)\}$, and $\{y = y(\tau_3)\}$.

Therefore, since $v^+_{-2}(t)$ cannot enter Σ we have $\int_{-\infty}^{T} y^2(s)\,ds > \text{Area}(\Sigma)$. The remaining step is to compute the area of Σ which can be done explicitly since Σ is defined entirely in terms of polynomials with rational coefficients. Indeed,

$$\text{Area}(\Sigma) = \text{Area}(\Sigma_1) + \text{Area}(\Sigma_2) + \text{Area}(\Sigma_3) > 2.21$$

which proves (3.4.24) and provides a contradiction when combined with (3.4.23). This proves (ii) in Ω_1. Note that small margin of error which is allowed here (after all the relevant numbers are, respectively, 2.21 and 2.18) is a reflection of the "almost heteroclinic" nature of the blue line in Figure 3.2 which is W_2^u. This forces us to be very precise about the Lyapunov functionals that we constructed above.

Next, we will establish (ii) in Ω_2. The relevant trajectory is $v_4^-(t)$ which has data $v_4^-(-\infty) = (x_4, 0)$. As before, we can show that the only possibilities for $v_4^-(t)$ are either that (ii) holds, or that there exists a time $T \in \mathbb{R} \cup \{\infty\}$ such that $v_4^-(T) = (x_2, y_4^-(T))$ where $y_4^-(t) \leq 0$ for all $-\infty < t \leq T$. We assume the latter holds and seek a contradiction. As in the proof of (ii) in Ω_1 we will construct a subset $\Sigma \subset \Omega_2$ so that the boundary, $\partial\Sigma$, is repulsive with respect to the forward flow $v_4^-(t)$. To construct Σ we define the polynomial

$$p(x) := \frac{3}{100} + \frac{15}{4}\left(x - \frac{11}{2}\right) + \frac{18}{89}\left(x - \frac{11}{2}\right)^2 - \frac{136}{181}\left(x - \frac{11}{2}\right)^3$$

and define

$$\Sigma := \{(x, y) \in \Omega_2 \,|\, 18/5 < x < 11/2,\, p(x) < y < 0\}$$

The function p is constructed in the same fashion as the Lyapunov functional for Ω_{-1} except that here we need only a 3rd order approximation. Indeed, the trajectory v_4^- is far from heteroclinic and thus provides us with a much larger margin for error as we seek a contradiction.

Again it suffices to show that the outward normal ν on $\partial\Sigma \cap \{y < 0\}$ satisfies $\nu \cdot N \geq 0$.

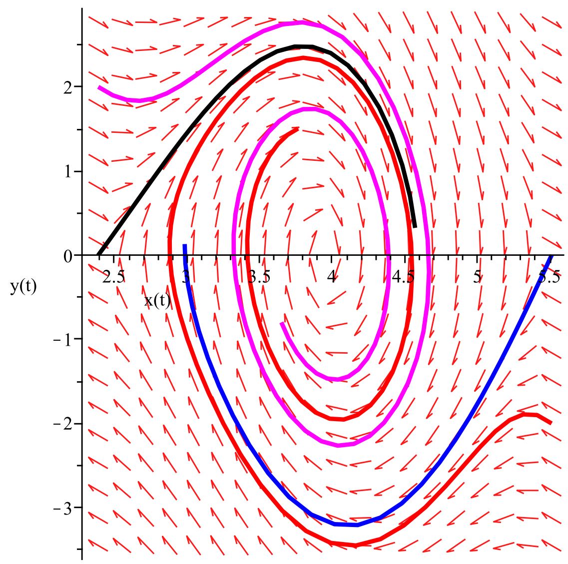

Figure 3.4: A schematic depiction of the flow in the second strip Ω_2.

We have $\nu = (p'(x), -1)^{\mathrm{tr}}$. And one can show that

$$\nu \cdot N = p(x)(1 + p'(x)) - f(x) > 0 \text{ for every } 18/5 < x < 11/2$$

Again, we use (3.4.20) to obtain,

$$\frac{1}{2} y^2(T) + \int_{-\infty}^{T} y^2(s)\, ds = F(x_2) - F(x_4) \approx 2.52841 < 2.6 \tag{3.4.25}$$

However, we have

$$\int_{-\infty}^{T} y^2(s)\, ds > \mathrm{Area}(\Sigma) > 3.8 \tag{3.4.26}$$

which contradicts (3.4.25). This completes the proof of Lemma 3.4.6 in Ω_1 and in Ω_2. We remark that the Lyapunov construction for Ω_2 is considerably easier than for Ω_1 as can be seen by Figure 3.4. Indeed, the unstable manifold W_4^u, which is depicted by the blue trajectory in Figure 3.4, is very far from being heteroclinic. $\square$

To prove Lemma 3.4.6 on Ω_ℓ for $\ell \geq 3$ we first shift and rescale (3.4.16) via the following renormalization. For each $j \in \mathbb{N}$, $\varepsilon \in \mathbb{R}$ we define ζ and η via

$$\begin{aligned} x(t) &=: \frac{2j-1}{4}\pi + \zeta(\varepsilon^{-1}t) \\ y(t) &=: \varepsilon^{-1}\,\eta(\varepsilon^{-1}t) \end{aligned} \tag{3.4.27}$$

Define $z_j := \frac{2j-1}{4}\pi$. Then (3.4.16) implies the following system of equations for ζ, η

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ -\varepsilon\eta + \varepsilon^2 f(z_j + \zeta) \end{pmatrix} \tag{3.4.28}$$

where $\dot{} = \frac{d}{ds}$ where $s = \varepsilon^{-1}t$. Observe that we have

$$f(z_j + \zeta) = (-1)^j \frac{29}{18} z_j \sin(2\zeta) + (-1)^{j+1} g(\zeta)$$

where $g(\zeta) := \frac{1}{4}\cos(2\zeta) - \frac{29}{18}\zeta\sin(2\zeta)$. Fix $j = 2\ell$ with $\ell \geq 2$ and set

$$\varepsilon := \sqrt{\frac{72}{29\pi(2j-1)}} \tag{3.4.29}$$

Note that $j \geq 4$ implies that $0 < \varepsilon < \frac{7}{20}$. Then (3.4.28) becomes

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \eta \\ \sin(2\zeta) - \varepsilon\eta - \varepsilon^2 g(\zeta) \end{pmatrix} \tag{3.4.30}$$

Note that (3.4.30) is the equation governing the motion of a *damped pendulum* with a small perturbative term $\varepsilon^2 g(\zeta)$, and in the limit as $\varepsilon \to 0$, (3.4.30) is exactly the the equation of a simple pendulum.

Let's rephrase the set-up of Lemma 3.4.6 in terms of this renormalization. First we examine how this affects the strip $\Omega_{j/2+1}$. We can write the zeros of f as

$$\begin{aligned} x_j &= z_j + \zeta_0 \\ x_{j+1} &= z_{j+1} + \zeta_1 = z_j + \frac{\pi}{2} + \zeta_1 \\ x_{j+2} &= z_{j+2} + \zeta_2 = z_j + \pi + \zeta_2 \end{aligned}$$

where $0 < \zeta_0 < \frac{\pi}{2} + \zeta_1 < \pi + \zeta_2$ are the first three positive zeros of

$$h(\zeta) := \sin(2\zeta) - \varepsilon^2 g(\zeta)$$

Hence the strip $\Omega_{j/2+1}$ becomes the strip $\tilde{\Omega} = [\zeta_0, \pi + \zeta_2] \times \mathbb{R}$. Note that the renormalization (3.4.27) does not affect the topological properties of the dynamics of (3.4.16) and hence the invariant manifolds associated to the equilibria of (3.4.16) in $\Omega_{j/2+1}$ become invariant manifolds associated to the equilibria of (3.4.30) in the strip $\tilde{\Omega}$. Denote by $W^u_{\zeta_0}$ and $W^u_{\zeta_2}$, the unstable invariant manifolds associated to the equilibria $(\zeta_0, 0)$ and $(\pi + \zeta_2, 0)$. Thus Lemma 3.4.6 in Ω_ℓ for $\ell \geq 3$ is equivalent to the following result. For simplicity, we again use t to denote time.

Lemma 3.4.7. *Denote by $v^+ = (\zeta^+, \eta^+)$ the unique solution of* (3.4.30) *with data in $W^u_{\zeta_0}$ such that there exists a $\tau_1 > 0$ large enough so that $\eta^+(t) > 0$ for all $t < -\tau_1$. And denote by $v^- = (\zeta^-, \eta^-)$ the unique solution in $W^u_{\zeta_2}$ such that there exists a $\tau_2 > 0$ large enough so that $\eta^-(t) < 0$ for all $t < -\tau_2$. Then, the following statements hold:*

(i) There exists $T_1 \in \mathbb{R}$ such that $v^+(T_1) = (p_1, 0)$ with $p_1 \in (\pi/2 + \zeta_1, \pi)$.

(ii) There exists $T_2 \in \mathbb{R}$ such that $v^-(T_2) = (p_2, 0)$ with $p_2 \in (\zeta_0, \pi/2 + \zeta_1)$.

Again, we let T_1, T_2 be minimal with these properties.

The proof of Lemma 3.4.7 will require a rather precise knowledge of the location of the zeros ζ_0 and $\pi + \zeta_2$ of $h(\zeta)$.

Lemma 3.4.8. *Set* $h(\zeta) = \sin(2\zeta) - \varepsilon^2 g(\zeta)$*. Then*

(a) There exists a function $a : [0, \frac{7}{20}] \to [-\frac{1}{3}, -\frac{1}{9}]$ *such that* h *has a zero at* $\zeta_0 = \zeta_0(\varepsilon) = \frac{1}{2}\varepsilon^2 g(0)(1 + a(\varepsilon)\varepsilon^4)$*.*

(b) There exists a function $c : [0, \frac{7}{20}] \to [10, 40]$ *such that* h *has a zero at* $\pi + \zeta_2 = \pi + \zeta_2(\varepsilon) = \pi + \frac{1}{2}\varepsilon^2 g(\pi)(1 - \frac{29}{18}\pi\varepsilon^2 + c(\varepsilon)\varepsilon^4)$*.*

In particular, $\zeta_0 > 0$ *and* $\zeta_2 > 0$*.*

We will momentarily postpone the proof of Lemma 3.4.8 and first establish Lemma 3.4.7.

Proof of Lemma 3.4.7. Again our main tool will be the following identity, which is deduced in the same manner as (3.4.20),

$$\begin{aligned}\frac{1}{2}(\eta^2(t_1) - \eta^2(t_0)) + \varepsilon \int_{t_0}^{t_1} \eta^2(s)\, ds &= \int_{t_0}^{t_1} \sin(2\zeta)\dot{\zeta}\, ds - \varepsilon^2 \int_{t_0}^{t_1} g(\zeta(s))\dot{\zeta}(s)\, ds \\ &= \frac{1}{2}\left(\cos(2\zeta(t_0)) - \cos(2\zeta(t_1))\right) \\ &\quad - \varepsilon^2(G(\zeta(t_1)) - G(\zeta(t_0)))\end{aligned} \tag{3.4.31}$$

where $G(x) := \frac{29}{36}x\cos(2x) - \frac{5}{18}\sin(2x)$ is a primitive of g.

First we prove (i). The only possibilities for the forward trajectory $v^+(t)$ are for (i) to hold, or for there to exist a time T, possibly infinite, such that $v^+(T) = (\pi, \eta^+(T))$ with $0 \leq \eta^+(t)$ for all $t \leq T$. In this latter case, (3.4.31) implies that

$$\begin{aligned}\frac{1}{2}\eta^2(T) + \varepsilon \int_{-\infty}^{T} \eta^2(s)\, ds &= \frac{1}{2}\left(\cos(2\zeta(t_0)) - 1)\right) - \varepsilon^2(G(\pi) - G(\zeta_0)) \\ &\leq -\varepsilon^2(G(\pi) - G(\zeta_0)) \leq 0\end{aligned}$$

which is a contradiction since the left-hand-side above is strictly positive.

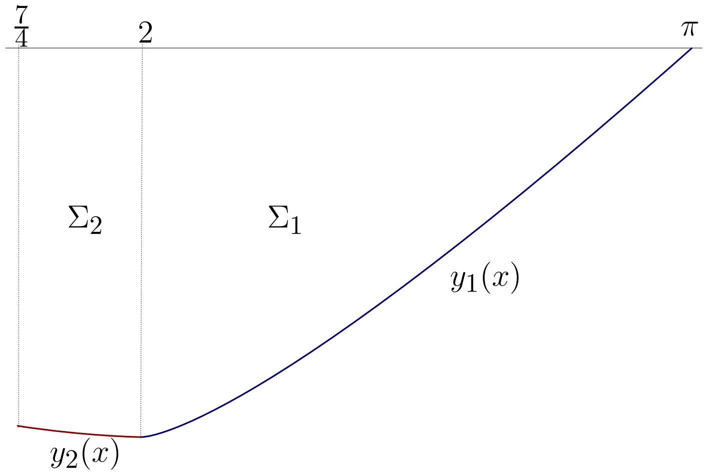

Figure 3.5: The region $\Sigma = \Sigma_1 \cup \Sigma_2$ pictured above has the property that $\partial\Sigma$ is repulsive with respect to the unstable manifold $W^u_{\zeta_2}$.

Now, assume (ii) fails. Then there exists a time $T \in \mathbb{R} \cup \{\infty\}$ such that $v_-(T) = (\zeta_0, \eta^-(T))$ with $\eta^-(t) \leq 0$ for every $t \leq T$. As in the proof of Lemma 3.4.6 (ii) for Ω_1 and Ω_2 we construct a region Σ in $\tilde{\Omega}$ so that the boundary $\partial\Sigma$ is repulsive with respect to the flow $v^-(t)$. Set

$$y_1(\zeta) := -\frac{5}{4}\sin(\zeta) \tag{3.4.32}$$

$$y_2(\zeta) = -\frac{5}{4}\sin(2)\sqrt{1 - \frac{25}{36}(\zeta - 2)^2} \tag{3.4.33}$$

Define $\Sigma = \Sigma_1 \cup \Sigma_2$ by

$$\Sigma_1 := \{(x, y) \in \tilde{\Omega} \,|\, 2 \leq x \leq \pi,\, y_1(x) \leq y \leq 0\} \tag{3.4.34}$$

$$\Sigma_2 := \{(x, y) \in \tilde{\Omega} \,|\, \frac{7}{4} \leq x < 2,\, y_2(x) \leq y \leq 0\} \tag{3.4.35}$$

The region Σ is depicted in Figure 3.5.

Once again we need to check that the outward normal vectors ν_1 on $\partial\Sigma_1$ and ν_2 on $\partial\Sigma_2$ satisfy $\nu_k \cdot \tilde{N} \geq 0$ for $k = 1, 2$, where

$$\tilde{N}(\zeta, \eta) = (\eta\,,\, \sin(2\zeta) - \varepsilon\eta - \varepsilon^2 g(\zeta))^{\mathrm{tr}}$$

Here $\nu_1 = (y_1'(\zeta), -1)^{\mathrm{tr}}$ and $\nu_2 = (y_2'(x), -1)^{\mathrm{tr}}$ and we have

$$\nu_1 \cdot \tilde{N} = -\frac{y_1(x)}{\beta^2} F_1(x, \varepsilon) \tag{3.4.36}$$

$$\nu_2 \cdot \tilde{N} = -\frac{y_2(x)}{\beta^2 \sin^2(2)} F_2(x, \varepsilon) \tag{3.4.37}$$

where, for $\alpha := \frac{6}{5}$ and $\beta := \frac{5}{4}$, F_1 and F_2 are defined by

$$F_1(x, \varepsilon) := 2g(x)\varepsilon^2 - 2\beta \sin(x)\varepsilon + (\beta^2 - 2)\sin(2x) \tag{3.4.38}$$

$$\begin{aligned} F_2(x, \varepsilon) := {} & g(x)\varepsilon^2 - \varepsilon \frac{\beta \sin(2)}{\alpha} \sqrt{\alpha^2 - (x-2)^2} \\ & - \frac{\beta^2 \sin^2(2)(x-2) + \alpha \sin(2x)}{\alpha^2} \end{aligned} \tag{3.4.39}$$

Observe that $y_1(x) \leq 0$ for $2 \leq x \leq \pi$, and $y_2(x) \leq 0$ for $\frac{7}{4} \leq x \leq 2$. Hence, the following lemma will suffice to conclude that $\nu_k \cdot \tilde{N} \geq 0$ for $k = 1, 2$.

Lemma 3.4.9. *Define F_1, F_2 as in* (3.4.38) *and* (3.4.39)*. Then*

(*A*) $F_1(x, \varepsilon) \geq 0$ *for every* $(x, \varepsilon) \in [2, \pi] \times [0, \frac{7}{20}]$.

(*B*) $F_2(x, \varepsilon) \geq 0$ *for every* $(x, \varepsilon) \in [\frac{7}{4}, 2] \times [0, \frac{7}{20}]$.

For the moment we assume Lemma 3.4.9 and observe that it implies that the boundary of Σ is repulsive with respect to the flow $v^-(t)$. By (3.4.31) we have the following identity

$$\begin{aligned} \frac{1}{2}\eta^2(T) + \varepsilon \int_{-\infty}^{T} \eta^2(s)\, ds = {} & \frac{1}{2}\left(\cos(2\pi + 2\zeta_2) - \cos(2\zeta_0)\right) \\ & - \varepsilon^2 (G(\pi + \zeta_2) - G(\zeta_0)) \end{aligned} \tag{3.4.40}$$

To arrive at a contradiction we carefully estimate the left and right-hand sides of (3.4.40).

By Lemma 3.4.8, we can expand the right hand side in powers of ε.

$$\begin{aligned}\frac{1}{2}\left(\cos(2\zeta_2) - \cos(2\zeta_0)\right) - \varepsilon^2(G(\pi+\zeta_2) - G(\zeta_0)) &= \frac{29\pi}{36}\varepsilon^2 - \frac{29\pi}{1152}\varepsilon^6 + O(\varepsilon^8) \\ &< \frac{29\pi}{36}\varepsilon^2\end{aligned} \tag{3.4.41}$$

for $0 \le \varepsilon \le \frac{7}{20}$.

On the other hand, as in the proof of Lemma 3.4.6 for Ω_1 and Ω_2, we have that

$$\varepsilon \int_{-\infty}^{T} \eta^2(s)\, ds > \varepsilon \text{Area}(\Sigma) = \varepsilon \left(- \int_{\frac{7}{4}}^{2} y_2(x)\, dx - \int_{2}^{\pi} y_1(x)\, dx \right) > \varepsilon \tag{3.4.42}$$

Finally, (3.4.40) then implies that $\varepsilon < \frac{29\pi}{36}\varepsilon^2$ which is a contradiction for $0 \le \varepsilon \le \frac{7}{20}$. Hence, assuming the results of Lemma 3.4.8 and Lemma 3.4.9, we have established Lemma 3.4.7 and therefore we have also completed the proof of Lemma 3.4.6. □

It remains to prove Lemma 3.4.8 and Lemma 3.4.9.

Proof of Lemma 3.4.8. For fixed a, we plug $\zeta_0(a,\varepsilon) = \frac{1}{2}\varepsilon^2 g(0)(1 + a\varepsilon^4)$ into h and expand in powers of ε about $\varepsilon = 0$. This gives

$$h(\zeta_0(a,\varepsilon)) = \left(\frac{1}{18} + \frac{a}{4}\right)\varepsilon^6 + O(\varepsilon^{10})$$

With this in mind we set $a_1 = -\frac{1}{3}$, and obtain

$$h(\zeta_0(-\frac{1}{3},\varepsilon)) = -\frac{1}{36}\varepsilon^6 + R_9(\varepsilon) \tag{3.4.43}$$

where $R_9(\varepsilon)$ is the ninth remainder term in Taylor's theorem. One can show that for $0 \le$

$\varepsilon \leq \frac{7}{20}$, we have

$$|R_9(\varepsilon)| \leq \sup_{0\leq|\xi|\leq\varepsilon} \left| \left(\frac{d}{d\xi}\right)^{10} h(\zeta_0(-\frac{1}{3},\xi)) \right| (10!)^{-1}\varepsilon^{10} \leq \varepsilon^{10}$$

Hence,

$$h(\zeta_0(-\frac{1}{3},\varepsilon)) \leq -\frac{1}{36}\varepsilon^6 + \varepsilon^{10} \leq -\frac{1}{36}\varepsilon^6 + \left(\frac{7}{20}\right)^4 \varepsilon^6 \leq 0$$

as long as $0 \leq \varepsilon \leq \frac{7}{20}$. Next we set $a = -\frac{1}{9}$ and we obtain

$$h(\zeta_0(-\frac{1}{9},\varepsilon)) = \frac{1}{36}\varepsilon^6 + R_9(\varepsilon)$$

Again, one can show that $|R_9(\varepsilon)| \leq \varepsilon^{10}$ for $0 \leq \varepsilon \leq \frac{7}{20}$ and hence

$$h(\zeta_0(-\frac{1}{9},\varepsilon)) \geq \frac{1}{36}\varepsilon^6 - \varepsilon^{10} \geq \frac{1}{36}\varepsilon^6 - \left(\frac{7}{20}\right)^4 \varepsilon^6 \geq 0$$

for $0 \leq \varepsilon \leq \frac{7}{20}$. This proves (a). We carry out the same procedure to prove (b). First, fix c and plug $\pi + \zeta_2(c,\varepsilon) = \pi + \frac{1}{2}\varepsilon^2 g(\pi)(1 - \frac{29}{18}\pi\varepsilon^2 + c\varepsilon^4)$ into h and expand in powers of ε about $\varepsilon = 0$. This gives,

$$h(\pi + \zeta_2(c,\varepsilon)) = \frac{(72 + 324c - 841\pi^2)}{1296}\varepsilon^6 + O(\varepsilon^8)$$

Now, fix $c = 10$. Then

$$h(\pi + \zeta_2(10,\varepsilon)) = \left(\frac{23}{9} - \frac{841\pi^2}{1296}\right)\varepsilon^6 + R_7(\varepsilon)$$

One can show that $|R_7(\varepsilon)| \le 20\varepsilon^8$ for $0 \le \varepsilon \le \frac{7}{20}$, and hence

$$h(\pi + \zeta_2(10, \varepsilon)) \le -3.8\varepsilon^6 + 20\varepsilon^8 \le -3.8\varepsilon^6 + 2.5\varepsilon^6 \le 0$$

as long as $0 \le \varepsilon \le \frac{7}{20}$. Finally, set $c = 40$. Then

$$h(\pi + \zeta_2(40, \varepsilon)) = \left(\frac{181}{18} - \frac{841\pi^2}{1296}\right)\varepsilon^6 + R_7(\varepsilon)$$

One can show that $|R_7(\varepsilon)| \le 60\varepsilon^8$ for $0 \le \varepsilon \le \frac{1}{8}$, and hence

$$h(\pi + \zeta_2(40, \varepsilon)) \ge 3.6\varepsilon^6 - 60\varepsilon^8 \ge 3.6\varepsilon^6 - \varepsilon^6 \ge 0$$

as long as $0 \le \varepsilon \le \frac{1}{8}$. To conclude, we note that the positivity of $h(\pi + \zeta_2(40, \varepsilon))$ on the compact interval $\varepsilon \in [\frac{1}{8}, \frac{7}{20}]$ is readily checked. $\square$

Proof of Lemma 3.4.9. Observe that for fixed, x, $F_1(x, \varepsilon)$ and $F_2(x, \varepsilon)$ are quadratic functions in ε and hence have real zeros for $\varepsilon \in [0, \frac{7}{20}]$ if and only if their associated discriminants are nonnegative. One can readily check that the discriminant associated to $F_1(x, \cdot)$ is negative for each $2 \le x \le \pi$. And the discriminant associated to $F_2(x, \cdot)$ is negative for each $\frac{7}{4} \le x \le 2$. Therefore, by continuity, F_1 has a fixed sign on $[2, \pi] \times [0, \frac{7}{20}]$ and F_2 has a fixed sign on $[\frac{7}{4}, 2] \times [0, \frac{7}{20}]$. Hence checking the positivity of F_1 and F_2 on their respective domains reduces to checking that they are positive at a single point. And, for example $F_1(\frac{5}{2}, \frac{1}{4}) \approx 0.54 > 0$ and $F_2(\frac{15}{8}, \frac{1}{4}) \approx .41 > 0$. $\square$

This concludes the proofs of Lemmas 3.4.3–3.4.7.

3.5 The higher topological classes

In this section we prove Theorem 3.1.2. By [5] we know that for each integer $n \geq 1$ there is a unique solution $Q = Q_n$ to the stationary problem

$$-Q'' - \frac{2}{r}Q' + \frac{\sin(2Q)}{r^2} = 0, \quad Q(1) = 0,\ Q'(1) > 0 \tag{3.5.1}$$

with the property that $\lim_{r\to\infty} Q_n(r) = n\pi$. Moreover, these Q_n are strictly increasing and satisfy

$$Q_n(r) = n\pi - O(r^{-2}) \ \text{ as } \ r \to \infty \tag{3.5.2}$$

Now fix any such Q_n for $n > 0$ and drop the subscript. Set $\psi(r) := \partial_\lambda Q(\lambda r)\Big|_{\lambda=1} = rQ'(r)$. Then $\psi(r) > 0$ for all $r \geq 1$ and $\psi(r) = O(r^{-2})$ as $r \to \infty$. Furthermore, ψ is a solution to the linearized elliptic problem

$$-\psi''(r) - \frac{2}{r}\psi'(r) + \frac{2}{r^2}\cos(2Q(r))\psi(r) = 0 \tag{3.5.3}$$

in $\mathbb{R}^3_*$, but it *does not* satisfy the Dirichlet condition at $r = 1$. As before, the 5-dimensional reduction reads

$$\varphi(r) := \frac{1}{r}\psi(r),\ (-\Delta_5 + V)\varphi = 0,\ V(r) = \frac{2}{r^2}(\cos(2Q(r)) - 1) \tag{3.5.4}$$

where Δ_5 is the Laplacian in $\mathbb{R}^5$. By the preceding, V is a real-valued, radial, bounded and smooth potential on $\mathbb{R}^5_*$ which decays like r^{-6} as $r \to \infty$ (and each derivative improves the decay by one power of r).

The operator $H := -\Delta + V = -\Delta_5 + V$ is self-adjoint with domain $\mathcal{D} := (H^2 \cap H^1_0)(\mathbb{R}^5_*)$. Its essential spectrum coincides with $[0, \infty)$ and that spectrum is purely absolutely continuous. As observed in [5], H has no negative spectrum. Indeed, if it did, then by a variational principle there would have to be a lowest eigenvalue $-E_*^2 < 0$ which is simple and with associated eigenfunction f_* which is smooth, radial, and does not change its sign on $r > 1$. We may assume that $f_* > 0$ whence $f_*'(1) > 0$. Then, with $\langle \cdot | \cdot \rangle$ being the L^2-pairing in $\mathbb{R}^5_*$,

$$-E_*^2 \langle f_* | \varphi \rangle = \langle H f_* | \varphi \rangle = |S^4| f_*'(1) \varphi(1) > 0 \tag{3.5.5}$$

which is a contradiction since the left-hand side is negative. It remains to analyze the threshold 0, which generally speaking can be either a resonance or an eigenvalue. Since we are in dimension 5, the former would mean that there exists $f \in \mathcal{D}$, $f \not\equiv 0$, with $|f(x)| \sim \frac{c}{|x|^3}$ as $x \to \infty$ (the decay here being that of the Newton kernel). However, in that case $f \in L^2$, whence we recover the well-known fact that zero energy can only be an eigenfunction, necessarily radial by our standing assumption. Thus, let $Hf = 0$, $f \in L^2$ radial. Then

$$0 = \langle Hf | \varphi \rangle = \langle f | H\varphi \rangle + |S^4| f'(1) \varphi(1) = |S^4| f'(1) \varphi(1) \tag{3.5.6}$$

which is a contradiction since $f(1) = 0$ precludes $f'(1) = 0$ (recall $\varphi(1) \neq 0$). In conclusion, H has no point spectrum (as already noted in [5]). For future reference we remark that the same argument as in (3.5.6) shows that there can be no solution $f \in L^2(\mathbb{R}^5_*)$ of $Hf = 0$, unless

$$f'(1) + 2f(1) = 0 \tag{3.5.7}$$

Of course φ satisfies this condition, as can be seen from the equation.

In order to prove Theorem 3.1.2 we need to establish Strichartz estimates for the wave equation exterior to the ball, perturbed by the radial potential V. Once this is done, Theorem 3.1.2 is an immediate consequence via a standard contraction argument. Henceforth, the *free problem* refers to the wave equation exterior to a ball in $\mathbb{R}^5$ with a Dirichlet condition at $r = 1$ as considered by [72]. By an *admissible Strichartz norm* for the free problem we mean any Strichartz norm as in [72] for solutions with $\dot{H}^1_0 \times L^2$-data excluding the L^2_t-endpoint.

Proposition 3.5.1. *Let $\|\cdot\|_X$ be an admissible Strichartz norm for the free problem. Let V be a potential as above and assume that $-\Delta + V$ has no point spectrum. Then any solution of*

$$\begin{aligned} \Box u + Vu = F, \quad & (t,x) \in (0,\infty) \times \mathbb{R}^5_* \\ u(1,t) = 0, \quad & t \geq 0, \\ (u(0), \dot{u}(0)) = (f,g) \in & \dot{H}^1_0 \times L^2(\mathbb{R}^5_*) \end{aligned} \tag{3.5.8}$$

with radial data satisfies

$$\|u\|_X \leq C\big(\|(f,g)\|_{\dot{H}^1_0 \times L^2} + \|F\|_{L^1_t L^2_x}\big) \tag{3.5.9}$$

with a constant $C = C(V)$.

Proof. The argument is a variant of the one in [64]. It suffices to consider $F = 0$ by Minkowski's inequality. Let $-\Delta$ be the Laplacian on $\mathbb{R}^5_*$ with domain $\mathcal{D} := H^2 \cap H^1_0(\mathbb{R}^5_*)$ on which it is self-adjoint (this incorporates the Dirichlet condition at $r = 1$). We claim that $A := (-\Delta)^{\frac{1}{2}}$ satisfies

$$\|A\, f\|_2 \simeq \|f\|_{\dot{H}^1_0} \tag{3.5.10}$$

for all $f \in C^\infty(\mathbb{R}^5)$ which are compactly supported in $\{x \in \mathbb{R}^5 \mid 1 < |x| < \infty\}$. Indeed, squaring both sides this is equivalent to

$$\langle -\Delta f | f \rangle = \|\nabla f\|_2^2$$

for all such f, which is obviously true. For any real-valued $u = (u_1, u_2) \in \dot{H}_0^1 \times L^2$ we set

$$U := Au_1 + iu_2$$

Then (3.5.10) implies that $\|U\|_2 \simeq \|(u_1, u_2)\|_{\mathcal{H}}$. Furthermore, u solves (3.5.8) if and only if

$$\begin{aligned} i\partial_t U &= AU + Vu \\ U(0) &= Af + ig \in L^2(\mathbb{R}^5_*) \end{aligned} \tag{3.5.11}$$

Then

$$U(t) = e^{-itA}U(0) - i\int_0^t e^{-i(t-s)A} V u(s)\, ds$$

By [72], with $P := A^{-1}\mathrm{Re}\,$,

$$\|Pe^{-itA}U(0)\|_X \le C\|U(0)\|_2$$

Factorize $V = V_1 V_2$ where the factors decay like r^{-3}. By the Christ-Kiselev lemma, see [72], and our exclusion of L^2_t, it suffices to bound

$$\begin{aligned} \Big\| P\int_{-\infty}^{\infty} e^{-i(t-s)A}\, V_1 V_2\, u(s)\, ds \Big\|_X &\le \|K\|_{L^2_{t,x}\to X} \|V_2\, u(s)\|_{L^2_{s,x}} \\ (KF)(t) &:= P\int_{-\infty}^{\infty} e^{-i(t-s)A} V_1 F(s)\, ds \end{aligned} \tag{3.5.12}$$

Now

$$\|KF\|_X \le \|Pe^{-itA}\|_{2\to X} \Big\| \int_{-\infty}^{\infty} e^{isA} V_1 F(s)\, ds \Big\|_2$$

The first factor on the right-hand side is some constant by [72]. We claim that the second one is bounded by $C\|F\|_{L^2_{t,x}}$. By duality, this claim is equivalent to the *local energy bound*

$$\|V_1\, e^{-itA}\phi\|_{L^2_{t,x}} \le C\|\phi\|_2 \tag{3.5.13}$$

relative to $L^2(\mathbb{R}^5_*)$. This is elementary to prove for radial ϕ (which suffices for us), using the distorted Fourier transform relative to $-\partial_{rr} + \frac{2}{r^2}$ on $L^2((1,\infty))$ with a Dirichlet condition at $r = 1$. Indeed, map any smooth radial $f = f(r) \in L^2(\mathbb{R}^5_*)$ onto the function $\tilde{f}(r) = r^2 f(r) \in L^2(1,\infty)$. Then

$$(-\Delta_5 f)(r) = r^{-2}(\mathcal{L}_0\tilde{f})(r), \qquad \mathcal{L}_0 = -\partial_{rr} + \frac{2}{r^2}$$

Associated with $\mathcal{L}_0$ there is a distorted Fourier basis $\phi_0(r;\lambda)$ that satisfies

$$\phi_0(1;\lambda) = 0, \qquad \mathcal{L}_0\phi_0(r;\lambda) = \lambda^2\phi_0(r;\lambda),$$

and such that for all $g \in L^2((1,\infty))$

$$\begin{aligned} \hat{g}(\lambda) &= \int_1^\infty \phi_0(r;\lambda)g(r)\,dr \\ g(r) &= \int_0^\infty \phi_0(r;\lambda)\hat{g}(\lambda)\,\rho_0(d\lambda) \\ \|g\|^2_{L^2(1,\infty)} &= \int_0^\infty |\hat{g}(\lambda)|^2\,\rho_0(d\lambda) \end{aligned} \tag{3.5.14}$$

where the integrals need to be interpreted in a suitable limiting sense. The real-valued functions $\phi_0(r;\lambda)$ and the positive measure $\rho_0(d\lambda) = \omega_0(\lambda)\,d\lambda$ are explicit, see Lemma 3.5.2 below. Moreover, it is shown there that

$$\sup_{r\ge 1,\,\lambda>0} |\phi_0(r;\lambda)|^2\omega_0(\lambda) \le C < \infty \tag{3.5.15}$$

Taking this for granted, we note that (3.5.13) is equivalent to the following estimate for

$f \in L^2((1,\infty))$

$$\int_{-\infty}^{\infty} \Big\| V_1 \int_0^\infty e^{-it\lambda} \phi_0(r;\lambda) \hat{f}(\lambda)\, \rho_0(d\lambda) \Big\|_2^2 \, dt \le C\|f\|_2^2 \tag{3.5.16}$$

Here we used that $A = \sqrt{\mathcal{L}_0}$ (in the half-line picture) is given by multiplication by λ on the Fourier side, and so e^{-itA} becomes $e^{-it\lambda}$. Expanding the left-hand side and carrying out the t-integration explicitly reduces this to the following statement:

$$\int_1^\infty V_1^2(r) \int_0^\infty \int_0^\infty \phi_0(r;\lambda)\phi_0(r;\mu)\hat{f}(\lambda)\overline{\hat{f}(\mu)}\delta(\lambda-\mu)\, \rho_0(d\lambda)\rho_0(d\mu)\, dr \le C\|f\|_2^2 \tag{3.5.17}$$

The left-hand side above is

$$= \int_1^\infty V_1^2(r) \int_0^\infty \phi_0(r;\lambda)^2 \hat{f}(\lambda)^2 \omega_0(\lambda)^2 \, d\lambda\, dr$$

In view of (3.5.15), (3.5.14), and $\int_1^\infty V_1^2(r)\, dr < \infty$, we obtain (3.5.16), and thus (3.5.13). This means that $\|K\|_{L^2_{t,x}\to X} \le C$, some finite constant.

For the second factor in (3.5.12) we claim the estimate

$$\|V_2\, u(t)\|_{L^2_{t,x}} \le C\|U(0)\|_2 = C\|(f,g)\|_{\dot{H}^1\times L^2} \tag{3.5.18}$$

valid for any solution of (3.5.8) with $F=0$. To prove it, we invoke the distorted Fourier transform relative to the self-adjoint operator $H := -\Delta + V$ on the domain $\mathcal{D}$ as defined above, restricted to radial functions. As before, conjugation by r^2 reduces matters to a half-line operator $\mathcal{L} := -\partial_{rr} + \frac{2}{r^2} + V$ on $L^2((1,\infty))$ with a Dirichlet condition at $r=1$. In analogy with $\mathcal{L}_0$, we show in Lemma 3.5.2 below that there exists a Fourier basis $\phi(r;\lambda)$ satisfying for all $\lambda \ge 0$

$$\mathcal{L}\phi(r;\lambda) = \lambda^2\phi(r;\lambda), \quad \phi(1;\lambda) = 0$$

and the correspondences

$$\begin{aligned} \hat{f}(\lambda) &:= \int_1^\infty \phi(r;\lambda) f(r)\, dr \\ f(r) &= \int_0^\infty \phi(r;\lambda)\hat{f}(\lambda)\, \rho(d\lambda) \\ \|f\|_{L^2(1,\infty)} &= \|\hat{f}\|_{L^2((0,\infty);\rho)} \end{aligned} \tag{3.5.19}$$

for a suitable positive measure $\rho(d\lambda) = \omega(\lambda)\, d\lambda$ on $(0,\infty)$. It is here that the assumptions on the spectrum of H enter crucially. Indeed, the absence of negative spectrum means that ρ is supported on $(0,\infty)$, and the absence of a zero eigenvalue implies that ω exhibits the same rate of decay as ω_0 as $\lambda \to 0+$. The exact property which emerges from all this and which underlies the proof of (3.5.18) is the following variant of (3.5.15), see Lemma 3.5.2,

$$\sup_{r\geq 1,\, \lambda>0} (\lambda r)^{-2} |\phi(r;\lambda)|^2 \omega(\lambda) \leq C < \infty \tag{3.5.20}$$

The local energy estimate (3.5.18) reduces to

$$\begin{aligned} &\int_{-\infty}^\infty \int_1^\infty \Big| V_2(r) \int_0^\infty \phi(r;\lambda)\big(\cos(t\lambda)\hat{f}(\lambda) + \lambda^{-1}\sin(t\lambda)\hat{g}(\lambda)\big)\, \rho(d\lambda)\Big|^2 \, drdt \\ &\leq C \int_1^\infty (|f'(r)|^2 + |g(r)|^2)\, dr \end{aligned}$$

Consider the case $g = 0$. Expanding and integrating out the left-hand side one obtains

$$\begin{aligned} &\frac{1}{2}\int_1^\infty \int_0^\infty V_2(r)^2 \phi(r;\lambda)^2 |\hat{f}(\lambda)|^2\, \omega(\lambda)^2\, d\lambda \\ &\leq C \int_1^\infty V_2(r)^2 r^2\, dr \int_0^\infty \lambda^2 |\hat{f}(\lambda)|^2\, \rho(d\lambda) \leq C\|\sqrt{\mathcal{L}} f\|_2^2 \leq C\|f'\|_2^2 \end{aligned} \tag{3.5.21}$$

where we used (3.5.20) to pass to the second inequality sign, and (3.5.10) to pass to the final inequality. The calculation for $f = 0$ is similar.

Putting everything together we obtain (3.5.18) and therefore also (3.5.9). $\square$

Now we turn to the technical statements concerning the distorted Fourier transforms for the half-line operators $\mathcal{L}_0 = -\partial_{rr} + \frac{2}{r^2}$ and $\mathcal{L} = \mathcal{L}_0 + V$ on $L^2((1,\infty))$, respectively, with a Dirichlet condition at $r = 1$. This is completely standard, see for example [28, Section 2], the first two chapters in [10], or Newton's survey [60]. But since these references do not treat the specific half-line problem that we are dealing with, and in order to keep this chapter self-contained, we include the details.

Lemma 3.5.2. *The half-line operators $\mathcal{L}_0$ and $\mathcal{L}$ admit Fourier bases satisfying* (3.5.14)*,* (3.5.15)*, and* (3.5.19)*,* (3.5.20)*, respectively. For $\mathcal{L}$ it is essential to assume that it has no point spectrum.*

Proof. For any $z \in \mathbb{C}$ denote by $\phi_0(r; z)$ and $\theta_0(r; z)$ the unique solutions of

$$\mathcal{L}_0\phi_0(\cdot; z) = z^2\phi_0(\cdot; z), \quad \mathcal{L}_0\theta_0(\cdot; z) = z^2\theta_0(\cdot; z)$$

with initial conditions

$$\phi_0(1; z) = 0, \phi_0'(1; z) = 1, \quad \theta_0(1; z) = 1, \theta_0'(1; z) = 0$$

These are entire in z, and satisfy $W(\theta_0(\cdot; z), \phi_0(\cdot; z)) = 1$ by construction. Here $W(f, g) = fg' - f'g$ is the Wronskian. Furthermore, since $\mathcal{L}_0$ is in the limit-point case at $r = \infty$, for any $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ there exists a unique solution $\psi_0(\cdot; z) \in L^2((1,\infty))$ to $\mathcal{L}_0\psi_0(\cdot; z) = z^2\psi_0(\cdot; z)$ with $\psi_0(1; z) = 1$. Writing

$$\psi_0(\cdot; z) = \theta_0(\cdot; z) + m_0(z)\phi_0(\cdot; z)$$

one finds that m_0 is analytic in $\operatorname{Im} z > 0$, as well as a Herglotz function ($\operatorname{Im} m(z) > 0$ in the upper half plane) and the spectral measure is determined by

$$\rho_0(d\lambda) = 2\lambda \mathrm{Im}\, m_0(\lambda + i0)\, d\lambda \tag{3.5.22}$$

It is common to refer to m_0 as the *Weyl-Titchmarsh function*, and to ψ as the *Weyl-Titchmarsh solution*.

For the specific case of $\mathcal{L}_0$ a fundamental system is of $\mathcal{L}_0 f = z^2 f$ is given by weighted Hankel functions $r^{\frac{1}{2}} H^{\pm}_{\frac{3}{2}}(zr)$. These functions are explicit linear combinations of $e^{\pm izr}$ with rational (in r) coefficients. Indeed, one verifies that

$$\begin{aligned}
\phi_0(r; z) &= (z^3 r)^{-1}\big[(1 + z^2 r)\sin(z(r-1)) - z(r-1)\cos(z(r-1))\big] \\
\theta_0(r; z) &= (z^3 r)^{-1}\big[(1 + z^2(r-1))\sin(z(r-1)) + (z^3 r - z(r-1))\cos(z(r-1))\big] \\
\psi_0(r; z) &= \frac{z + i/r}{z + i} e^{iz(r-1)} \\
m_0(z) &= \frac{i(z^2 - 1) - z}{z + i}
\end{aligned}$$

Note that while the first two lines are entire in z, the third and fourth are meromorphic in $\mathbb{C}$ and analytic in $\mathrm{Im}\, z \geq 0$. For the spectral measure we find that

$$\rho_0(d\lambda) = \frac{2\lambda^4}{1 + \lambda^2}\, d\lambda$$

To prove (3.5.15), we set $u := \lambda(r - 1)$ whence

$$\phi_0(r; \lambda) = \lambda^{-2}(u + \lambda)^{-1}\big[\sin u - u\cos u + \lambda(u + \lambda)\sin u\big]$$

If $\lambda > 1$, one checks that $\lambda\phi_0(r; \lambda) = O(1)$ uniformly in $u > 0$, whereas for $0 < \lambda < 1$ one has $\lambda^2\phi(r; \lambda) = O(1)$ for all $u > 0$. In fact, in both cases one gains a factor of u for small u.

These two bounds amount to

$$|\phi_0(r;\lambda)|\frac{\lambda^2}{1+\lambda} \le C\, \min(1, \lambda(r-1)) \qquad \forall\, r \ge 1, \lambda > 0$$

which is precisely (3.5.15). Notice that this estimate contains the $\mathcal{L}_0$-analogue of (3.5.20).

By standard perturbation theory we now transfer these results to $\mathcal{L}$, see [10] for more background. First, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we set

$$\widetilde{\psi}(r;\lambda) = \psi_0(r;\lambda) + \int_r^\infty G_0(r,r';\lambda)V(r')\widetilde{\psi}(r';\lambda)\, dr' \tag{3.5.23}$$

with the Green function

$$G_0(r,r';\lambda) := \frac{\psi_0(r;\lambda)\overline{\psi_0(r';\lambda)} - \psi_0(r';\lambda)\overline{\psi_0(r;\lambda)}}{W(\psi_0(\cdot;\lambda), \overline{\psi_0(\cdot;\lambda)})}$$

Evaluating at $r = \infty$ one sees that $W(\psi_0(\cdot;\lambda), \overline{\psi_0(\cdot;\lambda)}) = -2i\lambda^3/(1+\lambda^2) \neq 0$. To be specific,

$$G_0(r,r';\lambda) = \frac{1}{\lambda^2}(\frac{1}{r'} - \frac{1}{r})\cos(\lambda(r-r')) + \frac{\lambda^2 + \frac{1}{rr'}}{\lambda^2}\frac{\sin(\lambda(r'-r))}{\lambda} \tag{3.5.24}$$

whence for all $\lambda \neq 0$ and $1 < r < r' < \infty$,

$$|G_0(r,r';\lambda)| \le C_0\big(|\lambda|^{-1}\chi_{[|\lambda|>1]} + (r'-r+(r'-r)^3)\chi_{[0<|\lambda|<1]}\big) \tag{3.5.25}$$

By Volterra iteration we see that (3.5.23) has a unique solution $\widetilde{\psi}(r;\lambda)$ even for $\lambda = 0$ which satisfies for all $r \ge 1$

$$|\widetilde{\psi}(r;\lambda) - \psi_0(r;\lambda)| \le \exp\Big(C_0\int_r^\infty s^3|V(s)|\, ds\Big) - 1 \tag{3.5.26}$$

We used here that $\|\psi_0(\cdot;\lambda)\|_{L^\infty(1,\infty)} \leq 1$ for all λ. It follows that

$$\widetilde{\psi}(r;\lambda) = \psi_0(r;\lambda) + O(r^{-4}) \qquad r \to \infty \tag{3.5.27}$$

uniformly in λ. In particular, we conclude that

$$W\big(\widetilde{\psi}(\cdot;\lambda), \overline{\widetilde{\psi}(\cdot;\lambda)}\big) = W(\psi_0(\cdot;\lambda), \overline{\psi_0(\cdot;\lambda)}) = -\frac{2i\lambda^3}{1+\lambda^2} \tag{3.5.28}$$

whence $\widetilde{\psi}(r,\lambda) \neq 0$ for all $\lambda \neq 0$ and $r \geq 1$. Hence, we can find a (smooth) function $c(\lambda)$ for $\lambda \neq 0$ such that $\psi(r;\lambda) := c(\lambda)\widetilde{\psi}(r;\lambda)$ satisfies $\psi(1;\lambda) = 1$. Furthermore, the first estimate in (3.5.25) implies that

$$\widetilde{\psi}(r;\lambda) = \psi_0(r;\lambda) + O(\lambda^{-1}) \qquad \lambda \to \infty \tag{3.5.29}$$

uniformly in $r \geq 1$. This shows that $c(\lambda) = 1 + O(\lambda^{-1})$ as $\lambda \to \infty$ and that

$$2i\mathrm{Im}\, m(\lambda) = W(\overline{\psi(\cdot;\lambda)}, \psi(\cdot;\lambda)) = \frac{2i\lambda^3}{1+\lambda^2} + O(1) \qquad \lambda \to \infty$$

where m is the Weyl-Titchmarsh function for $\mathcal{L}$. In view of the universal property (3.5.22) one has for all $0 < \lambda_0 < \lambda < \infty$

$$C^{-1} \leq \lambda^{-1}\frac{d\rho}{d\lambda}(\lambda) \leq C \tag{3.5.30}$$

for some constant $C = C(\lambda_0)$. As far as the bounds on $\phi(r;\lambda)$ are concerned, one has

$$\phi(r;\lambda) = \frac{\mathrm{Im}\,\psi(r;\lambda)}{\mathrm{Im}\, m(\lambda)} \tag{3.5.31}$$

which immediately shows that for $\lambda > \lambda_0$,

$$\lambda|\phi(r;\lambda)| \leq C$$

To gain a factor $\lambda(r-1)$, observe that (3.5.23) implies that $\|\partial_r\psi(r;\lambda)\|_\infty \leq C(\lambda_0)\lambda$. In particular,

$$|\operatorname{Im}\psi(r;\lambda)| \leq |\psi(r;\lambda) - \psi(1;\lambda)| \leq C\lambda(r-1)$$

where $C = C(\lambda_0)$ as before. It remains to verify (3.5.19), (3.5.20) in the regime $0 < \lambda \ll 1$. It is of course here that the assumption on absence of a zero energy eigenvalue enters.

We begin with the zero energy solution, i.e., a fundamental system of solutions to $\mathcal{L}f = 0$. First, $\frac{1}{r}, r^2$ form such a system for $\mathcal{L}_0 f = 0$. Then

$$u_0(r) = r^{-1} - \int_r^\infty G_0(r,s)V(s)u_0(s)\,ds \tag{3.5.32}$$

with Green function

$$G_0(r,s) := \frac{1}{3}\frac{r^3 - s^3}{sr}$$

defines a solution of $\mathcal{L}u_0 = 0$. The Volterra iteration again converges and yields

$$u_0(r) = r^{-1}(1 + O(r^{-4})) \qquad r \to \infty \tag{3.5.33}$$

Here and in what follows, the $O(\cdot)$-terms can be differentiated in r (and λ where appropriate) with the expected effect. We leave the detailed verification of this property to the reader. By (3.5.7), both $u_0(1) \neq 0$ and $u_0'(1) \neq 0$. Another solution is given by

$$u_1(r) = u_0(r)\int_{r_0}^r u_0^{-2}(s)\,ds \tag{3.5.34}$$

for all $r > r_0$ where $r_0 \gg 1$ is chosen such that $u_0(r) > 0$ in that range. Inserting (3.5.33) into (3.5.34) yields

$$u_1(r) = \frac{1}{3}r^2(1 + O(r^{-4})) \qquad r \to \infty \tag{3.5.35}$$

Clearly, $\{u_0, u_1\}$ forms a fundamental system of $\mathcal{L}u = 0$ with $W(u_0, u_1) = 1$.

Next, define for all $r \geq 1$ and $0 < \lambda \ll 1$,

$$u_1(r; \lambda) = u_1(r) + \lambda^2 \int_1^r G(r, r')u_1(r'; \lambda)\, dr' \tag{3.5.36}$$

where

$$G(r, r') := u_1(r)u_0(r') - u_0(r)u_1(r')$$

Then (3.5.36) has a solution, which satisfies $\mathcal{L}u_1(\cdot; \lambda) = \lambda^2 u_1(\cdot; \lambda)$ and

$$u_1(r; \lambda) = u_1(r) + O(\lambda^2 r^2 (r - 1)^2)$$

as long as $\lambda^2 r^2 \ll 1$. Similarly, we define $u_0(r; \lambda)$ as

$$u_0(r; \lambda) = u_0(r) + \lambda^2 \Big(\int_1^r u_0(r)u_1(s)u_0(s; \lambda)\, ds + \int_r^{\varepsilon\lambda^{-1}} u_1(r)u_0(s)u_0(s; \lambda)\, ds \Big) \tag{3.5.37}$$

Here $\varepsilon > 0$ is a small absolute constant, which is to be determined. Notice that (3.5.37) is not a Volterra equation, but it can be solved by a contraction argument. Indeed, we set

$$u_0(r; \lambda) = u_0(r) + \lambda^2 r u_2(r; \lambda)$$

and reformulate (3.5.37) in the form $u_2 = Tu_2$ for some linear map $T = T_{\varepsilon,\lambda}$. Then one checks that for all $0 < \lambda \ll 1$ and a small but fixed $\varepsilon > 0$, the map T is a contraction in a ball of fixed size in the space $C([1,\varepsilon\lambda^{-1}])$. Consequently, there is a unique solution satisfying

$$|u_2(r;\lambda)| \le C \qquad \forall\, 1 \le r \le \varepsilon\lambda^{-1}$$

and all $0 < \lambda \ll 1$. Returning to (3.5.37), we see that this integral equation has a solution for all $1 \le r \le \varepsilon\lambda^{-1}$, which is also a solution of $\mathcal{L}u_0 = \lambda^2 u_0$, and which is of the form

$$u_0(r;\lambda) = u_0(r) + O(\lambda^2 r) \ \text{ on } \ [1,\varepsilon\lambda^{-1}]$$

Furthermore, $\{u_0(\cdot;\lambda), u_1(\cdot;\lambda)\}$ forms a fundamental system of $\mathcal{L}u = \lambda^2 u$ with

$$W(u_0(\cdot;\lambda), u_1(\cdot;\lambda)) = 1 + O(\lambda^2)$$

as $\lambda \to 0$, and $u_0(1;\lambda) \neq 0$ for small λ since $u_0(1) \neq 0$.

Consequently, for all $|\lambda| \ll 1$ one has (since $u_1(1;\lambda) = u_1(1)$)

$$\phi(r;\lambda) = c(\lambda)\Big(u_1(r;\lambda) - \frac{u_0(r;\lambda)}{u_0(1;\lambda)}u_1(1)\Big) \tag{3.5.38}$$

where $c(\lambda)$ is continuous with $|c(\lambda)| \simeq 1$. Indeed,

$$c(\lambda) = \Big(u_1'(1;\lambda) - \frac{u_0'(1;\lambda)}{u_0(1;\lambda)}u_1(1)\Big)^{-1} = \frac{u_0(1;\lambda)}{W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))}$$

By inspection, one has the bounds on $1 < r < \lambda^{-1}$,

$$|\phi(r;\lambda)| \le C\lambda^{-2}, \qquad |\partial_r\phi(r;\lambda)| \le C\lambda^{-1}$$

Indeed, u_1 satisfies these bounds, and u_0 better ones as can be seen directly from the Volterra equations (3.5.37), (3.5.36). Hence,

$$\lambda^2|\phi(r;\lambda)| \le C\min(1,\lambda(r-1)) \qquad \forall\, 1<r<\lambda^{-1} \tag{3.5.39}$$

as desired. To extend this bound to $r>\lambda^{-1}$, and in order to describe the spectral measure for small λ, we use $\tilde{\psi}$ from (3.5.23). In fact, writing

$$\phi(r;\lambda) = a(\lambda)\widetilde{\psi}(r;\lambda) + \bar{a}(\lambda)\overline{\widetilde{\psi}(r;\lambda)} \tag{3.5.40}$$

one has

$$a(\lambda) = \frac{W(\phi(\cdot;\lambda),\overline{\widetilde{\psi}(\cdot;\lambda)})}{W(\widetilde{\psi}(\cdot;\lambda),\overline{\widetilde{\psi}(\cdot;\lambda)})} = O(\lambda^{-3}) \tag{3.5.41}$$

For the denominator we used (3.5.28), whereas the numerator is evaluated at $r=\lambda^{-\frac{1}{2}}$, say which reduces matters to

$$W\big(\phi(\cdot;\lambda),\overline{\widetilde{\psi}(\cdot;\lambda)}\big) = c(\lambda)W\big(u_1(\cdot;\lambda),\overline{\psi_0(\cdot;\lambda)}\big) + o(1) = O(1) \qquad \lambda\to 0 \tag{3.5.42}$$

Inserting (3.5.41) into (3.5.40) one obtains $\sup_{r>1}\lambda^2|\phi(r;\lambda)| = O(1)$ as $\lambda\to 0$. Together with (3.5.39), this concludes the proof of (3.5.20).

Finally, in order to determine $\operatorname{Im} m(\lambda)$ for small λ, we use the relation (3.5.31), valid for all $r\ge 1$. We use it at $r=C$ a large constant to conclude that

$$\phi(r;\lambda)\asymp 1, \quad \operatorname{Im}\psi(r;\lambda)\asymp \operatorname{Im}\psi_0(r;\lambda)\asymp\lambda^3$$

which implies $\operatorname{Im} m(\lambda)\asymp\lambda^3$ and we are done. Here $a\asymp b$ means $C^{-1}<\frac{a}{b}<C$. $\square$

CHAPTER 4

$3D$ WAVE MAPS EXTERIOR TO A BALL: RELAXATION TO HARMONIC MAPS FOR ALL DATA AND FOR ALL DEGREES

In this chapter we describe all possible asymptotic dynamics for the 1-equivariant wave-map equation from

$$\mathbb{R}^{1+3}_{t,x} \setminus (\mathbb{R} \times B(0,1)) \to S^3$$

with a Dirichlet condition on the boundary of the ball $B(0,1)$, and data of finite energy for all degree classes, $n \geq 0$. To remind the reader, we are considering the Lagrangian

$$\mathcal{L}(U, \partial_t U) = \int_{\mathbb{R}^{1+3} \setminus (\mathbb{R} \times B(0,1))} \frac{1}{2}\Big(-|\partial_t U|_g^2 + \sum_{j=1}^{3} |\partial_j U|_g^2 \Big)\, dtdx$$

where g is the round metric on $\mathbb{S}^3$, and we only consider functions for which the boundary of the cylinder $\mathbb{R} \times B(0,1)$ gets mapped to a fixed point on $\mathbb{S}^3$, say the north pole. Under the usual 1-equivariance assumption the Euler-Lagrange equation associated with this Lagrangian becomes

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{4.0.1}$$

where $\psi(t, r)$ measures the angle from the north-pole on $\mathbb{S}^3$. The imposed Dirichlet boundary condition is then $\psi(t, 1) = 0$ for all $t \in \mathbb{R}$. In other words, we are considering the Cauchy problem

$$
\begin{aligned}
&\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1, \\
&\psi(t,1) = 0, \quad \forall\, t, \\
&\psi(0,r) = \psi_0(r), \quad \psi_t(0,r) = \psi_1(r)
\end{aligned}
\tag{4.0.2}
$$

The conserved energy is

$$
\mathcal{E}(\psi,\psi_t) = \int_1^\infty \frac{1}{2}\big(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\big) r^2\, dr \tag{4.0.3}
$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0,t_1)$ must satisfy $\psi(t,\infty) = n\pi$ for all $t \in I$ where $n \in \mathbb{Z}$ is fixed. We can restrict to the case $n \geq 0$ since this covers the entire range $n \in \mathbb{Z}$ by the symmetry $\psi \mapsto -\psi$. We call n the *degree*, and denote by $\mathcal{E}_n$ the connected component of the metric space of all $\vec{\psi} = (\psi_0,\psi_1)$ with $\mathcal{E}(\vec{\psi}) < \infty$ and fixed degree n (of course obeying the boundary condition at $r=1$), i.e.,

$$
\mathcal{E}_n := \{(\psi_0,\psi_1) \mid \mathcal{E}(\psi_0,\psi_1) < \infty,\ \psi_0(1) = 0,\ \lim_{r\to\infty} \psi_0(r) = n\pi\} \tag{4.0.4}
$$

The natural space to place the solution into for $n = 0$ is the *energy space* $\mathcal{H}_0 := (\dot{H}_0^1 \times L^2)(\mathbb{R}^3_*)$ with norm

$$
\|\vec{\psi}\|^2_{\mathcal{H}_0} := \int_1^\infty (\psi_r^2(r) + \psi_t^2(r))\, r^2\, dr, \qquad \vec{\psi} = (\psi,\psi_t) \tag{4.0.5}
$$

Here, $\mathbb{R}^3_* := \mathbb{R}^3 \backslash B(0,1)$ and $\dot{H}^1_0(\mathbb{R}^3_*)$ is the completion under the first norm on the right-hand side of (4.0.5) of the smooth radial functions on $\{x \in \mathbb{R}^3 \mid |x| > 1\}$ with compact support. For $n \geq 1$, we denote $\mathcal{H}_n := \mathcal{E}_n - (Q_n, 0)$ with "norm"

$$
\|\vec{\psi}\|_{\mathcal{H}_n} := \|\vec{\psi} - (Q_n,0)\|_{\mathcal{H}_0}
$$

The point of this notation is that the boundary condition at $r = \infty$ is $\vec{\psi} - (Q_n, 0)(r) \to 0$ as $r \to \infty$.

Our main result is as follows. It should be viewed as a verification of the *soliton resolution conjecture* for this particular case and completes the study of this model initiated in Chapter 3.

Theorem 4.0.3. *For any smooth energy data in $\mathcal{E}_n$ there exists a unique global and smooth solution to* (4.0.2) *which scatters to the harmonic map* $(Q_n, 0)$.

Scattering here means that on compact regions in space one has $(\psi, \psi_t)(t) - (Q_n, 0) \to (0, 0)$ in the energy topology, or alternatively

$$(\psi, \psi_t)(t) = (Q_n, 0) + (\varphi, \varphi_t)(t) + o_{\mathcal{H}_n}(1) \quad t \to \infty \tag{4.0.6}$$

where $(\varphi, \varphi_t) \in \mathcal{H}_0$ solves the linearized version of (4.0.2), i.e.,

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{2}{r^2}\varphi = 0, \; r \geq 1, \; \varphi(t, 1) = 0 \tag{4.0.7}$$

We would like to emphasize that only the scattering part of Theorem 4.0.3 is difficult.

In the previous chapter and in [53] the author, together with Wilhelm Schlag, established this theorem for degree zero, and also proved asymptotic stability of the Q_n for $n \geq 1$. Here we are able to treat data of all sizes in the higher degree case. As in [53] we employ the method of concentration compactness from [36, 37]. The main difference from [53] lies with the rigidity argument. While the virial identity was the key to rigidity in [53] for degree zero (which seems to be impossible for $n \geq 1$), here we follow an alternate route which was developed in a very different context in [23, 25] for the three-dimensional energy critical nonlinear focusing wave equation. To be specific we rely on the *exterior asymptotic energy* arguments developed there. A novel feature of our work is that we elucidate the role of the Newton potential as an obstruction to linear energy estimates exterior to a cone in odd dimensions; in particular we do this for dim = 5, which is what is needed for equivariant wave maps in $\mathbb{R}^3$. It is precisely this feature which allows us to adapt the rigidity blueprint from [23, 25] to the model under consideration.

Finally, let us mention that we expect the methods of this chapter to carry over to higher equivariance classes as well.

4.1 Preliminaries

In this section we discuss the harmonic maps Q_n, as well as the reduction of the equivariant wave maps equation to a semi-linear equation in $\mathbb{R}^5_* := \mathbb{R}^5 \setminus B(0,1)$ with a Dirichlet condition at $r = 1$.

4.1.1 Exterior Harmonic Maps

In each energy class, $\mathcal{E}_n$ there is a unique finite energy exterior harmonic map, $(Q, 0) = (Q_n, 0)$. In fact $(Q_n, 0)$ can be seen to have minimal energy in $\mathcal{E}_n$. An exterior harmonic map is a stationary solution of (4.0.2), i.e.,

$$Q_{rr} + \frac{2}{r}Q_r = \frac{\sin(2Q)}{r^2} \tag{4.1.1}$$

$$Q(1) = 0, \quad \lim_{r\to\infty} Q(r) = n\pi \tag{4.1.2}$$

Lemma 4.1.1. *For all $\alpha \in \mathbb{R}$ there exists a unique solution $Q_\alpha \in \dot{H}^1(\mathbb{R}^3_*)$ to (4.1.1) with*

$$Q_\alpha(r) = n\pi - \alpha r^{-2} + O(r^{-6})$$

The $O(\cdot)$ is determined by α, and vanishes for $\alpha = 0$. Moreover, there exists a unique α such that $Q_\alpha(1) = 0$, which we denote by α_0. One has $\alpha_0 > 0$.

The proof of Lemma 4.1.1 is standard so we just sketch an outline below. In order to study solutions to (4.1.1) it is convenient to introduce new variables $s = \log(r)$ and $\phi(s) = Q(r)$. With this change of variables we obtain an autonomous differential equation for ϕ, viz.,

$$\ddot{\phi} + \dot{\phi} = \sin(2\phi) \tag{4.1.3}$$

which is the equation for a damped pendulum. We can thus reduce matters to the phase portrait associated to (4.1.3). Setting $x(s) = \phi(s)$, $y(s) = \dot{\phi}(s)$ we rewrite (4.1.3) as the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2x) \end{pmatrix} =: X(x, y) \tag{4.1.4}$$

and we denote by Φ_s the flow associated to X. The equilibria of (4.1.4) occur at points $v_{k/2} = (\frac{k\pi}{2}, 0)$ where $k \in \mathbb{Z}$. For each $\frac{k}{2} = n \in \mathbb{Z}$ the flow has a saddle with eigenvalues $\lambda_+ = 1$, $\lambda_- = -2$, and the corresponding unstable and stable invariant subspaces for the linearized flow are given by the spans of $(1, \lambda_+) = (1, 1)$, respectively $(1, \lambda_-) = (1, -2)$. In a neighborhood $V \ni v_n = (n\pi, 0)$ one can define the 1-dimensional invariant unstable manifold

$$W_n^u = \{(x, y) \in V \mid \Phi_s(x, y) \to v_n \text{ as } s \to -\infty\}$$

and the 1-dimensional invariant stable manifold

$$W_n^{st} = \{(x, y) \in V \mid \Phi_s(x, y) \to v_n \text{ as } s \to \infty\}$$

which are tangent at v_n to the invariant subspaces of the linearized flow. In particular, for each n one can parameterize the stable manifold W_n^{st} by

$$\phi_{n,\alpha}(s) = n\pi - \alpha e^{-2s} + O(e^{-6s})$$

with the parameter α determining all the coefficients of higher order. This proves the existence of the Q_α in Lemma 4.1.1. One can show that if the parameter α satisfies $\alpha > 0$ then $\phi_{n,\alpha}(s)$ lies on the branch of the stable manifold which stays below $n\pi$ for all $s \in \mathbb{R}$, i.e., $\phi_{n,\alpha}(s) < n\pi$ for all $s \in \mathbb{R}$. If $\alpha = 0$ then $\phi_{n,\alpha}(s) = n\pi$ for all s. Finally, if $\alpha < 0$ then $\phi_{n,\alpha}(s) > n\pi$ for all $s \in \mathbb{R}$. Different choices of α correspond to translations in s along the respective branches of the stable manifold, which is what we mean by uniqueness in the statement of Lemma 4.1.1.

To prove the existence and uniqueness of α_0, we note that an analysis of the phase portrait shows that any trajectory with $\alpha > 0$ must have crossed the y-axis at some finite time s_0, and once it has crossed can never do so again. Note that if the parameter α satisfies $\alpha < 0$ then the trajectory can never cross the y-axis since in this case $\phi_{n,\alpha}(s) > n\pi$ for all $s \in \mathbb{R}$.

Now, fix any $\alpha_+ > 0$ and $\alpha_- < 0$. Passing back to the original variables we have three trajectories

$$\begin{aligned} Q_{n,\alpha_\pm}(r) &= n\pi - \alpha_\pm r^{-2} + O(r^{-6}) \\ Q_{n,0}(r) &= n\pi \end{aligned} \tag{4.1.5}$$

where $Q_{n,\alpha_+}(r)$ is a trajectory on the branch of W_n^{st} that increases to $n\pi$ as $r \to \infty$, and $Q_{n,\alpha_-}(r)$ is a trajectory on the branch of W_n^{st} that decreases to $n\pi$ as $r \to \infty$. Since the trajectory Q_{n,α_+} satisfies $Q_{n,\alpha_+}(r_0) = 0$ for some $r_0 > 0$, we can obtain our solution $Q_n(r)$

to (4.1.1) which satisfies (4.1.2) by rescaling $Q_{n,\alpha_+}(r)$ by $\lambda_0 > 0$, i.e., we set

$$Q_n(r) = Q_n^+(r/\lambda_0) = n\pi - \lambda_0^2 \alpha_+ r^{-2} + O(r^{-6})$$

where we note that $\lambda_0 > 0$ is uniquely chosen to ensure that the boundary condition $Q_n(1) = 0$ is satisfied. Note that such rescalings amount to a translation in the s-variable above. Setting $\alpha_0 := \lambda_0^2 \alpha_+$, the unique harmonic map $(Q_n(r), 0) \in \mathcal{E}_n$ therefore satisfies

$$Q_n(r) = n\pi - \alpha_0 r^{-2} + O(r^{-6}) \tag{4.1.6}$$

as claimed above.

4.1.2 5d Reduction

In the higher topological classes, $\mathcal{E}_n$ for $n \geq 1$, we linearize about $Q = Q_n$ by writing

$$\psi = Q + \varphi$$

where $Q = Q_n$ is the unique harmonic map and energy minimizer in $\mathcal{E}_n$. If $\vec{\psi} \in \mathcal{E}_n$ is a wave map, then $\vec{\varphi} \in \mathcal{H}_n$ satisfies

$$\begin{aligned}
&\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{2\cos(2Q)}{r^2}\varphi = Z(r, \varphi) \\
&Z(r, \varphi) := \frac{\cos(2Q)(2\varphi - \sin(2\varphi)) + 2\sin(2Q)\sin^2(\varphi)}{r^2} \\
&\varphi(t, 1) = 0, \ \varphi(t, \infty) = 0 \quad \forall t, \quad \vec{\varphi}(0) = (\psi_0 - Q, \psi_1)
\end{aligned} \tag{4.1.7}$$

The standard $5d$ reduction is given by setting $ru := \varphi$ and then $\vec{u}$ solves

$$\begin{aligned}
&u_{tt} - u_{rr} - \frac{4}{r}u_r + V(r)u = F(r,u) + G(r,u), \quad r \geq 1 \\
&u(t,1) = 0 \quad \forall t, \quad \vec{u}(0) = (u_0, u_1) \\
&V(r) := \frac{2(\cos(2Q) - 1)}{r^2} \\
&F(r,u) := 2\sin(2Q)\frac{\sin^2(ru)}{r^3} \\
&G(r,u) := \cos(2Q)\frac{(2ru - \sin(2ru))}{r^3}
\end{aligned} \tag{4.1.8}$$

We will consider radial initial data $(u_0, u_1) \in \mathcal{H} := \dot{H}_0^1 \times L^2(\mathbb{R}_*^5)$ where $\mathbb{R}_*^5 = \mathbb{R}^5 \setminus B(0,1)$,

$$\|(u_0, u_1)\|_{\mathcal{H}}^2 := \int_1^\infty ((\partial_r u_0(r))^2 + u_1^2(r))\, r^4\, dr \tag{4.1.9}$$

where $\dot{H}_0^1(\mathbb{R}_*^5)$ is the completion under the first norm on the right-hand side above of all smooth radial compactly supported functions on $\{x \in \mathbb{R}^5 \mid |x| > 1\}$. We remark that the potential

$$V(r) := \frac{2(\cos(2Q) - 1)}{r^2} \tag{4.1.10}$$

is real-valued, radial, bounded, smooth and by (4.1.6) satisfies

$$V(r) = O(r^{-6}) \text{ as } r \to \infty \tag{4.1.11}$$

Also, by (4.1.6) we can deduce that

$$\begin{aligned}
|F(r,u)| &\lesssim r^{-3}\,|u|^2 \\
|G(r,u)| &\lesssim |u|^3
\end{aligned} \tag{4.1.12}$$

For the remainder of the chapter we deal exclusively with $u(t,r)$ in $\mathbb{R}^5_*$ rather than the equivariant wave map angle $\psi(t,r)$. In fact, one can check that the Cauchy problem (4.0.2) with data $(\psi_0, \psi_1) \in \mathcal{E}_n$ is equivalent to (4.1.8). To see this let $\vec{\psi} \in \mathcal{E}_n$ and set

$$r\vec{u}(r) := (\psi_0(r) - Q_n(r), \psi_1(r)) \tag{4.1.13}$$

We claim that

$$\|\vec{\psi}\|_{\mathcal{H}_n} \simeq \|\vec{u}\|_{\mathcal{H}} \tag{4.1.14}$$

Indeed, setting $\varphi(r) := \psi_0(r) - Q_n(r)$ we see that

$$\int_1^\infty \varphi_r^2(r) r^2 \, dr \simeq \int_1^\infty u_r^2(r) r^4 \, dr \tag{4.1.15}$$

via Hardy's inequality and the relations

$$\varphi_r = ru_r + u = ru_r + \frac{\varphi}{r}$$

Therefore for each topological class $\mathcal{E}_n$ the map

$$\vec{\psi} \mapsto \frac{1}{r}(\psi_0(r) - Q_n(r), \psi_1(r))$$

is an isomorphism between the spaces $\mathcal{E}_n$ and $\mathcal{H}$ respectively.

In particular, we will prove the analogous formulation of Theorem 4.0.3 in the u-setting rather than the original one. Scattering in this context will mean that we approach a solution of (4.1.8) but with $V = F = G = 0$.

4.2 Small Data Theory and Concentration Compactness

4.2.1 Global existence and scattering for data with small energy

Here we give a brief review of the small data well-posedness theory for (4.1.8) that was developed in the previous chapter; see also [53]. As usual the small data theory rests on Strichartz estimates for the inhomogeneous linear, radial exterior wave equation with the potential V,

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{4}{r}u_r + V(r)u = h \\ &u(t,1) = 0 \quad \forall t \\ &\vec{u}(0) = (u_0, u_1) \in \mathcal{H} \end{aligned} \tag{4.2.1}$$

where $V(r)$ is as in (4.1.10). We define $S_V(t)$ to be the exterior linear propagator associated to (4.2.1). The conserved energy associated to (4.2.1) with $h = 0$ is given by

$$\mathcal{E}_L(u, u_t) = \frac{1}{2}\int_1^\infty (u_t^2 + u_r^2 + V(r)u^2)\, r^4\, dr$$

This energy has an important positive definiteness property: one has

$$\mathcal{E}_L(u, u_t) = \frac{1}{2}(\|u_t\|_2^2 + \langle Hu|u\rangle), \quad H = -\Delta + V \tag{4.2.2}$$

It is shown in [5, 53] that H is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^5_*)$ (with a Dirichlet condition at $r = 1$), and moreover, that the threshold energy zero is regular; in other words, if $Hf = 0$ where $f \in H^2 \cap \dot{H}^1_0$ then $f = 0$. It is now standard to conclude from this spectral information that for some constants $0 < c < C$,

$$c\|f\|^2_{\dot{H}^1_0} \leq \langle Hf|f\rangle \leq C\|f\|^2_{\dot{H}^1_0} \quad \forall\, f \in \dot{H}^1_0(\mathbb{R}^5_*) \tag{4.2.3}$$

We sometimes write $\|\vec{u}\|_{\mathcal{E}}^2 := \mathcal{E}_L(\vec{u})$, which satisfies

$$\|\vec{u}\|_{\mathcal{E}} \simeq \|\vec{u}\|_{\mathcal{H}} \quad \forall \vec{u} \in \mathcal{H} \tag{4.2.4}$$

In what follows we say a triple (p, q, γ) is admissible if

$$\begin{aligned} &p > 2, \;\; q \geq 2 \\ &\frac{1}{p} + \frac{5}{q} = \frac{5}{2} - \gamma \\ &\frac{1}{p} + \frac{2}{q} \leq 1 \end{aligned}$$

For the free exterior $5d$ wave, i.e., the case $V = 0$ in (4.2.1), Strichartz estimates were established in [33]. Although the estimates in [33] hold in more general exterior settings, we state only the specific example of these estimates that we need here.

Proposition 4.2.1. *[33] Let (p, q, γ) and (r, s, ρ) be admissible triples. Then any solution $\vec{v}(t)$ to*

$$\begin{aligned} &v_{tt} - v_{rr} - \frac{4}{r} v_r = h \\ &\vec{v}(0) = (f, g) \in \mathcal{H}(\mathbb{R}^5_*) \\ &v(t, 1) = 0 \quad \forall t \in \mathbb{R} \end{aligned} \tag{4.2.5}$$

with radial initial data satisfies

$$\| \, |\nabla|^{-\gamma} \, \nabla v\|_{L^p_t L^q_x} \lesssim \|(f, g)\|_{\mathcal{H}} + \| \, |\nabla|^{\rho} \, h\|_{L^{r'}_t L^{s'}_x} \tag{4.2.6}$$

In the previous chapter, the author and Wilhelm Schlag showed that in fact the same family of Strichartz estimates hold for (4.2.1).

Proposition 4.2.2. *Let (p, q, γ) and (r, s, ρ) be admissible triples. Then any solution $\vec{u}(t)$*

to (4.2.1) *with radial initial data satisfies*

$$\| \, |\nabla|^{-\gamma} \nabla u\|_{L^p_t L^q_x} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \| \, |\nabla|^{\rho} \, h\|_{L^{r'}_t L^{s'}_x} \tag{4.2.7}$$

With these Strichartz estimates the following small data, global well-posedness theory for (4.1.8) follows from the standard contraction argument.

Proposition 4.2.3. *The exterior Cauchy problem for* (4.1.8) *is globally well-posed in* $\mathcal{H} := \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$. *Moreover, a solution* u *scatters as* $t \to \infty$ *to a free wave, i.e., a solution* $\vec{u}_L \in \mathcal{H}$ *of*

$$\Box u_L = 0, \; r \geq 1, \; u_L(t,1) = 0, \; \forall t \geq 0 \tag{4.2.8}$$

if and only if $\|u\|_{\mathcal{S}} < \infty$ *where* $\mathcal{S} = L^3_t([0,\infty); L^6_x(\mathbb{R}^5_*))$. *In particular, there exists a constant* $\delta > 0$ *small so that if* $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, *then* u *scatters to free waves as* $t \to \pm\infty$.

Remark 7. We remark that in [53, Theorem 1.2], the conclusions of Proposition 4.2.3 were phrased in terms of the original wave map angle ψ where here the result is phased in terms of $u(t,r) := \frac{1}{r}(\psi(t,r) - Q_n(r))$. As we saw in Section 4.1 this passage to the $u-$formulation is allowed since the map $\vec{u} = \frac{1}{r}(\psi - Q_n, \psi_t)$ is an isomorphism between the energy class $\mathcal{E}_n$ and $\mathcal{H} := \dot{H_0}^1 \times L^2(\mathbb{R}^5_*)$, respectively.

We refer the reader to the previous chapter for the details regarding Proposition 4.2.2 and Proposition 4.2.3. For convenience, we recall how the scattering norm $L^3_t L^6_x$ is obtained. By Proposition 4.2.2, solutions to (4.2.1) satisfy

$$\|u\|_{L^3_t(\mathbb{R}; \dot{W}^{\frac{1}{2},3}_x(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{L^1_t L^2_x + L^{\frac{3}{2}}_t L^{\frac{30}{17}}_x} \tag{4.2.9}$$

As in the previous chapter, we claim the embedding $\dot{W}^{\frac{1}{2},3}_x \hookrightarrow L^6_x$ for radial functions in $r \geq 1$

in $\mathbb{R}^5_*$. Indeed, one checks via the fundamental theorem of calculus that $\dot{W}^{1,3}_x \hookrightarrow L^\infty_x$. More precisely,

$$|f(r)| \leq r^{-\frac{2}{3}} \|f\|_{\dot{W}^{1,3}_x} \tag{4.2.10}$$

Interpolating this with the embedding $L^3 \hookrightarrow L^3$ we obtain the claim. From (4.2.9) we infer the weaker Strichartz estimate

$$\|u\|_{L^3_t(\mathbb{R};L^6_x(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|h\|_{L^1_t(\mathbb{R};L^2_x(\mathbb{R}^5_*))+L^{\frac{3}{2}}_t(\mathbb{R};L^{\frac{30}{17}}_x(\mathbb{R}^5_*))} \tag{4.2.11}$$

which suffices for our purposes. Indeed, using (4.2.11) on the nonlinear equation (4.1.8) gives

$$\begin{aligned}
\|u\|_{L^3_t(\mathbb{R};L^6_x(\mathbb{R}^5_*))} &\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|F(r,u)+G(r,u)\|_{L^1_tL^2_x+L^{\frac{3}{2}}_tL^{\frac{30}{17}}_x} \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|r^{-3}u^2\|_{L^{\frac{3}{2}}_tL^{\frac{30}{17}}_x} + \|u^3\|_{L^1_tL^2_x} \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|r^{-3}\|_{L^\infty_tL^{\frac{30}{7}}_x}\|u^2\|_{L^{\frac{3}{2}}_tL^3_x} + \|u\|^3_{L^3_tL^6_x} \\
&\lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \|u\|^2_{L^3_tL^6_x} + \|u\|^3_{L^3_tL^6_x}
\end{aligned}$$

where we have estimated the size of the nonlinearity $h = F(r,u) + G(r,u)$ using (4.1.12). Thus for small initial data, $\|\vec{u}(0)\|_{\mathcal{H}} < \delta$, we obtain the global a priori estimate

$$\|u\|_{L^3_t(\mathbb{R};L^6_x(\mathbb{R}^5_*))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} \lesssim \delta \tag{4.2.12}$$

from which the small data scattering statement in Proposition 4.2.3 follows.

4.2.2 Concentration Compactness

We now formulate the concentration compactness principle relative to the linear wave equation with a potential, see (4.2.1) with $h = 0$. This is what we mean by "free" in Lemma 4.2.4. Note that this is a different meaning of "free" than the one used in Proposition 4.2.3. However, observe that any solution to (4.2.1) with $h = 0$, which is in $L_t^3 L_x^6$ must scatter to "free" waves, where "free" is in the sense of Proposition 4.2.3.

Lemma 4.2.4. *Let $\{u_n\}$ be a sequence of free radial waves bounded in $\mathcal{H} = \dot{H}_0^1 \times L^2(\mathbb{R}_*^5)$. Then after replacing it by a subsequence, there exist a sequence of free solutions v^j bounded in $\mathcal{H}$, and sequences of times $t_n^j \in \mathbb{R}$ such that for γ_n^k defined by*

$$u_n(t) = \sum_{1 \le j < k} v^j(t + t_n^j) + \gamma_n^k(t) \tag{4.2.13}$$

we have for any $j < k$,

$$\vec{\gamma}_n^k(-t_n^j) \rightharpoonup 0 \tag{4.2.14}$$

weakly in $\mathcal{H}$ as $n \to \infty$, as well as

$$\lim_{n \to \infty} |t_n^j - t_n^k| = \infty \tag{4.2.15}$$

and the errors γ_n^k vanish asymptotically in the sense that

$$\lim_{k \to \infty} \limsup_{n \to \infty} \|\gamma_n^k\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R} \times \mathbb{R}_*^5)} = 0 \quad \forall\ \frac{10}{3} < p < \infty \tag{4.2.16}$$

Finally, one has orthogonality of the free energy with a potential, cf. (4.2.4),

$$\|\vec{u}_n\|_{\mathcal{E}}^2 = \sum_{1\le j<k} \|\vec{v}^j\|_{\mathcal{E}}^2 + \|\vec{\gamma}_n^k\|_{\mathcal{E}}^2 + o(1) \tag{4.2.17}$$

as $n \to \infty$.

The proof is essentially identical with that of Lemma 3.3.2 in the previous chapter. In fact, instead of the Strichartz estimates for $\Box$ in $\mathbb{R}^5_*$ we use those from Proposition 4.2.2 above.

Applying this decomposition to the nonlinear equation requires a perturbation lemma which we now formulate. All spatial norms are understood to be on $\mathbb{R}^5_*$. The exterior propagator $S_V(t)$ is as above.

Lemma 4.2.5. *There are continuous functions* $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ *such that the following holds: Let* $I \subset \mathbb{R}$ *be an open interval (possibly unbounded),* $u, v \in C(I; \dot{H}^1_0) \cap C^1(I; L^2)$ *radial functions satisfying for some* $A > 0$

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|\vec{v}\|_{L^\infty(I;\mathcal{H})} + \|v\|_{L^3_t(I;L^6_x)} \le A$$
$$\|\mathrm{eq}(u)\|_{L^1_t(I;L^2_x)} + \|\mathrm{eq}(v)\|_{L^1_t(I;L^2_x)} + \|w_0\|_{L^3_t(I;L^6_x)} \le \varepsilon \le \varepsilon_0(A),$$

where $\mathrm{eq}(u) := (\Box + V)u - F(r,u) - G(r,u)$ *in the sense of distributions, and* $\vec{w}_0(t) := S_V(t-t_0)(\vec{u}-\vec{v})(t_0)$ *with* $t_0 \in I$ *arbitrary but fixed. Then*

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty_t(I;\mathcal{H})} + \|u - v\|_{L^3_t(I;L^6_x)} \le C_0(A)\varepsilon.$$

In particular, $\|u\|_{L^3_t(I;L^6_x)} < \infty$.

The proof of this lemma is essentially identical with that of Lemma 3.3.3 in the previous chapter. The only difference is that we use the propagator S_V instead of S_0.

4.2.3 Critical Element

We now turn to the proof of Theorem 4.0.3 following the concentration compactness methodology from [36, 37]. We begin by noting that Theorem 4.0.3 was proved in the regime of all energies slightly above the ground state energy $\mathcal{E}(Q_n, 0)$ in Theorem 3.1.2, see also Proposition 4.2.3 above. As usual, we assume that Theorem 4.0.3 fails and construct a *critical element* which is a non-scattering solution of minimal energy, E_*, which is necessarily strictly bigger than $\mathcal{E}(Q_n, 0)$. This is done in the following proposition on the level of the semi-linear formulation given by (4.1.8).

Proposition 4.2.6. *Suppose that Theorem 4.0.3 fails. Then there exists a nonzero energy solution to* (4.1.8) *(referred to as a critical element)* $\vec{u}_*(t)$ *for* $t \in \mathbb{R}$ *with the property that the trajectory*

$$\mathcal{K} := \{\vec{u}_*(t) \mid t \in \mathbb{R}\} \tag{4.2.18}$$

is pre-compact in $\mathcal{H}(\mathbb{R}^5_*)$*.*

Proof. Suppose that the theorem fails. Then there exists a bounded sequence of $\vec{\psi}_j = (\psi_{0,j}, \psi_{1,j}) \in \mathcal{E}_n$ with

$$\mathcal{E}(\vec{\psi}_j) \to E_* > 0 \tag{4.2.19}$$

and a bounded sequence $\vec{u}_j := (u_{0,j}, u_{1,j}) \in \mathcal{H}$ where $\vec{u}_j(r) = \frac{1}{r}(\vec{\psi}_j(r) - (Q(r), 0))$ with

$$\|u_j\|_{\mathcal{S}} \to \infty$$

where u_n denotes the global evolution of $\vec{u}_n$ of (4.1.8). We may assume that E_* is minimal with this property. Applying Lemma 4.2.4 to the free evolutions S_V of $\vec{u}_j(0)$ yields free waves v^i and times t_j^i as in (4.2.13). Let U^i be the nonlinear profiles of (v^i, t_j^i), i.e., those energy solutions of (4.1.8) which satisfy

$$\lim_{t \to t_\infty^i} \|\vec{v}^i(t) - \vec{U}^i(t)\|_{\mathcal{H}} \to 0$$

where $\lim_{j\to\infty} t_j^i = t_\infty^i \in [-\infty, \infty]$. The U^i exist locally around $t = t_\infty^i$ by the local existence and scattering theory, see Proposition 4.2.3. Note that here and throughout we are using the equivalence of norms in (4.2.4). Locally around $t = 0$ one has the following *nonlinear profile decomposition*

$$u_j(t) = \sum_{i<k} U^i(t + t_j^i) + \gamma_j^k(t) + \eta_j^k(t) \tag{4.2.20}$$

where $\|\vec{\eta}_j^k(0)\|_{\mathcal{H}} \to 0$ as $j \to \infty$. Now suppose that either there are two non-vanishing v^j, say v^1, v^2, or that

$$\limsup_{k\to\infty} \limsup_{j\to\infty} \|\vec{\gamma}_j^k\|_{\mathcal{E}} > 0 \tag{4.2.21}$$

Note that the left-hand side does not depend on time since γ_j^k is a free wave. By the minimality of E_* and the orthogonality of the nonlinear energy–which follows from (4.2.15) and (4.2.14)–each U^i is a global solution and scatters with $\|U^i\|_{L_t^3 L_x^6} < \infty$.

We now apply Lemma 4.2.5 on $I = \mathbb{R}$ with $u = u_j$ and

$$v(t) = \sum_{i<k} U^i(t + t_j^i) \tag{4.2.22}$$

That $\|\mathrm{eq}(v)\|_{L_t^1 L_x^2}$ is small for large n follows from (4.2.15). To see this, note that with

$N(v) := F(r,v) + G(r,v)$,

$$\begin{aligned}\mathrm{eq}(v) &= (\Box + V)v - F(r,v) - G(r,v)\\ &= \sum_{i<k} N(U^i(t+t^i_j)) - N\big(\sum_{i<k} U^i(t+t^i_j)\big)\end{aligned}$$

The difference on the right-hand side here only consists of terms which involve at least one pair of distinct i, i'. But then $\|\mathrm{eq}(v)\|_{L^1_t L^2_x} \to 0$ as $j \to \infty$ by (4.2.15). In order to apply Lemma 4.2.5 it is essential that

$$\limsup_{j\to\infty} \big\| \sum_{i<k} U^i(t+t^i_j) \big\|_{L^3_t L^6_x} \le A < \infty \tag{4.2.23}$$

uniformly in k, which follows from (4.2.15), (4.2.17), and Proposition 4.2.3. The point here is that the sum can be split into one over $1 \le i < i_0$ and another over $i_0 \le i < k$. This splitting is performed in terms of the energy, with i_0 being chosen such that for all $k > i_0$

$$\limsup_{j\to\infty} \sum_{i_0 \le i < k} \|\vec{U}^i(t^i_j)\|^2_{\mathcal{H}} \le \varepsilon_0^2 \tag{4.2.24}$$

where ε_0 is fixed such that the small data result of Proposition 4.2.3 applies. Clearly, (4.2.24) follows from (4.2.17). Using (4.2.15) as well as the small data scattering theory one now obtains

$$\begin{aligned}\limsup_{j\to\infty} \Big\| \sum_{i_0\le i<k} U^i(\cdot + t^i_j) \Big\|^3_{L^3_t L^6_x} &= \sum_{i_0\le i<k} \big\| U^i(\cdot) \big\|^3_{L^3_t L^6_x}\\ &\le C \limsup_{j\to\infty} \Big(\sum_{i_0 \le i<k} \|\vec{U}^i(t^i_j)\|^2_{\mathcal{H}} \Big)^{\frac{3}{2}}\end{aligned} \tag{4.2.25}$$

with an absolute constant C. This implies (4.2.23), uniformly in k.

Hence one can take k and j so large that Lemma 4.2.5 applies to (4.2.20) whence

$$\limsup_{j\to\infty} \|u_j\|_{L_t^3 L_x^6} < \infty$$

which is a contradiction. Thus, there can be only one nonvanishing v^i, say v^1, and moreover

$$\limsup_{j\to\infty} \|\vec{\gamma}_j^2\|_{\mathcal{H}} = 0 \tag{4.2.26}$$

Thus, if we let $\vec{\psi}^1$ be the wave map angle associated to $\vec{U}^1$ then we have $\mathcal{E}(\vec{\psi}^1) = E_*$. By the preceding, necessarily

$$\|U^1\|_{L_t^3 L_x^6} = \infty \tag{4.2.27}$$

Therefore, $U^1 =: u_*$ is the desired critical element. Suppose that

$$\|u_*\|_{L_t^3([0,\infty);L_x^6)} = \infty \tag{4.2.28}$$

Then we claim that

$$\mathcal{K}_+ := \{\vec{u}_*(t) \mid t \geq 0\}$$

is precompact in $\mathcal{H}$. If not, then there exists $\delta > 0$ so that for some infinite sequence $t_n \to \infty$ one has

$$\|\vec{u}_*(t_n) - \vec{u}_*(t_m)\|_{\mathcal{H}} > \delta \quad \forall\, n > m \tag{4.2.29}$$

Applying Lemma 4.2.4 to $U^1(t_n)$ one concludes via the same argument as before based on the minimality of E_* and (4.2.27) that

$$\vec{u}_*(t_n) = \vec{v}(\tau_n) + \vec{\gamma}_n(0) \tag{4.2.30}$$

where $\vec{v}$, $\vec{\gamma}_n$ are free waves in $\mathcal{H}$, and τ_n is some sequence in $\mathbb{R}$. Moreover, $\|\vec{\gamma}_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. If $\tau_n \to \tau_\infty \in \mathbb{R}$, then (4.2.30) and (4.2.29) lead to a contradiction. If $\tau_n \to \infty$, then

$$\|v(\cdot + \tau_n)\|_{L_t^3([0,\infty);L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies via the local wellposedness theory that $\|u_*(\cdot + t_n)\|_{L_t^3([0,\infty);L_x^6)} < \infty$ for all large n, which is a contradiction to (4.2.28). If $\tau_n \to -\infty$, then

$$\|v(\cdot + \tau_n)\|_{L_t^3((-\infty,0];L_x^6)} \to 0 \qquad \text{as } n \to \infty$$

implies that $\|u_*(\cdot + t_n)\|_{L_t^3((-\infty,0];L_x^6)} < C < \infty$ for all large n where C is some fixed constant. Passing to the limit yields a contradiction to (4.2.27) and (4.2.29) is seen to be false, concluding the proof of compactness of $\mathcal{K}_+$.

Finally, we need to make sure that $u_*(t)$ is precompact with respect to both $t \to +\infty$ and $t \to -\infty$, see (4.2.18). To achieve the latter, we extract another critical element from the sequence

$$\{\vec{u}_*(n)\}_{n=1}^{\infty} \subset \mathcal{H}$$

Indeed, by the compactness that we have already established we can pass to a strong limit $\vec{u}_n \to \vec{u}_\infty$ in $\mathcal{H}$, which has the same energy E_*. By construction, the nonlinear evolution (4.1.8) with data $\vec{u}_\infty$ has infinite $L_t^3 L_x^6$-norm in both time directions. Therefore, the same compactness argument as above concludes the proof. Indeed, the solution given by $\vec{u}_\infty$ is now our desired critical element. $\square$

In Section 4.4 we will show that u_* cannot exist. In order to do so, we need to develop another tool for the linear evolution.

4.3 The linear external energy estimates in $\mathbb{R}^5$

We now turn to our main new ingredient from the linear theory, which is Proposition 4.3.1. In order to motivate this result, we first review the analogous statements in dimensions $d = 1$ and $d = 3$.

Suppose $w_{tt} - w_{xx} = 0$ with smooth energy data $(w(0), \dot{w}(0)) = (f, g)$. Then by local energy conservation

$$\int_{x>a} \frac{1}{2}(w_t^2 + w_x^2)(0,x)\,dx - \int_{x>T+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx = \frac{1}{2}\int_0^T (w_t + w_x)^2(t, t+a)\,dt$$

for any $T > 0$ and $a \in \mathbb{R}$. Since $(\partial_t - \partial_x)(w_t + w_x) = 0$, we have that

$$\begin{aligned}
&\frac{1}{2}\int_0^T (w_t + w_x)^2(t, t+a)\,dt = \frac{1}{2}\int_0^T (w_t + w_x)^2(0, a+2t)\,dt \\
&= \frac{1}{4}\int_a^{a+2T} (w_t + w_x)^2(0,x)\,dx = \frac{1}{4}\int_a^{a+2T} (f_x + g)^2(x)\,dx
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{x>a} \frac{1}{2}(w_t^2 + w_x^2)(0,x)\,dx - \lim_{T\to\infty}\int_{x>T+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx \\
&= \frac{1}{4}\int_a^{\infty} (f_x + g)^2(x)\,dx
\end{aligned}$$

and thus

$$\begin{aligned}
&\min_{\pm}\Big[\int_{x>a} \frac{1}{2}((f_x^2 + g^2)(0,x)\,dx - \lim_{T\to\pm\infty}\int_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx\Big] \\
&\le \frac{1}{4}\int_a^{\infty} (f_x^2 + g^2)(x)\,dx
\end{aligned}$$

whence

$$\max_{\pm} \lim_{T\to\pm\infty} \int_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T,x)\,dx \geq \frac{1}{4}\int_a^\infty (f_x^2 + g^2)(x)\,dx \tag{4.3.1}$$

Here we used that $t \mapsto -t$ leaves f unchanged, but turns g into $-g$.

Given $\Box u = 0$ radial in three dimensions, $w(t,r) = ru(t,r)$ solves $w_{tt} - w_{rr} = 0$. Consequently, (4.3.1) gives the following estimate from [22, Lemma 4.2], see also [24, 23, 25]: for any $a \geq 0$ one has

$$\begin{aligned} &\max_{\pm} \lim_{T\to\pm\infty} \int_{r>|T|+a} \frac{1}{2}((ru)_r^2 + (ru_t)^2)(T,r)\,dr \\ &\geq \frac{1}{4}\int_{r>a} ((rf)_r^2 + (rg)^2)(r)\,dr \end{aligned} \tag{4.3.2}$$

where $u(0) = f$, $\dot{u}(0) = g$. The left-hand side of (4.3.2) equals

$$\max_{\pm} \lim_{T\to\pm\infty} \int_{r>|T|+a} \frac{1}{2}(u_r^2 + u_t^2)(T,r)\,r^2 dr \tag{4.3.3}$$

by the standard dispersive properties of the wave equation. The right-hand side, on the other hand, exhibits the following dichotomy: if $a = 0$, then it equals half of the full energy

$$\frac{1}{4}\int_0^\infty (f_r^2 + g^2)(r)\,r^2 dr$$

However, if $a > 0$, then integration by parts shows that it equals (ignoring the constant from the spherical measure in $\mathbb{R}^3$)

$$\frac{1}{4}\int_{r>a} (f_r^2 + g^2)(r) r^2\,dr - \frac{1}{4} a f^2(a) = \frac{1}{4}\|\pi_a^\perp(f,g)\|^2_{\dot{H}^1\times L^2(r>a)}$$

where $\pi_a^\perp = \mathrm{Id} - \pi_a$ and π_a is the orthogonal projection onto the line

$$\{(cr^{-1}, 0) \mid c \in \mathbb{R}\} \subset \dot{H}^1 \times L^2(r > a).$$

The appearance of this projection is natural, in view of the fact that the Newton potential r^{-1} in $\mathbb{R}^3$ yields an explicit solution to $\Box u = 0$, $u(0,r) = r^{-1}$, $\dot{u}(0,r) = 0$: indeed, one has $u(r,t) = r^{-1}$ in $r > |t| + a$ for which (4.3.3) vanishes. Since $r^{-1} \notin L^2(r > 1)$ no projection appears in the time component. In contrast, the Newton potential in $\mathbb{R}^5$, viz. r^{-3}, does lie in $H^1(r > a)$ for any $a > 0$. This explains why in $\mathbb{R}^5$ we need to project away from a plane rather than a line, see (4.3.4) and the end of the proof of Proposition 4.3.1.

Proposition 4.3.1. *Let $\Box u = 0$ in $\mathbb{R}^{1+5}_{t,x}$ with radial data $(f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$. Then with some absolute constant $c > 0$ one has for every $a > 0$*

$$\max_{\pm} \limsup_{t \to \pm\infty} \int_{r > a + |t|}^{\infty} (u_t^2 + u_r^2)(t,r) r^4 \, dr \geq c \|\pi_a^\perp (f,g)\|^2_{\dot{H}^1 \times L^2(r > a)} \tag{4.3.4}$$

where $\pi_a = \mathrm{Id} - \pi_a^\perp$ is the orthogonal projection onto the plane

$$\{(c_1 r^{-3}, c_2 r^{-3}) \mid c_1, c_2 \in \mathbb{R}\}$$

in the space $\dot{H}^1 \times L^2(r > a)$. The left-hand side of (4.3.4) vanishes for all data in this plane.

Remark 8. We note that by finite propagation speed Proposition 4.3.1 with $a > 1$ holds as well for solutions $v(t)$ to the free radial wave equation in $\mathbb{R} \times \mathbb{R}^5_*$ with a Dirichlet boundary condition at $r = 1$.

$$\begin{aligned} &v_{tt} - v_{rr} - \frac{4}{r} v_r = 0 \\ &\vec{v}(0) = (f,g) \\ &v(t,1) = 0 \quad \forall t \in \mathbb{R} \end{aligned} \tag{4.3.5}$$

Proof. By the basic energy estimate we may assume that f, g are compactly supported and smooth, say. We first note that it suffices to deal with data $(f, 0)$ and $(0, g)$ separately. Indeed, reversing the time direction keeps the former fixed, whereas the latter changes to $(0, -g)$. This implies that we may choose the time-direction so as to render the bilinear interaction term between the two respective solutions nonnegative on the left-hand side of (4.3.4).

We begin with data $(f, 0)$ and set $w(t, r) := r^{-1}(r^3 u(t, r))_r$, see [38]. Throughout this proof, the singularity at $r = 0$ plays no role due to the fact that $r \geq a + |t| \geq a > 0$. Then

$$w_{tt} - w_{rr} = r^2 \partial_r \big(u_{tt} - u_{rr} - \frac{4}{r} u_r\big) + 3r\big(u_{tt} - u_{rr} - \frac{4}{r} u_r\big) = 0$$

From the d'Alembert formula,

$$\limsup_{t\to\infty} \int_{a+t}^{\infty} w^2(t, r)\, dr \geq \frac{1}{4} \int_a^{\infty} w^2(0, r)\, dr$$

which is the same as

$$\limsup_{t\to\infty} \int_{a+t}^{\infty} (r^2 u_r(t, r) + 3ru(t, r))^2\, dr \geq \frac{1}{4} \int_a^{\infty} (r^2 f'(r) + 3rf(r))^2\, dr \tag{4.3.6}$$

By our assumption on the data, we have the point wise bound

$$|u(t, r)| \leq Ct^{-2} \chi_{[R-t \leq r \leq R+t]}$$

for $t \geq 1$ and some large R. Hence, (4.3.6) equals

$$\limsup_{t\to\infty} \int_{a+t}^{\infty} u_r^2(t, r) r^4\, dr \geq \frac{1}{4} \Big(\int_a^{\infty} r^4 f'(r)^2\, dr - 3a^3 f(a)^3 \Big) \tag{4.3.7}$$

where we integrated by parts on the right-hand side. Finally, one checks that

$$\tilde{f}(r) := f(r) - \frac{a^3}{r^3} f(a)$$

is the orthogonal projection perpendicular to r^{-3} in $\dot{H}^1(r > a)$ in $\mathbb{R}^5$ and that it satisfies

$$\int_a^\infty r^4 \tilde{f}'(r)^2 \, dr = \int_a^\infty r^4 f'(r)^2 \, dr - 3a^3 f(a)^3$$

which agrees with the right-hand side of (4.3.7) and concludes the proof of (4.3.4) for data $(f, 0)$.

For data $(0, g)$ we use the new dependent variable

$$v(t, r) := \int_r^\infty s \partial_t u(t, s) \, ds \tag{4.3.8}$$

By direct differentiation and integration by parts one verifies that v solves the 3-dimensional radial wave equation, viz.

$$v_{tt} - v_{rr} - \frac{2}{r} v_r = 0$$

Moreover, $v_t(0, r) = 0$. From the exterior energy estimate in dim $= 3$, i.e., (4.3.2),

$$\limsup_{t \to \infty} \int_{a+t}^\infty ((rv)_t^2 + (rv)_r^2)(t, r) \, dr \geq \frac{1}{2} \int_a^\infty ((rv)_t^2 + (rv)_r^2)(0, r) \, dr \tag{4.3.9}$$

where we have used the fact that for data $(v_0, 0)$ or $(0, v_1)$ the estimate (4.3.2) holds in both time directions. By our assumption on the data and stationary phase

$$|v(t, r)| \leq C t^{-1} \chi_{[r \leq R+t]}, \quad |v_t(t, r)| \leq C t^{-2} \chi_{[r \leq R+t]}$$

Hence (4.3.9) reduces to

$$\limsup_{t\to\infty} \int_{a+t}^{\infty} v_r^2(t,r)r^2\,dr \geq \frac{1}{2}\int_a^{\infty} (rh'(r)+h(r))^2\,dr \tag{4.3.10}$$

where $h(r) := \int_r^\infty sg(s)\,ds$. Inserting (4.3.8) on the left-hand side and integrating by parts on the right-hand side yields

$$\begin{aligned}\limsup_{t\to\infty} \int_{a+t}^{\infty} 2u_t^2(t,r)r^4\,dr &\geq \int_a^{\infty} h'(r)^2r^2\,dr - ah^2(a)\\ &= \int_a^{\infty} g(r)^2r^4\,dr - a\Big(\int_a^{\infty} \rho g(\rho)\,d\rho\Big)^2\end{aligned} \tag{4.3.11}$$

Finally, the right-hand side here is $\|\tilde{g}\|^2_{L^2(r>a)}$ where

$$\tilde{g}(r) := g(r) - ar^{-3}\int_a^{\infty} sg(s)\,ds$$

is the orthogonal projection perpendicular to r^{-3} in $L^2(r>a)$ in $\mathbb{R}^5$.

For data $(r^{-3},0)$ the solution equals r^{-3} on $r>t+a\geq a>0$ since r^{-3} is the Newton potential in $\mathbb{R}^5$. Similarly, data $(0,r^{-3})$ produce the solution tr^{-3} on the same region. In both cases, the left-hand side of (4.3.4) vanishes. $\square$

4.4 Rigidity Argument

In this section we will complete the proof of Thereom 4.0.3 by showing that a critical element as constructed in Section 4.2 does not exist. In particular, we prove the following proposition:

Proposition 4.4.1 (Rigidity Property). *Let $\vec{u}(t)\in\mathcal{H} := \dot{H}_0^1\times L^2(\mathbb{R}^5_*)$ be a global solution to* (4.1.8) *and suppose that the trajectory*

$$K := \{\vec{u}(t) \mid t\in\mathbb{R}\}$$

is pre-compact in $\mathcal{H}$. Then $\vec{u}(t) \equiv (0,0)$.

First note that the pre-compactness of K immediately implies that the energy of $\vec{u}(t)$ on the exterior cone $\{r \geq R + |t|\}$ vanishes as $|t| \to \infty$.

Corollary 4.4.2. *Let $\vec{u}(t)$ be as in Proposition 4.4.1. Then for any $R \geq 1$ we have*

$$\|\vec{u}(t)\|_{\mathcal{H}(r \geq R+|t|)} \to 0 \quad \textit{as} \quad |t| \to \infty. \tag{4.4.1}$$

The proof of Proposition 4.4.1 will proceed in several steps. The rough outline is to first use Corollary 4.4.2 together with Proposition 4.3.1 to determine the precise asymptotic behavior of $u_0(r) = u(0,r)$ and $u_1(r) = u_t(0,r)$ as $r \to \infty$. Namely, we show that

$$\begin{aligned} &r^3 u_0(r) = \ell_o + O(r^{-3}) \quad \text{as} \quad r \to \infty \\ &r \int_r^\infty u_1(\rho)\rho \, d\rho = O(r^{-1}) \quad \text{as} \quad r \to \infty \end{aligned} \tag{4.4.2}$$

We will then argue by contradiction to show that $\vec{u}(t,r) = (0,0)$ is the only possible solution that has both a pre-compact trajectory and initial data satisfying (4.4.2).

4.4.1 Step 1

We use the exterior estimates for the free radial wave equation in Proposition 4.3.1 together with Corollary 4.4.2 to deduce the following inequality for the pre-compact trajectory $\vec{u}(t)$.

Lemma 4.4.3. *There exists $R_0 > 1$ such that for every $R \geq R_0$ and for all $t \in \mathbb{R}$ we have*

$$\begin{aligned} \|\pi_R^\perp \vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3} &\|\pi_R \vec{u}(t)\|^2_{\mathcal{H}(r \geq R)} \\ &+ R^{-11/3}\|\pi_R \vec{u}(t)\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R \vec{u}(t)\|^6_{\mathcal{H}(r\geq R)} \end{aligned} \tag{4.4.3}$$

where again $P(R) := \{(k_1 r^{-3}, k_2 r^{-3}) \mid k_1, k_2 \in \mathbb{R}, \, r > R\}$, π_R denotes the orthogonal projection onto $P(R)$ and ${\pi_R}^\perp$ denotes the orthogonal projection onto the orthogonal com-

plement of the plane $P(R)$ in $\mathcal{H}(r > R; \mathbb{R}^5_*)$. We note that (4.4.3) *holds with a constant that is uniform in $t \in \mathbb{R}$.*

In order to prove Lemma 4.4.3 we need a preliminary result concerning the nonlinear evolution for a modified Cauchy problem which is adapted to capture the behavior of our solution $\vec{u}(t)$ only on the exterior cone $\{(t, r) \mid r \geq R + |t|\}$. Since we will only consider the evolution – and in particular the vanishing property (4.4.1) – on the exterior cone we can, by finite propagation speed, alter the nonlinearity and the potential term in (4.1.8) on the interior cone $\{1 \leq r \leq R + |t|\}$ without affecting the flow on the exterior cone. In particular, we can make the potential and the nonlinearity small on the interior of the cone so that for small initial data we can treat the potential and nonlinearity as small perturbations.

With this in mind, for every $R > 1$ we define $Q_R(t, r)$ by setting

$$Q_R(t, r) := \begin{cases} Q(R + |t|) & \text{for } \ 1 \leq r \leq R + |t| \\ Q(r) & \text{for } \ r \geq R + |t| \end{cases} \tag{4.4.4}$$

Next, set

$$V_R(t, r) := \begin{cases} 2(R + |t|)^{-2}(\cos(2Q_R(t, r)) - 1) & \text{for } \ 1 \leq r \leq R + |t| \\ 2r^{-2}(\cos(2Q(r)) - 1) & \text{for } \ r \geq R + |t| \end{cases}$$

$$F_R(t, r, h) := \begin{cases} 2(R + |t|)^{-3} \sin(2Q_R(t, r)) \sin^2((R + |t|)h) & \text{for } \ 1 \leq r \leq R + |t| \\ 2r^{-3} \sin(2Q(r)) \sin^2(rh) & \text{for } \ r \geq R + |t| \end{cases}$$

$$G(r, h) := r^{-3} \cos(2Q(r))(2rh - \sin(2rh)) \quad \forall\, r \geq 1$$

Note that for R large enough we have, using (4.1.6) and (4.1.11) that

$$|V_R(t,r)| \lesssim \begin{cases} (R+|t|)^{-6} \quad \text{for} \quad 1 \le r \le R+|t| \\ r^{-6} \quad \text{for} \quad r \ge R+|t| \end{cases} \tag{4.4.5}$$

$$|F_R(t,r,h)| \lesssim \begin{cases} (R+|t|)^{-3} \, |h(t,r)|^2 \quad \text{for} \quad 1 \le r \le R+|t| \\ r^{-3} \, |h(t,r)|^2 \quad \text{for} \quad r \ge R+|t| \end{cases} \tag{4.4.6}$$

$$|G(r,h)| \lesssim |h(t,r)|^3 \quad \text{for} \quad r \ge 1, \quad \forall t \in \mathbb{R} \tag{4.4.7}$$

We consider the modified Cauchy problem in $\mathbb{R} \times \mathbb{R}^5_*$:

$$\begin{aligned} &h_{tt} - h_{rr} - \frac{4}{r} h_r = \mathcal{N}_R(t,r,h) \\ &\mathcal{N}_R(t,r,h) := -V_R(t,r)h + F_R(t,r,h) + G(r,h) \\ &h(1,t) = 0 \quad \forall t \in \mathbb{R} \\ &\vec{h}(0) = (h_0, h_1) \in \mathcal{H} \end{aligned} \tag{4.4.8}$$

Lemma 4.4.4. *There exists $R_0 > 0$ and there exists $\delta_0 > 0$ small enough so that for all $R > R_0$ and all initial data $\vec{h}(0) = (h_0, h_1) \in \mathcal{H}$ with*

$$\|\vec{h}(0)\|^2_{\mathcal{H}} \le \delta_0$$

there exists a unique global solution $\vec{h}(t) \in \mathcal{H}$ to (4.4.8). In addition $\vec{h}(t)$ satisfies

$$\|h\|_{L^3_t L^6_x(\mathbb{R} \times \mathbb{R}^5_*)} \lesssim \|\vec{h}(0)\|_{\mathcal{H}} \lesssim \delta_0 \tag{4.4.9}$$

Moreover, if we let $h_L(t) := S_0(t)\vec{h}(0) \in \mathcal{H}$ denote the free linear evolution, i.e., solution *to* (4.3.5)*, of the data* $\vec{h}(0)$ *we have*

$$\sup_{t\in\mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}} \lesssim R^{-11/3}\|\vec{h}(0)\|_{\mathcal{H}} + R^{-11/6}\|\vec{h}(0)\|_{\mathcal{H}}^2 + \|\vec{h}(0)\|_{\mathcal{H}}^3 \tag{4.4.10}$$

Remark 9. Note that for each $t \in \mathbb{R}$,

$$\mathcal{N}_R(t, r, h) = -V(r)h + F(r, h) + G(r, h) \quad \forall r \geq R + |t| \tag{4.4.11}$$

where $V(r)$, $F(r, h)$, and $G(r, h)$ are as in (4.1.8). By finite propagation speed it is then immediate that solutions to (4.4.8) and (4.1.8) agree on the exterior cone $\{(t, r) \mid r \geq R+|t|\}$.

Proof of Lemma 4.4.4. The small data well-posedness theory, including estimate (4.4.9), follows from the usual contraction and continuity arguments based on the Strichartz estimates in Proposition 4.2.1. To prove (4.4.10) we note that by the Duhamel formula and Strichartz estimates we have

$$\begin{aligned}\|\vec{h}(t) - \vec{h}_L(t)\|_{\mathcal{H}} &\lesssim \|\mathcal{N}_R(\cdot, \cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} \\ &\lesssim \|V_R h\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} + \|F_R(\cdot, \cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} + \|G(\cdot, h)\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)}\end{aligned}$$

We can now estimate the three terms on the right-hand side above. First, we claim that

$$\|V_R h\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^5_*)} \lesssim \|V_R\|_{L^{\frac{3}{2}}_t L^3_x} \|h\|_{L^3_t L^6_x} \lesssim R^{-11/3}\|h\|_{L^3_t L^6_x}$$

To see this, we can use (4.4.5) to deduce that for each $t \in \mathbb{R}$

$$\begin{aligned}\|V_R\|_{L^3_x}^3 &\lesssim \int_1^{R+|t|} (R + |t|)^{-18} r^4\, dr + \int_{R+|t|}^{\infty} r^{-18} r^4\, dr \\ &\lesssim (R + |t|)^{-13}\end{aligned}$$

Therefore,

$$\|V_R\|_{L_t^{\frac{3}{2}} L_x^3} \lesssim \left(\int_{\mathbb{R}} (R + |t|)^{-13/2} \, dt \right)^{\frac{2}{3}} \lesssim R^{-11/3}$$

Similarly, we can show using (4.4.6) and (4.4.7) that

$$\|F_R(\cdot, \cdot, h)\|_{L_t^1 L_x^2(\mathbb{R} \times \mathbb{R}_*^5)} \lesssim R^{-11/6} \|h\|_{L_t^3 L_x^6}^2$$
$$\|G(\cdot, h)\|_{L_t^1 L_x^2(\mathbb{R} \times \mathbb{R}_*^5)} \lesssim \|h\|_{L_t^3 L_x^6}^3$$

which proves (4.4.10). □

We can now prove Lemma 4.4.3.

Proof of Lemma 4.4.3. We will first prove Lemma 4.4.3 for time $t = 0$. The fact that (4.4.3) holds at all times $t \in \mathbb{R}$ for $R > R_0$, with R_0 independent of t will follow from the pre-compactness of K.

For each $R \geq 1$, define truncated initial data $\vec{u}_R(0) = (u_{0,R}, u_{1,R})$ given by

$$\begin{aligned} u_{0,R}(r) &= \begin{cases} u_0(r) & \text{for} \quad r \geq R \\ \frac{u_0(R)}{R-1}(r-1) & \text{for} \quad r < R, \end{cases} \\ u_{1,R}(r) &= \begin{cases} u_1(r) & \text{for} \quad r \geq R \\ 0 & \text{for} \quad r < R. \end{cases} \end{aligned} \tag{4.4.12}$$

Observe that this truncated data has small energy for large R since

$$\|\vec{u}_R(0)\|_{\mathcal{H}} \lesssim \|\vec{u}(0)\|_{\mathcal{H}(r \geq R)}. \tag{4.4.13}$$

In particular, there exists $R_0 \geq 1$ so that for all $R \geq R_0$ we have

$$\|\vec{u}_R(0)\|_{\mathcal{H}} \leq \delta_0$$

where δ_0 is the small constant in Lemma 4.4.4. Let $\vec{u}_R(t)$ denote the solution to (4.4.8) given by Lemma 4.4.4 with data $\vec{u}_R(0)$ as in (4.4.12). Note that by finite propagation speed we have

$$\vec{u}_R(t,r) = \vec{u}(t,r) \quad \forall t \in \mathbb{R},\ \forall r \geq R + |t|$$

Also let $\vec{u}_{R,L}(t) = S_0(t)\vec{u}_R(0)$ denote the solution to free wave equation (4.3.5) with initial data $\vec{u}_R(0)$. Now, by the triangle inequality we obtain for each $t \in \mathbb{R}$

$$\begin{aligned} \|\vec{u}(t)\|_{\mathcal{H}(r\geq R+|t|)} = \|\vec{u}_R(t)\|_{\mathcal{H}(r\geq R+|t|)} \geq \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} \\ - \|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}} \end{aligned} \tag{4.4.14}$$

By (4.4.10) and (4.4.13) we can deduce that

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|\vec{u}_R(t) - \vec{u}_{R,L}(t)\|_{\mathcal{H}} &\lesssim R^{-11/3}\|\vec{u}_R(0)\|_{\mathcal{H}} + R^{-11/6}\|\vec{u}_R(0)\|_{\mathcal{H}}^2 + \|\vec{u}_R(0)\|_{\mathcal{H}}^3 \\ &\lesssim R^{-11/3}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)} + R^{-11/6}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)}^2 \\ &\quad + \|\vec{u}(0)\|_{\mathcal{H}(r\geq R)}^3 \end{aligned}$$

Therefore (4.4.14) gives

$$\begin{aligned} \|\vec{u}(t)\|_{\mathcal{H}(r\geq R+|t|)} \geq \|\vec{u}_{R,L}(t)\|_{\mathcal{H}(r\geq R+|t|)} - C_0 R^{-11/3}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)} \\ - C_0 R^{-11/6}\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)}^2 - C_0\|\vec{u}(0)\|_{\mathcal{H}(r\geq R)}^3 \end{aligned}$$

Letting t tend to either $\pm\infty$ – the choice determined by Proposition 4.3.1 – we can use Proposition 4.3.1 to estimate the right-hand side above and use Corollary 4.4.2 to see that the left-hand side above tends to zero, which gives

$$\|\pi_R^\perp \vec{u}_R(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3}\|\vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + R^{-11/3}\|\vec{u}(0)\|^4_{\mathcal{H}(r\geq R)} + \|\vec{u}(0)\|^6_{\mathcal{H}(r\geq R)}$$

after squaring both sides. Finally we note that by the definition of $\vec{u}_R(0)$,

$$\|\pi_R^\perp \vec{u}_R(0)\|^2_{\mathcal{H}(r\geq R)} = \|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}$$

Therefore,

$$\begin{aligned}\|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3} &\left(\|\pi_R \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right) \\ &+ R^{-11/3}\left(\|\pi_R \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right)^2 \\ &+ \left(\|\pi_R \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} + \|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)}\right)^3\end{aligned}$$

where we have used the orthogonality of the projection π_R to expand the right-hand side. To conclude the proof, simply choose R_0 large enough so that we can absorb all of the terms on the right-hand side involving $\pi^\perp$ into the left-hand side and deduce that

$$\begin{aligned}\|\pi_R^\perp \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3}&\|\pi_R \vec{u}(0)\|^2_{\mathcal{H}(r\geq R)} \\ &+ R^{-11/3}\|\pi_R \vec{u}(0)\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R \vec{u}(0)\|^6_{\mathcal{H}(r\geq R)}.\end{aligned}$$

This proves Lemma 4.4.3 for $t = 0$. To show that this inequality holds for all $t \in \mathbb{R}$ observe that by the pre-compactness of K we can choose $R_0 = R_0(\delta_0)$ so that

$$\|\vec{u}(t)\|_{\mathcal{H}(r\geq R)} \leq \delta_0 \tag{4.4.15}$$

uniformly in $t \in \mathbb{R}$. Now simply repeat the argument given above with the truncated initial data for time $t = t_0$ and $R \geq R_0$ defined by

$$u_{0,R,t_0}(r) = \begin{cases} u(t_0, r) & \text{for } \ r \geq R \\ \frac{u(t_0,R)}{R-1}(r-1) & \text{for } \ r < R, \end{cases}$$

$$u_{1,R,t_0}(r) = \begin{cases} u_t(t_0, r) & \text{for } \ r \geq R \\ 0 & \text{for } \ r < R. \end{cases}$$

This concludes the argument. $\square$

4.4.2 Step 2

In this step we will deduce the asymptotic behavior of $\vec{u}(0, r)$ as $r \to \infty$ described in (4.4.2). In particular we will establish the following result.

Lemma 4.4.5. *Let $\vec{u}(t)$ be as in Proposition 4.4.1 with $\vec{u}(0) = (u_0, u_1)$. Then there exists $\ell_0 \in \mathbb{R}$ such that*

$$r^3 u_0(r) \to \ell_0 \quad \textit{as} \quad r \to \infty \tag{4.4.16}$$

$$r \int_r^\infty u_1(\rho)\rho \, d\rho \to 0 \quad \textit{as} \quad r \to \infty \tag{4.4.17}$$

Moreover, we have the following estimates for the rates of convergence,

$$\left| r^3 u_0(r) - \ell_0 \right| = O(r^{-3}) \quad \textit{as} \quad r \to \infty \tag{4.4.18}$$

$$\left| r \int_r^\infty u_1(\rho)\rho \, d\rho \right| = O(r^{-1}) \quad \textit{as} \quad r \to \infty \tag{4.4.19}$$

To begin, we define

$$v_0(t,r) := r^3 u(t,r)$$
$$v_1(t,r) := r\int_r^\infty u_t(t,\rho)\rho\,d\rho \tag{4.4.20}$$

and for simplicity we will write $v_0(r) := v_0(0,r)$ and $v_1(r) := v_1(0,r)$. By direct computation one can show that

$$\|\pi_R^\perp \vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} = \int_R^\infty \left(\frac{1}{r}\partial_r v_0(t,r)\right)^2 dr + \int_R^\infty (\partial_r v_1(t,r))^2\,dr \tag{4.4.21}$$

$$\|\pi_R \vec{u}(t)\|^2_{\mathcal{H}(r\geq R)} = 3R^{-3}v_0^2(t,R) + R^{-1}v_1^2(t,R) \tag{4.4.22}$$

For convenience, we can rewrite the conclusions of Lemma 4.4.3 in terms of (v_0, v_1):

Lemma 4.4.6. *Let* (v_0, v_1) *be defined as in* (4.4.20). *There exists* $R_0 > 1$ *so that for all* $R > R_0$ *we have*

$$\begin{aligned}\int_R^\infty \left(\frac{1}{r}\partial_r v_0(t,r)\right)^2 dr + \int_R^\infty (\partial_r v_1(t,r))^2\,dr &\lesssim R^{-\frac{22}{3}}\left(3R^{-3}v_0^2(t,R) + R^{-1}v_1^2(t,R)\right)\\
&\quad + R^{-\frac{11}{3}}\left(3R^{-3}v_0^2(t,R) + R^{-1}v_1^2(t,R)\right)^2\\
&\quad + \left(3R^{-3}v_0^2(t,R) + R^{-1}v_1^2(t,R)\right)^3\\
&\lesssim R^{-\frac{31}{3}}v_0^2(t,R) + R^{-\frac{29}{3}}v_0^4(t,R) + R^{-9}v_0^6(t,R)\\
&\quad + R^{-\frac{25}{3}}v_1^2(t,R) + R^{-\frac{17}{3}}v_1^4(t,R) + R^{-3}v_1^6(t,R)\end{aligned}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We will use Lemma 4.4.6 to prove a difference estimate. First, let $\delta_1 > 0$ be a small number to be determined below with $\delta_1 \leq \delta_0$ where δ_0 is as in Lemma 4.4.4. Let R_1 be large enough so that for all $R \geq R_1$ we have

$$\begin{aligned} &\|\vec{u}(t)\|_{\mathcal{H}(r\geq R)} \leq \delta_1 \leq \delta_0 \quad \forall R \geq R_1, \quad \forall t \in \mathbb{R} \\ &R_1^{-\frac{11}{3}} \leq \delta_1 \end{aligned} \tag{4.4.23}$$

We note again that such an $R_1 = R_1(\delta_1)$ exists by the pre-compactness of K.

Corollary 4.4.7. *Let R_1 be as above. The for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$\begin{aligned} \left|v_0(t,r) - v_0(t,r')\right| \lesssim\, & r^{-\frac{11}{3}} \left|v_0(t,r)\right| + r^{-\frac{10}{3}} \left|v_0(t,r)\right|^2 + r^{-3} \left|v_0(t,r)\right|^3 \\ & + r^{-\frac{8}{3}} \left|v_1(t,r)\right| + r^{-\frac{4}{3}} \left|v_1(t,r)\right|^2 + \left|v_1(t,r)\right|^3 \end{aligned} \tag{4.4.24}$$

and

$$\begin{aligned} \left|v_1(t,r) - v_1(t,r')\right| \lesssim\, & r^{-\frac{14}{3}} \left|v_0(t,r)\right| + r^{-\frac{13}{3}} \left|v_0(t,r)\right|^2 + r^{-4} \left|v_0(t,r)\right|^3 \\ & + r^{-\frac{11}{3}} \left|v_1(t,r)\right| + r^{-\frac{7}{3}} \left|v_1(t,r)\right|^2 + r^{-1} \left|v_1(t,r)\right|^3 \end{aligned} \tag{4.4.25}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We will also need a trivial consequence of the preceding result which we state as another corollary for convenience.

Corollary 4.4.8. *Let R_1 be as above. The for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$\left|v_0(t,r) - v_0(t,r')\right| \lesssim \delta_1 \left|v_0(t,r)\right| + r\delta_1 \left|v_1(t,r)\right| \tag{4.4.26}$$

and

$$\left|v_1(t,r) - v_1(t,r')\right| \lesssim r^{-1}\delta_1 \left|v_0(t,r)\right| + \delta_1 \left|v_1(t,r)\right| \tag{4.4.27}$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

We remark that Corollary 4.4.8 follows immediately from Corollary 4.4.7 in light of (4.4.22) and (4.4.23).

Proof of Corollary 4.4.7. This is a simple consequence of Lemma 4.4.6. Indeed, for $r \geq R_1$ and $r' \in [r, 2r]$ we use Lemma 4.4.6 to see that

$$\begin{aligned}
\left|v_0(t,r) - v_0(t,r')\right|^2 &\leq \left(\int_r^{r'} |\partial_r v_0(t,\rho)| \, d\rho \right)^2 \\
&\leq \left(\int_r^{r'} \rho^2 \, d\rho \right) \left(\int_r^{r'} \left| \frac{1}{\rho} \partial_r v_0(t,\rho) \right|^2 \, d\rho \right) \\
&\lesssim r^3 \left(r^{-\frac{31}{3}} v_0^2(t,r) + r^{-\frac{29}{3}} v_0^4(t,r) + r^{-9} v_0^6(t,r) \right) \\
&+ r^3 \left(r^{-\frac{25}{3}} v_1^2(t,r) + r^{-\frac{17}{3}} v_1^4(t,r) + r^{-3} v_1^6(t,r) \right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left|v_1(t,r) - v_1(t,r')\right|^2 &\leq \left(\int_r^{r'} |\partial_r v_1(t,\rho)| \, d\rho \right)^2 \\
&\leq \left(\int_r^{r'} d\rho \right) \left(\int_r^{r'} |\partial_r v_1(t,\rho)|^2 \, d\rho \right) \\
&\lesssim r \left(r^{-\frac{31}{3}} v_0^2(t,r) + r^{-\frac{29}{3}} v_0^4(t,r) + r^{-9} v_0^6(t,r) \right) \\
&+ r \left(r^{-\frac{25}{3}} v_1^2(t,r) + r^{-\frac{17}{3}} v_1^4(t,r) + r^{-3} v_1^6(t,r) \right)
\end{aligned}$$

as claimed. □

The next step towards establishing Lemma 4.4.5 is to provide an upper bound on the growth rates of $v_0(t,r)$ and $v_1(t,r)$.

Claim 4.4.9. *Let $v_0(t,r)$, $v_1(t,r)$ be as in* (4.4.20)*. Then,*

$$|v_0(t,r)| \lesssim r^{\frac{1}{6}} \tag{4.4.28}$$

$$|v_1(t,r)| \lesssim r^{\frac{1}{18}} \tag{4.4.29}$$

uniformly in $t \in \mathbb{R}$.

Proof. First, note that it suffices to prove Claim 4.4.9 only for $t = 0$ since the ensuing argument relies exclusively on results in this section that hold uniformly in $t \in \mathbb{R}$. Fix $r_0 \geq R_1$ and observe that by (4.4.26), (4.4.27)

$$\left|v_0(2^{n+1}r_0)\right| \leq (1 + C_1\delta_1)\left|v_0(2^n r_0)\right| + (2^n r_0)C_1\delta_1\left|v_1(2^n r_0)\right| \tag{4.4.30}$$

$$\left|v_1(2^{n+1}r_0)\right| \leq (1 + C_1\delta_1)\left|v_1(2^n r_0)\right| + (2^n r_0)^{-1}C_1\delta_1\left|v_0(2^n r_0)\right| \tag{4.4.31}$$

To simply the exposition, we introduce the notation

$$a_n := \left|v_1(2^n r_0)\right| \tag{4.4.32}$$

$$b_n := (2^n r_0)^{-1}\left|v_0(2^n r_0)\right| \tag{4.4.33}$$

Then, combining (4.4.30) and (4.4.31) gives

$$\begin{aligned} a_{n+1} + b_{n+1} &\leq \left(1 + \frac{3}{2}C_1\delta_1\right)a_n + \left(\frac{1}{2} + \frac{3}{2}C_1\delta_1\right)b_n \\ &\leq \left(1 + \frac{3}{2}C_1\delta_1\right)(a_n + b_n) \end{aligned}$$

Arguing inductively we then see that for each n we have

$$(a_n + b_n) \leq \left(1 + \frac{3}{2}C_1\delta_1\right)^n (a_0 + b_0)$$

Choosing δ_1 small enough so that $(1 + \frac{3}{2}C_1\delta_1) \leq 2^{\frac{1}{18}}$ allows us to conclude that

$$a_n \leq C(2^n r_0)^{\frac{1}{18}} \tag{4.4.34}$$

where the constant $C > 0$ above depends on r_0 which is fixed. In light of (4.4.32) we have thus proved (4.4.29) for all $r = 2^n r_0$. Now define

$$c_n := |v_0(2^n r_0)| \tag{4.4.35}$$

By (4.4.22), (4.4.23), (4.4.24), and (4.4.34) we have

$$c_{n+1} \leq (1 + C_1\delta_1)c_n + C(2^n r_0)^{\frac{1}{6}}$$

Inductively, we can deduce that

$$\begin{aligned} c_n &\leq (1 + C_1\delta_1)^n c_0 + C r_0^{\frac{1}{6}} \sum_{k=1}^{n} (1 + C_1\delta_1)^{n-k} 2^{\frac{k-1}{6}} \\ &\leq C(2^n r_0)^{\frac{1}{6}} \end{aligned}$$

where we have used that $(1 + C_1\delta_1) \leq 2^{\frac{1}{18}}$, and again the constant $C > 0$ depends on r_0, which is fixed. This proves (4.4.28) for $r = 2^n r_0$. The general estimates (4.4.28) and (4.4.29) follow from the difference estimates (4.4.24) and (4.4.25). $\square$

Claim 4.4.10. *For each $t \in \mathbb{R}$ there exists a number $\ell_1(t) \in \mathbb{R}$ such that*

$$|v_1(t, r) - \ell_1(t)| = O(r^{-1}) \quad \textit{as} \quad r \to \infty \tag{4.4.36}$$

where the $O(\cdot)$ is uniform in t.

Proof. Again, it suffices to show this for $t = 0$. Let $r_0 \geq R_1$ where $R_1 > 1$ is as in (4.4.23).

By (4.4.25) and Claim 4.4.9 we have

$$\begin{aligned}
\left|v_1(2^{n+1}r_0) - v_1(2^n r_0)\right| &\lesssim (2^n r_0)^{-\frac{9}{2}} + (2^n r_0)^{-4} + (2^n r_0)^{-\frac{7}{2}} \\
&\quad + (2^n r_0)^{-\frac{65}{18}} + (2^n r_0)^{-\frac{20}{9}} + (2^n r_0)^{-\frac{5}{6}} \\
&\lesssim (2^n r_0)^{-\frac{5}{6}}
\end{aligned}$$

This implies that the series

$$\sum_n \left|v_1(2^{n+1}r_0) - v_1(2^n r_0)\right| < \infty$$

which in turn implies that there exists $\ell_1 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} v_1(2^n r_0) = \ell_1.$$

The fact that $\lim\limits_{r\to\infty} v_1(r) = \ell_1$ follows from the difference estimates (4.4.24), (4.4.25), and the growth estimates (4.4.28), (4.4.29). To establish the estimates on the rate of convergence in (4.4.36) we note that by the difference estimate (4.4.25) and the fact that we now know that $|v_1(r)|$ is bounded, for large enough r we have

$$\left|v_1(2^{n+1}r) - v_1(2^n r)\right| \lesssim (2^n r)^{-1}$$

Hence,

$$|v_1(r) - \ell_1| = \left|\sum_{n\geq 0}(v_1(2^{n+1}r) - v_1(2^n r))\right| \lesssim r^{-1}\sum_{n\geq 0} 2^{-n} \lesssim r^{-1}$$

as desired. $\square$

Next we show that the limit $\ell_1(t)$ is actually independent of t.

Claim 4.4.11. *The function $\ell_1(t)$ in Claim 4.4.10 is independent of t, i.e., $\ell_1(t) = \ell_1$ for all $t \in \mathbb{R}$.*

Proof. By the definition of $v_1(t,r)$ we have shown that

$$\ell_1(t) = r \int_r^\infty u_t(t,\rho)\rho \, d\rho + O(r^{-1})$$

Fix $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$. We will show that

$$\ell_1(t_2) - \ell_1(t_1) = 0$$

To see this observe that for each $R \geq R_1$ we have

$$\begin{aligned}
\ell_1(t_2) - \ell_1(t_1) &= \frac{1}{R}\int_R^{2R} (\ell_1(t_2) - \ell_1(t_1)) \, ds \\
&= \frac{1}{R}\int_R^{2R} \left(s\int_s^\infty (u_t(t_2,r) - u_t(t_1,r)) r \, dr \right) ds + O(R^{-1}) \\
&= \frac{1}{R}\int_R^{2R} \left(s\int_s^\infty \int_{t_1}^{t_2} u_{tt}(t,r) \, dt \, r \, dr \right) ds + O(R^{-1})
\end{aligned}$$

Using the fact that u is a solution to (4.1.8), we can rewrite the above integral as

$$\begin{aligned}
&= \int_{t_1}^{t_2} \frac{1}{R}\int_R^{2R} \left(s\int_s^\infty (r u_{rr}(t,r) + 4u_r(t,r)) \, dr \right) ds \, dt + \\
&+ \int_{t_1}^{t_2} \frac{1}{R}\int_R^{2R} \left(s\int_s^\infty (-rV(r)u(t,r) + rN(r,u(t,r))) \, dr \right) ds \, dt \\
&+ O(R^{-1}) \\
&= I + II + O(R^{-1})
\end{aligned} \tag{4.4.37}$$

To estimate I we integrate by parts:

$$
\begin{aligned}
I &= \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty \frac{1}{r^3} \partial_r (r^4 u_r(t,r))\, dr \right) ds\, dt \\
&= 3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty u_r(t,r)\, dr \right) ds\, dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} s^2 u_r(t,s)\, ds\, dt \\
&= -3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r\, u(t,r)\, dr\, dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r^2\, u_r(t,r)\, dr\, dt \\
&= - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r\, u(t,r)\, dr\, dt + \int_{t_1}^{t_2} (Ru(t,R) - 2Ru(t,2R))\, dt
\end{aligned}
\tag{4.4.38}
$$

Finally, we note that (4.4.28) and the definition of $v_0(t,r)$ give us

$$
r^3 \, |u(t,r)| = |v_0(t,r)| \lesssim r^{\frac{1}{6}} \tag{4.4.39}
$$

Using this estimate for $|u(t,r)|$ in the last line in (4.4.38) shows that

$$
I = |t_2 - t_1| \, O(R^{-\frac{11}{6}})
$$

To estimate II we can use (4.4.39) to see that for $r > R$ large enough

$$
\begin{aligned}
|-V(r)u(t,r) + N(r, u(t,r))| &\lesssim r^{-6} \, |u(t,r)| + r^{-3} \, |u(t,r)|^2 + |u(t,r)|^3 \\
&\lesssim r^{-6-\frac{17}{6}} + r^{-3-\frac{17}{3}} + r^{-\frac{17}{2}} \\
&\lesssim r^{-8}
\end{aligned}
$$

Hence,

$$
II \lesssim \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} s \int_s^\infty r^{-8}\, dr\, ds\, dt = |t_2 - t_1| \, O(R^{-6})
$$

Putting this together we get

$$|\ell_1(t_2) - \ell_1(t_1)| = O(R^{-1})$$

which implies that $\ell_1(t_2) = \ell_1(t_1)$. $\square$

We next show that ℓ_1 is necessarily equal to 0.

Claim 4.4.12. $\ell_1 = 0$.

Proof. Suppose $\ell_1 \neq 0$. We know that for all $R \geq R_1$ and for all $t \in \mathbb{R}$ we have

$$R \int_R^\infty u_t(t,r)\, r\, dr = \ell_1 + O(R^{-1})$$

where $O(\cdot)$ is uniform in t. Hence, for R large, the left-hand side above has the same sign as ℓ_1, for all t. Thus we can choose $R \geq R_1$ large enough so that for all $t \in \mathbb{R}$,

$$\left| R \int_R^\infty u_t(t,r)\, r\, dr \right| \geq \frac{|\ell_1|}{2}$$

Integrating from $t = 0$ to $t = T$ gives

$$\Big| \int_0^T R \int_R^\infty u_t(t,r)\, r\, dr\, dt \Big| \geq T \frac{|\ell_1|}{2}$$

However, we integrate in t on the left-hand side and use (4.4.39) to obtain

$$\begin{aligned} \left| R \int_R^\infty \int_0^T u_t(t,r)\, r\, dt\, dr \right| &= \left| R \int_R^\infty [u(T,r) - u(0,r)]\, r\, dr \right| \\ &\lesssim R \int_R^\infty r^{-\frac{11}{6}}\, dr \lesssim R^{\frac{1}{6}} \end{aligned}$$

Therefore for fixed large R we have

$$T\frac{|\ell_1|}{2} \lesssim R^{\frac{1}{6}}$$

which gives a contradiction by taking T large. $\square$

Now that we have shown that $v_1(r) \to 0$ as $r \to \infty$, we can prove that $v_0(r)$ also converges and complete the proof of Lemma 4.4.5.

Proof of Lemma 4.4.5. It remains to show that there exists $\ell_0 \in \mathbb{R}$ such that

$$|v_0(r) - \ell_0| = O(r^{-3}) \quad \text{as} \quad r \to \infty \tag{4.4.40}$$

Using the difference estimate (4.4.24) as well as (4.4.28) and the fact that $|v_1(r)| \lesssim r^{-1}$ for $r \geq R_1$ we have for $r_0 \geq R_1$

$$\begin{aligned}\Big|v_0(2^{n+1}r_0) - v_0(2^n r_0)\Big| &\lesssim (2^n r_0)^{-\frac{11}{3}}(2^n r_0)^{\frac{1}{6}} + (2^n r_0)^{-\frac{10}{3}}(2^n r_0)^{\frac{1}{3}} + (2^n r_0)^{-3}(2^n r_0)^{\frac{1}{2}} \\ &+ (2^n r_0)^{-\frac{8}{3}}(2^n r_0)^{-1} + (2^n r_0)^{-\frac{4}{3}}(2^n r_0)^{-2} + (2^n r_0)^{-3} \\ &\lesssim (2^n r_0)^{-\frac{5}{2}}\end{aligned}$$

Hence,

$$\sum_{n\geq 0} \Big|v_0(2^{n+1}r_0) - v_0(2^n r_0)\Big| < \infty$$

and therefore there exists $\ell_0 \in \mathbb{R}$ so that

$$\lim_{n\to\infty} v_0(2^n r_0) = \ell_0$$

By the difference estimate (4.4.24) and the fact that $v_1(r) \to 0$ we can conclude that in fact $\lim_{r\to\infty} v_0(r) = \ell_0$. To establish the convergence rate, we note that since we now know that

$|v_0(r)|$ is bounded we have the improved difference estimate

$$\left| v_0(2^{n+1}r) - v_0(2^n r) \right| \lesssim (2^n r)^{-3} \tag{4.4.41}$$

which holds for all $r \geq R$. Therefore,

$$|v_0(r) - \ell_0| = \left| \sum_{n \geq 0} (v_0(2^{n+1}r) - v_0(2^n r)) \right| \lesssim r^{-3} \sum_{n \geq 0} 2^{-3n} \tag{4.4.42}$$

as claimed. □

4.4.3 Step 3

Finally, we complete the proof of Proposition 4.4.1 by showing that $\vec{u}(t) = (0,0)$. We divide this argument into two separate cases depending on whether the number ℓ_0 found in the previous step is zero or nonzero.

Case 1: $\ell_0 = 0$ implies $\vec{u}(0) = (0,0)$:

In this case we show that if $\ell_0 = 0$, then $\vec{u}(t) = (0,0)$.

Lemma 4.4.13. *Let $\vec{u}(t)$ be as in Proposition 4.4.1 and let ℓ_0 be as in Lemma 4.4.5. Suppose that $\ell_0 = 0$. Then $\vec{u}(t) = (0,0)$.*

We begin by showing that if $\ell_0 = 0$ then (u_0, u_1) must be compactly supported.

Claim 4.4.14. *Let ℓ_0 be as in Lemma 4.4.5. If $\ell_0 = 0$ then (u_0, u_1) must be compactly supported.*

Proof. The assumption $\ell_0 = 0$ means that

$$\begin{aligned} |v_0(r)| &= O(r^{-3}) \quad \text{as} \quad r \to \infty \\ |v_1(r)| &= O(r^{-1}) \quad \text{as} \quad r \to \infty \end{aligned} \tag{4.4.43}$$

Therefore, for $r_0 \geq R_1$ we have

$$|v_0(2^n r_0)| + |v_1(2^n r_0)| \lesssim (2^n r_0)^{-3} + (2^n r_0)^{-1} \lesssim (2^n r_0)^{-1} \tag{4.4.44}$$

On the other hand, using the difference estimates (4.4.24)–(4.4.27) as well as our assumption (4.4.43) we obtain

$$\begin{aligned} \left|v_0(2^{n+1} r_0)\right| &\geq (1 - C_1\delta_1)\,|v_0(2^n r_0)| - C_1(2^n r_0)^{-2}\,|v_1(2^n r_0)| \\ \left|v_1(2^{n+1} r_0)\right| &\geq (1 - C_1\delta_1)\,|v_1(2^n r_0)| - C_1(2^n r_0)^{-4}\,|v_0(2^n r_0)| \end{aligned}$$

This means that

$$\left|v_0(2^{n+1} r_0)\right| + \left|v_1(2^{n+1} r_0)\right| \geq (1 - C_1\delta_1 - C_1 r_0^{-2})\,(|v_0(2^n r_0)| + |v_1(2^n r_0)|)$$

Choose r_0 large enough and δ_1 small enough so that $C_1(\delta_1 + r_0^{-2}) < \frac{1}{4}$. Arguing inductively we can conclude that

$$|v_0(2^n r_0)| + |v_1(2^n r_0)| \geq \left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|)$$

Estimating the left hand side above using (4.4.44) gives

$$\left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 2^{-n} r_0^{-1}$$

which means that

$$\left(\frac{3}{2}\right)^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 1$$

Hence $\vec{v}(r_0) := (v_0(r_0), v_1(r_0)) = (0, 0)$. But then (4.4.22) implies that

$$\|\pi_{r_0}\vec{u}(0)\|_{\mathcal{H}(r\geq r_0)} = 0$$

Using Lemma 4.4.3 we can also deduce that

$$\|\pi_{r_0}^{\perp}\vec{u}(0)\|_{\mathcal{H}(r\geq r_0)} = 0$$

and hence

$$\|\vec{u}(0)\|_{\mathcal{H}(r\geq r_0)} = 0$$

which conclude the proof since $\lim_{r\to\infty} u_0(r) = 0$. $\square$

Proof of Lemma 4.4.13. Assume that $\ell_0 = 0$. Then by Claim 4.4.14, (u_0, u_1) is compactly supported. We assume that $(u_0, u_1) \neq (0, 0)$ and argue by contradiction. In this case we can find $\rho_0 > 1$ so that

$$\rho_0 := \inf\{\rho \,:\, \|\vec{u}(0)\|_{\mathcal{H}(r\geq\rho)} = 0\}$$

Let $\varepsilon > 0$ small to be determined below and find $1 < \rho_1 < \rho_0$, $\rho_1 = \rho_1(\varepsilon)$ so that

$$0 < \|\vec{u}(0)\|^2_{\mathcal{H}(r\geq\rho_1)} \leq \varepsilon \leq \delta_1^2$$

where $\delta_1 > 0$ is as in (4.4.23). With (v_0, v_1) as in (4.4.20) we have

$$\int_{\rho_1}^{\infty}\left(\frac{1}{r}\partial_r v_0(r)\right)^2 dr + \int_{\rho_1}^{\infty}(\partial_r v_1(r))^2\, dr + 3\rho_1^{-3}v_0^2(\rho_1) + \rho_1^{-1}v_1^2(\rho_1) =$$
$$= \|\pi_{\rho_1}^{\perp}\vec{u}(0)\|^2_{\mathcal{H}(r\geq\rho_1)} + \|\pi_{\rho_1}\vec{u}(0)\|^2_{\mathcal{H}(r\geq\rho_1)} = \|\vec{u}(0)\|^2_{\mathcal{H}(r\geq\rho_1)} < \varepsilon \quad (4.4.45)$$

By Lemma 4.4.6 we also have

$$\int_{\rho_1}^{\infty}\left(\frac{1}{r}\partial_r v_0(r)\right)^2 dr+\int_{\rho_1}^{\infty}(\partial_r v_1(r))^2\,dr\lesssim \rho_1^{-\frac{31}{3}}v_0^2(\rho_1)+\rho_1^{-\frac{29}{3}}v_0^4(\rho_1)+\rho_1^{-9}v_0^6(\rho_1)$$
$$+\rho_1^{-\frac{25}{3}}v_1^2(\rho_1)+\rho_1^{-\frac{17}{3}}v_1^4(\rho_1)+\rho_1^{-3}v_1^6(\rho_1)\quad(4.4.46)$$

Arguing as in Corollary 4.4.8 and using the fact that $v_0(\rho_0)=v_1(\rho_0)=0$ gives

$$|v_0(\rho_1)|=|v_0(\rho_1)-v_0(\rho_0)|\lesssim\varepsilon\,|v_0(\rho_1)|+\rho_1\varepsilon\,|v_1(\rho_1)|\quad(4.4.47)$$

and

$$|v_1(\rho_1)|=|v_1(\rho_1)-v_1(\rho_0)|\lesssim\rho_1^{-1}\varepsilon\,|v_0(\rho_1)|+\varepsilon\,|v_1(\rho_1)|\quad(4.4.48)$$

Plugging (4.4.47) into (4.4.48) gives

$$|v_1(\rho_1)|\lesssim\rho_1^{-1}\varepsilon^2\,|v_0(\rho_1)|+\varepsilon(1+\varepsilon)\,|v_1(\rho_1)|$$

which means that for ε small enough we have

$$|v_1(\rho_1)|\lesssim\rho_1^{-1}\varepsilon^2\,|v_0(\rho_1)|\quad(4.4.49)$$

Putting this estimate back into (4.4.47) we obtain

$$|v_0(\rho_1)|\lesssim\varepsilon\,|v_0(\rho_1)|+\varepsilon^3\,|v_0(\rho_1)|\lesssim\varepsilon(1+\varepsilon^2)\,|v_0(\rho_1)|$$

which implies that $v_0(\rho_1)=0$ as long as ε is chosen small enough. By (4.4.49) we can conclude that $v_1(\rho_1) = 0$ as well. By (4.4.46) and (4.4.45) we then have that

$$\|\vec{u}(0)\|_{\mathcal{H}(r \geq \rho_1)} = 0$$

which is a contradiction since $\rho_1 < \rho_0$. $\square$

We next consider the case $\ell_0 \neq 0$.

Case 2: $\ell_0 \neq 0$ is impossible.

In this final step we show that the case $\ell_0 \neq 0$ is impossible. Indeed we prove that if $\ell_0 \neq 0$ then our original wave map $\vec{\psi}(t)$ is equal to a rescaled solution Q_{ℓ_0} to (4.1.1) that does not satisfy the Dirichlet boundary condition, $Q_{\ell_0}(1) \neq 0$, which is a contradiction since $\psi(t, 1) = 0$ for all $t \in \mathbb{R}$.

We have shown that

$$r^3 u_0(r) = \ell_0 + O(r^{-3})$$

Recall that $ru_0(r) = \varphi_0(r) = \psi_0(r) - Q(r)$ and that

$$Q(r) = n\pi - \frac{\alpha_0}{r^2} + O(r^{-6})$$

where $\alpha_0 > 0$ is *uniquely* determined by the boundary condition $Q(1) = 0$. Hence,

$$\psi_0(r) = n\pi - \frac{\alpha_0 - \ell_0}{r^2} + O(r^{-5}) \tag{4.4.50}$$

By Lemma 4.1.1 there is a solution $Q_{\alpha_0 - \ell} \in \dot{H}^1(\mathbb{R}^3_*)$ to (4.1.1) satisfying

$$Q_{\alpha_0 - \ell_0}(r) = n\pi - \frac{\alpha_0 - \ell_0}{r^2} + O(r^{-6}) \tag{4.4.51}$$

and from here out we write $Q_{\ell_0} := Q_{\alpha_0 - \ell_0}$. Note, by Lemma 4.1.1, $\ell_0 \neq 0$ implies that

$$Q_{\ell_0}(1) \neq 0$$

Indeed, recall from the discussion following Lemma 4.1.1 that if $\alpha_0 - \ell_0 > 0$ then Q_{ℓ_0} is a nontrivial rescaling of the harmonic map Q and hence no longer satisfies the boundary condition. If $\alpha_0 - \ell_0 = 0$ then $Q_{\ell_0}(r) = n\pi$ for all r. Finally, we recall that $\alpha_0 - \ell_0 < 0$ implies that $Q_{\ell_0}(r) > n\pi$ for all r. Now set

$$\begin{aligned} u_{\ell_0,0}(r) &:= \frac{1}{r}(\psi_0(r) - Q_{\ell_0}(r)) \\ u_{\ell_0,1}(r) &:= \frac{1}{r}\psi_1(r) \end{aligned} \tag{4.4.52}$$

For each $t \in \mathbb{R}$ define $u_{\ell_0}(t,r) := \frac{1}{r}(\psi(t,r) - Q_{\ell_0}(r))$. We record a few properties of $\vec{u}_{\ell_0} := (u_{\ell_0}, \partial_t u_{\ell_0})$. Note that by construction we have

$$\begin{aligned} v_{\ell_0,0}(r) &:= r^3 u_{\ell_0}(r) = O(r^{-3}) \quad \text{as} \quad r \to \infty \\ v_{\ell_0,1}(r) &:= r\int_r^\infty \rho\, u_{\ell_0,1}(\rho)\, d\rho = O(r^{-1}) \quad \text{as} \quad r \to \infty \end{aligned} \tag{4.4.53}$$

Also, $\vec{u}_{\ell_0}(t)$ satisfies

$$\partial_{tt} u_{\ell_0} - \partial_{rr} u_{\ell_0} - \frac{4}{r}\partial_r u_{\ell_0} = -V_{\ell_0}(r) u + N_{\ell_0}(r, u_{\ell_0}) \tag{4.4.54}$$

where

$$\begin{aligned} V_{\ell_0}(r) &:= \frac{2(\cos(2Q_{\ell_0}) - 1)}{r^2} \\ N_{\ell_0}(r, u_{\ell_0}) &:= \cos(2Q_{\ell_0})\frac{(2ru_{\ell_0} - \sin(2ru_{\ell_0}))}{r^3} + 2\sin(2Q_{\ell_0})\frac{\sin^2(ru_{\ell_0})}{r^3} \end{aligned} \tag{4.4.55}$$

Crucially, we remark that $\vec{u}_{\ell_0}(t)$ inherits the compactness property from $\vec{\psi}(t)$. Indeed, the trajectory

$$\tilde{K} := \{\vec{u}_{\ell_0}(t) \mid t \in \mathbb{R}\}$$

is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$. However, since we have assumed that $\ell_0 \neq 0$ we see that

$$u_{\ell_0}(t,1) = \psi_0(t,1) - Q_{\ell_0}(1) = -Q_{\ell_0}(1) \neq 0. \tag{4.4.56}$$

On the other hand, below we will show that $\vec{u}_{\ell_0} = (u_{\ell_0}, \partial_t u_{\ell_0}) = (0,0)$ which contradicts (4.4.56).

Lemma 4.4.15. *Suppose $\ell_0 \neq 0$. Let $\vec{u}(t)$ be as in Proposition 4.4.1 and define $\vec{u}_{\ell_0}$ as in* (4.4.52). *Then $\vec{u}_{\ell_0} = (0,0)$.*

The argument that we will use to prove Lemma 4.4.15 is nearly identical to the one presented in the previous steps to reach the desired conclusion for $\ell_0 = 0$ and we omit many details here.

We start by showing that $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ must be compactly supported. As before we can argue as in the proof of Lemma 4.4.3, by modifying (4.4.54) inside the interior cone $\{(t,r) \mid 1 \leq r \leq R + |t|\}$, and using the linear exterior estimates in Proposition 4.3.1 to produce the same type of inequality as (4.4.3).

Lemma 4.4.16. *There exists $R_0 > 1$ so that for all $R \geq R_0$ we have*

$$\begin{aligned} \|\pi_R^{\perp} \vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq R)} \lesssim R^{-22/3} &\|\pi_R \vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq R)} \\ &+ R^{-11/3}\|\pi_R \vec{u}_{\ell_0}\|^4_{\mathcal{H}(r\geq R)} + \|\pi_R \vec{u}_{\ell_0}\|^6_{\mathcal{H}(r\geq R)} \end{aligned} \tag{4.4.57}$$

where again $P(R) := \{(k_1 r^{-3}, k_2 r^{-3}) \mid k_1, k_2 \in \mathbb{R}, r > R\}$, π_R denotes the orthogonal projection onto $P(R)$ and $\pi_R{}^{\perp}$ denotes the orthogonal projection onto the orthogonal com-

plement of the plane $P(R)$ in $\mathcal{H}(r > R; \mathbb{R}^5_*)$.

We remark that the proof of Lemma 4.4.16 follows exactly as the proof of Lemma 4.4.3 where we simply replace Q with Q_{ℓ_0} and $\vec{u}$ with $\vec{u}_{\ell_0}$ in the arguments given for the proof of Lemma 4.4.3. We note that since the trajectory $\tilde{K}$ is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$, $\vec{u}_{\ell_0}$ satisfies the conclusions of Corollary 4.4.2, namely for each $R > 1$ we have

$$\|\vec{u}_{\ell_0}(t)\|_{\mathcal{H}(r \geq R + |t|)} \to 0 \quad \text{as} \quad |t| \to \infty$$

where the condition $R > 1$ allows the interchange of the norms $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^5_*)$ and $\dot{H}^1 \times L^2(\mathbb{R}^5_*)$. With $(v_{\ell_0,0}, v_{\ell_0,1})$ defined as in (4.4.53) we can then conclude that for all $R > R_0$ large enough we have

$$\begin{aligned}
\int_R^\infty \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_R^\infty (\partial_r v_{\ell_0,1}(r))^2 \, dr &\lesssim R^{-\frac{31}{3}} v_{\ell_0,0}^2(R) + R^{-\frac{29}{3}} v_{\ell_0,0}^4(R) \\
&\quad + R^{-9} v_{\ell_0,0}^6(R) + R^{-\frac{25}{3}} v_{\ell_0,1}^2(R) \\
&\quad + R^{-\frac{17}{3}} v_{\ell_0,1}^4(R) + R^{-3} v_{\ell_0,1}^6(R) \\
&\lesssim R^{-7}(v_{\ell_0,0}^2(R) + v_{\ell_0,1}^2(R))
\end{aligned} \tag{4.4.58}$$

where the first inequality follows by rewriting (4.4.57) in terms of $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$ and the last line following from the known decay estimates in (4.4.53). Next, mimicking the proof of Corollary 4.4.7 we can again establish difference estimates using (4.4.58). Indeed, for all $R_0 \leq r \leq r' \leq 2r$ we have

$$\begin{aligned}
\left|v_{\ell_0,0}(r) - v_{\ell_0,0}(r')\right|^2 &\lesssim r^{-4}(v_{\ell_0,0}^2(r) + v_{\ell_0,1}^2(r)) \\
\left|v_{\ell_0,1}(r) - v_{\ell_0,1}(r')\right|^2 &\lesssim r^{-6}(v_{\ell_0,0}^2(r) + v_{\ell_0,1}^2(r))
\end{aligned} \tag{4.4.59}$$

In terms of the vector $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$ we then have

$$\left|\vec{v}_{\ell_0}(r) - \vec{v}_{\ell_0}(r')\right| \lesssim r^{-2} \left|\vec{v}_{\ell_0}(r)\right| \tag{4.4.60}$$

Hence for fixed $r_0 \geq R_0$ large enough we can deduce that

$$\left|\vec{v}_{\ell_0}(2^{n+1}r_0)\right| \gtrsim \frac{3}{4} \left|\vec{v}_{\ell_0}(2^n r_0)\right|$$

Therefore for each n,

$$\left|\vec{v}_{\ell_0}(2^n r_0)\right| \gtrsim \left(\frac{3}{4}\right)^n \left|\vec{v}_{\ell_0}(r_0)\right|$$

On the other hand, by (4.4.53) we have

$$\left|\vec{v}_{\ell_0}(2^n r_0)\right| \lesssim (2^n r_0)^{-1}$$

Combining the last two lines we see that

$$\left(\frac{3}{2}\right)^n \left|\vec{v}_{\ell_0}(r_0)\right| \lesssim 1,$$

which implies that $\vec{v}_{\ell_0}(r_0) = (0,0)$. By (4.4.58) we can deduce that

$$\int_{r_0}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2 \, dr = 0$$

Therefore,

$$\begin{aligned}
&\|\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq r_0)} = \\
&\quad = \int_{r_0}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2 \, dr + 3r_0^{-3} v^2_{\ell_0,0}(r_0) + r_0^{-1} v^2_{\ell_0,1}(r_0) = 0
\end{aligned}$$

which means that $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ is compactly supported. We conclude by showing that $\vec{u}_{\ell_0} = (0,0)$.

Proof of Lemma 4.4.15. The proof is nearly identical to the proof of Lemma 4.4.14. Suppose

$$(\partial_r u_{\ell_0,0}, u_{\ell_0,1}) \neq (0,0)$$

and we argue by contradiction. By the preceding arguments $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ is compactly supported. Then we can define $\rho_0 > 1$ by

$$\rho_0 := \inf\{\rho \,:\, \|\vec{u}_{\ell_0}\|_{\mathcal{H}(r\geq\rho)} = 0\}$$

Let $\varepsilon > 0$ small to be determined below and find $1 < \rho_1 < \rho_0$, $\rho_1 = \rho_1(\varepsilon)$ so that

$$0 < \|\vec{u}_{\ell_0}\|_{\mathcal{H}(r\geq\rho_1)} \leq \varepsilon$$

We then have

$$\begin{aligned}\int_{\rho_1}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r v_{\ell_0,1}(r))^2\, dr + 3\rho_1^{-3} v_{\ell_0,0}^2(\rho_1) + \rho_1^{-1} v_{\ell_0,1}^2(\rho_1) = \\ = \|\pi_{\rho_1}^{\perp}\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} + \|\pi_{\rho_1}\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} = \|\vec{u}_{\ell_0}\|^2_{\mathcal{H}(r\geq\rho_1)} < \varepsilon \end{aligned} \tag{4.4.61}$$

By (4.4.58) we also have

$$\begin{aligned}\int_{\rho_1}^{\infty} \left(\frac{1}{r}\partial_r v_{\ell_0,0}(r)\right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r v_{\ell_0,1}(r))^2\, dr \lesssim \rho_1^{-\frac{31}{3}} v_{\ell_0,0}^2(\rho_1) + \rho_1^{-\frac{29}{3}} v_{\ell_0,0}^4(\rho_1) + \\ + \rho_1^{-9} v_{\ell_0,0}^6(\rho_1) + \rho_1^{-\frac{25}{3}} v_{\ell_0,1}^2(\rho_1) + \rho_1^{-\frac{17}{3}} v_{\ell_0,1}^4(\rho_1) + \rho_1^{-3} v_{\ell_0,1}^6(\rho_1)\end{aligned} \tag{4.4.62}$$

Arguing as in Corollary 4.4.8 and using the fact that $v_0(\rho_0) = v_1(\rho_0) = 0$ gives

$$\left|v_{\ell_0,0}(\rho_1)\right| = \left|v_{\ell_0,0}(\rho_1) - v_{\ell_0,0}(\rho_0)\right| \lesssim \varepsilon \left|v_{\ell_0,0}(\rho_1)\right| + \rho_1 \varepsilon \left|v_{\ell_0,1}(\rho_1)\right| \tag{4.4.63}$$

and

$$\left|v_{\ell_0,1}(\rho_1)\right| = \left|v_{\ell_0,1}(\rho_1) - v_{\ell_0,1}(\rho_0)\right| \lesssim \rho_1^{-1} \varepsilon \left|v_{\ell_0,0}(\rho_1)\right| + \varepsilon \left|v_{\ell_0,1}(\rho_1)\right| \tag{4.4.64}$$

Plugging (4.4.63) into (4.4.64) gives

$$\left|v_{\ell_0,1}(\rho_1)\right| \lesssim \rho_1^{-1} \varepsilon^2 \left|v_{\ell_0,0}(\rho_1)\right| + \varepsilon(1+\varepsilon) \left|v_{\ell_0,1}(\rho_1)\right|$$

which means that for ε small enough we have

$$\left|v_{\ell_0,1}(\rho_1)\right| \lesssim \rho_1^{-1} \varepsilon^2 \left|v_{\ell_0,0}(\rho_1)\right| \tag{4.4.65}$$

Putting this estimate back into (4.4.63) we obtain

$$\left|v_{\ell_0,0}(\rho_1)\right| \lesssim \varepsilon \left|v_{\ell_0,0}(\rho_1)\right| + \varepsilon^3 \left|v_{\ell_0,0}(\rho_1)\right| \lesssim \varepsilon(1+\varepsilon^2) \left|v_{\ell_0,0}(\rho_1)\right|$$

which implies that $v_{\ell_0,0}(\rho_1) = 0$ as long as ε is chosen small enough. By (4.4.65) we can conclude that $v_{\ell_0,1}(\rho_1) = 0$ as well. By (4.4.62) and (4.4.61) we then have that

$$\|\vec{u}_{\ell_0}\|_{\mathcal{H}(r \geq \rho_1)} = 0$$

which is a contradiction since $\rho_1 < \rho_0$. Therefore, $(\partial_r u_{\ell_0,0}, u_{\ell_0,1}) = (0,0)$ Since $u_{\ell_0}(r) \to 0$ as $r \to \infty$ we can also conclude that $(u_{\ell_0,0}, u_{\ell_0,1}) = (0,0)$. $\square$

4.4.4 Proof of Proposition 4.4.1 and Proof of Theorem 4.0.3

For clarity, we summarize what we have done in the proof of Proposition 4.4.1.

Proof of Proposition 4.4.1. Let $\vec{u}(t)$ be a solution to (4.1.8) and suppose that the trajectory

$$K = \{\vec{u}(t) \mid t \in \mathbb{R}\}$$

is pre-compact in $\mathcal{H}$. We recall that

$$r\vec{u}(t,r) = \vec{\psi}(t,r) - (Q_n(r), 0)$$

where $\vec{\psi}(t) \in \mathcal{H}_n$ is a degree n wave map, i.e., a solution to (4.0.2). By Lemma 4.4.5 there exists $\ell_0 \in \mathbb{R}$ so that

$$\left| r^3 u_0(r) - \ell_0 \right| = O(r^{-3}) \quad \text{as} \quad r \to \infty \tag{4.4.66}$$

$$\left| r \int_r^\infty u_1(\rho)\rho \, d\rho \right| = O(r^{-1}) \quad \text{as} \quad r \to \infty \tag{4.4.67}$$

If $\ell_0 \neq 0$ then by Lemma 4.4.15, $\psi(0,r) = Q_{\ell_0}$ where Q_{ℓ_0} is defined in (4.4.51). However, this is impossible since $Q_{\ell_0}(1) \neq 0$, which contradicts the Dirichlet boundary condition $\psi(t,1) = 0$ for all $t \in \mathbb{R}$.

Hence, $\ell_0 = 0$. Then by Lemma 4.4.13 we can conclude that $\vec{u}(0) = (0,0)$, which proves Proposition 4.4.1. □

The proof of Theorem 4.0.3 is now complete. We conclude by summarizing the argument.

Proof of Theorem 4.0.3. Suppose that Theorem 4.0.3 fails. Then by Proposition 4.2.6 there exists a critical element, that is, a nonzero solution $\vec{u}_*(t) \in \mathcal{H}$ to (4.1.8) such that the trajectory $K = \{\vec{u}_*(t) \mid t \in \mathbb{R}\}$ is pre-compact in $\mathcal{H}$. However, Proposition 4.4.1 implies that any such solution is necessarily identically equal to $(0,0)$, which contradicts the fact that the critical element $\vec{u}_*(t)$ is nonzero. $\square$

CHAPTER 5

CLASSIFICATION OF $2D$ EQUIVARIANT WAVE MAPS TO POSITIVELY CURVED TARGETS: PART I

5.1 Introduction

In this chapter we consider energy critical equivariant wave maps. We restrict out attention to the corotational case $\ell = 1$, and study maps $U : (\mathbb{R}^{1+2}, \eta) \to (\mathbb{S}^2, g)$, where g is the round metric on $\mathbb{S}^2$. In spherical coordinates,

$$(\psi, \omega) \mapsto (\sin\psi\cos\omega, \sin\psi\sin\omega, \cos\psi),$$

on $\mathbb{S}^2$, the metric g is given by the matrix $g = \mathrm{diag}(1, \sin^2(\psi))$. In the 1-equivariant setting, we thus require our wave map, U, to have the form

$$U(t, r, \omega) = (\psi(t, r), \omega) \mapsto (\sin\psi(t, r)\cos\omega,\ \sin\psi(t, r)\sin\omega,\ \cos\psi(t, r)),$$

where (r, ω) are polar coordinates on $\mathbb{R}^2$. In this case, the Cauchy problem (1.1.8) reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1). \end{aligned} \tag{5.1.1}$$

cp flat In this equivariant setting, the conservation of energy becomes

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr = \text{const.} \tag{5.1.2}$$

Any $\psi(r,t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(t,0) = m\pi$ and $\psi(t,\infty) = n\pi$ for all $t \in I$, where m, n are fixed integers. This requirement splits the energy space into disjoint classes according to this topological condition. The wave map evolution preserves these classes.

In light of this discussion, the natural spaces in which to consider Cauchy data for (5.1.1) are the energy classes

$$\mathcal{H}_{m,n} := \{(\psi_0, \psi_1) \,|\, \mathcal{E}(\psi_0, \psi_1) < \infty \quad \text{and} \quad \psi_0(0) = m\pi,\, \psi_0(\infty) = n\pi\}. \tag{5.1.3}$$

We will mainly consider the spaces $\mathcal{H}_{0,n}$ and we denote these by $\mathcal{H}_n := \mathcal{H}_{0,n}$. In this case we refer to n as the degree of the map. We also define $\mathcal{H} = \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n$ to be the full energy space.

In the analysis of 1-equivariant wave maps to the sphere, an important role is played by the harmonic map, Q, given by stereographic projection. In spherical coordinates, Q is given by $Q(r) = 2\arctan(r)$ and is a solution to

$$Q_{rr} + \frac{1}{r} Q_r = \frac{\sin(2Q)}{2r^2}. \tag{5.1.4}$$

One can show via an explicit calculation that $(Q, 0)$ is an element of $\mathcal{H}_1$, i.e., Q has finite energy and sends the origin in $\mathbb{R}^2$ to the north pole and spacial infinity to the south pole. In fact, the energy $\mathcal{E}(Q) := \mathcal{E}(Q, 0) = 4$ is minimal in $\mathcal{H}_1$ and simple phase space analysis shows that, up to a rescaling, $(Q, 0)$ is the unique, nontrivial, 1-equivariant harmonic map to the sphere in $\mathcal{H}_1$. Note the slight abuse of notation above in that we will denote the energy of the element $(Q, 0) \in \mathcal{H}_1$ by $\mathcal{E}(Q)$ rather than $\mathcal{E}(Q, 0)$.

It has long been understood that in the energy-critical setting, the geometry of the target should play a decisive role in determining the asymptotic behavior of wave maps. For equivariant wave maps, global well-posedness for all smooth data was established by Struwe in [76] in the case where the target manifold does not admit a non-constant finite energy harmonic sphere. This extended the results of Shatah, Tahvildar-Zadeh [70], and Grillakis [30], where global well-posedness was proved for targets satisfying a geodesic convexity condition. Recently, global well-posedness, including scattering, has been established in the full (non-equivariant), energy critical wave maps problem in a remarkable series of works [49], [74], [75], [79], for targets that do not admit finite energy harmonic spheres, completing the program developed in [81], [78].

However, finite-time blow-up can occur in the case of compact targets that admit non-constant harmonic spheres. Because we are working in the equivariant, energy critical setting, blow-up can only occur at the origin and in an energy concentration scenario which amounts to a breakdown in regularity. Moreover, in [76], Struwe showed that if a solution is C^∞ before a regularity breakdown occurs, then such a scenario can only happen by the bubbling off of a non-constant harmonic map.

In particular, Struwe showed that if a solution, $\psi(t,r)$, with smooth initial data $\vec{\psi}(0) = (\psi(0), \dot{\psi}(0))$, breaks down at $t = 1$, then the energy concentrates at the origin and there is a sequence of times $t_j \nearrow 1$ and scales $\lambda_j > 0$ with $\lambda_j \ll 1 - t_j$ so that the rescaled sequence of wave maps

$$\vec{\psi}_j(t,r) := \Big(\psi(t_j + \lambda_j t, \lambda_j r), \lambda_j \dot{\psi}\left(t_j + \lambda_j t, \lambda_j r\right)\Big)$$

converges *locally* to $\pm Q(r/\lambda_0)$ in the space-time norm $H^1_{\mathrm{loc}}((-1,1)\times\mathbb{R}^2;\mathbb{S}^2)$ for some $\lambda_0 > 0$. Further evidence of finite time blow up for equivariant wave maps to the sphere was provided by Cote, [14]. Recently, explicit blow-up solutions have been constructed in [63] for equivariance classes $\ell \geq 4$ and in the 1-equivariant case in [50], [51] and [62]. In [50], Krieger, Schlag, and Tataru constructed explicit blow-up solutions with prescribed blow-up rates $\lambda(t) = (1-t)^{1+\nu}$ for $\nu > \frac{1}{2}$ although it is believed that all rates with $\nu > 0$ are possible as well. In [51], a similar result is given for the radial, energy critical Yang Mills equation. In [62], Rodnianski and Raphaël give a description of stable blow-up dynamics for equivariant wave maps and the radial, energy critical Yang Mills equation in an open set about Q in a stronger topology than the energy.

Our goal in this chapter is twofold. On one hand, we study the asymptotic behavior of solutions to (5.1.1) with data in the "zero" topological class, i.e., $\vec{\psi}(0) \in \mathcal{H}_0$, below a sharp energy threshold, namely $2\mathcal{E}(Q)$. Additionally, we seek to classify the behavior of wave maps of topological degree one, i.e., those with data $\vec{\psi} \in \mathcal{H}_1$, that blow up in finite time with energies below the threshold $3\mathcal{E}(Q)$. In particular, we show that blow-up profiles exhibited in the works [50], [63] and [62] are universal in this energy regime in a precise sense described below in Section 5.1.2.

5.1.1 Global existence and scattering for wave maps in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$

We begin with a description of our results in the degree zero case. In [76], Struwe's work implies that solutions $\vec{\psi}(t)$ to (5.1.1) with data $\vec{\psi}(0) \in \mathcal{H}_0$ are global in time if $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$. This follows directly from the fact that wave maps in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$ stay bounded away from the south pole and hence cannot converge, even locally, to a degree one rescaled harmonic map, thus ruling out blow-up. Recently, the Cote, Kenig, and Merle, [17], extended this result to include scattering to zero in the regime, $\vec{\psi}(0) \in \mathcal{H}_0$ and $\mathcal{E}(\vec{\psi}) \le \mathcal{E}(Q) + \delta$ for small $\delta > 0$. It was conjectured in [17] that scattering should also hold for all energies up to $2\mathcal{E}(Q)$. This conjecture is a refined version of what is usually called *threshold conjecture*, adapted to the case of topologically trivial equivariant data. It is implied by the recent work of Sterbenz and Tataru in [74], [75] when one considers their results in the equivariant setting with topologically trivial data. Here we give an alternate proof of this *refined threshold conjecture* in the equivariant setting based on the concentration compactness/rigidity method of Kenig and Merle, [36], [37]. In particular, we prove the following:

Theorem 5.1.1 (Global Existence and Scattering in $\mathcal{H}_0$ below $2\mathcal{E}(Q)$)**.** *For any smooth data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$, there exists a unique global evolution $\vec{\psi} \in C^0(\mathbb{R}; \mathcal{H}_0)$. Moreover, $\vec{\psi}(t)$ scatters to zero in the sense that the energy of $\vec{\psi}(t)$ on any arbitrary, but fixed compact region vanishes as $t \to \infty$. In other words, one has*

$$\vec{\psi}(t) = \vec{\varphi}(t) + o_{\mathcal{H}}(1) \quad as \quad t \to \infty \tag{5.1.5}$$

where $\vec{\varphi} \in \mathcal{H}$ solves the linearized version of (5.1.1)*, i.e.,*

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0 \tag{5.1.6}$$

Furthermore, this result is sharp in $\mathcal{H}_0$ in sense that $2\mathcal{E}(Q)$ is a true threshold. Indeed for all $\delta > 0$ there exists data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}) \leq 2\mathcal{E}(Q) + \delta$, such that $\vec{\psi}$ blows up in finite time.

Remark 10. Characterizing the possible dynamics at the threshold, $\vec{\psi} \in \mathcal{H}_0$, $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(Q)$ and above $\mathcal{E}(\vec{\psi}) > 2\mathcal{E}(Q)$, remain open questions.

Remark 11. We briefly remark that Theorem 5.1.1 holds with the same assumptions and conclusions for data $\vec{\psi} \in \mathcal{H}_{n,n}$ where $\mathcal{H}_{n,n}$ is defined as in (5.1.3). Indeed, the spaces $\mathcal{H}_0$ and $\mathcal{H}_{n,n}$ are isomorphic via the map $(\psi_0, \psi_1) \mapsto (\psi_0 + n\pi, \psi_1)$. Also, we can replace the words "smooth finite energy data" in Theorem 5.1.1 with just "finite energy data" using the well-posedness theory for (5.1.1), see for example [17].

As mentioned above, Theorem 5.1.1 is established by the concentration compactness – rigidity method of Kenig and Merle in [36] and [37]. The novel aspect of our implementation of this method lies in the development of a robust rigidity theory for wave maps $\vec{U}(t)$ with trajectories that are pre-compact in the energy space up to certain time-dependent modulations. We note that the following theorem is independent of both the topological class and the energy of the wave map.

Theorem 5.1.2 (Rigidity)**.** *Let $\vec{U}(t,r,\omega) = ((\psi(t,r),\omega),(\dot{\psi}(t,r),0)) \in \mathcal{H}$ be a solution to* (5.1.1) *and let $I_{\max}(\psi) = (T_-(\psi), T_+(\psi))$ be the maximal interval of existence. Suppose that there exists $A_0 > 0$ and a continuous function $\lambda : I_{\max} \to [A_0, \infty)$ such that the set*

$$\tilde{K} := \left\{ \left(U\left(t, \frac{r}{\lambda(t)}, \omega\right), \frac{1}{\lambda(t)} \partial_t U\left(t, \frac{r}{\lambda(t)}, \omega\right)\right) \Big| \, t \in I_{max} \right\} \tag{5.1.7}$$

is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^2;\mathbb{S}^2)$. Then, $I_{\max} = \mathbb{R}$ and either $U \equiv 0$ or $U : \mathbb{R}^2 \to \mathbb{S}^2$ is an equivariant harmonic map, i.e., $U(t,r,\omega) = (\pm Q(r/\tilde{\lambda}),\omega)$ for some $\tilde{\lambda} > 0$.

Remark 12. To establish Theorem 5.1.1 we only need a version of Theorem 5.1.2 that deals with data in $\mathcal{H}_0$ below $2\mathcal{E}(Q)$. This rigidity result in $\mathcal{H}_0$ is given in Theorem 5.4.1 below, and states that any solution $\vec{\psi} \in \mathcal{H}_0$ with a pre-compact rescaled trajectory must be identically zero. The full result in Theorem 5.1.2 is established for its own interest. In fact, we use the conclusions of Theorem 5.1.1 in order to deduce the full classification of pre-compact solutions given in Theorem 5.1.2. Alternatively, we can prove Theorem 5.1.2 using the scattering result of [17, Theorem 1], and deduce Theorem 5.4.1 as a corollary. We have chosen the former approach here to illustrate the independence of our stronger rigidity results from the variational arguments given in [17, Lemma 7].

5.1.2 Classification of blow-up solutions in $\mathcal{H}_1$ with energies below $3\mathcal{E}(Q)$

We now turn to the issue of describing blow-up for wave maps in $\mathcal{H}_1$, i.e., those maps $\vec{\psi}(t)$ with $\psi(t,0) = 0$ and $\psi(t,\infty) = \pi$. From here on out, any wave map that is assumed to blow-up will be also be assumed to do so at time $t = 1$. As mentioned above, the recent works [50] and [62] construct explicit blow-up solutions $\psi(t) \in \mathcal{H}_1$. In [50], the blow up solutions constructed there exhibit a decomposition of the form

$$\psi(t,r) = Q(r/\lambda(t)) + \epsilon(t,r) \tag{5.1.8}$$

where the concentration rate satisfies $\lambda(t) = (1-t)^{1+\nu}$ for $\nu > \frac{1}{2}$, and $\epsilon(t) \in \mathcal{H}_0$ is small and regular. Here we consider the converse problem. Namely, if blow-up does occur for a solution $\vec{\psi}(t) \in \mathcal{H}_1$, in which energy regime, and in what sense does such a decomposition always hold?

The works of Struwe, in [76] for the equivariant case, and Sterbenz, Tataru in [75] for the full wave map problem, give a partial answer to this question. As mentioned above, they show that if blow-up occurs, then along a sequence of times, a sequence of rescaled versions of the original wave map converge *locally* to Q in the space-time norm $H^1_{\text{loc}}((-1,1) \times \mathbb{R}^2; \mathbb{S}^2)$. However working locally removes any knowledge of the topology of the wave map, which is determined by the behavior of the map at spacial infinity. In this chapter we seek to strengthen the results in [76] and [75] in the equivariant setting by working globally in space in the *energy topology*. Here we are forced to account for the topological restrictions of a degree one wave map, and in fact we use these restrictions, along with our degree zero theory, to our advantage.

In particular, we make the following observation. If a wave map $\psi(t) \in \mathcal{H}_1$ blows up at $t = 1$ then the local convergence results of Struwe in [76] allow us to extract the blow up profile $\pm Q_{\lambda_n} := \pm Q(\cdot/\lambda_n)$ at least along a sequence of times $t_n \to 1$. If $\vec{\psi}$ has energy below $3\mathcal{E}(Q)$ the profile must be $+Q(\cdot/\lambda_n)$, and since $Q \in \mathcal{H}_1$ as well we thus have $\psi(t_n) - Q_{\lambda_n} \in \mathcal{H}_0$. Since this object should converge locally to zero, the energy of the difference should be roughly the difference of the energies, at least for large n. Hence, if $\psi(t)$ has energy below $3\mathcal{E}(Q)$ the difference $\psi(t_n) - Q_{\lambda_n}$ is degree zero and has energy below $2\mathcal{E}(Q)$. By Theorem 5.1.1, we then suspect that the blow-up profile already extracted is indeed universal in this regime and that a decomposition of the form (5.1.8) should indeed hold, excluding the possibility of any different dynamics, such as more bubbles forming. We prove the following result:

Theorem 5.1.3 (Classification of blow-up solutions in $\mathcal{H}_1$ with energies below $3\mathcal{E}(Q)$). *Let*

$\vec{\psi}(t) \in \mathcal{H}_1$ be a smooth solution to (5.1.1) blowing up at time $t = 1$ with

$$\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q).$$

Then, there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \eta$, and a decomposition

$$\vec{\psi}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + \vec{\epsilon}(t) \tag{5.1.9}$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

Remark 13. In the companion work [16] we address the question of *global solutions* $\psi(t) \in \mathcal{H}_1$ in the regime $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$. We can show that in this case we have a decomposition and convergence as in (5.1.9) with $\lambda(t) \ll t$ as $t \to \infty$. This will give us a complete classification of the possible dynamics in $\mathcal{H}_1$ for energies below $3\mathcal{E}(Q)$. Of course, our results do not give information about the precise rates $\lambda(t)$. We also would like to mention the recent results of Bejenaru, Krieger, and Tataru [3], regarding wave maps in $\mathcal{H}_1$, where they prove asymptotic orbital stability for a co-dimension two class of initial data which is "close" to Q_λ with respect to a stronger topology than the energy.

Remark 14. Theorem 5.1.3 is reminiscent of the recent results proved by Duyckaerts, Kenig, and Merle in [22], [21], for the energy critical focusing semi-linear wave equation in $\mathbb{R}^{1+3}$. In fact, the techniques developed in these works provided important ideas for the proof of Theorem 5.1.3. The situation for wave maps is somewhat different, however, as the geometric nature of the problem provides some key distinctions. The most notable of these distinctions is that the underlying linear theory for wave maps of degree zero is not nearly as strong as that of a semi-linear wave in $\mathbb{R}^{1+3}$, which causes serious problems. Indeed, as demonstrated in [18], the strong lower bound on the exterior energy in [22, Lemma 4.2] *fails* for general initial data in even dimensions. This difficulty is overcome by the fact that there is no self-similar blow-up for for energy critical equivariant wave maps, see e.g., [68], which can be shown directly due to the non-negativity of the energy density.

In addition, our degree zero result and the rigid topological restrictions of the problem allow us to extend the conclusions of Theorem 5.1.3 all the way up to $3\mathcal{E}(Q)$ instead of just slightly above the energy of the harmonic map $\mathcal{E}(Q) + \delta$, for $\delta > 0$ small, as is the case in [22], [21]. This large enegy result is similar in nature to the results for the $3d$ semi-linear radial wave equation in [24], when, in the notation from [24], $J_0 = 1$.

Remark 15. The results in [22], [21] have recently been extended by Duyckaerts, Kenig, and Merle in [24] and [23]. In [23], a classification of solutions to the radial, energy critical, focusing semi-linear wave equation in $\mathbb{R}^{1+3}$ of *all* energies is given in the sense that only three scenarios are shown to be possible; (1) type I blow-up; (2) type II blow-up with the solution decomposing into a sum of blow-up profiles arising from rescaled solitons plus a radiation term; or (3) the solution is global and decomposes into a sum of rescaled solitons plus a radiation term as $t \to \infty$.

5.1.3 Remarks on the proofs of the main results

In addition to the methods originating in [36], [37] and [22], [21], the work in this chapter rests explicitly on several developments in the field over the past two decades. Here we provide a quick guide to the work on which our results lie:

Results used in the proof of Theorem 5.1.1

- Theory of equivariant wave maps developed in the nineties in the works of Shatah, Tahvildar-Zadeh, [70], [71], including the use of virial identities to prove energy decay estimates.

- The concentration compactness decomposition of Bahouri-Gérard, [1].

- Lemma 2 in [17] which relates energy constraints to L^∞ estimates for equivariant wave maps. In particular, if a degree zero map has energy less than $2\mathcal{E}(Q)$, then the evolution, $\psi(t,r)$, is bounded uniformly below π. In addition, although only a weaker small data result such as [68, Theorem 8.1] is needed, we use the global existence and scattering result for degree one wave maps with energy below $\mathcal{E}(Q)+\delta$ for small $\delta > 0$, which was established in [18, Theorem 1].

- Hélein's theorem on the regularity of harmonic maps which says that a weakly harmonic map is, in fact, harmonic, [32].

Results used in the proof of Theorem 5.1.3

- The virial identity and the corresponding energy decay estimates in [70].

- Struwe's characterization of blow-up, [76, Theorem 2.2], which gives H^1_{loc} convergence along a sequence of times to Q if blow-up occurs. This allows us, a priori, to identify and extract the blow-up profile Q_{λ_n} along a sequence of times, t_n, which is absolutely crucial in our argument since we can then work with degree zero maps once Q_{λ_n} has been subtracted from the degree one maps $\psi(t_n)$.

- The concentration compactness decomposition of Bahouri-Gérard, [1].

- The new results on the free radial $4d$ wave equation established by the Cote, Kenig and Schlag in [18].

- The decomposition of degree one maps which have energy slightly above Q and the stability of this decomposition under the wave map evolution for a period of time inversely proportional to the proximity of the data to Q in the energy space established by Cote [14].

As we outline in the appendix, the proofs of Theorem 5.1.1, Theorem 5.1.2, and Theorem 5.1.3 extend easily to energy critical 1-equivariant wave maps with more general targets. In addition, the proofs of Theorem 5.1.2 and Theorem 5.1.1 apply equally well to the equivariance classe $\ell = 2$ and the $4d$ equivariant Yang-Mills system after suitable modifications. One should also be able to deduce these results for the equivariance classes $\ell \geq 3$ once a suitable small data theory is established for these equations, which are similar in nature to the even dimensional energy critical semi-linear wave equations in high dimensions treated in [8] – the difficulty here resides in the low fractional power in the nonlinearity.

However, the method we used to prove Theorem 5.1.3 only works, as developed here, for odd equivariance classes, $\ell = 1, 3, 5, \ldots$, and does not work when one considers even equivariance classes, $\ell = 2, 4, 6, \ldots$, or the $4d$ equivariant Yang-Mills system in this context. This failure of our technique arises in the linear theory in [18] for even dimensions, which provides favorable estimates for our proof scheme only when ℓ is odd. Since the $4d$ equivariant Yang-Mills system corresponds roughly to a 2-equivarant wave map, this falls outside the scope of our current method as well. To be more specific, one can identify the linearized ℓ-equivariant wave map equation with the $2\ell + 2$-dimensional free radial wave equation. In the final stages of the proof of Theorem 5.1.3, and in particular Corollary 5.5.8, we require the exterior energy estimate

$$\|f\|_{\dot{H}^1} \lesssim \|S(t)(f, 0)\|_{\dot{H}^1 \times L^2(r \geq t)} \quad \text{for all} \quad t \geq 0$$

where $S(t)$ is the the free radial wave evolution operator. In [18], this estimate is shown to be true in even dimensions $4, 8, 12, \ldots$, and false in dimensions $2, 6, 10, \ldots$. Without this estimate, our proof would show compactness of the error term in our decomposition in a certain suitable Strichartz space but not in the energy space. Therefore, the full conclusion of Theorem 5.1.3 remains open for the $4d$ equivariant Yang-Mills system and the ℓ-equivariant wave map equation when ℓ is even.

5.1.4 Structure of this Chapter

The outline of this chapter is as follows. In Section 5.2 we establish the necessary preliminaries needed for the rest of the work. We include a brief review of the results of Shatah, Tahvildhar-Zadeh, [70] and Struwe [76]. We also recall the concentration compactness decomposition of Bahouri, Gérard [1] and adapt their theory to case of equivariant wave maps to the sphere. In particular, we deduce a Pythagorean expansion of the nonlinear wave map energy of such a decomposition at a fixed time. This type of result is crucial in the concentration compactness/rigidity method of [36], [37]. We also establish an appropriate nonlinear profile decomposition.

In Section 5.3 we give a brief outline of the concentration compactness/rigidity method that is used to prove Theorem 5.1.1. In Section 5.4 we prove Theorem 5.1.2, which allows us to complete the proof of Theorem 5.1.1.

Finally, in Section 5.5 we establish Theorem 5.1.3, which relies crucially on the linear theory developed in [18].

5.1.5 Notation and Conventions

We will interchangeably use the notation $\psi_t(t,r)$ and $\dot{\psi}(t,r)$ to refer to the derivative with respect to the time variable t of the function $\psi(t,r)$.

The notation $X \lesssim Y$ means that there exists a constant $C > 0$ such that $X \leq CY$. Similarly, $X \simeq Y$ means that there exist constants $0 < c < C$ so that $cY \leq X \leq CY$.

5.2 Preliminaries

We define the energy space

$$\mathcal{H} = \{\vec{U} \in \dot{H}^1 \times L^2(\mathbb{R}^2; \mathbb{S}^2) \,|\, U \circ \rho = \rho \circ U, \;\; \forall \rho \in SO(2)\}.$$

$\mathcal{H}$ is endowed with the norm

$$\mathcal{E}(\vec{U}(t)) = \|\vec{U}(t)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^2;\mathbb{S}^2)} = \int_{\mathbb{R}^2} (|\partial_t U|_g^2 + |\nabla U|_g^2)\, dx. \tag{5.2.1}$$

As noted in the introduction, by our equivariance condition we can write $U(t, r, \omega) = (\psi(t, r), \omega)$ and the energy of a wave map becomes

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr = \text{const}. \tag{5.2.2}$$

We also define the localized energy as follows: Let $r_1, r_2 \in [0, \infty)$. Then we set

$$\mathcal{E}_{r_1}^{r_2}(\vec{\psi}(t)) := \int_{r_1}^{r_2} \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr.$$

Following Shatah and Struwe, [68], we set

$$G(\psi) := \int_0^\psi |\sin\rho|\ d\rho. \tag{5.2.3}$$

Observe that for any $(\psi, 0) \in \mathcal{H}_n$ and for any $r_1, r_2 \in [0, \infty)$ we have

$$\begin{aligned} |G(\psi(r_2)) - G(\psi(r_1))| &= \left|\int_{\psi(r_1)}^{\psi(r_2)} |\sin\rho|\ d\rho\right| \\ &= \left|\int_{r_1}^{r_2} |\sin(\psi(r))|\, \psi_r(r)\, dr\right| \le \frac{1}{2}\mathcal{E}_{r_1}^{r_2}(\psi, 0) \end{aligned} \tag{5.2.4}$$

5.2.1 *Properties of degree zero wave maps*

As in [17], let $\alpha \in [0, 2\mathcal{E}(Q)]$ and define the set $V(\alpha) \subset \mathcal{H}_0$:

$$V(\alpha) := \{(\psi_0, \psi_1) \in \mathcal{H}_0 \,|\, \mathcal{E}(\psi_0, \psi_1) < \alpha\}$$

We claim that for every $\alpha \in [0, 2\mathcal{E}(Q)]$, $V(\alpha)$ is naturally endowed with the norm

$$\|(\psi_0, \psi_1)\|_{H \times L^2}^2 = \int_0^\infty \left(\psi_1^2 + (\psi_0)_r^2 + \frac{\psi_0^2}{r^2} \right) r \, dr \tag{5.2.5}$$

To see this, we recall the following lemma proved in [17].

Lemma 5.2.1. *[17, Lemma 2] There exists an increasing function $K : [0, 2\mathcal{E}(Q)) \to [0, \pi)$ such that*

$$|\psi(r)| \leq K(\mathcal{E}(\vec{\psi})) < \pi \quad \forall \vec{\psi} \in \mathcal{H}_0 \quad \textit{with} \quad \mathcal{E}(\psi) < 2\mathcal{E}(Q) \tag{5.2.6}$$

Moreover, for each $\alpha \in [0, 2\mathcal{E}(Q)]$ we have

$$\mathcal{E}(\psi_0, \psi_1) \simeq \|(\psi_0, \psi_1)\|_{H \times L^2} \tag{5.2.7}$$

for every $(\psi_0, \psi_1) \in V(\alpha)$, with the constant above depending only on α.

When considering Cauchy data for (5.1.1) in the class $\mathcal{H}_0$ the formulation in (5.1.1) can be modified in order to take into account the strong repulsive potential term that is hidden in the nonlinearity:

$$\frac{\sin(2\psi)}{2r^2} = \frac{\psi}{r^2} + \frac{\sin(2\psi) - 2\psi}{2r^2} = \frac{\psi}{r^2} + \frac{O(\psi^3)}{r^2}$$

Indeed, the presence of the strong repulsive potential $\frac{1}{r^2}$ indicates that the linearized operator of (5.1.1) has more dispersion than the 2-dimensional wave equation. In fact, it has the same dispersion as the 4-dimensional wave equation as the following standard reduction shows.

Setting $\psi = ru$ we are led to this equation for u:

$$u_{tt} - u_{rr} - \frac{3}{r}u_r + \frac{\sin(2ru) - 2ru}{2r^3} = 0 \tag{5.2.8}$$
$$\vec{u}(0) = (u_0, u_1).$$

The nonlinearity above has the form $N(u, r) = u^3 Z(ru)$ where Z is a smooth, bounded, even function and the linear part is the radial d'Alembertian in $\mathbb{R}^{1+4}$. The linearized version of (5.2.8) is just the free radial wave equation in $\mathbb{R}^{1+4}$, namely

$$v_{tt} - v_{rr} - \frac{3}{r}v_r = 0. \tag{5.2.9}$$

Observe that for $\vec{\psi}(0) \in \mathcal{H}_0$ we have that

$$\mathcal{E}(\vec{\psi}(0)) \leq \|\vec{\psi}\|^2_{H\times L^2} := \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\psi^2}{r^2}\right) r\,dr = \int_0^\infty (u_t^2 + u_r^2)\, r^3\,dr. \tag{5.2.10}$$

If, in addition, we assume that $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$ then, by Lemma 5.2.1 we also have the opposite inequality

$$\|\vec{u}(0)\|^2_{\dot{H}^1\times L^2} = \|\vec{\psi}(0)\|^2_{H\times L^2} \lesssim \mathcal{E}(\vec{\psi}(0)). \tag{5.2.11}$$

Therefore, when considering initial data $(\psi_0, \psi_1) \in V(\alpha)$ for $\alpha \leq 2\mathcal{E}(Q)$ the Cauchy problem (5.1.1) is equivalent to the Cauchy problem for (5.2.8) for radial initial data $(r\psi_0, r\psi_1) =: \vec{u}(0) \in \dot{H}^1 \times L^2(\mathbb{R}^4)$.

The following exterior energy estimates for the $4d$ free radial wave equation established by Cote, Kenig, and Schlag [18] will play a key role in our analysis:

Proposition 5.2.2. *[18, Corollary 5] Let $S(t)$ denote the free evolution operator for the $4d$*

radial wave equation, (5.2.9)*. There exists* $\alpha_0 > 0$ *such that for all* $t \geq 0$ *we have*

$$\|S(t)(f, 0)\|_{\dot{H}^1 \times L^2(r \geq t)} \geq \alpha_0 \|f\|_{\dot{H}^1} \tag{5.2.12}$$

for all radial data $(f, 0) \in \dot{H}^1 \times L^2$*.*

The point here is that this same result applies to the linearized version of the wave map equation:

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0 \tag{5.2.13}$$

with initial data $\vec{\varphi}(0) = (\varphi_0, 0)$. Indeed we have the following:

Corollary 5.2.3. *Let* $W(t)$ *denote the linear evolution operator associated to* (5.2.13)*. Then there exists* $\beta_0 > 0$ *such that for all* $t \geq 0$ *we have*

$$\|W(t)(\varphi_0, 0)\|_{H \times L^2(r \geq t)} \geq \beta_0 \|\varphi_0\|_H \tag{5.2.14}$$

for all radial initial data $(\varphi_0, 0) \in H \times L^2$*.*

Proof. Let $\vec{\varphi}(t) = W(t)(\varphi_0, 0)$ be the linear evolution of the smooth radial data $(\varphi_0, 0) \in H \times L^2$. Define $\vec{v}(t)$ by $\varphi(t, r) = rv(t, r)$. Then $\vec{v}(t) \in \dot{H}^1 \times L^2(\mathbb{R}^4)$ and is a solution to (5.2.9) with initial data $(v_0, 0) = (\frac{\varphi_0}{r}, 0)$. Next observe that for all $A \geq 0$ we have

$$\begin{aligned} \|v(t)\|^2_{\dot{H}^1(r \geq A)} = \int_A^\infty v_r^2(t, r)\, r^3\, dr = \int_A^\infty \left(\frac{\varphi_r(t, r)}{r} - \frac{\varphi(t, r)}{r^2} \right)^2 r^3\, dr \\ \leq 2\|\varphi(t)\|^2_{H(r \geq A)} \end{aligned}$$

Similarly we can show that $\|\varphi(t)\|^2_{H(r \geq A)} \leq 2\|v(t)\|^2_{\dot{H}^1(r \geq A)}$. Therefore using (5.2.12) on

$v(t)$ we obtain

$$\|\vec{\varphi}(t)\|^2_{H\times L^2(r\geq t)} \geq \frac{1}{2}\|v(t)\|^2_{\dot{H}^1(r\geq t)} \geq \frac{\alpha_0^2}{2}\|v_0\|^2_{\dot{H}^1} = \frac{\alpha_0^2}{2}\|\varphi_0\|^2_H$$

which proves (5.2.14) with $\beta_0 = \frac{\alpha_0}{\sqrt{2}}$. $\square$

5.2.2 *Properties of degree one wave maps*

Now, suppose $\vec{\psi} = (\psi_0, \psi_1) \in \mathcal{H}_1$. This means that $\psi(0) = 0$ and $\psi(\infty) = \pi$. The $H \times L^2$ norm of $\vec{\psi}$ is no longer finite, but we do have the following comparison:

Lemma 5.2.4. *Let $\vec{\psi} = (\psi_0, 0) \in \mathcal{H}_1$ be smooth and let $r_0 \in [0, \infty)$. Then there exists $\alpha > 0$ such that*

(a) If $\mathcal{E}_0^{r_0}(\vec{\psi}) < \alpha$, then

$$\|\psi\|^2_{H(r\leq r_0)} \lesssim \mathcal{E}_0^{r_0}(\vec{\psi}). \tag{5.2.15}$$

(b) If $\mathcal{E}_{r_0}^{\infty}(\vec{\psi}) < \alpha$, then

$$\|\psi(\cdot) - \pi\|^2_{H(r\geq r_0)} \lesssim \mathcal{E}_{r_0}^{\infty}(\vec{\psi}). \tag{5.2.16}$$

Proof. We prove only the second estimate as the proof of the first is similar. Since $G(\pi) = 2$, by (5.2.4) we have for all $r \in [r_0, \infty)$ that

$$|G(\psi(r)) - 2| \leq \frac{1}{2}\mathcal{E}_r^{\infty}(\psi, 0) < \frac{\alpha}{2}.$$

Since G is continuous and increasing this means that $\psi(r) \in [\pi - \varepsilon(\alpha), \pi + \varepsilon(\alpha)]$ where $\varepsilon(\rho) \to 0$ as $\rho \to 0$. Hence for α small enough we have the estimate $\sin^2(\psi(r)) \geq \frac{1}{2}|\psi(r) - \pi|^2$ for all $r \in [r_0, \infty]$ and the estimate (5.2.16) follows by integrating this. $\square$

Let $Q(r) := 2\arctan(r)$. Note that $(Q,0) \in \mathcal{H}_1$ is the unique (up to scaling) time-independent, solution to (5.1.1) in $\mathcal{H}_1$. Indeed, Q has minimal energy in $\mathcal{H}_1$ and $\mathcal{E}(Q,0) = 4$. One way to see this is to note that Q satisfies $rQ_r(r) = \sin(Q)$ and hence for any $0 \le a \le b < \infty$ we have

$$G(Q(b)) - G(Q(a)) = \int_a^b |\sin(Q(r))|\, Q_r(r)\, dr = \frac{1}{2}\mathcal{E}_a^b(Q,0) \tag{5.2.17}$$

Letting $a \to 0$ and $b \to \infty$ we obtain $\mathcal{E}(Q,0) = 2G(\pi) = 4$. To see that $\mathcal{E}(Q,0)$ is indeed minimal in $\mathcal{H}_1$, observe that we can factor the energy as follows:

$$\begin{aligned}\mathcal{E}(\psi,\psi_t) &= \int_0^\infty \psi_t^2\, r\, dr + \int_0^\infty \left(\psi_r - \frac{\sin(\psi)}{r}\right)^2 r\, dr + 2\int_0^\infty \sin(\psi)\psi_r\, dr\\ &= \int_0^\infty \psi_t^2\, r\, dr + \int_0^\infty \left(\psi_r - \frac{\sin(\psi)}{r}\right)^2 r\, dr + 2\int_{\psi(0)}^{\psi(\infty)} \sin(\rho)\, d\rho\end{aligned}$$

Hence, in $\mathcal{H}_1$ we have

$$\mathcal{E}(\psi,\psi_t) \ge \int_0^\infty \psi_t^2\, r\, dr + 4 = \int_0^\infty \psi_t^2\, r\, dr + \mathcal{E}(Q) \tag{5.2.18}$$

We shall also require a decomposition from [14] which amounts to the coercivity of the energy near to ground state Q, up to the scaling symmetry.

Lemma 5.2.5. *[14, Proposition 2.3] There exists a function $\delta : (0,\infty) \to (0,\infty)$ such that $\delta(\alpha) \to 0$ as $\alpha \to 0$ and such that the following holds: Let $\vec{\psi} = (\psi, 0) \in \mathcal{H}_1$. Define*

$$\alpha := \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q) > 0$$

Then there exists $\lambda \in (0,\infty)$ such that

$$\|\psi - Q(\cdot/\lambda)\|_H \le \delta(\alpha)$$

Note that one can choose $\lambda > 0$ so that $\mathcal{E}_0^\lambda(\vec{\psi}) = \mathcal{E}_0^1(Q) = \mathcal{E}(Q)/2$.

We will also need the following consequence of Lemma 5.2.5 that is also proved in [14].

Corollary 5.2.6. *[14, Corollary 2.4] Let $\rho_n, \sigma_n \to \infty$ be two sequences such that $\rho_n \ll \sigma_n$. Let $\vec{\psi}_n(t) \in \mathcal{H}_1$ be a sequence of wave maps defined on time intervals $[0, \rho_n]$ and suppose that*

$$\|\vec{\psi}_n(0) - (Q, 0)\|_{H\times L^2} \le \frac{1}{\sigma_n}.$$

Then

$$\sup_{t\in[0,\rho_n]} \|\vec{\psi}_n(t) - (Q,0)\|_{H\times L^2} = o_n(1) \quad \text{as} \quad n \to \infty$$

Remark 16. We refer the reader to the proof of [14, Corollary 2.4] and the remark immediately following it for a detailed proof of Corollary 5.2.6. We have phrased the above result in terms of sequences of wave maps because this is the form in which it will be applied in Section 5.5. Also, we note that in [14] the notation $\|\cdot\|_H^2$ is used to denote the nonlinear energy, $\mathcal{E}(\cdot)$, of a map, whereas here $\|\cdot\|_H$ is defined as in (5.2.5). Both Lemma 5.2.5 and Corollary 5.2.6 hold with either definition.

5.2.3 Properties of blow-up solutions

Now let $\vec{\psi}(t) \in \mathcal{H}$ be a wave map with maximal interval of existence

$$I_{\max}(\vec{\psi}) = (T_-(\vec{\psi}), T_+(\vec{\psi})) \neq \mathbb{R}.$$

By translating in time, we can assume that $T_+(\vec{\psi}) = 1$. We recall a few facts that we will need in our argument. From the work of Shatah and Tahvildar-Zadeh [70], we have the following results:

Lemma 5.2.7. *[70, Lemma 2.2] For any $\lambda \in (0,1]$ we have*

$$\mathcal{E}^{1-t}_{\lambda(1-t)}(\vec{\psi}(t)) = \int_{\lambda(1-t)}^{1-t} \left(\psi_t^2(t,r) + \psi_r^2(t,r) + \frac{\sin^2(\psi(t,r))}{r^2}\right) r\, dr \to 0 \quad as \quad t \to 1 \tag{5.2.19}$$

Lemma 5.2.8. *[70, Corollary 2.2] Let $\vec{\psi}(t) \in \mathcal{H}$ be a solution to (5.1.1) such that $I_{\max}(\vec{\psi})$ is a finite interval. Without loss of generality we can assume $T_+(\vec{\psi}) = 1$. Then we have*

$$\frac{1}{1-t}\int_t^1 \int_0^{1-s} \dot{\psi}^2(s,r)\, r\, dr\, ds \to 0 \quad as \quad t \to 1 \tag{5.2.20}$$

As in [22], we can use Lemma 5.2.8 to establish the following result. The proof is identical to the argument given in [22, Corollary 5.3] so we do not reproduce it here.

Corollary 5.2.9. *[22, Corollary 5.3] Let $\psi(t) \in \mathcal{H}$ be a solution to (5.1.1) such that $T_+(\vec{\psi}) = 1$. Then, there exists a sequence of times $\{t_n\} \nearrow 1$ such that for every n and for every $\sigma \in (0, 1-t_n)$, we have*

$$\frac{1}{\sigma}\int_{t_n}^{t_n+\sigma} \int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \frac{1}{n} \tag{5.2.21}$$

$$\int_0^{1-t_n} \dot{\psi}^2(t_n,r)\, r\, dr \le \frac{1}{n} \tag{5.2.22}$$

Note that (5.2.22) follows from (5.2.21) by letting $\sigma \to 0$ in (5.2.21) and recalling the continuity of the map $t \mapsto \dot{\psi}(t,\cdot)$ from $[0,1) \to L^2$.

We now recall a result of Struwe, [76], which will be essential in our argument for degree 1.

Theorem 5.2.10. *[76, Theorem 2.1] Let $\psi(t) \in \mathcal{H}$ be a smooth solution to (5.1.1) such that $T_+(\vec{\psi}) = 1$. Let $\{t_n\} \nearrow 1$ be defined as in Corollary 5.2.9. Then there exists a sequence*

$\{\lambda_n\}$ with $\lambda_n = o(1-t_n)$ so that the following results hold: Let

$$\vec{\psi}_n(t,r) := (\psi(t_n + \lambda_n t, \lambda_n r), \lambda_n \dot{\psi}(t_n + \lambda_n t, \lambda_n r)) \tag{5.2.23}$$

be the wave map evolutions associated to the data $\vec{\psi}_n(r) := \vec{\psi}(t_n, \lambda_n r)$. And denote by $U_n(t,r,\omega) := (\psi_n(t,r),\omega)$ the full wave maps. Then,

$$U_n(t,r,\omega) \to U_\infty(r,\omega) \quad \textit{in} \quad H^1_{loc}((-1,1)\times\mathbb{R}^2;\mathbb{S}^2) \tag{5.2.24}$$

where U_∞ is a smooth, non-constant, 1-equivariant, time independent solution to (1.1.8), and hence $U_\infty(r,\omega) = (\pm Q(r/\lambda_0),\omega)$ for some $\lambda_0 > 0$. We further note that after passing to a subsequence, $U_n(t,r,\omega) \to U_\infty(r,\omega)$ locally uniformly in $(-1,1)\times(\mathbb{R}^2 - \{0\})$.

Moreover, with the times t_n and scales λ_n as above, we have

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt = o_n(1). \tag{5.2.25}$$

Remark 17. We note that we have altered the selection procedure by which the sequence of times t_n is chosen in the proof of Theorem 5.2.10. In [76], after defining a scaling factor $\lambda(t)$, Struwe uses Lemma 5.2.8 to select a sequence of times t_n via an argument involving Vitali's covering theorem, and he sets $\lambda_n := \lambda(t_n)$. Here we do something different. Given Lemma 5.2.8 we use the argument in [22, Corollary 5.3] to find a sequence $t_n \to 1$ so that (5.2.21) and (5.2.22) hold. Now we choose the scales $\lambda(t)$ as in Struwe and for each n we set $\sigma = \lambda_n := \lambda(t_n)$ and we establish (5.2.25), which is exactly [76, Lemma 3.3]. The rest of the proof of Theorem 5.2.10 now proceeds exactly as in [76].

We will also need the following consequences of Theorem 5.2.10:

Lemma 5.2.11. *Let $\psi(t) \in \mathcal{H}$ be a solution to (5.1.1) such that $T_+(\vec{\psi}) = 1$. Let $\{t_n\} \nearrow 1$*

and $\{\lambda_n\}$ *be chosen as in Theorem 5.2.10. Define* $\psi_n(t,r)$, $\pm Q(r/\lambda_0)$ *as in* (5.2.23). *Then*

$$\psi_n \mp Q(\cdot/\lambda_0) \to 0 \quad as \quad n \to \infty \quad in \quad L^2_t((-1,1); H_{loc}) \tag{5.2.26}$$

where H *is defined as in* (5.2.5).

Proof. We prove the case where the convergence in Theorem 5.2.10 is to $+Q(r/\lambda_0)$. Let $Q_{\lambda_0}(r) = Q(r/\lambda_0)$. By Theorem 5.2.10, we know that

$$\begin{aligned}\int_{\mathbb{R}^{1+2}} \left(|\partial_t \psi_n(t,r)|^2 + \left|\partial_r(\psi_n(t,r) - Q_{\lambda_0}(r))\right|^2 \right) \chi(t,r)\, r\, dr\, dt \\ + \int_{\mathbb{R}^{1+2}} \left|\psi_n(t,r) - Q_{\lambda_0}(r)\right|^2 \chi(t,r)\, r\, dr\, dt \longrightarrow 0 \quad \text{as} \quad n \to \infty\end{aligned} \tag{5.2.27}$$

for all $\chi \in C_0^\infty((-1,1)\times\mathbb{R}^2)$, radial in space. Hence to prove (5.2.26), it suffices to show that

$$\int_{\mathbb{R}^{1+2}} \frac{\left|\psi_n(t,r) - Q_{\lambda_0}(r)\right|^2}{r^2} \chi(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty \tag{5.2.28}$$

for all χ as above. Next, note that if for fixed $\delta > 0$, $\chi(t,r)$ satisfies $\text{supp}(\chi(t,\cdot)) \subset [\delta, \infty)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{1+2}} \frac{\left|\psi_n(t,r) - Q_{\lambda_0}(r)\right|^2}{r^2} \chi(t,r)\, r\, dr\, dt \\ &\qquad \le \delta^{-2} \int_{\mathbb{R}^{1+2}} \left|\psi_n(t,r) - Q_{\lambda_0}(r)\right|^2 \chi(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

with the convergence in the last line following from (5.2.27). Hence, from here out we only need to consider χ with $\text{supp}\chi(t,\cdot) \subset [0,1]$. Referring to Struwe's argument in [76, Proof of Theorem 2.1, (ii)], we note that by construction, λ_n and λ_0 are such that

$$\mathcal{E}_0^1(\vec{\psi}_n(t)) < \varepsilon_1, \quad \mathcal{E}_0^1(Q_{\lambda_0}) < \varepsilon_1$$

uniformly in $|t| \le 1$ and uniformly in n, where $\varepsilon_1 > 0$ is a fixed constant that we can choose to be as small as we want. Recalling that for each t, $\psi(t,0) = Q(0) = 0$ and using (5.2.4), this implies that

$$|G(\psi_n(t,r))| \le \frac{1}{2}\varepsilon_1, \quad \left|G(Q_{\lambda_0}(r))\right| \le \frac{1}{2}\varepsilon_1$$

for all $r \in [0,1]$. In particular, we can choose ε_1 small enough so that

$$|\psi_n(t,r)| < \frac{\pi}{8}, \quad \left|Q_{\lambda_0}(r)\right| < \frac{\pi}{8}$$

for all $r \in [0,1]$. Using the above line we then can conclude that there exists $c > 0$ such that

$$(\psi_n(t,r) - Q(r/\lambda_0))(\sin(2\psi_n(t,r)) - \sin(2Q_{\lambda_0}(r))) \ge c(\psi_n(t,r) - Q(r/\lambda_0))^2 \tag{5.2.29}$$

for all $r \in [0,1]$, and $|t| \le 1$. Consider the equation

$$(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r)(\psi_n(t,r) - Q_{\lambda_0}(r)) = \frac{\sin(2\psi_n(t,r)) - \sin(2Q_{\lambda_0}(r))}{r^2}.$$

Now, let $\chi \in C_0^\infty((-1,1) \times \mathbb{R}^2)$ satisfy $\operatorname{supp}(\chi(t,\cdot)) \subset [0,1]$. Multiply the above equation by $(\psi_n(t,r) - Q_{\lambda_0}(r))\chi(t,r)$, and integrate over $\mathbb{R}^{1+2}$. Then, integrating by parts and using the strong local convergence in (5.2.27) we can deduce that

$$\int_{\mathbb{R}^{1+2}} \frac{(\sin(2\psi_n(t,r)) - \sin(2Q_{\lambda_0}(r)))(\psi_n(t,r) - Q(r/\lambda_0))}{r^2} \chi(t,r)\, r\, dr\, dt \to 0$$

as $n \to \infty$. The lemma then follows by combining the above line with (5.2.29). $\square$

Lemma 5.2.12. *Let $\psi(t) \in \mathcal{H}$ be a wave map that blows up at time $t = 1$. Then, there* *exists a sequence of times $\bar{t}_n \to 1$ and a sequence of points $r_n \in [0, 1 - \bar{t}_n)$ such that*

$$\psi(\bar{t}_n, r_n) \to \pm\pi \quad as \quad n \to \infty \tag{5.2.30}$$

Proof. If not, then there exists a $\delta_0 > 0$ such that for every time $t \in [0, 1)$ we have $|\psi(t, r)| \in \mathbb{R} - [\pi - \delta_0, \pi + \delta_0]$ for all $r \in [0, 1 - t)$. Now let t_n, λ_n and $\psi_n(t, r)$ and $\pm Q_{\lambda_0}$ be as in Theorem 5.2.10 and Lemma 5.2.11. Choose $0 < R_1 < R_2 < \infty$ so that $\left|Q_{\lambda_0}(r)\right| > \pi - \frac{\delta_0}{2}$ for $r \in [R_1, R_2]$ and choose N large enough so that $[\lambda_n R_1, \lambda_n R_2] \subset [0, 1 - t_n - \lambda_n t)$ for all $t \in [0, 1]$ and for all $n \geq N$. This implies that

$$\left|\psi_n(t, r) \mp Q_{\lambda_0}(r)\right| \geq \frac{\delta_0}{2} \quad \forall n \geq N, \quad \forall r \in [R_1, R_2], \tag{5.2.31}$$

and for all $t \in [0, 1]$. But this provides an immediate contradiction with the convergence in (5.2.26). ☐

Corollary 5.2.13. *Let $\psi(t) \in \mathcal{H}_1$ be a wave map that blows up at time $t = 1$ such that $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$. Recall that $\vec{\psi}(t) \in \mathcal{H}_1$ means that $\psi(t, 0) = 0, \psi(t, \infty) = \pi$. Then we have*

$$\psi_n - Q(\cdot/\lambda_0) \to 0 \quad as \quad n \to \infty \quad in \quad L^2_t((-1, 1); H_{loc}), \tag{5.2.32}$$

with $\psi_n(t, r)$, t_n, and λ_n defined as in Theorem 5.2.10. In addition, there exists another sequence of times $\bar{t}_n \to 1$ and a sequence of points $r_n \in [0, 1 - \bar{t}_n)$ such that

$$\psi(\bar{t}_n, r_n) \to \pi \quad as \quad n \to \infty \tag{5.2.33}$$

Proof. We use the energy bound $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$ to eliminate the possibility that the convergence in Theorem 5.2.10 is to $-Q(r/\lambda_0)$ instead of to $+Q(r/\lambda_0)$. Suppose that in fact we had in (5.2.26) that $\psi_n + Q(\cdot/\lambda_n) \to 0$ in $L^2_t((-1, 1); H_{\text{loc}})$. Lemma 5.2.12 then gives a sequence of times $\bar{t}_n \to 1$ and a sequence $r_n \in [0, 1 - \bar{t}_n)$ such that

$$\psi(\bar{t}_n, r_n) \to -\pi \tag{5.2.34}$$

as $n \to \infty$. Now recall that $\vec{\psi}(t) \in \mathcal{H}_1$. Using the above along with (5.2.4) we see that

$$2\mathcal{E}(Q) = 8 \leftarrow 2\,|G(\psi(\bar{t}_n, r_n)) - 2| \leq \mathcal{E}_{r_n}^{\infty}(\psi(\bar{t}_n), 0))$$

On the other hand, we can use (5.2.34) and (5.2.4) again to see that

$$\mathcal{E}(Q) = 4 \leftarrow 2\,|G(\psi(\bar{t}_n, r_n))| \leq \mathcal{E}_0^{r_n}(\psi(\bar{t}_n), 0)$$

Putting this together we see that we must have $\mathcal{E}(\vec{\psi}) \geq 3\mathcal{E}(Q)$ which contradicts our initial assumption on the energy. $\square$

5.2.4 Profile Decomposition

Another essential ingredient of our argument is the profile decomposition of Bahouri and Gerard [1]. Here we restate the main results of [1] and then adapt these results to the case of $2d$ equivariant wave maps to the sphere of topological degree zero. In fact the results for the $4d$ wave equation stated here first appeared in [9] as the decomposition in [1] was performed only in dimension 3. In particular, we recall the following result:

Theorem 5.2.14. *[1, Main Theorem] [9, Theorem 1.1] Consider a sequence of data* $\vec{u}_n \in \dot{H}^1 \times L^2(\mathbb{R}^4)$ *such that* $\|u_n\|_{\dot{H}^1 \times L^2} \leq C$*. Then, up to extracting a subsequence, there exists a sequence of free 4d radial waves* $\vec{V}_L^j \in \dot{H}^1 \times L^2$*, a sequence of times* $\{t_n^j\} \subset \mathbb{R}$*, and sequence* *of scales* $\{\lambda_n^j\} \subset (0,\infty)$*, such that for* $\vec{w}_n^k$ *defined by*

$$u_{n,0}(r) = \sum_{j=1}^{k} \frac{1}{\lambda_n^j} V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) + w_{n,0}^k(r) \tag{5.2.35}$$

$$u_{n,1}(r) = \sum_{j=1}^{k} \frac{1}{(\lambda_n^j)^2} \dot{V}_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) + w_{n,1}^k(r) \tag{5.2.36}$$

we have, for any $j \le k$*, that*

$$(\lambda_n^j w_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot), (\lambda_n^j)^2 w_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot)) \rightharpoonup 0 \quad \text{weakly in} \quad \dot{H}^1 \times L^2(\mathbb{R}^4). \tag{5.2.37}$$

In addition, for any $j \neq k$ *we have*

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} \to \infty \quad \text{as} \quad n \to \infty. \tag{5.2.38}$$

Moreover, the errors $\vec{w}_n^k$ *vanish asymptotically in the sense that if we let* $w_{n,L}^k(t) \in \dot{H}^1 \times L^2$ *denote the free evolution, (i.e., solution to* (5.2.9)*), of the data* $\vec{w}_n^k \in \dot{H}^1 \times L^2$*, we have*

$$\limsup_{n\to\infty} \left\| w_{n,L}^k \right\|_{L_t^\infty L_x^4 \cap L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.2.39}$$

Finally, we have the almost-orthogonality of the $\dot{H}^1 \times L^2$ *norms of the decomposition:*

$$\|\vec{u}_n\|_{\dot{H}^1\times L^2}^2 = \sum_{1\le j\le k} \|\vec{V}_L^j(-t_n^j/\lambda_n^j)\|_{\dot{H}^1\times L^2}^2 + \|\vec{w}_n^k\|_{\dot{H}^1\times L^2}^2 + o_n(1) \tag{5.2.40}$$

as $n \to \infty$*.*

The norms appearing in (5.2.39) are dispersive and examples of Strichartz estimates, see Lindblad, Sogge [55] and Sogge's book [73] for more background and details. For our purposes here, it will often be useful to rephrase the above decomposition in the framework of the $2d$ linear wave equation (5.1.6). Using the right-most equality in (5.2.10) together with the identifications

$$\begin{aligned}
&\psi_n(r) = r u_n(r) \\
&\varphi_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) = \frac{r}{\lambda_n^j} V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) \\
&\gamma_n^k(r) = r w_n^k,
\end{aligned}$$

we see that Theorem 5.2.14 directly implies the following decomposition for sequences $\vec{\psi}_n \in \mathcal{H}_0$ with uniformly bounded $H \times L^2$ norms. In particular, by (5.2.11), the following corollary holds for all sequences $\vec{\psi}_n \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\psi}_n) \le C < 2\mathcal{E}(Q)$.

Corollary 5.2.15. *Consider a sequence of data $\vec{\psi}_n \in \mathcal{H}_0$ that is uniformly bounded in $H \times L^2$. Then, up to extracting a subsequence, there exists a sequence of linear waves $\vec{\varphi}_L^j \in \mathcal{H}_0$, (i.e., solutions to (5.1.6)), a sequence of times $\{t_n^j\} \subset \mathbb{R}$, and a sequence of scales $\{\lambda_n^j\} \subset (0,\infty)$, such that for $\vec{\gamma}_n^k$ defined by*

$$\psi_{n,0}(r) = \sum_{j=1}^{k} \varphi_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) + \gamma_{n,0}^k(r) \tag{5.2.41}$$

$$\psi_{n,1}(r) = \sum_{j=1}^{k} \frac{1}{\lambda_n^j} \dot{\varphi}_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) + \gamma_{n,1}^k(r) \tag{5.2.42}$$

we have, for any $j \le k$, that

$$(\gamma_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot), \lambda_n^j \gamma_n^k(\lambda_n^j t_n^j, \lambda_n^j \cdot)) \rightharpoonup 0 \quad \textit{weakly in} \quad H \times L^2. \tag{5.2.43}$$

In addition, for any $j \ne k$ we have

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} \to \infty \quad \textit{as} \quad n \to \infty. \tag{5.2.44}$$

Moreover, the errors $\vec{\gamma}_n^k$ vanish asymptotically in the sense that if we let $\gamma_{n,L}^k(t) \in \mathcal{H}_0$ denote the linear evolution, (i.e., solution to (5.1.6)*) of the data $\vec{\gamma}_n^k \in \mathcal{H}_0$, we have*

$$\limsup_{n\to\infty} \left\| \frac{1}{r}\gamma_{n,L}^k \right\|_{L_t^\infty L_x^4 \cap L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} \to 0 \quad \textit{as} \quad k \to \infty. \tag{5.2.45}$$

Finally, we have the almost-orthogonality of the $H \times L^2$ norms of the decomposition:

$$\|\vec{\psi}_n\|_{H\times L^2}^2 = \sum_{1\le j\le k} \|\vec{\varphi}_L^j(-t_n^j/\lambda_n^j)\|_{H\times L^2}^2 + \|\vec{\gamma}_n^k\|_{H\times L^2}^2 + o_n(1) \tag{5.2.46}$$

as $n \to \infty$.

In order to apply the concentration-compactness/rigidity method developed by Kenig and Merle in [36], [37], we need the following "Pythagorean decomposition" of the nonlinear energy (5.2.2):

Lemma 5.2.16. *Consider a sequence $\vec{\psi}_n \in \mathcal{H}_0$ and a decomposition as in Corollary 5.2.15. Then this Pythagorean decomposition holds for the energy of the sequence:*

$$\mathcal{E}(\vec{\psi}_n) = \sum_{j=1}^k \mathcal{E}(\vec{\varphi}_L^j(-t_n^j/\lambda_n^j)) + \mathcal{E}(\vec{\gamma}_n^k) + o_n(1) \tag{5.2.47}$$

as $n \to \infty$.

Proof. By (5.2.46), it suffices to show for each k that

$$\int_0^\infty \frac{\sin^2(\psi_n)}{r}\,dr = \sum_{j=1}^k \int_0^\infty \frac{\sin^2\left(\varphi_L^j(-t_n^j/\lambda_n^j)\right)}{r}\,dr + \int_0^\infty \frac{\sin^2\left(\gamma_n^k\right)}{r}\,dr + o_n(1).$$

We will need the following simple inequality:

$$\left|\sin^2(x+y) - \sin^2(x) - \sin^2(y)\right| = \left|-2\sin^2(x)\sin^2(y) + \frac{1}{2}\sin(2x)\sin(2y)\right| \lesssim |x|\,|y|\,. \tag{5.2.48}$$

Since at some point we will need to make use dispersive estimates for the $4d$ linear wave equation the argument is clearer if, at this point, we pass back to the $4d$ formulation. Recall that this means we set

$$\begin{aligned}
&\psi_n(r) = r u_n(r)\\
&\varphi_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j) = \frac{r}{\lambda_n^j} V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j)\\
&\gamma_n^k(r) = r w_n^k.
\end{aligned}$$

Since we have fixed k, we can, by an approximation argument, assume that all of the profiles $V^j(0,\cdot)$ are smooth and supported in the same compact set, say $B(0,R)$. We seek to prove that

$$\left|\int_0^\infty \frac{\sin^2(r u_n)}{r}\,dr - \sum_{j=1}^k \int_0^\infty \frac{\sin^2\left(\frac{r}{\lambda_n^j} V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j)\right)}{r}\,dr - \int_0^\infty \frac{\sin^2\left(r w_n^k\right)}{r}\,dr\right| = o_n(1).$$

Using the inequality (5.2.48) $k-1$ times, we can reduce our problem to showing the following two estimates:

$$\int_0^\infty \frac{\left|V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j)\right|}{(\lambda_n^j)} \frac{\left|V_L^i(-t_n^i/\lambda_n^i, r/\lambda_n^i)\right|}{(\lambda_n^i)} \, r\,dr = o_n(1) \quad \text{for } i \neq j \tag{5.2.49}$$

$$\int_0^\infty \frac{\left|V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j)\right|}{(\lambda_n^j)} \left|w_n^k(r)\right| \, r\,dr = o_n(1) \quad \text{for } j \leq k. \tag{5.2.50}$$

From here the proof proceeds on a case by case basis where the cases are determined by which pseudo-orthogonality condition is satisfied in (5.2.44).

Case 1*:* $\lambda_n^i \simeq \lambda_n^j$.

In this case we may assume, without loss of generality, that $\lambda_n^j = \lambda_n^i = 1$ for all n. By (5.2.44) we then must have that $\left|t_n^i - t_n^j\right| \to \infty$ as $n \to \infty$. This means that either $\left|t_n^i\right|$ or $\left|t_n^j\right|$, or both tend to ∞ as $n \to \infty$. To prove (5.2.49) we rely on the $\langle t \rangle^{-\frac{3}{2}}$ point-wise decay of free waves in $\mathbb{R}^4$. Indeed, we have

$$\begin{aligned}
&\int_0^\infty \left|V_L^j(-t_n^j, r)\right| \left|V_L^i(-t_n^i, r)\right| \, r\,dr \\
&\leq \left(\int_0^{R+\left|t_n^j\right|} \left|V_L^j(-t_n^j, r)\right|^2 \, r\,dr \right)^{\frac{1}{2}} \left(\int_0^{R+\left|t_n^i\right|} \left|V_L^i(-t_n^i, r)\right|^2 \, r\,dr \right)^{\frac{1}{2}} \\
&\lesssim \left\langle t_n^j \right\rangle^{-1/2} \left\langle t_n^i \right\rangle^{-1/2} = o_n(1).
\end{aligned}$$

Next we prove (5.2.50). First suppose that $\left|t_n^j\right| \to \infty$. Then we have

$$\begin{aligned}
\int_0^\infty \left|V_L^j(-t_n^j, r)\right| \left|w_n^k(r)\right| \, r\,dr &\leq \left(\int_0^{R+\left|t_n^j\right|} \left|V_L^j(-t_n^j, r)\right|^2 \, r\,dr \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^\infty \left|w_n^k(r)\right|^2 \, r\,dr \right)^{\frac{1}{2}} \\
&\lesssim \|w_n^k\|_{\dot{H}^1} \left\langle t_n^j \right\rangle^{-\frac{1}{2}} = o_n(1)
\end{aligned}$$

where the second inequality follows from the point-wise decay of free waves in $\mathbb{R}^4$ and Hardy's inequality. Finally consider the case where $\left|t_n^j\right| \le C$. Then we can assume, after passing to a subsequence and translating the profile, that $t_n^j = 0$ for every n. In this case, then we know that $w_n^k \rightharpoonup 0$ weakly in $\dot{H}^1$ and hence $w_n^k \to 0$ strongly in, e.g., $L^3_{loc}(\mathbb{R}^4)$ as $n \to \infty$. And we have

$$\begin{aligned}\int_0^\infty \left|V_L^j(0,r)\right| \left|w_n^k(r)\right| r\,dr &\le \left(\int_0^R \left|V_L^j(0,r)\right|^{\frac{3}{2}} dr\right)^{\frac{2}{3}} \left(\int_0^R \left|w_n^k(r)\right|^3 r^3\,dr\right)^{\frac{1}{3}} \\ &\le C(R)\|w_n^k\|_{L^3(B(0,R))} = o_n(1).\end{aligned}$$

Case 2: $\mu_n^{ij} = \frac{\lambda_n^i}{\lambda_n^j} \to 0$ *and* $\frac{\left|t_n^j\right|}{\lambda_n^j} + \frac{\left|t_n^i\right|}{\lambda_n^i} \le C$ *as* $n \to \infty$.

We can assume, by translating the profiles, that $t_n^i = t_n^j = 0$ for all n. We begin by establishing (5.2.49).

Changing variables we have

$$\begin{aligned}&\int_0^\infty \frac{\left|V_L^j(0,r/\lambda_n^j)\right|}{(\lambda_n^j)} \frac{\left|V_L^i(0,r/\lambda_n^i)\right|}{(\lambda_n^i)} r\,dr = \int_0^R \left|V^j(0,r)\right| \mu_n^{ij} \left|V^i(0,\mu_n^{ij} r)\right| r\,dr \\ &\le \left(\int_0^R \left|V_L^j(0,r)\right|^2 r\,dr\right)^{\frac{1}{2}} \left(\int_0^R (\mu_n^{ij})^2 \left|V_L^i(0,\mu_n^{ij} r)\right|^2 r\,dr\right)^{\frac{1}{2}} \\ &\le C \left(\int_0^{R\mu_n^{ij}} \left|V_L^i(0,r)\right|^2 r\,dr\right)^{\frac{1}{2}} = o_n(1),\end{aligned}$$

where the last line follows from the fact that $R\mu_n^{ij} \to 0$ as $n \to \infty$. Next we prove (5.2.50).

Again, we change variables to obtain

$$\int_0^\infty \frac{\left|V_L^j(-t_n^j/\lambda_n^j, r/\lambda_n^j)\right|}{(\lambda_n^j)} \left|w_n^k(r)\right| r\,dr = \int_0^R \left|V_L^j(0,r)\right| \lambda_n^j \left|w_n^k(\lambda_n^j r)\right| r\,dr$$
$$\leq \left(\int_0^R \left|V_L^j(0,r)\right|^{\frac{3}{2}} r\,dr\right)^{\frac{2}{3}} \left(\int_0^R (\lambda_n^j)^3 \left|w_n^k(\lambda_n^j r)\right|^3 r^3\,dr\right)^{\frac{1}{3}} = o_n(1),$$

where the last line tends to 0 as $n \to \infty$ since (5.2.37) implies that $\lambda_n^j w_n^k(\lambda_n^j \cdot) \to 0$ in $L^3_{loc}(\mathbb{R}^4)$.

Cases 3*:* $\mu_n^{ij} = \frac{\lambda_n^i}{\lambda_n^j} \to 0$, $\frac{\left|t_n^j\right|}{\lambda_n^j} + \frac{\left|t_n^i\right|}{\lambda_n^i} \to \infty$

This remaining case can be handled by combining the techniques demonstrated in *Case* 1 and *Case* 2 using either the point-wise decay of free waves or (5.2.37) when applicable. We leave the details to the reader. □

We will state the remaining results in this section in the $4d$ setting for simplicity. The transition back to the $2d$ setting is straight-forward and is omitted.

Next, we exhibit the existence of a non-linear profile decomposition as in [1]. We will employ the following notation: For a profile decomposition as in (5.2.35) with profiles $\{V_L^j\}$ and parameters $\{t_n^j, \lambda_n^j\}$ we will denote by $\{V^j\}$ the non-linear profiles associated to $\{V_L^j(-t_n^j/\lambda_n^j), \dot{V}_L^j(-t_n^j/\lambda_n^j)\}$, i.e., the unique solution to (5.2.8) such that for all $-t_n^j/\lambda_n^j \in I_{\max}(V^j)$ we have

$$\lim_{n\to\infty} \left\|\vec{V}^j(-t_n^j/\lambda_n^j) - \vec{V}_L^j(-t_n^j/\lambda_n^j)\right\|_{\dot{H}^1 \times L^2} = 0$$

The existence of the non-linear profiles follows immediately from the local well-posedness theory for (5.2.8) developed in [17] in the case that $-t_n^j/\lambda_n^j \to \tau_\infty^j \in \mathbb{R}$. If $-t_n^j/\lambda_n^j \to \pm\infty$ then the existence of the nonlinear profile follows from the existence of wave operators for (5.2.8).

We will make use of the following result on several occasions.

Proposition 5.2.17. *Let $\vec{u}_n \in \dot{H}^1 \times L^2$ be a uniformly bounded sequence with a profile decomposition as in Theorem 5.2.14. Assume that the nonlinear profiles V^j associated to the linear profiles V_L^j all exist globally and scatter in the sense that*

$$\|V^j\|_{L_t^3(\mathbb{R};L_x^6)} < \infty.$$

Let $\vec{u}_n(t)$ denote the solution of (5.2.8) *with initial data $\vec{u}_n$. Then, for n large enough, $\vec{u}_n(t,r)$ exists globally in time and scatters with*

$$\limsup_{n\to\infty} \|u_n\|_{L_t^3(\mathbb{R};L_x^6)} < \infty.$$

Moreover, the following non-linear profile decomposition holds:

$$u_n(t,r) = \sum_{j=1}^{k} \frac{1}{\lambda_n^j} V^j\left(\frac{t-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) + w_{n,L}^k(t,r) + z_n^k(t,r) \tag{5.2.51}$$

with $w_{n,L}^k(t,r)$ as in (5.2.39) *and*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \left(\|z_n^k\|_{L_t^3 L_x^6} + \|\vec{z}_n^k\|_{L_t^\infty \dot{H}^1 \times L^2} \right) = 0. \tag{5.2.52}$$

The proof of Proposition 5.2.17 is similar to the the proof of [22, Proposition 2.8] and we give a sketch of the argument below. In the current formulation, the argument is easier than the one given in [22] since here we make the simplifying assumption that all of the non-linear profiles exist globally and scatter. We also refer the reader to [53, Proof of Proposition 3.1] where the essential elements of the argument are carried out in an almost identical setting.

The main ingredient in the proof of Proposition 5.2.17 is the following non-linear perturbation lemma which we will also make use of later as well. For the proof of the perturbation lemma we refer the reader to [37, Theorem 2.20], and [53, Lemma 3.3]. In the latter reference a detailed proof in an almost identical setting is provided which can be applied verbatim here.

Lemma 5.2.18. *[37, Theorem 2.20] [53, Lemma 3.3] There are continuous functions $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval, (possibly unbounded), $u, v \in C^0(I;\dot{H}^1(\mathbb{R}^4)) \cap C^1(I;L^2(\mathbb{R}^4))$ radial functions satisfying for some $A > 0$*

$$\begin{aligned} &\|\vec{u}\|_{L^\infty(I;\dot{H}^1\times L^2)} + \|\vec{v}\|_{L^\infty(I;\dot{H}^1\times L^2)} + \|v\|_{L^3_t(I;L^6_x)} \leq A \\ &\|eq(u)\|_{L^1_t(I;L^2_x)} + \|eq(v)\|_{L^1_t(I;L^2_x)} + \|w_0\|_{L^3_t(I;L^6_x)} \leq \varepsilon \leq \varepsilon_0(A) \end{aligned}$$

where $eq(u) := \Box u + u^3 Z(ru)$ in the sense of distributions, and $\vec{w}_0(t) := S(t-t_0)(\vec{u}-\vec{v})(t_0)$ with $t_0 \in I$ arbitrary, but fixed and S denoting the free wave evolution operator in $\mathbb{R}^{1+4}$. Then,

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty_t(I;\dot{H}^1\times L^2)} + \|u - v\|_{L^3_t L^6_x} \leq C_0(A)\varepsilon$$

In particular, $\|u\|_{L^3_t(I;L^6_x)} < \infty$.

Proof of Proposition 5.2.17. Set

$$v_n^k(t,r) = \sum_{j=1}^{k} \frac{1}{\lambda_n^j} V^j\left(\frac{t-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)$$

We would like to apply Lemma 5.2.18 to u_n and v_n^k for large n and we need to check that the conditions of Lemma 5.2.18 are satisfied for these choices. First note that $\mathrm{eq}(u_n) = 0$. We claim that $\|\mathrm{eq}(v_n^k)\|_{L^1_t L^2_x}$ is small for large n. To see this, observe that

$$\mathrm{eq}(v_n^k) = \sum_{j=1}^{k} N\left(V_n^j(t,r)\right) - N\left(\sum_{j=1}^{k} V_n^j(t,r)\right)$$

where we have used the notation $V_n^j(t,r) := \frac{1}{\lambda_n^j} V^j\left(\frac{t-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)$ and $N(v) = v^3 Z(rv)$ as in (5.2.8). Using the simple inequality

$$\begin{aligned}
&\left|\frac{\sin(2ru)+\sin(2rv)-\sin(2r(u+v))}{2r^3}\right| \\
&\qquad = \left|\frac{2\sin(2ru)\sin^2(rv)+2\sin(2rv)\sin^2(ru)}{2r^3}\right| \lesssim u^2\,|v| + v^2\,|u| \quad (5.2.53)
\end{aligned}$$

together with the pseudo-orthogonality of the times and scales in (5.2.38) and arguing as in the proof of Lemma 5.2.16 we obtain $\|\mathrm{eq}(v_n^k)\|_{L_t^1 L_x^2} \to 0$ as $n \to \infty$ for any fixed k. Next it is essential that

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^{k} V_n^j \right\|_{L_t^3 L_x^6} \le A < \infty \tag{5.2.54}$$

uniformly in k, which will follow from the small data theory together with (5.2.40). The point here is that the sum can be split into one over $1 \le j \le j_0$ and another over $j_0 \le j \le k$. The splitting is performed in terms of the free energy, with j_0 being chosen so that

$$\limsup_{n\to\infty} \sum_{j_0<j\le k} \|V_L^j(-t_n^j/\lambda_n^j)\|^2_{\dot H^1\times L^2} < \delta_0^2$$

where δ_0 is chosen so that the small data theory applies. Using again (5.2.38) as well as the small data scattering theory one now obtains

$$\begin{aligned}
\limsup_{n\to\infty} \left\| \sum_{j_0<j\le k} V_n^j \right\|^3_{L_t^3 L_x^6} &= \sum_{j_0<j\le k} \|V^j\|^3_{L_t^3 L_x^6} \\
&\le C \limsup_{n\to\infty} \left(\sum_{j_0<j\le k} \|V_L^j(-t_n^j/\lambda_n^j)\|^2_{\dot H^1\times L^2} \right)^{\frac{3}{2}}
\end{aligned}$$

with an absolute constant C. This implies (5.2.54). Now the desired result follows directly from Lemma 5.2.18. $\square$

In Section 5.5 we will require a few additional results from [18]. We restate these results here for completeness. First, we note that for a profile decomposition as in Theorem 5.2.14, the Pythagorean decompositions of the free energy remain valid even after a space localization. In particular we have the following:

Proposition 5.2.19. *[18, Corollary 8] Consider a sequence of radial data $\vec{u}_n \in \dot{H}^1 \times L^2(\mathbb{R}^4)$ such that $\|u_n\|_{\dot{H}^1 \times L^2} \leq C$, and a profile decomposition of this sequence as in Theorem 5.2.14. Let $\{r_n\} \subset (0, \infty)$ be any sequence. Then we have*

$$\|\vec{u}_n\|^2_{\dot{H}^1 \times L^2(r \geq r_n)} = \sum_{1 \leq j \leq k} \|\vec{V}^j_L(-t^j_n/\lambda^j_n)\|^2_{\dot{H}^1 \times L^2(r \geq r_n/\lambda^j_n)} + \|\vec{w}^k_n\|^2_{\dot{H}^1 \times L^2(r \geq r_n)} + o_n(1)$$

as $n \to \infty$.

Next, we will need a fact about solutions to the free $4d$ radial wave equation that is also established in [18]. The following result is the analog of [22, Claim 2.11] adapted to $\mathbb{R}^4$. In [22] it is proved in odd dimensions only.

Lemma 5.2.20. *[18, Lemma 11] [22, Claim 2.11] Let $\vec{w}_n(0) = (w_{n,0}, w_{n,1})$ be a uniformly bounded sequence in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ and let $\vec{w}_n(t) \in \dot{H}^1 \times L^2(\mathbb{R}^4)$ be the corresponding sequence of radial 4d free waves. Suppose that*

$$\|w_n\|_{L^3_t L^6_x} \to 0$$

as $n \to \infty$. Let $\chi \in C^\infty_0(\mathbb{R}^4)$ be radial so that $\chi \equiv 1$ on $|x| \leq 1$ and supp$\chi \subset \{|x| \leq 2\}$. Let $\{\lambda_n\} \subset (0, \infty)$ and consider the truncated data

$$\vec{v}_n(0) := \varphi(r/\lambda_n)\vec{w}_n(0),$$

where either $\varphi = \chi$ or $\varphi = 1 - \chi$. Let $\vec{v}_n(t)$ be the corresponding sequence of free waves. Then

$$\|v_n\|_{L_t^3 L_x^6} \to 0 \quad as \quad n \to \infty.$$

5.3 Outline of the Proof of Theorem 5.1.1

The proof of Theorem 5.1.1 follows from the concentration-compactness/rigidity method developed by the Kenig and Merle in [36], [37]. This method provides a framework for establishing global existence and scattering results for a large class of nonlinear dispersive equations. We begin with a brief outline of the argument adapted to our current situation. For data $\vec{\psi}(0) \in \mathcal{H}_0$ denote by $\vec{\psi}(t)$ the nonlinear evolution to (5.1.1) associated to $\vec{\psi}(0)$. Define the set

$$\mathcal{S} := \{\vec{\psi}(0) \in \mathcal{H}_0 \,|\, \vec{\psi}(t) \text{ exists globally and scatters to zero as } t \to \pm\infty\} \tag{5.3.1}$$

Our goal is then to prove that

$$\{\vec{\psi}(0) \in \mathcal{H}_0 \,|\, \mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)\} \subset \mathcal{S}$$

This will be accomplished by establishing the following three steps. First, we recall the following global existence and scattering result proved in [17], for data in $\mathcal{H}_0$ with energy $\leq \mathcal{E}(Q)$.

Theorem 5.3.1. *[17, Theorem 1 and Corollary 1] There exists a small $\delta > 0$ with the following property. Let $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_0$ be such that $\mathcal{E}(\vec{\psi}) < \mathcal{E}(Q) + \delta$. Then, there exists a unique global evolution $\vec{\psi} \in C^0(\mathbb{R}; \mathcal{H}_0)$ to (5.1.1) which scatters to zero in the sense of (5.1.5).*

This shows that $\mathcal{S}$ is not empty. We remark that Theorem 5.3.1 gives more than what is needed for the rest of the argument. A small data global existence and scattering result such as [17, Theorem 2] would suffice to show that $\mathcal{S}$ is not empty. In fact, the proof of Theorem 5.1.1, and in particular Theorem 4.1 provide an independent alternative to the proof of scattering below $\mathcal{E}(Q) + \delta$ given in [17].

Next, we argue by contradiction. Assume that Thereom 5.1.1 fails and suppose that $\mathcal{E}(Q) < \mathcal{E}^* < 2\mathcal{E}(Q)$ is the minimal energy level at which a failure to the conclusions of Theorem 5.1.1 occurs. We then combine the concentration compactness decomposition given in Corollary 5.2.15, the nonlinear perturbation theory in Lemma 5.2.18, and the nonlinear profile decomposition in Proposition 5.2.17 to extract a so-called critical element, i.e., a nonzero solution $\vec{\psi}_* \in C^0(I_{\max}(\vec{\psi}_*); \mathcal{H}_0)$ to (5.1.1) whose trajectory in $\mathcal{H}_0$ is pre-compact up to certain time-dependent scaling factors arising due to the scaling symmetry of the equation. Here $I_{\max}(\vec{\psi})$ is the maximal interval of existence of $\vec{\psi}_*$. To be specific, we can deduce the following proposition:

Proposition 5.3.2. *[17, Proposition* 2 *and Proposition* 3*] Suppose that Theorem* 1 *fails and let $\mathcal{E}^*$ be defined as above. Then, there exists a nonzero solution $\vec{\psi}_*(t) \in \mathcal{H}_0$ to* (5.1.1)*, (referred to as a the critical element), defined on its maximal interval of existence $I_{\max}(\vec{\psi}_*) \ni 0$, with*

$$\mathcal{E}(\vec{\psi}_*) = \mathcal{E}^* < 2\mathcal{E}(Q)$$

Moreover, there exists $A_0 > 0$, and a continuous function $\lambda : I_{\max} \to [A_0, \infty)$ such that the set

$$K := \left\{ \left(\psi_* \left(t, \frac{r}{\lambda(t)} \right), \frac{1}{\lambda(t)} \dot{\psi}_* \left(t, \frac{r}{\lambda(t)} \right) \right) \, \Big| \, t \in I_{\max} \right\} \tag{5.3.2}$$

is pre-compact in $H \times L^2$.

Remark 18. As noted above, the Cauchy problem (5.1.1), for data $\vec{\psi}(0) \in V(\alpha)$ with $\alpha \leq$

$2\mathcal{E}(Q)$ is equivalent to the Cauchy problem for the $4d$ nonlinear radial wave equation, (5.2.8), via the identification $ru = \psi$. Hence, it suffices to carry out the small data global existence and scattering argument, as well as the concentration compactness decomposition and the extraction of a critical element on the the level of the $4d$ equation (5.2.8) for u. We remark that in this setting, scattering in the sense of (5.1.5) is equivalent to $\|u\|_{\mathcal{X}(\mathbb{R}^{1+4})} < \infty$ where $\mathcal{X}$ is a suitably chosen Strichartz norm. For example, $\mathcal{X} = L^3_t L^6_x$ will do.

Remark 19. In the proof of Theorem 5.1.1, the requirement that $\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(Q)$ arises in the concentration compactness procedure. Indeed, in order to ensure that the critical element $\vec{\psi}_*$ described in Proposition 5.3.2 lies in $\mathcal{H}_0$ one needs to require that any sequence of data $\{\vec{\psi}_n(0)\}$ with energies converging from below to the minimal energy level $\mathcal{E}_*$, also have uniformly bounded $H \times L^2$ norms. This is only guaranteed when $\mathcal{E}_* < 2\mathcal{E}(Q)$ by Lemma 5.2.1. In this case, one obtains a sequence of data $\vec{u}_n(0)$, via the identification $ru_n = \psi_n$, that is uniformly bounded in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ and on which one is free to perform the concentration compactness decomposition as in [1] and extract a critical element $\vec{u}_*$ as in [37], [17]. We can then define $\vec{\psi}_* := r\vec{u}_*$.

Remark 20. For the proof that the function $\lambda(t)$ described in Proposition 5.3.2 can be taken to be continuous, we refer the reader to [37, Lemma 4.6] and [36, Remark 5.4]. The fact that we can assume that λ is bounded from below follows verbatim from the arguments given in [22, Section 6, Step 3]. See also, [37, Proof of Theorem 7.1] and [36, Proof of Theorem 5.1].

The final step, referred to as the rigidity argument, consists of showing any solution $\vec{\psi}(t) \in \mathcal{H}_0$ with the aforementioned compactness properties must be identically zero, which provides the contradiction. This part of the concentration compactness/rigidity method is what allows us to extend the result in [17] to all energies below $2\mathcal{E}(Q)$ and we will thus carry out the proof in detail in the next section.

5.3.1 Sharpness of Theorem 5.1.1 in $\mathcal{H}_0$

Before we begin the rigidity argument, we first show that Theorem 5.1.1 is indeed sharp in $\mathcal{H}_0$ by demonstrating the following claim: for all $\delta > 0$ there exist data $\vec{\psi}(0) \in \mathcal{H}_0$ with $\mathcal{E}(\psi) \leq 2\mathcal{E}(Q) + \delta$, such that the corresponding wave map evolution, $\psi(t)$, blows up in finite time. This follows easily from the blow-up constructions of [50] or [62].

Fix $\delta_0 > 0$. By [50] or [62] we can choose data $\vec{u}(0) \in \mathcal{H}_1$ such that

$$\mathcal{E}(\vec{u}(0)) \leq \mathcal{E}(Q) + \delta, \quad \delta \ll \delta_0$$

such that the corresponding wave map evolution $\vec{u}(t) \in \mathcal{H}_1$ blows up at time $t = 1$. In other words, the energy of $\vec{u}(t)$ concentrates in the backwards light cone, $K(1,0) := \{(t,r) \in [0,1] \times [0,1] \mid r \leq 1-t\}$, emanating from the point $(1,0) \in \mathbb{R} \times [0,\infty]$, i.e.,

$$\lim_{t \nearrow 1} \mathcal{E}_0^{1-t}(\vec{u}(t)) \geq \mathcal{E}(Q)$$

where $\mathcal{E}_a^b(u,v) = \int_a^b (u_r^2 + v^2 + \frac{\sin^2(u)}{r^2})\, r\, dr$. Now define $\vec{\psi}(0) \in \mathcal{H}_0$ as follows:

$$\psi(0,r) = \begin{cases} u(0,r) & \text{if} \quad r \leq 2 \\ \pi - Q(\lambda r) & \text{if} \quad r \geq 2 \end{cases} \tag{5.3.3}$$

where $\lambda > 0$ is chosen so that $\pi - Q(2\lambda) = u(0,2)$. We note that the existence of such a λ follows form the fact that we can ensure that $u(0,r) > 0$ for $r > 1$. To see this, observe that since $\vec{u}(t)$ blows up at time $t = 1$ and thus must concentrate at least $\mathcal{E}(Q)$ inside the light cone we can deduce by the monotonicity of the energy that $\mathcal{E}_0^1(\vec{u}(0)) \geq \mathcal{E}(Q)$. Now choose $\delta < \mathcal{E}(Q)$. If we have $u(0,r) \leq 0$ for any $r > 1$ we would need at least $\mathcal{E}_r^\infty(u(0),0) \geq \mathcal{E}(Q)$ to ensure that $u(0,\infty) = \pi$. This follows from the minimality of $\mathcal{E}(Q)$ in $\mathcal{H}_1$. However $\mathcal{E}_r^\infty(u(0),0) \leq \delta < \mathcal{E}(Q)$.

Now observe that

$$\mathcal{E}(\vec{\psi}(0)) = \mathcal{E}_0^2(\vec{u}(0)) + \mathcal{E}_2^\infty(\pi - Q) \le \mathcal{E}(\vec{u}) + \mathcal{E}(Q) \le 2\mathcal{E}(Q) + \delta. \tag{5.3.4}$$

Let $\vec{\psi}(t)$ denote the wave map evolution of the data $\vec{\psi}(0)$. By the finite speed of propagation, we have that $\vec{\psi}(t, r) = \vec{u}(t, r)$ for all $(t, r) \in K(0, 1)$ and hence

$$\lim_{t \nearrow 1} \mathcal{E}_0^{1-t}(\vec{\psi}(t)) = \lim_{t \nearrow 1} \mathcal{E}_0^{1-t}(\vec{u}(t)) \ge \mathcal{E}(Q) \tag{5.3.5}$$

which means that $\vec{\psi}(t)$ blows up at $t = 1$ as desired. Note that if one wishes to construct blow-up data in $\mathcal{H}_0$ that maintains the smoothness of $u(0)$, one can simply smooth out $\vec{\psi}(0, r)$ in a small neighborhood of the point $r = 2$ using an arbitrarily small amount of energy.

We again remark that the questions of determining the possible dynamics at the threshold, $\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(Q)$, and above it, $\mathcal{E}(\vec{\psi}) > 2\mathcal{E}(Q)$, are not addressed here and remain open.

5.4 Rigidity

In this section we prove Theorem 5.1.2 and complete the proof of Theorem 5.1.1. We begin by establishing a rigidity theory in $\mathcal{H}_0$ which will allow us to deduce Theorem 5.1.1. We then use the conclusions of Theorem 5.1.1 together with the proof of Theorem 5.4.1 to establish Theorem 5.1.2.

Theorem 5.4.1 (Rigidity in $\mathcal{H}_0$). *Let $\vec{\psi}(t) \in \mathcal{H}_0$ be a solution to* (5.1.1) *and let $I_{\max}(\psi) = (T_-(\psi), T_+(\psi))$ be the maximal interval of existence. Suppose that there exist $A_0 > 0$ and a continuous function $\lambda : I_{\max} \to [A_0, \infty)$ such that the set*

$$K := \left\{ \left(\psi\left(t, \frac{r}{\lambda(t)}\right), \frac{1}{\lambda(t)} \dot{\psi}\left(t, \frac{r}{\lambda(t)}\right) \right) \Big| \, t \in I_{max} \right\} \tag{5.4.1}$$

is pre-compact in $H \times L^2$. *Then,* $I_{\max} = \mathbb{R}$ *and* $\psi \equiv 0$.

We begin by recalling the following virial identity:

Lemma 5.4.2. *Let* $\chi_R(r) = \chi(r/R) \in C_0^\infty(\mathbb{R})$ *satisfy* $\chi(r) = 1$ *on* $[-1,1]$ *with* $\operatorname{supp}(\chi) \subset [-2,2]$. *Suppose that* $\vec{\psi}$ *is a solution to* (5.1.1) *on some interval* $I \ni 0$. *Then, for all* $T \in I$ *we have*

$$\left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle \Big|_0^T = -\int_0^T \int_0^\infty \dot{\psi}^2 \, r \, dr \, dt + \int_0^T O(\mathcal{E}_R^\infty(\vec{\psi}(t))) \, dt. \tag{5.4.2}$$

Proof. Since $\vec{\psi}$ is a solution to (5.1.1) we have

$$\begin{aligned}
\frac{d}{dt}\left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle &= \left\langle \chi_R \ddot{\psi} \,|\, r\psi_r \right\rangle + \left\langle \chi_R \dot{\psi} \,|\, r\dot{\psi}_r \right\rangle \\
&= \left\langle \chi_R(\psi_{rr} + \frac{1}{r}\psi_r - \frac{\sin(2\psi)}{2r^2}) \,\Big|\, r\psi_r \right\rangle + \left\langle \chi_R \dot{\psi} \,|\, r\dot{\psi}_r \right\rangle \\
&= \int_0^\infty \frac{1}{2}\partial_r(\psi_r^2)\chi_R \, r^2 \, dr + \int_0^\infty \psi_r^2 \chi_R \, r \, dr \\
&\quad - \int_0^\infty \frac{1}{2}\partial_r(\sin^2(\psi))\chi_R \, dr + \int_0^\infty \frac{1}{2}\partial_r(\dot{\psi}^2)\chi_R \, r^2 \, dr \\
&= -\int_0^\infty \dot{\psi}^2 r \, dr + \int_0^\infty (1-\chi_R)\dot{\psi}^2 \, r \, dr \\
&\quad - \frac{1}{2}\int_0^\infty \left(\dot{\psi}^2 + \psi_r^2 - \frac{\sin^2(\psi)}{r^2}\right)\chi_R' \, r^2 \, dr.
\end{aligned}$$

Observe that

$$\left| \int_0^\infty (1-\chi_R)\dot{\psi}^2 \, r \, dr \right| \lesssim \mathcal{E}_R^\infty(\vec{\psi}).$$

Finally, noting that $\chi_R'(r) = \frac{1}{R}\chi'(r/R)$, we obtain

$$\begin{aligned}
&\left| \int_0^\infty \frac{1}{2}\left(\dot{\psi}^2 + \psi_r^2 - \frac{\sin^2(\psi)}{r^2}\right)\chi_R' \, r^2 \, dr \right| \\
&\qquad\qquad \lesssim \int_R^{2R} \left(\dot{\psi}^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) \frac{r}{R}\chi'\left(\frac{r}{R}\right) r \; dr \lesssim \mathcal{E}_R^\infty(\vec{\psi}).
\end{aligned}$$

Hence we can conclude that

$$\frac{d}{dt}\left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle = -\int_0^\infty \dot{\psi}^2 \, r\, dr \, + O(\mathcal{E}_R^\infty(\vec{\psi}(t))).$$

An integration from 0 to T proves the lemma. $\square$

With the virial identity (5.4.2), we can begin the proof of Theorem 5.4.1. This will be done in several steps and is inspired by the arguments in [22, Proof of Theorem 2]. To begin, we recall from [37] that any wave map with a pre-compact trajectory in $H \times L^2$ as in (5.4.1) that blows up in finite time is supported on the backwards light cone.

Lemma 5.4.3. *[37, Lemma 4.7 and Lemma 4.8] Let $\vec{\psi}(t) \in \mathcal{H}_0$ be a solution to* (5.1.1) *such that $I_{\max}(\vec{\psi})$ is a finite interval. Without loss of generality we can assume $T_+(\vec{\psi}) = 1$. Suppose there exists a continuous function $\lambda : I_{\max} \to (0, \infty)$ so that K, as defined in* (5.4.1)*, is pre-compact in $H \times L^2$. Then*

$$0 < \frac{C_0(K)}{1-t} \leq \lambda(t). \tag{5.4.3}$$

And, for every $t \in [0, 1)$ we have

$$\mathrm{supp}(\vec{\psi}(t)) \in [0, 1-t). \tag{5.4.4}$$

We can now begin the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1.

Step 1:

First we show that $I_{\max}(\psi) = \mathbb{R}$. Assume that $T_+(\vec{\psi}) < \infty$ and we proceed by contradiction. Without loss of generality, we may assume that $T_+(\vec{\psi}) = 1$. By Lemma 5.4.3, we can deduce that $0 < \frac{C_0(K)}{1-t} \leq \lambda(t)$ and $\mathrm{supp}(\vec{\psi}(t)) \in [0, 1-t)$. In addition, we know, by [76] or an argument in [71, Lemma 2.2], that self similar blow-up for $2d$ wave maps is ruled out. This implies that there exists a sequence $\{\tau_n\} \subset (0,1)$ with $\tau_n \to 1$ such that

$$\frac{1}{\lambda(\tau_n)(1-\tau_n)} < 1 \quad \text{as} \quad n \to \infty. \tag{5.4.5}$$

Hence, we can extract a further subsequence $\{t_n\} \to 1$ and apply Corollary 5.2.9 with $\sigma = \frac{1}{\lambda(t_n)}$ to obtain, for every n, the bound

$$\lambda(t_n) \int_{t_n}^{t_n + \frac{1}{\lambda(t_n)}} \int_0^\infty \dot{\psi}^2(t,r)\, r\, dr\, dt \leq \frac{1}{n}. \tag{5.4.6}$$

Note that above we have used the fact that $\operatorname{supp}(\vec{\psi}(t)) \in [0, 1-t)$. Next, with t_n as above, define a sequence in $\mathcal{H}_0$ by setting

$$\vec{\psi}_n(0) = (\psi_n^0, \psi_n^1) := \left(\psi\left(t_n, \frac{r}{\lambda(t_n)}\right), \frac{1}{\lambda(t_n)} \dot{\psi}\left(t_n, \frac{r}{\lambda(t_n)}\right)\right).$$

The nonlinear evolutions associated to our sequence

$$\vec{\psi}_n(t) := \left(\psi\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right), \frac{1}{\lambda(t_n)} \dot{\psi}\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right)\right)$$

are then solutions to (5.1.1) with $\mathcal{E}(\vec{\psi}_n) = \mathcal{E}(\vec{\psi})$. Observe that

$$\int_0^1 \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{5.4.7}$$

Indeed, by (5.4.6) we have that

$$\int_0^1 \int_0^\infty \dot{\psi}_n^2\, r\, dr\, dt = \lambda(t_n) \int_{t_n}^{t_n + \frac{1}{\lambda(t_n)}} \int_0^\infty \dot{\psi}^2(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

We now proceed as follows. By the compactness of K we can find $\vec{\psi}_\infty(0) = (\psi_\infty^0, \psi_\infty^1) \in \mathcal{H}_0$

and a subsequence of $\{\vec{\psi}_n(0)\}$ such that we have strong convergence

$$\vec{\psi}_n(0) \to \vec{\psi}_\infty(0) \quad \text{as} \quad n \to \infty \tag{5.4.8}$$

in $H \times L^2$. Note that this also implies strong convergence in the energy topology, i.e., $\vec{\psi}_n(0) \to \vec{\psi}_\infty(0)$ in $\mathcal{H}_0$. In particular, we have

$$\mathcal{E}(\vec{\psi}_\infty(0)) = \mathcal{E}(\vec{\psi}_n(0)) = \mathcal{E}(\vec{\psi}). \tag{5.4.9}$$

Now, let $\vec{\psi}_\infty(t) \in \mathcal{H}_0$ denote the forward solution to (5.1.1) with initial data $\vec{\psi}_\infty(0)$ on its maximal interval of existence $[0, T_+(\psi_\infty))$. Choose $T_0 \in (0, T_+(\psi_\infty))$ with $T_0 \leq 1$.

Using Lemma 5.2.18 for the equivalent 4-dimensional wave equation (5.2.8), the strong convergence of $\vec{\psi}_n(0)$ to $\vec{\psi}_\infty(0)$ in $H \times L^2$ implies that for large n, the nonlinear evolutions $\vec{\psi}_n(t)$ and $\vec{\psi}_\infty(t)$ remain uniformly close in $H \times L^2$ for $t \in [0, T_0]$. Indeed, we have

$$\sup_{t \in [0, T_0]} \|\vec{\psi}_n(t) - \vec{\psi}_\infty(t)\|_{H \times L^2} = o_n(1). \tag{5.4.10}$$

Hence, combining (5.4.7) with (5.4.10) we have

$$\begin{aligned} 0 \leftarrow \int_0^1 \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt &\geq \int_0^{T_0} \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt \\ &= \int_0^{T_0} \int_0^\infty \dot{\psi}_\infty^2(t,r)\, r\, dr\, dt + o_n(1). \end{aligned}$$

Therefore we have $\dot{\psi}_\infty \equiv 0$ on $[0, T_0]$. Since $\psi = 0$ is the unique harmonic map in $\mathcal{H}_0$ we necessarily have that $\psi_\infty \equiv 0$. But, by (5.4.9) we then have $0 = \mathcal{E}(\vec{\psi}_\infty) = \mathcal{E}(\vec{\psi}_n) = \mathcal{E}(\vec{\psi})$. Hence $\vec{\psi} \equiv 0$, which contradicts our assumption that $\psi \neq 0$ blows up at time $t = 1$.

Step 2: By Step 1, we have reduced the proof of Theorem 5.4.1 to the case $I_{\max} = \mathbb{R}$, and hence $\lambda : \mathbb{R} \to [A_0, \infty)$. By time symmetry we can, without loss of generality, work with nonnegative times only and thus consider $\lambda(t) : [0, \infty) \to [A_0, \infty)$.

First note that since K is pre-compact in $H \times L^2$ and since $\lambda(t) \geq A_0$ we have that for all $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that for every $t \in [0, \infty)$

$$\mathcal{E}^{\infty}_{R(\varepsilon)}(\vec{\psi}(t)) < \varepsilon. \tag{5.4.11}$$

Also, observe that for all $T > 0$ we have

$$\left| \left\langle \chi_R \dot{\psi} \,|\, r\psi_r \right\rangle \Big|_0^T \right| \lesssim R\mathcal{E}(\vec{\psi}). \tag{5.4.12}$$

Now, fix $\varepsilon > 0$ and fix R large enough so that $\sup_{t\geq 0} \mathcal{E}^{\infty}_R(\vec{\psi}) < \varepsilon$. Then, Lemma 5.4.2 together with (5.4.12) implies that for all $T \in [0, \infty)$ we have

$$\frac{1}{T}\int_0^T \int_0^\infty \dot{\psi}^2 \, r\, dr\, dt \lesssim \frac{R}{T}\mathcal{E}(\vec{\psi}) + \varepsilon.$$

This shows that

$$\frac{1}{T}\int_0^T \int_0^\infty \dot{\psi}^2 \, r\, dr\, dt \to 0 \quad \text{as} \quad T \to \infty. \tag{5.4.13}$$

Next, we claim that there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty} \lambda(t_n) \int_{t_n}^{t_n + \frac{1}{\lambda(t_n)}} \left(\int_0^\infty \dot{\psi}^2 \, r\, dr \right) dt = 0. \tag{5.4.14}$$

To see this, we begin by defining a sequence τ_n as follows. Set

$$\tau_0 = 0, \quad \tau_{n+1} := \tau_n + \frac{1}{\lambda(\tau_n)} = \sum_{k=0}^{n} \frac{1}{\lambda(\tau_k)}.$$

First we establish that $\tau_n \to \infty$ as $n \to \infty$. If not, then up to a subsequence we would have

$\tau_n \to \tau_\infty < \infty$. This would imply that

$$\tau_\infty = \sum_{k=0}^{\infty} \frac{1}{\lambda(\tau_k)} < \infty$$

which means that $\lim_{k\to\infty} \dfrac{1}{\lambda(\tau_k)} = 0$. But this is impossible since $\lambda(\tau_k) \to \lambda(\tau_\infty) < \infty$ by the continuity of λ.

Now, suppose that (5.4.14) fails for all subsequences $\{t_n\} \subset \{\tau_n\}$. Then there exists $\varepsilon > 0$ such that for all k,

$$\int_{\tau_k}^{\tau_{k+1}} \left(\int_0^\infty \dot{\psi}^2 \, r \, dr \right) dt \geq \varepsilon \frac{1}{\lambda(\tau_k)}.$$

Summing both sides above from 1 to n gives

$$\int_0^{\tau_{n+1}} \left(\int_0^\infty \dot{\psi}^2 \, r \, dr \right) dt \geq \varepsilon \sum_{k=1}^{n} \frac{1}{\lambda(\tau_k)} = \varepsilon \tau_{n+1}$$

which contradicts (5.4.13). Hence there exists a sequence $\{t_n\}$ such that (5.4.14) holds. Moreover, since $\lambda(t) \geq A_0 > 0$ for all $t \geq 0$ we can extract a further subsequence, still denoted by $\{t_n\}$, such that (5.4.14) holds and all the intervals $[t_n, t_n + \frac{1}{\lambda(t_n)}]$ are disjoint.

Next, with t_n as above, define a sequence in $\mathcal{H}_0$ by setting

$$\vec{\psi}_n(0) = (\psi_n^0, \psi_n^1) := \left(\psi\left(t_n, \frac{r}{\lambda(t_n)}\right), \frac{1}{\lambda(t_n)} \dot{\psi}\left(t_n, \frac{r}{\lambda(t_n)}\right) \right).$$

The nonlinear evolutions associated to our sequence

$$\vec{\psi}_n(t) := \left(\psi\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right), \frac{1}{\lambda(t_n)} \dot{\psi}\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right) \right)$$

are then global solutions to (5.1.1) with $\mathcal{E}(\vec{\psi}_n) = \mathcal{E}(\vec{\psi})$. Observe that

$$\int_0^1 \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{5.4.15}$$

Indeed, by (5.4.14) we have that

$$\int_0^1 \int_0^\infty \dot{\psi}_n^2\, r\, dr\, dt = \lambda(t_n) \int_{t_n}^{t_n + \frac{1}{\lambda(t_n)}} \left(\int_0^\infty \dot{\psi}^2\, r\, dr \right) dt \to 0 \quad \text{as} \quad n \to \infty.$$

We now proceed as follows. By the pre-compactness of K we can find $\vec{\psi}_\infty(0) = (\psi_\infty^0, \psi_\infty^1) \in \mathcal{H}_0$ and a subsequence of $\{\vec{\psi}_n(0)\}$ such that we have strong convergence

$$\vec{\psi}_n(0) \to \vec{\psi}_\infty(0) \quad \text{as} \quad n \to \infty \tag{5.4.16}$$

in $H \times L^2$. Note that this also implies strong convergence in the energy topology, i.e., $\vec{\psi}_n(0) \to \vec{\psi}_\infty(0)$ in $\mathcal{H}_0$. In particular, we have

$$\mathcal{E}(\vec{\psi}_\infty(0)) = \mathcal{E}(\vec{\psi}_n(0)) = \mathcal{E}(\vec{\psi}). \tag{5.4.17}$$

Now, let $\vec{\psi}_\infty(t) \in \mathcal{H}_0$ denote the forward solution to (5.1.1) with initial data $\vec{\psi}_\infty(0)$ on its maximal interval of existence $[0, T_+(\psi_\infty))$. Choose $T_0 \in (0, T_+(\psi_\infty))$ with $T_0 \leq 1$.

Using Lemma 5.2.18 for the 4-dimensional wave equation (5.2.8), the strong convergence of $\vec{\psi}_n(0)$ to $\vec{\psi}_\infty(0)$ in $H \times L^2$ implies that for large n the nonlinear evolutions $\vec{\psi}_n(t)$ and $\vec{\psi}_\infty(t)$ remain uniformly close in $H \times L^2$ in $t \in [0, T_0]$. Indeed, we have

$$\sup_{t \in [0,T_0]} \|\vec{\psi}_n(t) - \vec{\psi}_\infty(t)\|_{H \times L^2} = o_n(1). \tag{5.4.18}$$

Hence, combining (5.4.15) with (5.4.18) we have

$$
\begin{aligned}
0 \leftarrow \int_0^1 \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt &\geq \int_0^{T_0} \int_0^\infty \dot{\psi}_n^2(t,r)\, r\, dr\, dt \\
&= \int_0^{T_0} \int_0^\infty \dot{\psi}_\infty^2(t,r)\, r\, dr\, dt + o_n(1).
\end{aligned}
$$

Therefore we have $\dot{\psi}_\infty \equiv 0$ on $[0, T_0]$. Since $\psi = 0$ is the unique harmonic map in $\mathcal{H}_0$ we necessarily have that $\psi_\infty \equiv 0$. But, by (5.4.17) we then have $0 = \mathcal{E}(\psi_\infty, 0) = \mathcal{E}(\vec{\psi}_n) = \mathcal{E}(\vec{\psi})$. Hence $\vec{\psi} \equiv 0$ as desired. □

We can now complete the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Suppose that Theorem 5.1.1 fails. Then by Proposition 5.3.2 there would exist a nonzero critical element $\vec{\psi}_*$ that satisfies the assumptions of Theorem 5.4.1. But by Theorem 5.4.1, $\vec{\psi}_* \equiv 0$, which is a contradiction. □

To conclude, we prove Theorem 5.1.2.

Proof of Theorem 5.1.2.

Step 1: First we show that $I_{\max}(\vec{U}) = \mathbb{R}$. We argue by contradiction. Assume that $T_+(\vec{U}) < \infty$. Without loss of generality, we may assume that $T_+(\vec{U}) = 1$.

Applying the exact same argument as in Step 1 of the proof of Theorem 5.4.1 up to (5.4.7) we can construct a sequence of solutions $\vec{U}_n(t) \in \dot{H}^1 \times L^2(\mathbb{R}^2; S^2)$ to (5.1.1) such that

$$
\vec{U}_n(0) = (U_n^0, U_n^1) := \left(U\left(t_n, \frac{r}{\lambda(t_n)}, \omega\right), \frac{1}{\lambda(t_n)} \partial_t U\left(t_n, \frac{r}{\lambda(t_n)}, \omega\right)\right)
$$

with $\mathcal{E}(\vec{U}_n) = \mathcal{E}(\vec{U})$ and

$$
\int_0^1 \int_{\mathbb{R}^2} \partial_t U_n^2(t)\, dx\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{5.4.19}
$$

From this we obtain the following conclusions:

(i) Extracting a subsequence we have $U_n \rightharpoonup U_\infty$ weakly in $\dot{H}^1_{\text{loc}}([0,1]\times\mathbb{R}^2;S^2)$ and hence $\vec{U}_\infty(t)$ is a weak solution to (5.1.1) on $[0,1]$.

(ii) By the pre-compactness of $\tilde{K}$ we can, in fact, ensure that $\vec{U}_n(0) \to \vec{U}_\infty(0)$ strongly in $\dot{H}^1 \times L^2(\mathbb{R}^2;S^2)$. This implies that

$$\mathcal{E}(\vec{U}_\infty) = \mathcal{E}(\vec{U}_n) = \mathcal{E}(\vec{U}) \tag{5.4.20}$$

(iii) By (5.4.19) we can deduce that $\dot{U}_\infty \equiv 0$ on $[0,1]$.

Putting this all together, we have a time independent weak solution $\vec{U}_\infty \in \mathcal{H}$ to (5.1.1) for $t\in[0,1]$. By Hélein's Theorem [32, Theorem 1] we know that U_∞ is, in fact, harmonic. Since $U = 0$ and $U = (\pm Q, \omega)$ are the unique harmonic maps up to scaling in $\mathcal{H}$ we necessarily have that either $U_\infty = 0$ or $\vec{U}_\infty(r,\omega) = (Q(\tilde{\lambda}\cdot),\omega)$ for some $\tilde{\lambda} > 0$. Hence, by (5.4.20), we can deduce that either $\mathcal{E}(\vec{U}) = 0$ or $\mathcal{E}(\vec{U}) = \mathcal{E}(Q,0)$. The former case implies that $U \equiv 0$. If the latter case occurs, then $U(t)$ can either be an element of $\mathcal{H}_0$, $\mathcal{H}_1$, or of $\mathcal{H}_{-1}$ since all the higher topological classes, $\mathcal{H}_n$ for $|n| > 1$, require more energy. If $U(t)\in\mathcal{H}_0$ then it is global in time and scatters by Theorem 5.1.1. If $U(t)\in\mathcal{H}_1$ or $\mathcal{H}_{-1}$ then we have $U(t,r,\omega) = (\pm Q(\tilde{\lambda} r),\omega)$ for some $\tilde{\lambda} > 0$ since $(Q,0)$, respectively $(-Q,0)$, uniquely minimizes the the energy in $\mathcal{H}_1$, respectively $\mathcal{H}_{-1}$. In either case, this provides a contradiction to our assumption that $I_{\max} \neq \mathbb{R}$.

Step 2:

Again we apply the exact same argument given in Step 2 of the proof of Theorem 5.4.1 and we construct a sequence of solutions $\vec{U}_n(t) \in \dot{H}^1 \times L^2(\mathbb{R}^2;S^2)$ to (5.1.1) such that

$$\vec{U}_n(0) = (U_n^0, U_n^1) := \left(U\left(t_n, \frac{r}{\lambda(t_n)}, \omega\right), \frac{1}{\lambda(t_n)}\partial_t U\left(t_n, \frac{r}{\lambda(t_n)}, \omega\right)\right)$$

with $\mathcal{E}(\vec{U}_n) = \mathcal{E}(\vec{U})$ and

$$\int_0^1 \int_{\mathbb{R}^2} \partial_t U_n^2(t)\, dx\, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{5.4.21}$$

We thus obtain the following conclusions:

(i) Extracting a subsequence we have $U_n \rightharpoonup U_\infty$ weakly in $\dot{H}^1_{\text{loc}}([0,1] \times \mathbb{R}^2; S^2)$ and hence $\vec{U}_\infty(t)$ is a weak solution to (5.1.1) on $[0,1]$.

(ii) By the pre-compactness of $\tilde{K}$ we can extract a further subsequence with $\vec{U}_n(0) \to \vec{U}_\infty(0)$ strongly in $\dot{H}^1 \times L^2(\mathbb{R}^2; S^2)$. This implies that

$$\mathcal{E}(\vec{U}_\infty) = \mathcal{E}(\vec{U}_n) = \mathcal{E}(\vec{U}) \tag{5.4.22}$$

(iii) By (5.4.21) we can deduce that $\dot{U}_\infty \equiv 0$ on $[0,1]$.

Putting this all together, we have a time independent weak solution $\vec{U}_\infty \in \mathcal{H}$ to (5.1.1) for $t \in [0,1]$. By Hélein's Theorem [32, Theorem 1] we know that U_∞ is, in fact, harmonic. Since $U = 0$ and $U = (\pm Q, \omega)$ are the unique harmonic maps up to scaling in $\mathcal{H}$ we necessarily have that either $U_\infty = 0$ or $\vec{U}_\infty(r,\omega) = (\pm Q(\tilde{\lambda}\cdot), \omega)$ for some $\tilde{\lambda} > 0$. Hence by (5.4.22) we can deduce that either $\mathcal{E}(\vec{U}) = 0$ or $\mathcal{E}(\vec{U}) = \mathcal{E}(Q, 0)$. The former case implies that $U \equiv 0$. Arguing as in the conclusion to Step 1, the latter case implies that either $U(t) \in \mathcal{H}_0$ or $U(t) \in \mathcal{H}_{\pm 1}$. If $U(t) \in \mathcal{H}_{\pm 1}$, then $U(t, r, \omega) = (\pm Q(\tilde{\lambda} r), \omega)$ for some $\tilde{\lambda} > 0$. If $\vec{U}(t) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{U}) = \mathcal{E}(Q)$, then Theorem 5.1.1 shows that $\vec{U}(t)$ is global in time and scatters to 0 as $t \to \infty$ in $\dot{H}^1 \times L^2(\mathbb{R}^2; S^2)$ in the sense that the energy of $\vec{U}(t)$ goes to 0 as $t \to \infty$ on any fixed but compact set $V \subset \mathbb{R}^2$. Finally, we observe that the pre-compactness of $\tilde{K}$ renders such a scattering result impossible.

We thus conclude that either $U \equiv 0$ or $U(t, r, \omega) = (\pm Q(\tilde{\lambda} r), \omega)$ for some $\tilde{\lambda} > 0$ proving Theorem 5.1.2. $\square$

5.5 Universality of the blow-up profile for degree one wave maps with energy below $3\mathcal{E}(Q)$

In this section we prove Theorem 5.1.3. We start by first deducing the conclusions of Theorem 5.1.3 along a sequence of times. To be specific, we establish the following proposition:

Proposition 5.5.1. *Let $\vec{\psi}(t) \in \mathcal{H}_1$ be a solution to (5.1.1) blowing up at time $t = 1$ with*

$$\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q)$$

Then there exists a sequence of times $t_n \to 1$, a sequence of scales $\lambda_n = o(1 - t_n)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$, and a decomposition

$$(\psi(t_n), \dot{\psi}(t_n)) = (\varphi_0, \varphi_1) + \left(Q\left(\frac{\cdot}{\lambda_n}\right), 0\right) + \vec{\varepsilon}(t_n) \tag{5.5.1}$$

such that $\vec{\varepsilon}(t_n) \in \mathcal{H}_0$ and $\vec{\varepsilon}(t_n) \to 0$ in $H \times L^2$ as $n \to \infty$.

Most of this section will be devoted to the proof of Proposition 5.5.1. We will proceed in several steps, the first being the extraction of the radiation term.

5.5.1 Extraction of the radiation term

In this subsection we construct what we will call the radiation term, $\vec{\varphi} = (\varphi_0, \varphi_1)$, in the decomposition (5.5.1).

To begin, let $\bar{t}_n \to 1$ and $r_n \in (0, 1 - \bar{t}_n]$ be chosen as in Corollary 5.2.13. We make the

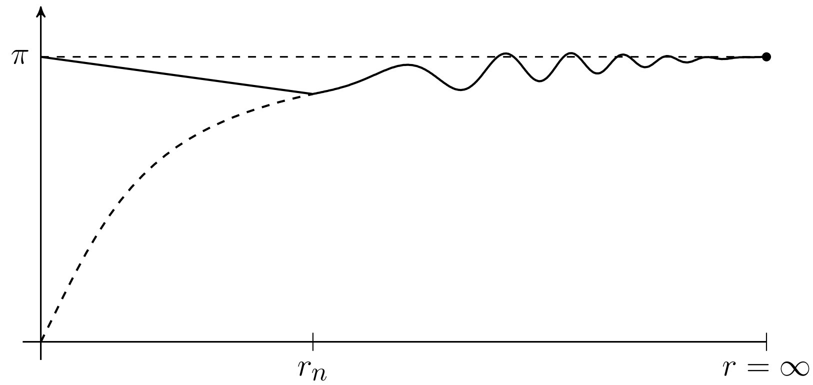

Figure 5.1: The solid line represents the graph of the function $\phi_n^0(\cdot)$ for fixed n, defined in (5.5.2). The dotted line is the piece of the function $\psi(\bar{t}_n, \cdot)$ that is chopped at $r = r_n$ in order to linearly connect to π, which ensures that $\vec{\phi}_n \in \mathcal{H}_{1,1}$.

following definition:

$$\phi_n^0(r) = \begin{cases} \pi - \frac{\pi - \psi(\bar{t}_n, r_n)}{r_n} r \quad \text{if} \quad 0 \leq r \leq r_n \\ \psi(\bar{t}_n, r) \quad \text{if} \quad r_n \leq r < \infty \end{cases} \tag{5.5.2}$$

$$\phi_n^1(r) = \begin{cases} 0 \quad \text{if} \quad 0 \leq r \leq r_n \\ \dot{\psi}(\bar{t}_n, r) \quad \text{if} \quad r_n \leq r < \infty \end{cases} \tag{5.5.3}$$

We claim that $\vec{\phi}_n := (\phi_n^0, \phi_n^1)$ forms a bounded sequence in the energy space $\mathcal{H}$–in fact, the sequence is in $\mathcal{H}_{1,1}$ which is defined in (5.1.3). To see this we start with the claim that

$$\mathcal{E}_{r_n}^{\infty}(\vec{\phi}_n) = \mathcal{E}_{r_n}^{\infty}(\vec{\psi}(\bar{t}_n)) \leq \eta + o_n(1). \tag{5.5.4}$$

Indeed, since $\psi(\bar{t}_n, r_n) \to \pi$ we have $G(\psi(\bar{t}_n, r_n)) \to 2 = \frac{1}{2}\mathcal{E}(Q)$ as $n \to \infty$. Therefore, by (5.2.4) have

$$\mathcal{E}_0^{r_n}(\psi(\bar{t}_n), 0) \geq 2G(\psi(\bar{t}_n, r_n)) \geq \mathcal{E}(Q) - o_n(1)$$

for large n which proves (5.5.4) since $\mathcal{E}_{r_n}^{\infty}(\vec{\psi}(\bar{t}_n)) = \mathcal{E}_0^{\infty}(\vec{\psi}(\bar{t}_n)) - \mathcal{E}_0^{r_n}(\vec{\psi}(\bar{t}_n))$.

We can also directly compute $\mathcal{E}_0^{r_n}(\phi_n^0, 0)$. Indeed,

$$\mathcal{E}_0^{r_n}(\phi_n^0, 0) = \int_0^{r_n} \left(\frac{\pi - \psi(\bar{t}_n, r_n)}{r_n} \right)^2 r \, dr + \int_0^{r_n} \frac{\sin^2 \left(\frac{\pi - \psi(\bar{t}_n, r_n)}{r_n} r \right)}{r} \, dr$$

$$\leq C \left| \pi - \psi(\bar{t}_n, r_n) \right|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\mathcal{E}(\vec{\phi}_n) \leq \eta + o_n(1)$. This means that for large enough n we have the uniform estimates $\mathcal{E}(\vec{\phi}_n) \leq C < 2\mathcal{E}(Q)$. Therefore, by Theorem 5.1.1, (which holds with exactly the same statement in $\mathcal{H}_{1,1}$ as in $\mathcal{H}_0 = \mathcal{H}_{0,0}$), we have that the wave map evolution $\vec{\phi}_n(t) \in \mathcal{H}_{1,1}$ with initial data $\vec{\phi}_n$ is global in time and scatters to π as $t \to \pm\infty$. We define $\vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_{1,1}$ by

$$\phi_0(r) := \begin{cases} \pi \quad \text{if} \quad r = 0 \\ \phi_n(1 - \bar{t}_n, r) \quad \text{if} \quad r > 2(1 - \bar{t}_n) \end{cases} \tag{5.5.5}$$

$$\phi_1(r) := \begin{cases} 0 \quad \text{if} \quad r = 0 \\ \dot{\phi}_n(1 - \bar{t}_n, r) \quad \text{if} \quad r > 2(1 - \bar{t}_n) \end{cases} \tag{5.5.6}$$

We need to check first that $\vec{\phi}$ is well-defined. First recall that by definition

$$\vec{\phi}_n(r) = \vec{\psi}(\bar{t}_n, r) \quad \forall r \geq 1 - \bar{t}_n$$

since $r_n \leq 1 - \bar{t}_n$. Using the finite speed of propagation of the wave map flow, see e.g., [68], we can then deduce that for all $t \in [0, 1)$ we have

$$\vec{\phi}_n(t - \bar{t}_n, r) = \vec{\psi}(t, r) \quad \forall \, r \geq 1 - \bar{t}_n + |t - \bar{t}_n|$$

Now let $m > n$ and thus $\bar{t}_m > \bar{t}_n$. The above implies that

$$\vec{\phi}_n(\bar{t}_m - \bar{t}_n, r) = \vec{\psi}(\bar{t}_m, r) = \vec{\phi}_n(r) \quad \forall\, r \geq 1 - \bar{t}_n + |\bar{t}_m - \bar{t}_n|$$

Therefore, using the finite speed of propagation again we can conclude that

$$\vec{\phi}_n(1 - \bar{t}_n, r) = \vec{\phi}_m(1 - \bar{t}_m, r) \quad \forall\, r > 2(1 - \bar{t}_n)$$

proving that $\vec{\phi}$ is well-defined. Next we claim that

$$\mathcal{E}(\vec{\phi}) \leq \eta \tag{5.5.7}$$

Indeed, observe that by monotonicity of the energy on light cones, see e.g. [68], we have

$$\mathcal{E}^{\infty}_{2(1-\bar{t}_n)}(\vec{\phi}) = \mathcal{E}^{\infty}_{2(1-\bar{t}_n)}(\vec{\phi}_n(1 - \bar{t}_n)) \leq \mathcal{E}^{\infty}_{1-\bar{t}_n}(\vec{\phi}_n(0)) \leq \mathcal{E}(\vec{\phi}_n(0)) \leq \eta + o_n(1)$$

and then (5.5.7) follows by taking $n \to \infty$ above. Now, let $\vec{\phi}(t) \in \mathcal{H}_{1,1}$ denote the wave map evolution of $\vec{\phi}$. Since $\vec{\phi} \in \mathcal{H}_{1,1}$ and $\mathcal{E}(\vec{\phi}) \leq \eta < 2\mathcal{E}(Q)$ we can deduce by Theorem 5.1.1 that $\vec{\phi}(t)$ is global in time and scatters as $t \to \pm\infty$. Our final observation regarding $\phi(t)$ is that for all $t \in [0, 1)$ we have

$$\vec{\phi}(t, r) = \vec{\psi}(t, r) \quad \forall\, r > 1 - t$$

This follows immediately from the definition of $\vec{\phi}$ and the finite speed of propagation. To be specific, fix $t_0 \in [0, 1)$ and $r_0 > 1 - t$. Since $\bar{t}_n \to 1$ we can choose n large enough so that $r_0 > 2(1 - \bar{t}_n) + 1 - t_0$. Then observe that by finite speed of propagation and the fact that

$\vec{\phi}(r) = \vec{\phi}_n(1 - \bar{t}_n, r)$ for all $r > 2(1 - \bar{t}_n)$ we have

$$\vec{\phi}(t_0, r) = \phi_n(t_0 - \bar{t}_n, r) = \vec{\psi}(t_0, r) \quad \forall\, r > r_0 > 2(1 - \bar{t}_n) + 1 - t_0$$

and in particular for $r = r_0$.

Finally, we define our radiation term $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ by setting

$$\varphi_0(r) := \phi_0 - \pi \tag{5.5.8}$$

$$\varphi_1(r) := \phi_1. \tag{5.5.9}$$

We denote by $\vec{\varphi}(t) \in \mathcal{H}_0$ the global wave map evolution of $\vec{\varphi}$. We gather the results established above in the following lemma:

Lemma 5.5.2. *Let $\vec{\varphi}$ be defined as in* (5.5.8), (5.5.9). *Then, $\varphi \in \mathcal{H}_0$ and $\mathcal{E}(\vec{\varphi}) \le \eta < 2\mathcal{E}(Q)$. Denote by $\vec{\varphi}(t)$ the wave map evolution of $\vec{\varphi}$. Then $\vec{\varphi}(t) \in \mathcal{H}_0$ is global in time and scatters to zero as $t \to \pm\infty$ and we have*

$$\vec{\varphi}(t, r) + \pi = \vec{\psi}(t, r) \quad \forall\ (t, r) \in \{(t, r) \,|\, t \in [0, 1), r \in (1 - t, \infty)\} \tag{5.5.10}$$

Now define

$$\vec{a}(t, r) := \vec{\psi}(t, r) - \vec{\varphi}(t, r). \tag{5.5.11}$$

We use Lemma 5.5.2 to show that $\vec{a}(t)$ has the following properties:

Lemma 5.5.3. *Let $\vec{a}(t)$ be defined as in* (5.5.11). *Then $a(t) \in \mathcal{H}_1$ for all $t \in [0, 1)$ and*

$$\operatorname{supp}(a_r(t), \dot{a}(t)) \in [0, 1 - t). \tag{5.5.12}$$

Moreover we have

$$\lim_{t\to 1}\mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}). \tag{5.5.13}$$

Proof. First observe that (5.5.12) follows immediately from (5.5.10). Next we prove (5.5.13). First observe since $\vec{\varphi}(t) \in \mathcal{H}_0$ is a global wave map with $\mathcal{E}(\vec{\varphi}) < 2\mathcal{E}(Q)$ we have

$$\sup_{t\in[0,1]} \|\vec{\varphi}(t)\|_{H\times L^2(r\le\delta)} \to 0 \quad \text{as} \quad \delta \to 0,$$

which implies in particular that

$$\|\vec{\varphi}(t)\|_{H\times L^2(r\le 1-t)} \to 0 \tag{5.5.14}$$

as $t \to 1$. Next we see that

$$\begin{aligned}
\mathcal{E}(\vec{a}(t)) &= \int_0^{1-t} \left(|\psi_t(t) - \varphi_t(t)|^2 + |\psi_r(t) - \varphi_r(t)|^2 + \frac{\sin^2(\psi(t) - \varphi(t))}{r^2} \right) r\,dr \\
&= \mathcal{E}_0^{1-t}(\vec{\psi}(t) + \int_0^{1-t} (-2\psi_t(t)\varphi(t) - 2\psi_r(t)\varphi_r(t))\; r\,dr \\
&+ \int_0^{1-t} \left(\varphi_t^2(t) + \varphi_r^2(t)\right) r\,dr + \int_0^{1-t} \frac{\sin^2(\psi(t) - \varphi(t)) - \sin^2(\psi(t))}{r}\,dr \\
&= \mathcal{E}_0^{1-t}(\vec{\psi}(t)) + C\mathcal{E}(\vec{\psi})\|\vec{\varphi}(t)\|_{H\times L^2(r\le 1-t)} + C\|\vec{\varphi}(t)\|^2_{H\times L^2(r\le 1-t)} \\
&= \mathcal{E}_0^{1-t}(\vec{\psi}(t)) + o(1) \quad \text{as} \quad t \to 1,
\end{aligned}$$

where on the last line two lines we used (5.5.14) and the fact that

$$\left|\sin^2(x+y) - \sin^2(x)\right| \le 2\,|\sin(x)|\,|y| + 2\,|y|^2\,. \tag{5.5.15}$$

Finally, by Lemma 5.5.2 we observe that for all $t \in [0, 1)$ we have

$$\mathcal{E}^{\infty}_{1-t}(\vec{\psi}(t)) = \mathcal{E}^{\infty}_{1-t}(\vec{\varphi}(t)).$$

Hence,

$$\mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}(t)) - \mathcal{E}^{\infty}_{1-t}(\vec{\varphi}(t)) + o(1) \quad \text{as} \quad t \to 1,$$

which completes the proof. $\square$

5.5.2 Extraction of the blow-up profile

Next, we use Struwe's result, Theorem 5.2.10, to extract a sequence of properly rescaled harmonic maps. At this point we note that we can, after a suitable rescaling and time translation assume, without loss of generality, that the scale λ_0 in Theorem 5.2.10 satisfies $\lambda_0 = 1$. We prove the following result:

Proposition 5.5.4. *Let $\vec{a}(t) \in \mathcal{H}_1$ be defined as in (5.5.11) and let α_n be any sequence with $\alpha_n \to \infty$. Then there exists a sequence of times $\tau_n \to 1$ and a sequence of scales $\lambda_n = o(1 - \tau_n)$ and $\alpha_n \lambda_n < 1 - \tau_n$ such that*

(a) As $n \to \infty$ we have

$$\int_0^{\infty} \dot{a}^2(\tau_n, r)\, r\, dr \leq \frac{1}{n}. \tag{5.5.16}$$

(b) As $n \to \infty$ we have

$$\int_0^{\alpha_n \lambda_n} \left(\left| a_r(\tau_n, r) - \frac{Q_r(r/\lambda_n)}{\lambda_n} \right|^2 + \frac{|a(\tau_n, r) - Q(r/\lambda_n)|^2}{r^2} \right) r\, dr \leq \frac{1}{n}. \tag{5.5.17}$$

(c) As $n \to \infty$ we also have

$$\mathcal{E}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \leq \eta + o_n(1), \tag{5.5.18}$$

which implies that for large enough n we have

$$\mathcal{E}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \leq C < 2\mathcal{E}(Q).$$

Proof. We begin by establishing (5.5.16) and (5.5.17). The basis for the argument is Theorem 5.2.10. Indeed, by Theorem 5.2.10 and Corollary 5.2.13 there exists a sequence of times $t_n \to 0$ and a sequence of scales $\lambda_n = o(1 - t_n)$ such that for any $B \geq 0$ we have

$$\frac{1}{\lambda_n} \int_{t_n}^{t_n+\lambda_n} \int_0^{1-t} \dot{\psi}^2(t, r)\, r\, dr\, dt \to 0$$
$$\frac{1}{\lambda_n} \int_{t_n}^{t_n+\lambda_n} \int_0^{B\lambda_n} \left(\left| \psi_r(t, r) - \frac{Q_r(r/\lambda_n)}{\lambda_n} \right|^2 + \frac{|\psi(t, r) - Q(r/\lambda_n)|^2}{r^2} \right) r\, dr\, dt \to 0$$

as $n \to \infty$. Next observe that since $\vec{\varphi}(t) \in \mathcal{H}_0$ is a global wave map with $\mathcal{E}(\vec{\varphi}) < 2\mathcal{E}(Q)$, we can use the monotonicity of the energy on light cones to deduce that

$$\sup_{t_n \leq t \leq 1} \mathcal{E}_0^{1-t}(\vec{\varphi}(t)) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.5.19}$$

The above then implies that

$$\sup_{t_n \leq t \leq 1} \|\vec{\varphi}(t)\|_{H \times L^2(r \leq 1-t)} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.5.20}$$

By (5.5.11), Lemma 5.5.3 we then have

$$\begin{aligned}\frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt &= \frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{1-t} \left|\dot{\psi}(t,r)-\dot{\varphi}(t,r)\right|^2 r\, dr\, dt \\ &\lesssim \frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt \\ &\quad + \frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{1-t} \dot{\varphi}^2(t,r)\, r\, dr\, dt \to 0.\end{aligned}$$

Using (5.5.20) it is also immediate that

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{B\lambda_n}\left(\left|a_r(t,r)-\frac{Q_r(r/\lambda_n)}{\lambda_n}\right|^2+\frac{|a(t,r)-Q(r/\lambda_n)|^2}{r^2}\right) r\, dr\, dt \to 0.$$

Now, define

$$\begin{aligned}s(B,n) := \frac{1}{\lambda_n}&\int_{t_n}^{t_n+\lambda_n}\int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt \\ &+ \frac{1}{\lambda_n}\int_{t_n}^{t_n+\lambda_n}\int_0^{B\lambda_n}\left(\left|a_r(t,r)-\frac{Q_r(r/\lambda_n)}{\lambda_n}\right|^2+\frac{|a(t,r)-Q(r/\lambda_n)|^2}{r^2}\right) r\, dr\, dt.\end{aligned}$$

We know that for all $B \geq 0$ we have $s(B,n) \to 0$ as $n \to \infty$. Let $\alpha_n \to \infty$. Then there exists a subsequence $\sigma(n)$ such that $s(\alpha_n, \sigma(n)) \to 0$ as $n \to \infty$ with $\alpha_n \lambda_{\sigma(n)} < 1 - t_{\sigma(n)}$. To see this let $N(B,\delta)$ be defined so that for $n \geq N(B,\delta)$ we have $s(B,n) \leq \delta$ and then set $\sigma(n) := N(\alpha_n, 1/n)$. Note that we necessarily have $\alpha_n\lambda_{\sigma(n)} < 1 - t_{\sigma(n)}$. Then we can extract $\tau_{\sigma(n)} \in [t_{\sigma(n)}, t_{\sigma(n)} + \lambda_{\sigma(n)}]$ so that after relabeling we have

$$\begin{aligned}\int_0^\infty &\dot{a}^2(\tau_n,r)\, r\, dr \\ &+ \int_0^{\alpha_n\lambda_n}\left(\left|a_r(\tau_n,r)-\frac{Q_r(r/\lambda_n)}{\lambda_n}\right|^2+\frac{|a(\tau_n,r)-Q(r/\lambda_n)|^2}{r^2}\right) r\, dr \leq \frac{1}{n}\end{aligned}$$

for every n which proves (5.5.16) and (5.5.17).

Lastly, we establish (5.5.18). To see this, let τ_n and λ_n be as in (5.5.16) and (5.5.17). Observe that

$$\begin{aligned}\mathcal{E}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0) &= \mathcal{E}_0^{\alpha_n\lambda_n}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \\ &\quad + \mathcal{E}_{\alpha_n\lambda_n}^{1-\tau_n}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \\ &\quad + \mathcal{E}_{1-\tau_n}^{\infty}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)).\end{aligned}$$

First, observe that (5.5.16) and (5.5.17) directly imply that

$$\mathcal{E}_0^{\alpha_n\lambda_n}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) = o_n(1) \tag{5.5.21}$$

as $n \to \infty$. Next we observe that

$$\mathcal{E}_{\alpha_n\lambda_n}^{\infty}(Q(\cdot/\lambda_n)) = \mathcal{E}_{\alpha_n}^{\infty}(Q) = o_n(1). \tag{5.5.22}$$

Using (5.5.22) and the fact that $\vec{a}(\tau_n, r) = (\pi, 0)$ for every $r \in [1 - \tau_n, \infty)$, we have that

$$\begin{aligned}\mathcal{E}_{1-\tau_n}^{\infty}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) &= \mathcal{E}_{1-\tau_n}^{\infty}((\pi, 0) - (Q(\cdot/\lambda_n), 0)) \\ &\leq \mathcal{E}_{\alpha_n\lambda_n}^{\infty}(Q(\cdot/\lambda_n)) = o_n(1).\end{aligned}$$

Hence it suffices to show that

$$\mathcal{E}_{\alpha_n\lambda_n}^{1-\tau_n}(\vec{a}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \leq \eta + o_n(1). \tag{5.5.23}$$

Applying (5.5.22) again we see that the above reduces to showing that

$$\mathcal{E}_{\alpha_n\lambda_n}^{1-\tau_n}(\vec{a}(\tau_n)) \leq \eta + o_n(1).$$

Now combine the following two facts. One the one hand, for large n, (5.5.13) implies that

$$\mathcal{E}(\vec{a}(\tau_n)) \leq \mathcal{E}(\vec{\psi}) + o_n(1).$$

On the other hand, (5.5.16) and (5.5.17) give that $\mathcal{E}_0^{\alpha_n \lambda_n}(\vec{a}(\tau_n)) = \mathcal{E}(Q) - o_n(1)$. Putting this all together we obtain (5.5.23). $\square$

In the next section we will also need the following consequence of Proposition 5.5.4.

Lemma 5.5.5. *Let α_n, λ_n, and τ_n be defined as in Proposition 5.5.4. Let $\beta_n \to \infty$ be any other sequence such that $\beta_n \leq c_0 \alpha_n$ for all n, for some $c_0 < 1$. Then for every $0 < c_1 < C_2$ such that $C_2 c_0 \leq 1$ there exists $\tilde{\beta}_n$ with $c_1 \beta_n \leq \tilde{\beta}_n \leq C_2 \beta_n$ such that*

$$\psi(\tau_n, \tilde{\beta}_n \lambda_n) \to \pi \quad as \quad n \to \infty \tag{5.5.24}$$

Proof. We first observe that we can combine (5.5.17) and (5.5.14) to conclude that

$$\|\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)\|_{H \times L^2(r \leq \alpha_n \lambda_n)} \to 0 \tag{5.5.25}$$

as $n \to \infty$. Now, suppose (5.5.24) fails. Then there exists $\delta_0 > 0$, $\beta_n \to \infty$ with $\beta_n \leq c_0 \alpha_n$, and $c_1 < C_2$, and a subsequence so that

$$\forall n \quad \psi(\tau_n, \lambda_n r) \notin [\pi - \delta_0, \pi + \delta_0] \quad \forall r \in [c_1 \beta_n, C_2 \beta_n]$$

Now, since $\beta_n \to \infty$ we can choose n large enough so that

$$Q(r) \in [\pi - \delta_0/2, \pi) \quad \forall r \in [c_1 \beta_n, C_2 \beta_n]$$

Putting this together we have that

$$\int_{c_1\beta_n}^{C_2\beta_n} \frac{|\psi(\tau_n, \lambda_n r) - Q(r)|^2}{r}\, dr \geq \left(\frac{C_2 - c_1}{2c_1}\right)^2 \delta_0^2$$

But this directly contradicts (5.5.25) since $C_2\beta_n \leq \alpha_n$ for every n. $\square$

5.5.3 Compactness of the error

For the remainder of this section, α_n, τ_n and λ_n will all be defined by Proposition 5.5.4. Next, we define $\vec{b}_n \in \mathcal{H}_0$ as follows:

$$b_{n,0}(r) = a(\tau_n, r) - Q(r/\lambda_n) \tag{5.5.26}$$

$$b_{n,1}(r) = \dot{a}(\tau_n, r) = o_n(1) \quad \text{in} \quad L^2. \tag{5.5.27}$$

Our goal in this section is to complete the proof of Proposition 5.5.1 by showing that $\vec{b}_n \to 0$ in the energy space. Indeed we prove the following result:

Proposition 5.5.6. *Define* $\vec{b}_n = (b_{n,0}, b_{n,1})$ *as in* (5.5.26), (5.5.27). *Then*

$$\|\vec{b}_n\|_{H\times L^2} \to 0 \tag{5.5.28}$$

as $n \to \infty$.

The first step in the proof of Proposition 5.5.6 is to show that the sequence $\vec{b}_n$ does not contain any nonzero profiles. The proof of this step is reminiscent of an argument given in [22, Section 5] and in particular [22, Proposition 5.1]. Here the situation has been simplified as we have already extracted the large profile $Q(\cdot/\lambda_n)$ by means of Struwe's theorem.

Observe that by Proposition 5.5.4 we have

$$\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q)$$

for n large enough. Denote by $\vec{b}_n(t) \in \mathcal{H}_0$ the wave map evolution with data $\vec{b}_n \in \mathcal{H}_0$. Since $\mathcal{E}(\vec{b}_n) \le C < 2\mathcal{E}(Q)$ for large n, we know from Theorem 5.1.1 that $\vec{b}_n(t) \in \mathcal{H}_0$ is global and scatters to zero as $t \to \pm\infty$.

Proposition 5.5.7. *Let $\vec{b}_n \in \mathcal{H}_0$ and the corresponding global wave map $\vec{b}_n(t) \in \mathcal{H}_0$ be defined as above. Then there exists a decomposition*

$$\vec{b}_n(t,r) = \vec{b}_{n,L}(t,r) + \vec{\theta}_n(t,r) \tag{5.5.29}$$

where $\vec{b}_{n,L}(t,r)$ satisfies the linear wave equation

$$\partial_{tt} b_{n,L} - \partial_{rr} b_{n,L} - \frac{1}{r}\partial_r b_{n,L} + \frac{1}{r^2} b_{n,L} = 0 \tag{5.5.30}$$

with initial data $\vec{b}_{n,L}(0,r) = (b_{n,0}, 0)$. Moreover, $b_{n,L}$ and $\vec{\theta}_n$ satisfy

$$\left\| \frac{1}{r} b_{n,L} \right\|_{L^3_t(\mathbb{R}; L^6_x(\mathbb{R}^4))} \longrightarrow 0 \tag{5.5.31}$$

$$\|\vec{\theta}_n\|_{L^\infty_t(\mathbb{R}; H \times L^2)} + \left\| \frac{1}{r}\theta_n \right\|_{L^3_t(\mathbb{R}; L^6_x(\mathbb{R}^4))} \longrightarrow 0 \tag{5.5.32}$$

as $n \to \infty$.

Before beginning the proof of Proposition 5.5.7 we deduce the following corollary which will be an essential ingredient in the proof of Proposition 5.5.6.

Corollary 5.5.8. *Let $\vec{b}_n(t)$ be defined as in Proposition 5.5.7. Suppose that there exists a constant δ_0 and a subsequence in n so that $\|b_{n,0}\|_H \ge \delta_0$. Then there exists $\alpha_0 > 0$ such that for all $t > 0$ and all n large enough we have*

$$\|\vec{b}_n(t)\|_{H \times L^2(r \ge t)} \ge \alpha_0 \delta_0 \tag{5.5.33}$$

Proof. First note that since $\vec{b}_{n,L}$ satisfies the linear wave equation (5.5.30) with initial data

$\vec{b}_{n,L}(0) = (b_{n,0}, 0)$ we know by Corollary 5.2.3 that there exists a constant $\beta_0 > 0$ so that for each $t \geq 0$ we have

$$\|\vec{b}_{n,L}(t)\|_{H \times L^2(r \geq t)} \geq \beta_0 \|b_{n,0}\|_H$$

On the other hand, by Proposition 5.5.7 we know that

$$\|\vec{b}_n(t) - \vec{b}_{n,L}(t)\|_{H \times L^2(r \geq t)} \leq \|\vec{\theta}_n(t)\|_{H \times L^2} = o_n(1)$$

Putting these two facts together gives

$$\begin{aligned}\|\vec{b}_n(t)\|_{H \times L^2(r \geq t)} &\geq \|b_{n,L}(t)\|_{H \times L^2(r \geq t)} - o_n(1)\\ &\geq \beta_0 \|b_{n,0}\|_H - o_n(1)\end{aligned}$$

This yields (5.5.33) by passing to a suitable subsequence and taking n large enough. $\square$

To prove Proposition 5.5.7 we will first pass to the standard $4d$ representation in order to perform a profile decomposition on the sequence $\vec{b}_n$. Up to extracting a subsequence, $\vec{b}_n \in \mathcal{H}_0$ forms a uniformly bounded sequence with $\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q)$. By Lemma 5.2.1 and the right-most equality in (5.2.10), the sequence $\vec{u}_n = (u_{n,0}, u_{n,1})$ defined by

$$u_{n,0}(r) = \frac{b_{n,0}(r)}{r} \tag{5.5.34}$$

$$u_{n,1}(r) = \frac{b_{n,1}(r)}{r} = o_n(1) \quad \text{in} \quad L^2(\mathbb{R}^4) \tag{5.5.35}$$

is uniformly bounded in $\dot{H}^1 \times L^2(\mathbb{R}^4)$. By Theorem 5.2.14 we can perform the following profile decomposition on the sequence $\vec{u}_n$:

$$u_{n,0}(r) = \sum_{j\leq k} \frac{1}{\lambda_n^j} V_L^j \left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right) + w_{n,0}^k(0,r) \tag{5.5.36}$$

$$u_{n,1}(r) = \sum_{j\leq k} \frac{1}{(\lambda_n^j)^2} \dot{V}_L^j \left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right) + w_{n,1}^k(0,r) \tag{5.5.37}$$

where each $\vec{V}_L^j$ is a free radial wave in $4d$ and where we have for $j \neq k$:

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} \to \infty \quad \text{as} \quad n \to \infty \tag{5.5.38}$$

Moreover, if we denote by $\vec{w}_{n,L}^k(t)$ the free evolution of $\vec{w}_n^k$ we have for $j \leq k$ that

$$\left(\lambda_n^j w_{n,L}^k(\lambda_n^j t_n^j, \lambda_n^j \cdot), (\lambda_n^j)^2 \dot{w}_{n,L}^k(\lambda_n^j t_n^j, \lambda_n^j \cdot) \right) \rightharpoonup 0 \in \dot{H}^1 \times L^2 \quad \text{as} \quad n \to \infty \tag{5.5.39}$$

$$\limsup_{n\to\infty} \|w_{n,L}^k\|_{L_t^3 L_x^6} \to 0 \quad \text{as} \quad k \to \infty \tag{5.5.40}$$

Finally,

$$\|\vec{u}_n\|_{\dot{H}^1\times L^2}^2 = \sum_{j\leq k} \left\| \vec{V}_L^j \left(\frac{-t_n^j}{\lambda_n^j} \right) \right\|_{\dot{H}^1\times L^2}^2 + \|\vec{w}_n^k(0)\|_{\dot{H}^1\times L^2}^2 + o_n(1) \tag{5.5.41}$$

It is also convenient to rephrase the above profile decomposition in the $2d$ formulation. We have

$$b_{n,0}(r) = \sum_{j\leq k} \varphi_L^j \left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right) + \gamma_{n,0}^k(r) \tag{5.5.42}$$

$$b_{n,1}(r) = \sum_{j\leq k} \frac{1}{\lambda_n^j} \dot{\varphi}_L^j \left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right) + \gamma_{n,1}^k(r), \tag{5.5.43}$$

where

$$\varphi_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) := \frac{r}{\lambda_n^j} V_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)$$
$$\gamma_n^k(r) := r w_{n,0}^k(r).$$

and similarly for the time derivatives.

We make the following crucial observation about the scales λ_n^j. By Proposition 5.5.4 we have as $n \to \infty$ that

$$\mathcal{E}_0^{\alpha_n \lambda_n}(b_{n,0}, 0) \to 0, \tag{5.5.44}$$

$$\mathcal{E}_{1-\tau_n}^{\infty}(b_{n,0}, 0) \to 0. \tag{5.5.45}$$

Note that we also have that if $\beta_n \to \infty$ is any other sequence with $\beta_n \leq \alpha_n$ then

$$\mathcal{E}_0^{\beta_n \lambda_n}(b_{n,0}, 0) \to 0. \tag{5.5.46}$$

We can combine (5.5.44) and (5.5.45) with Proposition 5.2.19 to conclude that for each scale λ_n^j corresponding to a nonzero profile φ^j we have

$$\lambda_n \ll \lambda_n^j \leq 1 - \tau_n \tag{5.5.47}$$

at least for n large. In particular,

$$\lambda_n^j \to 0 \quad \text{as} \quad n \to \infty \quad \text{for every } j. \tag{5.5.48}$$

The proof of Proposition 5.5.7 will consist of a sequence of steps designed to show that each of the profiles $\vec{V}_L^j$ (or equivalently the $\vec{\phi}_L^j$) must be identically zero.

Our first goal is to show that all of the time sequences $\{t_{n,j}\}$ can be taken to be $\equiv 0$ and that then the initial velocities of the profiles vanish, i.e., $\dot{V}_L^j(0,r) \equiv 0$ for each j. This is an easy consequence of the following lemma:

Lemma 5.5.9. *In the decomposition* (5.5.36)*,* (5.5.37) *we must have*

$$\limsup_{n\to\infty} \left| \frac{t_n^j}{\lambda_n^j} \right| < \infty \quad \forall\, j \in \mathbb{N} \tag{5.5.49}$$

Corollary 5.5.10. *In the decomposition* (5.5.36)*,* (5.5.37) *we can assume, without loss of generality, that $t_n^j = 0$ for every n and for every j. And, in addition we then have*

$$\dot{V}_L^j(0,r) \equiv 0 \quad \textit{for every} \quad j.$$

Proof of Corollary 5.5.10. Since all of the sequences t_n^j/λ_n^j are bounded, we can assume (by translating the profiles) that $t_{n,j} \equiv 0$ for all j and for all n. In the case when $t_n^j = 0$ for all j, it is easy to see that, besides (5.5.41) the following Pythagorean expansion also holds

$$o_n(1) = \|u_{n,1}\|_{L^2}^2 = \sum_{j\leq k} \left\| \dot{V}_L^j(0) \right\|_{L^2}^2 + \|w_{n,1}^k(0)\|_{L^2}^2 + o_n(1). \tag{5.5.50}$$

from which it is immediate that $V_1^j := \dot{V}_L^j(0) = 0$ for every j. □

We now move to the proof of Lemma 5.5.9. We follow closely the argument in [20], however since there are a few technical differences, we reproduce the proof here.

Note that one way of viewing Corollary 5.5.10 is that, under the hypothesis, one has ability to pass from (5.5.41) to (5.5.50). For a profile decomposition of a general sequence $(v_{n,0}, v_{n,1})$ in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ with $\|v_{n,1}\|_{L^2} = o_n(1)$ this is not possible due to the following example: Let $\vec{V}_L(t)$ be any nonzero free wave and let $s_n \to \infty$ be any sequence of times. Let

$v_{n,0} := 2V_L(s_n)$ and $v_{n,1} = 0$. Then

$$v_{n,0} = V_L^1(-s_n^1) + V_L^2(-s_n^2), \quad v_{n,1} = 0 \tag{5.5.51}$$

where

$$V_L^1(t) := V_L(t),\ s_n^1 := -s_n, \quad V_L^2(t) := V_L(-t),\ s_n^2 := s_n$$

is a profile decomposition which does not satisfy

$$0 = \|u_{n,1}\|_{L^2}^2 \neq \|\dot{V}_L^1(-s_n^1)\|_{L^2}^2 + \|\dot{V}_L^2(-s_n^2)\|_{L^2}^2 + o_n(1)$$

With this example in mind, the first step towards proving Lemma 5.5.9 is to show that such time-symmetric profiles are the only type that can arise with diverging parameters $t_n^j/\lambda_n^j \to \pm\infty$, for a sequence $(v_{n,0}, v_{n,1})$ in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ with $\|v_{n,1}\|_{L^2} = o_n(1)$.

First we establish the following claim. Denote by $\vec{S}(t)$ the free wave propagator in $\mathbb{R}^{1+4}$, i.e., for data (f, g) we set

$$S(t)(f,g) = \cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g,$$
$$\vec{S}(t)(f,g) := (S(t)(f,g),\, \partial_t S(t)(f,g))$$

Claim 5.5.11. *[20, Claim 2] Let $\{f_n, g_n\}$ be a bounded sequence of radial functions in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ and let $A_n >$ be any sequence so that*

$$\|g_n\|_{L^2(r \geq A_n)} \to 0 \quad as \quad n \to \infty \tag{5.5.52}$$

Let t_n be a time sequence so that $|t_n|/A_n \to \infty$ as $n \to \infty$. If

$$\vec{S}(-t_n)(f_n, g_n) \rightharpoonup (V_0, V_1) \in \dot{H}^1 \times L^2$$

then,

$$\vec{S}(t_n)(f_n, g_n) \rightharpoonup (V_0, -V_1) \in \dot{H}^1 \times L^2$$

Proof. The proof follows closely the argument given in [20], but here we crucially use [18, Theorem 4] in place of [22, Lemma 4.1]. Denote by $\langle \cdot, \cdot \rangle_{\dot{H}^1 \times L^2}$ the inner product in $\dot{H}^1 \times L^2$. Given any radial $(h_0, h_1) \in C_0^\infty \times C_0^\infty(\mathbb{R}^4)$ we have

$$\begin{aligned}\left\langle \vec{S}(-t_n)(f_n, g_n),\, (h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} &= \left\langle (f_n, g_n),\, \vec{S}(t_n)(h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} \\ &= \left\langle (f_n, -g_n),\, \vec{S}(t_n)(h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} + o_n(1) \quad \text{as} \quad n \to \infty\end{aligned}$$

We note that the last inequality above is due our assumptions on g_n. Indeed, by [18, Theorem 4] (which says roughly that radial free waves radiate most of their energy near the light cone) and since $|t_n|/A_n \to \infty$, we have

$$\left\langle (0, g_n),\, \vec{S}(t_n)(h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} = o_n(1) \quad \text{as} \quad n \to \infty$$

Using the fact that for any data (f, g) we have

$$\vec{S}(-t)(f, -g) = (S(t)(f, g), -\partial_t S(t)(f, g))$$

we obtain

$$\left\langle \vec{S}(-t_n)(f_n, g_n),\, (h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} = \left\langle \vec{S}(-t_n)(f_n, -g_n),\, (h_0, h_1) \right\rangle_{\dot{H}^1 \times L^2} + o_n(1)$$
$$= \left\langle \vec{S}(t_n)(f_n, g_n),\, (h_0, -h_1) \right\rangle_{\dot{H}^1 \times L^2} + o_n(1) \quad \text{as} \quad n \to \infty$$

which completes the proof. $\square$

Claim 5.5.12. *Let $(v_{n,0}, v_{n,1})$ be a bounded sequence of radial functions in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ such that*

$$\|v_{n,1}\|_{L^2} \to 0 \quad \textit{as} \quad n \to \infty. \tag{5.5.53}$$

Then, after passing to a subsequence, there exists a profile decomposition with free waves V_L^j and parameters $\{t_n^j, \lambda_n^j\}$ so that for a fixed $j \in \mathbb{N}$ we have either

$$t_n^j = 0, \quad \forall n \quad \textit{and} \quad \dot{V}_L^j(0) = 0 \tag{5.5.54}$$

or

$$\frac{t_n^j}{\lambda_n^j} \to \pm\infty \quad \textit{as} \quad n \to \infty \tag{5.5.55}$$

and there exists $k \neq j$ so that

$$V_L^k(t) = V_L^j(-t) \quad \textit{and} \quad \forall n\; t_n^j = -t_n^k, \quad \lambda_n^k = \lambda_n^j \tag{5.5.56}$$

Proof. Fix and $j \in \mathbb{N}$. Recall from [1] that the profile $\vec{V}_L^j$ with parameters $\{t_n^j, \lambda_n^j\}$ is defined by the weak limit

$$\vec{S}(t_n^j/\lambda_n^j)(\lambda_n^j v_{n,0}(\lambda_n^j \cdot), (\lambda_n^j)^2 v_{n,1}(\lambda_n^j \cdot)) \rightharpoonup \vec{V}_L^j(0) \in \dot{H}^1 \times L^2 \tag{5.5.57}$$

Now, we can assume without loss of generality that either $t_n^j = 0$ for all n or that (5.5.55) holds. If $t_{n,j} = 0$ then (5.5.53) and (5.5.57) show that $\partial_t V_L(0) = 0$. In the latter case, we can use Claim 5.5.11 to extract the weak limit

$$\vec{S}(-t_n^j/\lambda_n^j)(\lambda_n^j v_{n,0}(\lambda_n^j \cdot), (\lambda_n^j)^2 v_{n,1}(\lambda_n^j \cdot)) \rightharpoonup (V_L^j(0), -\partial_t V_L^j(0)) \in \dot{H}^1 \times L^2 \tag{5.5.58}$$

This gives us the existence of the kth profile V_L^k for some k precisely as in (5.5.56). $\square$

We can now prove Lemma 5.5.9.

Proof of Lemma 5.5.9. We argue by contradiction. Passing to the $2d$ formulation, assume that there exists a $j_0 \geq 1$ so that $\varphi_L^j \neq 0$ and $-t_n^{j_0}/\lambda_{n,j_0} \to +\infty$. By Claim 5.5.12 and after reordering the profiles we can assume that

$$\varphi_L^{j_0+1}(t) = \varphi_L^{j_0}(-t) \ \text{ and } \ t_n^{j_0+1} = -t_n^{j_0}, \ \ \lambda_n^{j_0+1} = \lambda_n^{j_0}$$

Recall that in Proposition 5.5.4 the time sequence τ_n was chosen so that for every n we have

$$\int_0^\infty \dot{a}^2(\tau_n, r)\, r\, dr \leq \frac{1}{n}$$

Our first observation is that there is considerable flexibility in the choice of τ_n in Proposition 5.5.4. In fact, we claim that there exists a number $\tau_0 \in (0, 1]$ so that

$$\int_0^\infty \dot{a}^2(\tau_n + \lambda_n^{j_0}\tau_0, r)\, r\, dr \to 0 \ \text{ as } \ n \to \infty. \tag{5.5.59}$$

To prove (5.5.59), we first show that there exists a sequence $\varepsilon_n \to 0$ so that

$$\frac{1}{\lambda_n^{j_0}} \int_{\tau_n}^{\tau_n+\lambda_n^{j_0}} \int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt = \varepsilon_n \tag{5.5.60}$$

Recalling that $\vec{a}(t) = \vec{\psi}(t) - \vec{\varphi}(t)$ and using the global regularity of φ we see that it suffices to show that

$$\frac{1}{\lambda_n^{j_0}} \int_{\tau_n}^{\tau_n+\lambda_n^{j_0}} \int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt = o_n(1) \quad \text{as} \quad n \to \infty \tag{5.5.61}$$

Note from the proof of Proposition 5.5.4 that $\tau_n \in [t_n, t_n + \lambda_n]$, where t_n is as in Corollary 5.2.9. We also have $\tau_n + \lambda_n^{j_0} < 1$. From this we infer that

$$\tau_n + \lambda_n^{j_0} \le t_n + \min\{1 - t_n, \lambda_n^{j_0} + \lambda_n\}$$

Setting $\sigma_n = \min\{1 - t_n, \lambda_n^{j_0} + \lambda_n\}$ we see that

$$\begin{aligned} \frac{1}{\lambda_n^{j_0}} \int_{\tau_n}^{\tau_n+\lambda_n^{j_0}} \int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt &\le \frac{1}{\lambda_n^{j_0}} \int_{t_n}^{t_n+\sigma_n} \int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt \\ &\lesssim \frac{1}{\sigma_n} \int_{t_n}^{t_n+\sigma_n} \int_0^{1-t} \dot{\psi}^2(t,r)\, r\, dr\, dt = o_n(1) \end{aligned}$$

where we have used Corollary 5.2.9 and (5.5.47) in the last line.

Next, let

$$E_n := \left\{ \tau \in [0,1] \mid \int_0^\infty \dot{a}^2(\tau_n + \lambda_n^{j_0}\tau, r)\, r\, dr \ge \varepsilon_n^{\frac{1}{4}} \right\}$$

We have

$$\begin{aligned}\varepsilon_n &= \frac{1}{\lambda_n^{j_0}} \int_{\tau_n}^{\tau_n+\lambda_n^{j_0}} \int_0^\infty \dot{a}^2(t,r)\, r\, dr\, dt = \int_0^1 \int_0^\infty \dot{a}^2(\tau_n + \lambda_n^{j_0} t, r)\, r\, dr\, dt \\ &\geq |E_n|\, \varepsilon_n^{\frac{1}{2}}\end{aligned}$$

This implies that $|E_n| \to 0$ as $n \to \infty$. Passing to a subsequence, we can assume that $|E_n| \leq 2^{-n-2}$ so that

$$\left| \bigcup_{n\geq 0} E_n \right| \leq \frac{1}{2} \tag{5.5.62}$$

It follows that 50% of all $\tau_0 \in (0,1]$ satisfy (5.5.59). Choosing any such τ_0 proves (5.5.59).

Now, recall the from the definition of $\vec{b}_n$ we have

$$\vec{\psi}(\tau_n) = Q(\cdot/\lambda_n) + \vec{\varphi}(\tau_n) + \sum_{j\leq k} \varphi_{L,n}^j(0) + \vec{\gamma}_n^k \tag{5.5.63}$$

where we write $\vec{\varphi}_{n,L}$ for the modulated linear profiles, i.e.,

$$\vec{\varphi}_{L,n}^j(t,r) = \left(\varphi_L^j \left(\frac{t - t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right), \frac{1}{\lambda_n^j} \dot{\varphi}_L^j \left(\frac{t - t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j} \right) \right)$$

Now, using (5.5.44), (5.5.47) and [22, Appendix B], choose a sequence $\overline{\lambda}_n \to 0$ such that

$$\begin{gathered}\overline{\lambda}_n \ll \alpha_n \lambda_n, \quad \lambda_n \ll \overline{\lambda}_n \ll \lambda_n^{j_0} \\ \overline{\lambda}_n \ll \lambda_n^j \text{ or } \lambda_n^j \ll \overline{\lambda}_n \quad \forall j > 1.\end{gathered}$$

Now set

$$\overline{\beta}_n = \frac{\overline{\lambda}_n}{\lambda_n} \to \infty$$

and we note that $\overline{\beta}_n \ll \alpha_n$ and $\overline{\lambda}_n = \overline{\beta}_n \lambda_n$. Therefore, up to replacing $\overline{\beta}_n$ by a sequence $\tilde{\beta}_n \simeq \overline{\beta}_n$ and $\overline{\lambda}_n$ by $\overline{\overline{\lambda}}_n := \tilde{\beta}_n \lambda_n$, we have by Lemma 6.3.11 and a slight abuse of notation that

$$\psi(\tau_n, \overline{\lambda}_n) \to \pi \quad \text{as} \quad n \to \infty. \tag{5.5.64}$$

We define the set

$$\mathcal{J}^1_{\text{ext}} := \{ j \geq 1 \mid \overline{\lambda}_n \ll \lambda_n^j \}.$$

Note that by construction $j_0 \in \mathcal{J}^1_{\text{ext}}$.

Next, with $\overline{\lambda}_n$ as above we define $(\overline{f}_{n,0}, \overline{f}_{n,1})$ as follows:

$$\overline{f}_{n,0}(r) := \begin{cases} \pi - \frac{\pi - \psi(\tau_n, \overline{\lambda}_n)}{\overline{\lambda}_n} r & \text{if} \quad 0 \leq r \leq \overline{\lambda}_n \\ \psi(\tau_n, r) & \text{if} \quad \overline{\lambda}_n \leq r \end{cases}$$
$$\overline{f}_{n,1}(r) := \dot{\psi}(\tau_n, r)$$

Then $(\overline{f}_{n,0}, \overline{f}_{n,1}) \in \mathcal{H}_{1,1}$. Now let $\chi \in C_0^\infty$ be defined so that $\chi(r) \equiv 1$ for all $r \in [2, \infty)$ and $\operatorname{supp}(\chi) \subset [1, \infty)$. We define $\vec{\overline{\psi}}_n = (\overline{\psi}_{n,0}, \overline{\psi}_{n,1}) \in \mathcal{H}_0$ as follows:

$$\overline{\psi}_{n,0} := \chi(2r/\tilde{\lambda}_n)(\overline{f}_{n,0}(r) - \pi)$$
$$\overline{\psi}_{n,1} := \chi(2r/\tilde{\lambda}_n)\overline{f}_{n,1}(r)$$

By construction for n large enough we have $\mathcal{E}(\vec{\tilde{\psi}}_n) \leq C < 2\mathcal{E}(Q)$ (for a proof of this fact we refer the reader to the proof of Lemma 5.5.13 for a similar arguement which applies verbatim here). It follows from Theorem 4.0.3 that for each n, the wave map evolution $\vec{\overline{\psi}}_n(t) \in \mathcal{H}_0$ of the data $\vec{\overline{\psi}}_n$ is global in time and scatters to zero as $t \to \pm\infty$. And by the finite speed of propagation, it is immediate that for all t such that $0 \leq \tau_n + t < 1$ we have

$$\vec{\overline{\psi}}_n(t, r) + (\pi, 0) = \vec{\psi}(\tau_n + t, r) \quad \forall r \geq \overline{\lambda}_n + |t| \, . \tag{5.5.65}$$

We also define

$$\vec{\overline{\gamma}}^k_{n,L}(0, r) := \chi(2r/\tilde{\lambda}_n)\vec{\gamma}^k_{n,L}(0, r)$$

Now observe that we can combine (5.5.63) and Proposition 5.2.19 to obtain the following decomposition:

$$\vec{\overline{\psi}}_n(r) = \vec{\varphi}(\tau_n, r) + \sum_{j \in \mathcal{J}^1_{\text{ext}}, \, j \leq k} \vec{\varphi}^j_{L,n}(0) + \vec{\overline{\gamma}}^k_{n,L}(0, r) + o_n(1) \tag{5.5.66}$$

where the $o_n(1)$ above is in the sense of $H \times L^2$. Using Proposition 5.2.17, Lemma 5.2.18, and Lemma 5.2.16 we can find a corresponding nonlinear profile decomposition

$$\vec{\overline{\psi}}_n(t, r) = \vec{\varphi}(\tau_n + t, r) + \sum_{j \in \mathcal{J}^1_{\text{ext}}, \, j \leq k} \vec{\varphi}^j_n(t, r) + \vec{\overline{\gamma}}^k_{n,L}(0, r) + \vec{\overline{\theta}}^k_n(t, r) \tag{5.5.67}$$

where

$$\lim_{k \to \infty} \limsup_{n \to \infty} \left\| \vec{\overline{\theta}}^k_n \right\|_{L^\infty(H \times L^2)} = 0$$

For the precise details on how to deduce (5.5.67) we again refer to the proof of Lemma 5.5.13.

Next, we evaluate (5.5.67) at the time $t = \lambda_n^{j_0}\tau_0$ note that one can extract a *linear* profile decomposition $(\vec{\overline{V}}_L^j, \overline{t}_n^j, \overline{\lambda}_n^j)$ from the sequence $\vec{\overline{\psi}}(\lambda_n^{j_0}\tau_0)$ where the parameters are given by

$$\overline{t}_n^j = t_n^j - \lambda_n^{j_0}, \quad \overline{\lambda}_n^j = \lambda_n^j \tag{5.5.68}$$

Note that the profiles corresponding to the indices j_0 and j_0+1 are precisely $\overline{\varphi}_L^{j_0}(t) = \varphi_L^{j_0}(t)$ and $\overline{\varphi}_L^{j_0+1}(t) = \varphi_L^{j_0+1}(t) = \varphi_L^{j_0}(-t)$. In addition to this we note that by (5.5.65)

$$\vec{\overline{\psi}}_n(\lambda_n^{j_0}\tau_0, r) + (\pi, 0) = \vec{\psi}(\tau_n + \lambda_n^{j_0}\tau_0, r) \quad \forall r \geq \overline{\lambda}_n + \lambda_n^{j_0}\tau_0.$$

Next we apply Claim 5.5.11 with $A_n = \overline{\lambda}_n/\lambda_n^{j_0} + \tau_0$ and $t_n = t_n^{j_0}/\lambda_n^{j_0}$ and

$$(f_n, g_n) = (\overline{\psi}(\lambda_n^{j_0}\tau_0, \lambda_n \cdot), \frac{1}{\lambda_n^{j_0}}\partial_t\overline{\psi}(\lambda_n^{j_0}\tau_0, \lambda_n^{j_0})).$$

By our choice of $\overline{\lambda}_n$ we see that $|t_n|/A_n \to \infty$ and hence

$$\begin{aligned}\text{weak} - \lim_{n\to\infty} \vec{S}(t_n^{j_0}/\lambda_n^{j_0})(f_n, g_n) &= \text{weak} - \lim_{n\to\infty} \vec{S}(\tau_0)\vec{S}(\overline{t}_n^{j_0}/\overline{\lambda}_n^{j_0})(f_n, g_n)\\ &= (\varphi_L^{j_0}(\tau_0), \partial_t\varphi_L^{j_0}(\tau_0))\end{aligned}$$

as well as

$$\begin{aligned}\text{weak} - \lim_{n\to\infty} \vec{S}(-t_n^{j_0}/\lambda_n^{j_0})(f_n, g_n) &= \text{weak} - \lim_{n\to\infty} \vec{S}(\tau_0)\vec{S}(\overline{t}_n^{j_0+1}/\overline{\lambda}_n^{j_0+1})(f_n, g_n)\\ &= (\overline{\varphi}_L^{j_0+1}(\tau_0), \partial_t\overline{\varphi}_L^{j_0+1}(\tau_0)) = (\varphi_L^{j_0}(-\tau_0), -\partial_t\varphi_L^{j_0}(-\tau_0))\end{aligned}$$

But the above implies that

$$\varphi_L^{j_0}(t) = \varphi_L^{j_0}(t + 2\tau_0).$$

Since $\vec{\varphi}_L^j$ is a solution to the linear wave equation the above implies that $\varphi_L^{j_0}$ can only be identically 0, which contradicts the assumption that $\varphi_L^{j_0}$ is nonzero. $\square$

Now, using Corollary 5.5.10 we can rewrite our profile decomposition in the $2d$ formulation as follows.

$$b_{n,0}(r) = \sum_{j \leq k} \varphi^j \left(0, \frac{r}{\lambda_n^j}\right) + \gamma_n^k(r) \tag{5.5.69}$$

$$b_{n,1}(r) = o_n(1) \quad \text{in} \quad L^2, \tag{5.5.70}$$

where

$$\varphi^j \left(0, \frac{r}{\lambda_n^j}\right) := \frac{r}{\lambda_n^j} V_L^j \left(0, \frac{r}{\lambda_n^j}\right)$$

$$\gamma_n^k(r) := r w_{n,0}^k(r).$$

Note that in addition to the Pythagorean expansions given in (5.5.41) we also have the following almost-orthogonal decomposition of the nonlinear energy given by Lemma 5.2.16:

$$\mathcal{E}(\vec{b}_n) = \sum_{j \leq k} \mathcal{E}(\varphi^j(0), 0) + \mathcal{E}(\gamma_n^k, 0) + o_n(1). \tag{5.5.71}$$

Note that $\varphi^j, \gamma_n^k \in \mathcal{H}_0$ for every j, for every n, and for every k. Using the fact that $\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q)$, (5.5.71) and Theorem 4.0.3 imply that, for every j, the nonlinear wave map evolution of the data $(\varphi^j(0, r/\lambda_n^j), 0)$ given by

$$\vec{\varphi}_n^j(t, r) = \left(\varphi^j \left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right), \frac{1}{\lambda_n^j} \dot{\varphi}^j \left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)\right) \tag{5.5.72}$$

is global in time and scatters as $t \to \pm\infty$. Moreover we have the following nonlinear profile decomposition given by Proposition 5.2.17:

$$\vec{b}_n(t,r) = \sum_{j \le k} \vec{\varphi}_n^j(t,r) + \vec{\gamma}_{n,L}^k(t,r) + \vec{\theta}_n^k(t,r) \tag{5.5.73}$$

where $\vec{b}_n(t,r)$ are the global wave map evolutions of the data $\vec{b}_n$ and $\vec{\gamma}_{n,L}^k(t,r)$ is the linear evolution of $(\gamma_n^k, 0)$. Finally, by (5.2.52), we have

$$\limsup_{n\to\infty} \left(\|\vec{\theta}_n^k\|_{L_t^\infty(H\times L^2)} + \left\| \frac{1}{r}\theta_n^k \right\|_{L_t^3(\mathbb{R};L_x^6(\mathbb{R}^4))} \right) \to 0 \quad \text{as} \quad k \to \infty. \tag{5.5.74}$$

Now, recall that our goal is to prove that $\varphi^j = 0$ for every j. Now, let k_0 be the index corresponding to the first nonzero profile φ^{k_0}. Without loss of generality, we can assume that $k_0 = 1$. Using (5.5.44), (5.5.47) and [22, Appendix B], we can find a sequence $\tilde{\lambda}_n \to 0$ such that

$$\begin{aligned}
&\tilde{\lambda}_n \ll \alpha_n \lambda_n \\
&\lambda_n \ll \tilde{\lambda}_n \ll \lambda_n^1 \\
&\tilde{\lambda}_n \ll \lambda_n^j \text{ or } \lambda_n^j \ll \tilde{\lambda}_n \quad \forall j > 1.
\end{aligned}$$

Now define

$$\beta_n = \frac{\tilde{\lambda}_n}{\lambda_n} \to \infty$$

and we note that $\beta_n \ll \alpha_n$ and $\tilde{\lambda}_n = \beta_n \lambda_n$. Therefore, up to replacing β_n by a sequence $\tilde{\beta}_n \simeq \beta_n$ and $\tilde{\lambda}_n$ by $\tilde{\tilde{\lambda}}_n := \tilde{\beta}_n \lambda_n$, we have by Lemma 5.5.5 and a slight abuse of notation that

$$\psi(\tau_n, \tilde{\lambda}_n) \to \pi \quad \text{as} \quad n \to \infty. \tag{5.5.75}$$

We define the set

$$\mathcal{J}_{\text{ext}} := \{j \geq 1 \mid \tilde{\lambda}_n \ll \lambda_n^j\}.$$

Note that by construction $1 \in \mathcal{J}_{\text{ext}}$. The next step consists of establishing the following claim:

Lemma 5.5.13. *Let φ^1, λ_n^1 be defined as above. Then for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}\left|\sum_{j\in\mathcal{J}_{ext},j\leq k}\dot{\varphi}_n^j(t,r)+\dot{\gamma}_{n,L}^k(t,r)\right|^2 r\,dr\,dt = o_n^k \tag{5.5.76}$$

where $\lim\limits_{k\to\infty}\limsup\limits_{n\to\infty} o_n^k = 0$. Also, for all $j > 1$ and for all $\varepsilon > 0$ we have

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}(\dot{\varphi}_n^j)^2(t,r)\,r\,drdt \to 0 \quad \textit{as} \quad n\to\infty. \tag{5.5.77}$$

Note that (5.5.76) and (5.5.77) together directly imply the following result:

Corollary 5.5.14. *Let φ^1 be as in Lemma 5.5.13. Then for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}\left|\dot{\varphi}_n^1(t,r)+\dot{\gamma}_{n,L}^k(t,r)\right|^2 r\,dr\,dt = o_n^k \tag{5.5.78}$$

where $\lim\limits_{k\to\infty}\limsup\limits_{n\to\infty} o_n^k = 0$.

Proof of Lemma 5.5.13. We begin by proving (5.5.76). First recall that by the definition of $\vec{b}_n$ we have the following decomposition

$$\vec{\psi}(\tau_n,r) = (Q(r/\lambda_n),0)+\vec{\varphi}(\tau_n,r)+\sum_{j\leq k}(\varphi^j(0,r/\lambda_n^j),0)+\vec{\gamma}_{n,L}^k(0,r) \tag{5.5.79}$$

Next, with $\tilde{\lambda}_n$ as above we define $\vec{f}_n = (f_{n,0}, f_{n,1})$ as follows:

$$f_{n,0}(r) := \begin{cases} \pi - \frac{\pi - \psi(\tau_n, \tilde{\lambda}_n)}{\tilde{\lambda}_n} r & \text{if} \quad 0 \le r \le \tilde{\lambda}_n \\ \psi(\tau_n, r) & \text{if} \quad \tilde{\lambda}_n \le r \end{cases}$$

$$f_{n,1}(r) := \dot{\psi}(\tau_n, r)$$

Then $\vec{f}_n \in \mathcal{H}_{1,1}$. Now let $\chi \in C_0^\infty$ be defined so that $\chi(r) \equiv 1$ for all $r \in [2, \infty)$ and $\text{supp}(\chi) \subset [1, \infty)$. We define $\vec{\tilde{\psi}}_n = (\tilde{\psi}_{n,0}, \tilde{\psi}_{n,1}) \in \mathcal{H}_0$ as follows:

$$\tilde{\psi}_{n,0} := \chi(2r/\tilde{\lambda}_n)(f_{n,0}(r) - \pi)$$

$$\tilde{\psi}_{n,1} := \chi(2r/\tilde{\lambda}_n) f_{n,1}(r)$$

We claim that for n large enough we have $\mathcal{E}(\vec{\tilde{\psi}}_n) \le C < 2\mathcal{E}(Q)$. To see this, observe that

$$\mathcal{E}(\vec{\tilde{\psi}}_n) = \mathcal{E}_{\tilde{\lambda}_n/2}^{\tilde{\lambda}_n}(\vec{\tilde{\psi}}_n) + \mathcal{E}_{\tilde{\lambda}_n}^{\infty}(\vec{\psi}(\tau_n)). \tag{5.5.80}$$

Using (5.5.75) and (5.2.4), we note that we have $\mathcal{E}_0^{\tilde{\lambda}_n}(\vec{\psi}(\tau_n)) \ge \mathcal{E}(Q) - o_n(1)$ which in turn implies that

$$\mathcal{E}_{\tilde{\lambda}_n}^{\infty}(\vec{\psi}(\tau_n)) \le \eta + o_n(1).$$

We can again use the fact that $\psi(\tau_n, \tilde{\lambda}_n) \to \pi$ and (5.5.44) to deduce that $\mathcal{E}_{\tilde{\lambda}_n/2}^{\tilde{\lambda}_n}(\vec{\tilde{\psi}}_n) = o_n(1)$. Putting these facts into (5.5.80) we obtain the claim since, by assumption, $\eta < 2\mathcal{E}(Q)$.

Now, since $\vec{\tilde{\psi}}_n \in \mathcal{H}_0$ satisfies $\mathcal{E}(\vec{\tilde{\psi}}_n) \le C < 2\mathcal{E}(Q)$, Theorem 5.1.1 implies that for each n, the wave map evolution $\vec{\tilde{\psi}}_n(t) \in \mathcal{H}_0$ of the data $\vec{\tilde{\psi}}_n$ is global in time and scatters to zero as $t \to \pm\infty$. And by the finite speed of propagation, it is immediate that for all t such that

$0 \leq \tau_n + t < 1$ we have

$$\vec{\tilde{\psi}}_n(t, r) + (\pi, 0) = \vec{\psi}(\tau_n + t, r) \quad \forall r \geq \varepsilon\lambda_n^1 + |t| \tag{5.5.81}$$

as long as n is large enough to ensure that $\tilde{\lambda}_n \leq \varepsilon\lambda_n^1$. We also define

$$\vec{\tilde{\gamma}}_{n,L}^k(0, r) := \chi(2r/\tilde{\lambda}_n)\vec{\gamma}_{n,L}^k(0, r)$$

Now observe that we can combine (5.5.79) and Proposition 5.2.19 to to obtain the following decomposition:

$$\vec{\tilde{\psi}}_n(r) = \vec{\varphi}(\tau_n, r) + \sum_{j \in \mathcal{J}_{\text{ext}}, \, j \leq k} (\varphi^j(0, r/\lambda_n^j), 0) + \vec{\tilde{\gamma}}_{n,L}^k(0, r) + o_n(1) \tag{5.5.82}$$

where the $o_n(1)$ above is in the sense of $H \times L^2$. By Lemma 5.2.20 we have that

$$\limsup_{n \to \infty} \left\| \frac{1}{r} \tilde{\gamma}_{n,L}^k \right\|_{L_t^3 L_x^6(\mathbb{R}^{1+4})} \to 0 \quad \text{as} \quad k \to \infty$$

since if the above did not hold we could find subsequences n_ℓ and k_ℓ such that for all ℓ we have

$$\left\| \frac{1}{r} \tilde{\gamma}_{n_\ell, L}^{k_\ell} \right\|_{L_t^3 L_x^6(\mathbb{R}^{1+4})} \geq \varepsilon \quad \text{and} \quad \lim_{\ell \to \infty} \left\| \frac{1}{r} \gamma_{n_\ell, L}^{k_\ell} \right\|_{L_t^3 L_x^6(\mathbb{R}^{1+4})} = 0$$

which would directly contradict Lemma 5.2.20. Hence, if we ignore the $o_n(1)$ term, the right-hand side of (5.5.82) is a profile decomposition in the sense of Corollary 5.2.15. Therefore, by Proposition 5.2.17, and Lemma 5.2.18, we can find $\vec{\tilde{\theta}}_n^k(t, r)$ with

$$\lim_{k \to \infty} \limsup_{n \to \infty} \left\| \vec{\tilde{\theta}}_n^k(t, r) \right\|_{L_t^\infty(H \times L^2)} = 0$$

such that the following nonlinear profile decomposition holds:

$$\vec{\tilde{\psi}}_n(t,r) = \vec{\varphi}(\tau_n + t, r) + \sum_{j\in\mathcal{J}_{\rm ext}\,,\, j\le k} \vec{\varphi}_n^j(t,r) + \vec{\tilde{\gamma}}_{n,L}^k(t,r) + \vec{\tilde{\theta}}_n^k(t,r) \tag{5.5.83}$$

To be precise, (5.5.83) is proved as follows: Define

$$\vec{\breve{\psi}}_n(r) = \vec{\varphi}(\tau_n, r) + \sum_{j\in\mathcal{J}_{\rm ext}\,,\, j\le k} (\varphi^j(0, r/\lambda_n^j), 0) + \vec{\tilde{\gamma}}_{n,L}^k(0,r) \tag{5.5.84}$$

As mentioned above, this is a profile decomposition in the sense of Corollary 5.2.15 and $\mathcal{E}(\vec{\breve{\psi}}_n) < C \le 2\mathcal{E}(Q)$. By Proposition 5.2.17 we then have the following nonlinear profile decomposition for the wave maps evolutions $\vec{\breve{\psi}}_n(t,\cdot) \in \mathcal{H}_0$:

$$\begin{aligned}
&\vec{\breve{\psi}}_n(t,r) = \vec{\varphi}(\tau_n + t, r) + \sum_{j\in\mathcal{J}_{\rm ext}\,,\, j\le k} \vec{\varphi}_n^j(t,r) + \vec{\tilde{\gamma}}_{n,L}^k(t,r) + \vec{\breve{\theta}}_n^k(t,r)\\
&\lim_{k\to\infty} \limsup_{n\to\infty} \left\| \vec{\breve{\theta}}_n^k(t,r) \right\|_{L_t^\infty(H\times L^2)} = 0
\end{aligned}$$

Now, by our perturbation theory, i.e., Lemma 5.2.18, we can deduce (5.5.83) since $\|\vec{\breve{\psi}}_n(0) - \vec{\tilde{\psi}}_n(0)\|_{H\times L^2} = o_n(1)$.

Next, we combine (5.5.83) with (5.5.81) to conclude that

$$\vec{\psi}(\tau_n + t, r) - (\pi, 0) - \vec{\varphi}(\tau_n + t, r) = \sum_{j\in\mathcal{J}_{\rm ext},\, j\le k} \vec{\varphi}_n^j(t,r) + \vec{\gamma}_{n,L}^k(t,r) + \vec{\tilde{\theta}}_n^k(t,r)$$

for all $t + \tau_n < 1$ and $r \ge \varepsilon\lambda_n^1 + t$ for n large enough so that $\tilde{\lambda}_n \le \varepsilon\lambda_n^1$. Using the above we can finally conclude that

$$
\begin{aligned}
\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}\left|\sum_{j\in\mathcal{J}_{\text{ext}}\,,j\le k}\dot{\varphi}_n^j(t,r)+\dot{\gamma}_{n,L}^k(t,r)\right|^2 r\,dr\,dt \\
&\le \frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^{\infty}\dot{a}^2(\tau_n+t,r)\,r\,dr\,dt+o_n^k \\
&\le \frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_0^{\infty}\dot{a}^2(\tau_n+t,r)\,r\,dr\,dt+o_n^k \\
&= \frac{1}{\lambda_n^1}\int_{\tau_n}^{\tau_n+\lambda_n^1}\int_0^{\infty}\dot{a}^2(t,r)\,r\,dr\,dt+o_n^k \\
&\le \frac{1}{\lambda_n^1}\int_{\tau_n}^{\tau_n+\lambda_n^1}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt+\sup_{t\ge\tau_n}\mathcal{E}_0^{1-t}(\vec{\varphi}(t))+o_n^k=o_n^k. \quad (5.5.85)
\end{aligned}
$$

To justify the last line above we need to show that

$$
\frac{1}{\lambda_n^1}\int_{\tau_n}^{\tau_n+\lambda_n^1}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt=o_n(1)
$$

On the one hand, by our construction in the proof of Proposition 5.5.4 we have $\tau_n\in[t_n,t_n+\lambda_n]$ where t_n is as in Corollary 5.2.9 and Theorem 5.2.10. On the other hand, note that $\tau_n+\lambda_n^1<1$. Putting these facts together we infer that

$$
\tau_n+\lambda_n^1\le t_n+\min\{1-t_n,\lambda_n^1+\lambda_n\}
$$

Therefore, if we define $\sigma:=\min\{1-t_n,\lambda_n^1+\lambda_n\}$ we have

$$
\begin{aligned}
\frac{1}{\lambda_n^1}\int_{\tau_n}^{\tau_n+\lambda_n^1}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt &\le \frac{1}{\lambda_n^1}\int_{t_n}^{t_n+\sigma}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt \\
&\lesssim \frac{1}{\sigma}\int_{t_n}^{t_n+\sigma}\int_0^{1-t}\dot{\psi}^2(t,r)\,r\,dr\,dt=o_n(1)
\end{aligned}
$$

where the last line above follows from Corollary 5.2.9. Note that we have used the fact that

$\lambda_n \ll \lambda_n^1$ in the second inequality above. This proves (5.5.76).

Next we prove (5.5.77). Recall that for $j \neq 1$ we have either $\mu_n^j := \frac{\lambda_n^1}{\lambda_n^j} \to 0$ or $\mu_n^j \to \infty$. Suppose the former occurs. Then

$$\begin{aligned}
\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_0^\infty (\dot{\varphi}_n^j)^2(t,r)\, r\, drdt &= \frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_0^\infty \frac{1}{(\lambda_n^j)^2}(\dot{\varphi}^j)^2\left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) r\, drdt \\
&= \frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_0^\infty (\dot{\varphi}^j)^2\left(\frac{t}{\lambda_n^j}, r\right) r\, drdt \\
&= \frac{1}{\mu_n^j}\int_0^{\mu_n^1}\int_0^\infty (\dot{\varphi}^j)^2\,(t,r)\ r\, drdt \\
&\longrightarrow \int_0^\infty (\dot{\varphi}^j)^2\,(0,r)\ r\, drdt = 0
\end{aligned}$$

Now suppose that $\mu_n^j \to \infty$. Then, changing variables as above, we have

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^\infty (\dot{\varphi}_n^j)^2(t,r)\, r\, drdt = \frac{1}{\mu_n^j}\int_0^{\mu_n^1}\int_{\varepsilon\mu_n^j+t}^\infty (\dot{\varphi}^j)^2\,(t,r)\ r\, drdt \tag{5.5.86}$$

Now note that by monotonicity of the energy on exterior cones we have that for all $\delta > 0$ there exists $M > 0$ such that for all $t \in [0, \infty)$ we have

$$\int_{M+t}^\infty (\dot{\varphi}^j)^2\,(t,r)\ r\, dr \leq \delta$$

This implies that the right-hand side of (5.5.86) tends to 0 as $n \to \infty$. $\square$

We can now conclude the proof Proposition 5.5.7.

Proof of Proposition 5.5.7. We first show that all of the profiles φ^j in the decomposition (5.5.69) must be identically 0. We argue by contradiction. As above we assume that $\varphi^1 \neq 0$. By Corollary 5.5.14 we know that for all $\varepsilon > 0$ we have

$$\frac{1}{\lambda_n^1}\int_0^{\lambda_n^1}\int_{\varepsilon\lambda_n^1+t}^\infty \left|\dot{\varphi}_n^1(t,r) + \dot{\gamma}_{n,L}^k(t,r)\right|^2 r\, dr\, dt = o_n^k$$

as $n \to \infty$ for any $k > 1$. Changing variables this implies that

$$\int_0^1 \int_{\varepsilon+t}^{\infty} \left|\dot{\varphi}^1(t,r) + \lambda_n^1 \dot{\gamma}_{n,L}^k(\lambda_n^1 t, \lambda_n^1 r)\right|^2 r\, dr\, dt = o_n^k \tag{5.5.87}$$

Now consider the mapping $H \times L^2 \to \mathbb{R}$ defined by

$$(f_0, f_1) \mapsto \int_0^1 \int_{\varepsilon+t} \dot{\varphi}^1(t,r)\dot{f}(t,r)\, r\, dr\, dt$$

where $\vec{f}(t,r)$ is the solution to the linear wave equation

$$f_{tt} - f_{rr} - \frac{1}{r} f_r + \frac{1}{r^2} f = 0$$

with initial data (f_0, f_1). This is a continuous linear functional on $H \times L^2$. Now, by (5.5.39) we have

$$\left(\gamma_{n,L}^k(\lambda_n^1 \cdot), \lambda_n^1 \dot{\gamma}_{n,L}^k(\lambda_n^1 \cdot)\right) \rightharpoonup 0 \text{ in } H \times L^2 \quad \text{as} \quad n \to \infty$$

Hence, for all $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \int_0^1 \int_{\varepsilon+t}^{\infty} \dot{\varphi}^1(t,r) \lambda_n^1 \dot{\gamma}_{n,L}^k(\lambda_n^1 t, \lambda_n^1 r)\, r\, dr\, dt = 0$$

Combining the above line with (5.5.87) we conclude that for all $\varepsilon > 0$ we have

$$\int_0^1 \int_{\varepsilon+t}^{\infty} \left|\dot{\varphi}^1(t,r)\right|^2 r\, dr\, dt = 0$$

Letting ε tend to 0 we obtain

$$\int_0^1 \int_t^{\infty} \left|\dot{\varphi}^1(t,r)\right|^2 r\, dr\, dt = 0$$

Therefore $\dot{\varphi}^1(t,r)=0$ if $r\geq t$ and $t\in[0,1]$. Let Ω denote the region in $[0,1]\times\mathbb{R}^2$ exterior to the light cone

$$\Omega=\{(t,x)\in[0,1]\times\mathbb{R}^2 \mid |x|\geq t\}$$

If we let $U^1(t,x)=(\varphi^1(t,r),\omega)$ denote the full equivariant wave map (here $x=(r,\omega)$ in polar coordinates on $\mathbb{R}^2$) then we have $(t,x)\in\Omega\Rightarrow U^1(t,x)=U_0^1(x)$. Hence $U_0^1(x)$ is a finite energy equivariant harmonic map on $\mathbb{R}^2-\{0\}$. By Sacks-Uhlenbeck [65] we can extend U_0^1 to a smooth equivariant harmonic map from $\mathbb{R}^2\to S^2$. But since $\varphi^1\in\mathcal{H}_0$, U_0^1 must be identically equal to 0, since 0 is the unique harmonic map in the topological class $\mathcal{H}_0$. But this contradicts the fact that we assumed $\varphi^1\neq 0$.

To complete the proof of Proposition 5.5.7 we note that we have now concluded that all the profiles in the decomposition (5.5.69) must be identically zero. Hence, we have $\gamma_n^k(r)=b_{n,0}(r)$, $\vec{\gamma}_{n,L}^k=: b_{n,L}$, and $\vec{\theta}_n^k=\vec{\theta}_n$ and we can rewrite (5.5.73) as follows:

$$\vec{b}_n(t,r)=\vec{b}_{n,L}(t,r)+\vec{\theta}_n(t,r) \tag{5.5.88}$$

Finally, (5.5.31) and (5.5.32) are satisfied because of (5.5.40) and (5.5.74). $\square$

We can now prove Proposition 5.5.6.

Proof of Proposition 5.5.6. Assume that Proposition 5.5.6 fails. Then up to extracting a subsequence, we can find $\delta_0>0$ so that

$$\|b_{n,0}\|_H\geq\delta_0 \tag{5.5.89}$$

for every n. With this assumption we seek a contradiction. We begin by rescaling. Set

$$\mu_n:=\frac{\lambda_n}{1-\tau_n}.$$

Since $\lambda_n = o(1-\tau_n)$ we have $\mu_n \to 0$ as $n \to \infty$. Now define the rescaled wave maps

$$g_n(t,r) := \psi(\tau_n + (1-\tau_n)t, (1-\tau_n)r)$$
$$h_n(t,r) := \varphi(\tau_n + (1-\tau_n)t, (1-\tau_n)r).$$

Then $\vec{g}_n(t) \in \mathcal{H}_1$ is a wave map defined on the interval $[-\frac{\tau_n}{1-\tau_n}, 1)$, and $\vec{h}_n(t) \in \mathcal{H}_0$ is global in time and scatters to 0. We then have

$$a(\tau_n + (1-\tau_n)t, (1-\tau_n)r) = g_n(t,r) - h_n(t,r).$$

Similarly, define

$$\tilde{b}_{n,0}(r) := b_{n,0}((1-\tau_n)r)$$
$$\tilde{b}_{n,1}(r) := (1-\tau_n)b_{n,1}((1-\tau_n)r)$$

and the corresponding rescaled wave map evolutions

$$\tilde{b}_n(t,r) := b_n((1-\tau_n)t, (1-\tau_n)r)$$
$$\partial_t\tilde{b}_n(t,r) := (1-\tau_n)\dot{b}_n((1-\tau_n)t, (1-\tau_n)r).$$

Observe that we have the decomposition

$$g_n(0,r) = h_n(0,r) + Q\left(\frac{r}{\mu_n}\right) + \tilde{b}_{n,0}(r) \tag{5.5.90}$$
$$\dot{g}_n(0,r) = \dot{h}_n(0,r) + \tilde{b}_{n,1}(r). \tag{5.5.91}$$

Note that by (5.5.12) we have $\tilde{b}_{n,0} = \pi - Q(\cdot/\mu_n)$ on $[1, \infty)$ and hence

$$\|\tilde{b}_{n,0}\|_{H(r\geq 1)} \to 0 \tag{5.5.92}$$

as $n \to \infty$.

Now, observe that the regularity properties of $\varphi(t)$ imply that

$$\lim_{\rho \to 0} \sup_n \|\vec{h}_n(0)\|_{H \times L^2(r \leq \rho/(1-\tau_n))} = 0 \tag{5.5.93}$$

Hence, for fixed large K, (to be chosen precisely later), we can find $r_0 > 0$ so that

$$\sup_n \|\vec{h}_n(0)\|_{H \times L^2(r \leq \frac{3r_0}{(1-\tau_n)})} \leq \frac{\delta_0}{K}, \tag{5.5.94}$$

where δ_0 is as in (5.5.89). Now, recall that $\alpha_n \to \infty$ has been fixed. Using Lemma 5.5.5 we can choose $\gamma_n \to \infty$ with

$$\gamma_n \ll \alpha_n$$

such that

$$g_n(0, \gamma_n \mu_n) \to \pi \quad \text{as} \quad n \to \infty$$

Now define $\delta_n \to 0$ by

$$|g_n(0, \gamma_n \mu_n) - \pi| =: \delta_n \to 0$$

Finally we choose $\beta_n \to \infty$ so that

$$\beta_n \le \min\{\sqrt{\gamma_n}, \delta_n^{-1/2}, \sqrt{n}\}$$
$$g_n(0, \beta_n\mu_n/2) \to \pi \quad \text{as} \quad n \to \infty \tag{5.5.95}$$

We make the following claims:

(i) As $n \to \infty$ we have

$$\|\vec{g}_n(-\beta_n\mu_n/2) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\beta_n\mu_n)} \to 0 \tag{5.5.96}$$

(ii) For each n, on the interval $r \in [\beta_n\mu_n, \infty)$ we have

$$\begin{aligned}\vec{g}_n\left(-\frac{\beta_n\mu_n}{2}, r\right) - (\pi, 0) = \vec{h}_n\left(-\frac{\beta_n\mu_n}{2}, r\right) + \vec{\tilde{b}}_n\left(-\frac{\beta_n\mu_n}{2}, r\right)\\ + \vec{\tilde{\theta}}_n\left(-\frac{\beta_n\mu_n}{2}, r\right),\\ \|\vec{\tilde{\theta}}_n\|_{L^\infty_t(H\times L^2)} \to 0\end{aligned} \tag{5.5.97}$$

We first prove (5.5.96). Note that by Proposition 5.5.4 we have

$$\|(\tilde{b}_{n,0}, \tilde{b}_{n,1})\|_{H\times L^2(r\le\alpha_n\mu_n)} \le \frac{1}{n} \to 0. \tag{5.5.98}$$

Using (5.5.93) together with $\alpha_n\lambda_n \le 1 - \tau_n \to 0$ as well as (5.5.98) and the decomposition (5.5.90) we can then deduce that

$$\|\vec{g}_n(0) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\gamma_n\mu_n)} \le \frac{2}{n} \to 0.$$

Unscale the above by setting $\tilde{g}_n(t,r) = g_n(\mu_n t, \mu_n r)$ and observe that,

$$\|(\tilde{g}_n(0), \partial_t \tilde{g}_n(0)) - (Q(\cdot), 0)\|_{H\times L^2(r\le\gamma_n)} \le \frac{2}{n} \to 0.$$

Now using Corollary 5.2.6 and the finite speed of propagation we claim that we have

$$\|(\tilde{g}_n(-\beta_n/2), \partial_t \tilde{g}_n(-\beta_n/2)) - (Q(\cdot), 0)\|_{H\times L^2(r\le\beta_n)} = o_n(1). \tag{5.5.99}$$

To see this, we need to show that Corollary 5.2.6 applies. Indeed define

$$\hat{g}_{n,0}(r) := \begin{cases} \pi \quad \text{if} \quad r \ge 2\gamma_n \\ \pi + \frac{\pi - \tilde{g}_n(0,\gamma_n)}{\gamma_n}(r - 2\gamma_n) \quad \text{if} \quad \gamma_n \le r \le 2\gamma_n \\ \tilde{g}_n(0,r) \quad \text{if} \quad r \le \gamma_n \end{cases}$$

$$\hat{g}_{n,1}(r) = \begin{cases} \partial_t \tilde{g}_n(0,r) \quad \text{if} \quad r \le \gamma_n \\ 0 \quad \text{if} \quad r \ge \gamma_n \end{cases}$$

Then, by construction we have $\vec{\hat{g}}_n \in \mathcal{H}_1$, and since

$$\|\vec{\hat{g}}_n - (\pi, 0)\|_{H\times L^2(\gamma_n \le r \le 2\gamma_n)} \le C\delta_n$$

we then can conclude that

$$\begin{aligned} \|\vec{\hat{g}}_n - (Q,0)\|_{H\times L^2} &\le \|\vec{\hat{g}}_n - (Q,0)\|_{H\times L^2(r\le\gamma_n)} + \|\vec{\hat{g}}_n - (\pi,0)\|_{H\times L^2(\gamma_n\le r\le 2\gamma_n)} \\ &\quad + \|(\pi,0) - (Q,0)\|_{H\times L^2(r\ge\gamma_n)} \\ &\le C\left(\frac{1}{n} + \delta_n + \gamma_n^{-1}\right) \end{aligned}$$

Now, given our choice of β_n, (5.5.99) follows from Corollary 5.2.6 and the finite speed of propagation. Rescaling (5.5.99) we have

$$\|(g_n(-\beta_n\mu_n/2), \partial_t g_n(-\beta_n\mu_n/2)) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\beta_n\mu_n)} \to 0.$$

This proves (5.5.96). Also note that by monotonicity of the energy on interior cones and the comparability of the energy and the $H \times L^2$ in $\mathcal{H}_0$ for small energies, we see that (5.5.98) implies that

$$\|(\tilde{b}_n(-\beta_n\mu_n/2), \partial_t\tilde{b}_n(-\beta_n\mu_n/2))\|_{H\times L^2(r\le\beta_n\mu_n)} \to 0 \tag{5.5.100}$$

Next we prove (5.5.97). First we define

$$\hat{g}_{n,0}(r) = \begin{cases} \pi - \frac{\pi - g_n(0,\mu_n\beta_n/2)}{\frac{1}{2}\mu_n\beta_n} r \quad \text{if} \quad r \le \beta_n\mu_n/2 \\ g_n(0,r) \quad \text{if} \quad r \ge \beta_n\mu_n/2 \end{cases}$$

$$\hat{g}_{n,1}(r) = \dot{g}_n(0,r)$$

Then, let $\chi \in C^\infty([0,\infty))$ be defined so that $\chi(r) \equiv 1$ on the interval $[2,\infty)$ and supp$\chi \subset [1,\infty)$. Define

$$\vec{\breve{g}}_n(r) := \chi(4r/\beta_n\mu_n)(\vec{\hat{g}}_n(r) - (\pi, 0))$$
$$\vec{\breve{b}}_n(r) := \chi(4r/\beta_n\mu_n)\vec{\tilde{b}}_n(r)$$

and observe that we have the following decomposition

$$\vec{\breve{g}}_n(r) = \vec{h}_n(0,r) + \vec{\breve{b}}_n(r) + o_n(1).$$

where the $o_n(1)$ is in the sense of $H \times L^2$ – here we also have used that $\beta_n\lambda_n \to 0$ together with (5.5.93). Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of Corollary 5.2.15 because of Proposition 5.5.7 and Lemma 5.2.20. We can then consider the nonlinear profiles. Note that by construction we have $\vec{\tilde{g}}_n \in \mathcal{H}_0$ and as usual, we can use (5.5.95) to show that $\mathcal{E}(\vec{\tilde{g}}_n) \leq C < 2\mathcal{E}(Q)$ for large n. The corresponding wave map evolution $\vec{\tilde{g}}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t \to \pm\infty$ by Theorem 5.1.1. We also need to check that $\mathcal{E}(\vec{\tilde{b}}_n) \leq C < 2\mathcal{E}(Q)$. Note that by construction and the definition of $\tilde{b}_n$, we have

$$
\begin{aligned}
\mathcal{E}(\vec{\tilde{b}}_n) &\leq \mathcal{E}(\vec{\tilde{\tilde{b}}}_n) + O\left(\int_0^\infty \frac{4r^2}{\beta_{n,0}^2\mu_n^2}(\chi')^2(4r/\beta_n\mu_n)\frac{b_n^2((1-\tau_n)r)}{r}\,dr\right) \\
&\quad + \int_{\beta_n\mu_n/2}^{\beta_n\mu_n} \frac{\sin^2(\chi(4r/\beta_n\mu_n)b_{n,0}((1-\tau_n)r))}{r}\,dr \\
&\leq \mathcal{E}(\vec{\tilde{\tilde{b}}}_n) + O\left(\int_{\beta_n\lambda_n/2}^{\beta_n\lambda_n} \frac{b_{n,0}^2(r)}{r}\,dr\right) \\
&= \mathcal{E}(\vec{\tilde{\tilde{b}}}_n) + o_n(1) \leq C < 2\mathcal{E}(Q),
\end{aligned}
$$

where the last line follows from Proposition 5.5.4 and the definition of $b_{n,0}$, since $\beta_n \ll \alpha_n$.

Arguing as in the proof of (5.5.83), we can use Proposition 5.5.7, Proposition 5.2.17 and Lemma 5.2.18 to obtain the following nonlinear profile decomposition

$$
\begin{aligned}
&\vec{\tilde{g}}_n(t,r) = \vec{h}_n(t,r) + \vec{\tilde{b}}_n(t,r) + \vec{\tilde{\theta}}_n(t,r) \\
&\|\vec{\tilde{\theta}}_n\|_{L_t^\infty(H\times L^2)} \to 0
\end{aligned}
$$

Finally observe that by construction and the finite speed of propagation we have

$$
\begin{aligned}
&\vec{\tilde{g}}_n(t,r) = \vec{g}_n(t,r) - \pi \\
&\vec{\tilde{b}}_n(t,r) = \vec{\tilde{\tilde{b}}}_n(t,r)
\end{aligned}
$$

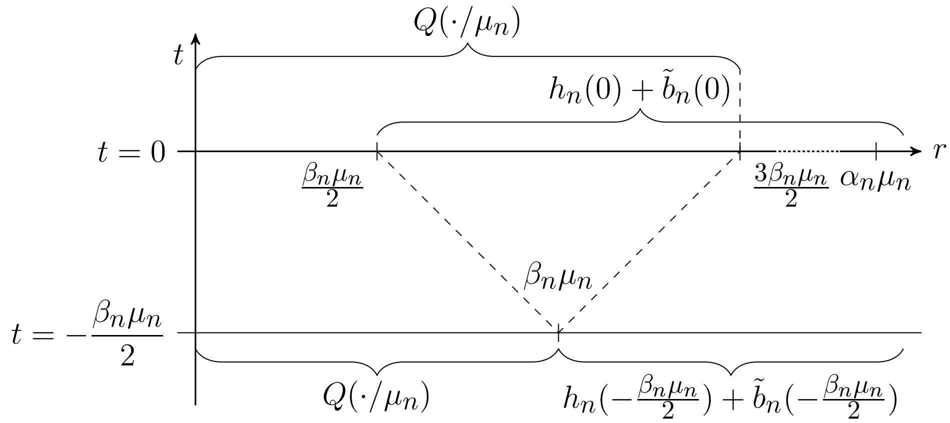

Figure 5.2: A schematic depiction of the evolution of the decomposition (5.5.90) from time $t = 0$ up to $t = -\frac{\beta_n\mu_n}{2}$. At time $t = -\frac{\beta_n\mu_n}{2}$ the decomposition (5.5.101) holds.

for all $t \in [-\tau_n/(1-\tau_n), 1)$ and $r \in [\beta_n\mu_n/2 + |t|, \infty)$. Therefore, in particular we have

$$\vec{g}_n(-\beta_n\mu_n/2, r) - (\pi, 0) = \vec{h}_n(-\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(-\beta_n\mu_n/2, r) + \vec{\tilde{\theta}}_n(\beta_n\mu_n/2, r)$$

for all $r \in [\beta_n\mu_n, \infty)$ which proves (5.5.97).

We can combine (5.5.96), (5.5.97), (5.5.100), and (5.5.93) together with the monotonicity of the energy on interior cones to obtain the decomposition

$$\begin{aligned} \vec{g}_n(-\beta_n\mu_n/2, r) &= (Q(r/\mu_n), 0) + \vec{h}_n(-\beta_n\mu_n/2, r) \\ &\quad + \vec{\tilde{b}}_n(-\beta_n\mu_n/2, r) + \vec{\tilde{\theta}}_n(r) \end{aligned} \tag{5.5.101}$$

$$\|\vec{\tilde{\theta}}_n\|_{H\times L^2} \to 0 \tag{5.5.102}$$

Now define

$$s_n := -\frac{r_0}{1-\tau_n}.$$

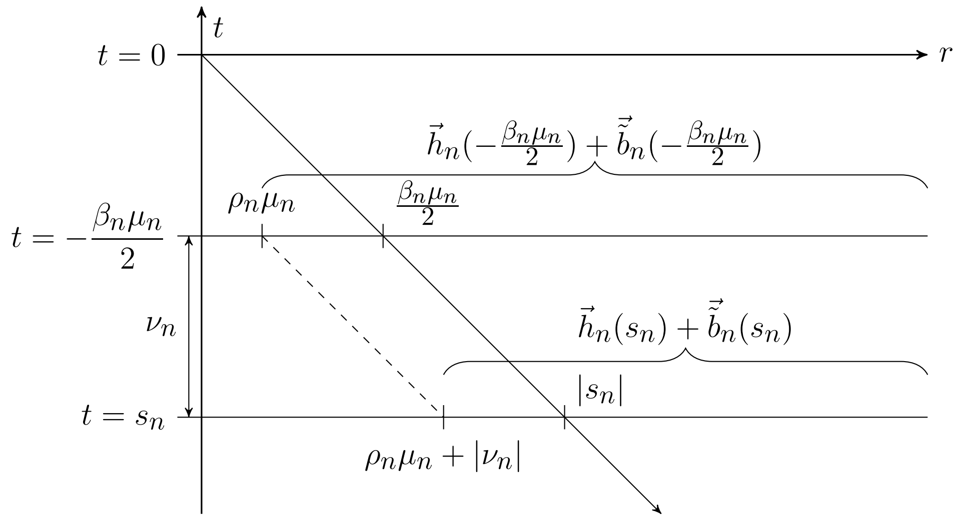

Figure 5.3: A schematic depiction of the evolution of the decomposition (5.5.101) up to time s_n. On the interval $[|s_n|, +\infty)$, the decomposition (5.5.103) holds.

The next step is to prove the following decomposition at time s_n:

$$\vec{g}(s_n, r) - (\pi, 0) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(s_n, r) + \vec{\zeta}_n(r) \quad \forall r \in [|s_n|, \infty) \tag{5.5.103}$$

$$\|\vec{\zeta}_n\|_{H\times L^2} \to 0 \tag{5.5.104}$$

We proceed as in the proof of (5.5.97). By (5.5.96) we can argue as in the proof of Lemma 5.5.5 and find $\rho_n \to \infty$ with $\rho_n \ll \beta_n$ so that

$$g_n(-\beta_n\mu_n/2,\, \rho_n\mu_n) \to \pi \quad \text{as} \quad n \to \infty \tag{5.5.105}$$

Define

$$\hat{f}_{n,0}(r) = \begin{cases} \pi - \frac{\pi - g_n(-\beta_n\mu_n/2,\, \rho_n\mu_n)}{\rho_n\mu_n} r & \text{if} \quad r \le \rho_n\mu_n \\ g_n(-\beta_n\mu_n/2, r) & \text{if} \quad r \ge \rho_n\mu_n \end{cases}$$

$$\hat{f}_{n,1}(r) = \dot{g}_n(-\beta_n\mu_n/2, r)$$

Let $\chi \in C^\infty$ be as above and set

$$\vec{f}_n(r) := \chi(2r/\rho_n\mu_n)(\vec{\tilde{f}}_n(r) - (\pi, 0))$$
$$\vec{\tilde{b}}_n(r) := \chi(2r/\rho_n\mu_n)\vec{\tilde{b}}_n(-\beta_n\mu_n/2, r)$$

Observe that we have the following decomposition:

$$\vec{f}_n(r) = \vec{h}_n(-\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(r) + o_n(1).$$

where the $o_n(1)$ above is in the sense of $H \times L^2$. Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of Corollary 5.2.15 because of Proposition 5.5.7 and Lemma 5.2.20. We can then consider the nonlinear profiles. Note that by construction we have $\vec{f}_n \in \mathcal{H}_0$ and, as usual, we can use (5.5.105) to show that $\mathcal{E}(\vec{f}_n) \leq C < 2\mathcal{E}(Q)$ for large n. The corresponding wave map evolution $\vec{f}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t \to \pm\infty$ by Theorem 5.1.1.

As in the proof of (5.5.97) it is also easy to show that $\mathcal{E}(\vec{\tilde{b}}_n) \leq C < 2\mathcal{E}(Q)$ where here we use (5.5.100) instead of Proposition 5.5.4.

Arguing as in the proof of (5.5.83) we can use Proposition 5.2.17 and Lemma 5.2.18 to obtain the following nonlinear profile decomposition

$$\vec{f}_n(t, r) = \vec{h}_n(-\beta_n\mu_n/2 + t, r) + \vec{\tilde{b}}_n(t, r) + \vec{\tilde{\zeta}}_n(t, r)$$
$$\|\vec{\tilde{\zeta}}_n\|_{L^\infty_t(H\times L^2)} \to 0$$

In particular, for

$$\nu_n := s_n + \beta_n\mu_n/2$$

we have

$$\vec{f}_n(\nu_n, r) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(\nu_n, r) + \vec{\tilde{\zeta}}_n(\nu_n, r).$$

By the finite speed of propagation we have that

$$\begin{aligned} \vec{f}_n(\nu_n, r) &= \vec{g}_n(s_n, r) \\ \vec{\tilde{b}}_n(\nu_n, r) &= \vec{\tilde{b}}_n(s_n, r) \end{aligned}$$

as long as $r \geq \rho_n\mu_n + |\nu_n|$. Using the fact that $\rho_n \ll \beta_n$ we have that $|s_n| \geq \rho_n\mu_n + |\nu_n|$ and hence,

$$\vec{g}_n(s_n, r) - (\pi, 0) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(s_n, r) + \vec{\tilde{\zeta}}_n(\nu_n, r) \quad \forall r \in [|s_n|, \infty).$$

Setting $\vec{\zeta}_n := \vec{\tilde{\zeta}}_n(\nu_n)$ we obtain (5.5.103) and (5.5.104). Now, combine (5.5.104), (5.5.94), and the monotonicity of the energy on light cones for the evolution of $\vec{h}_n$, we obtain:

$$\|\vec{g}_n(s_n) - (\pi, 0) - \vec{\tilde{b}}_n(s_n)\|_{H \times L^2(|s_n| \leq r \leq 2|s_n|)} \leq \frac{C\delta_0}{K} \tag{5.5.106}$$

for n large enough. By Corollary 5.5.8 and (5.5.89), there exists $\beta_0 > 0$ so that for all $t \in \mathbb{R}$ we have

$$\|\vec{\tilde{b}}_n(t)\|_{H \times L^2(r \geq |t|)} \geq \beta_0\delta_0$$

By (5.5.92) and the monotonicity of the energy on cones we have

$$\|\vec{\tilde{b}}_n(t)\|_{H \times L^2(r \geq |t|+1)} \to 0$$

as $n \to \infty$. Therefore we have

$$\|\vec{\tilde{b}}_n(t)\|_{H \times L^2(|t| \le r \le 1+|t|)} \ge \frac{\beta_0 \delta_0}{2}$$

for n large enough and for all $t \in \mathbb{R}$. Hence setting $t = s_n$ we see that the above and (5.5.106) imply in particular that

$$\|\vec{g}_n(s_n) - (\pi, 0)\|_{H \times L^2(|s_n| \le r \le 1+|s_n|)} \ge \frac{\beta_0 \delta_0}{4} > 0$$

for n, K large enough. Un-scaling this we obtain

$$\|\vec{\psi}(\tau_n - r_0) - (\pi, 0)\|_{H \times L^2(r_0 \le r \le r_0 + (1-\tau_n))} \ge \frac{\beta_0 \delta_0}{4} > 0.$$

However this contradicts the fact the $\psi(t, r)$ cannot concentrate any energy at the point $(1 - r_0, r_0) \in [0, 1) \times [0, \infty)$ with $r_0 > 0$. This concludes the proof of Proposition 5.5.1. $\square$

We can now finish the proof of Theorem 5.1.3.

Proof of Theorem 5.1.3. Let $\vec{a}(t)$ be defined as in (5.5.11). Recall that by Lemma 5.5.3 we have

$$\lim_{t \to 1} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}) \tag{5.5.107}$$

Over the course of the proof of Proposition 5.5.1 we have found a sequence of times $\tau_n \to 1$ so that

$$\mathcal{E}(\vec{a}(\tau_n)) \to \mathcal{E}(Q)$$

as $n \to \infty$. Since $\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta$ this implies that $\mathcal{E}(\vec{\varphi}) = \eta$ since the right hand side of

(5.5.107) is independent of t. This then implies that

$$\lim_{t\to 1} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(Q)$$

We now use the variational characterization of Q to show that in fact $\|\dot{a}(t)\|_{L^2} \to 0$ as $t \to 1$. To see this observe that since $a(t) \in \mathcal{H}_1$ we can deduce by (5.2.18) that

$$\mathcal{E}(Q) \leftarrow \mathcal{E}(a(t), \dot{a}(t)) \geq \int_0^\infty \dot{a}^2(t,r)\, r\, dr + \mathcal{E}(Q)$$

Next observe that the decomposition in Lemma 5.2.5 provides us with a function $\lambda : (0,\infty) \to (0,\infty)$ such that

$$\|a(t,\cdot) - Q(\cdot/\lambda(t))\|_H \leq \delta(\mathcal{E}(a(t),0) - \mathcal{E}(Q)) \to 0$$

This also implies that

$$\mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)), 0)) \to 0 \tag{5.5.108}$$

as $t \to 1$. Since $t \mapsto a(t)$ is continuous in H for $t \in [0,1)$ it follows from Lemma 5.2.5 that $\lambda(t)$ is continuous on $[0,1)$. Therefore we have established that

$$\vec{\psi}(t) - \vec{\varphi}(t) - (Q(\cdot/\lambda(t)), 0) \to 0 \quad \text{in} \quad H \times L^2 \quad \text{as} \quad t \to 1$$

It remains to show that $\lambda(t) = o(1-t)$. This follows immediately from the support properties of $\nabla_{t,r} a$ and from (5.5.108). To see this observe that $a(t,r) - Q(r/\lambda(t)) = \pi - Q(r/\lambda(t))$ on $[1-t,\infty)$. Thus,

$$\mathcal{E}^{\infty}_{\frac{1-t}{\lambda(t)}}(Q) = \mathcal{E}^{\infty}_{1-t}(\pi - Q(\cdot/\lambda(t))) \leq \mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)), 0)) \to 0.$$

But this then implies that $\frac{1-t}{\lambda(t)} \to \infty$ as $t \to 1$. This completes the proof. $\square$

5.6 Appendix: Higher Equivariance classes and more general targets

5.6.1 1-*equivariant wave maps to more general targets*

Theorem 5.1.1, Theorem 5.1.2, and Theorem 5.1.3 can be extended to a larger class of equations, namely equivariant wave maps to general, rotationally symmetric compact targets. To be specific, each of these theorems holds in the case that the target manifold M is a surface of revolution with the metric given in polar coordinates, $(\rho, \omega) \in [0, \infty) \times S^1$, by $ds^2 = d\rho^2 + g^2(\rho) d\omega^2$ where $g : \mathbb{R} \to \mathbb{R}$ is a smooth, odd, function with $g(0) = 0$, $g'(0) = 1$. In addition, in order to ensure the existence of stationary solutions to the corresponding equivariant wave map equation we need to require that there exists $C > 0$ such such that $g(C) = 0$ and we let C^* be minimal with this property. We also assume that $g'(C^*) = -1$ and that g is periodic with period $2C^*$. In this case, the nonlinear wave equation of interest is given by

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{f(\psi)}{r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \end{aligned} \tag{5.6.1}$$

where $f(\psi) = g(\psi)g'(\psi)$. The conserved energy for this problem is given by

$$\mathcal{E}(\vec{\psi}(t)) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{g^2(\psi)}{r^2} \right) r\, dr = \text{const.}$$

To see how this extension works, we note that the small data well-posedness theory for (5.6.1) is given in [17, Theorem 2]. One then needs replacements for the estimates involving the sin function in the proof of the orthogonality of the nonlinear energy, the proof of the nonlinear perturbation theory, and later in estimates involving the energy of $\vec{a}(t)$, namely (5.2.48), (5.2.53), and (5.5.15). But, the same type of estimates for g are easily established using the assumptions we have made on g and its derivatives and simple calculus.

For more details regarding more general metrics we refer the reader to [17]. Note that since we do not rely on [17, Lemma 7] we are able to eliminate their condition [17, (A3)].

5.6.2 Higher equivariance classes and the 4*d-equivariant Yang-Mills system*

We can also consider higher equivariance classes, $\ell > 1$. Restricting our attention again to the case $g(\rho) = \sin(\rho)$, the Cauchy problem for ℓ equivariant wave maps reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \ell^2 \frac{\sin(2\psi)}{2r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \end{aligned} \tag{5.6.2}$$

For ℓ-equivariant wave maps of topological degree zero we can, as in the 1-equivariant case, consider the reduction $\psi =: r^\ell u$ and we obtain the following Cauchy problem for u:

$$u_{tt} - u_{rr} - \frac{2\ell + 1}{r} u_r = u^{1+2/\ell} Z(r^\ell u) \tag{5.6.3}$$

with

$$Z(\rho) := \frac{\ell^2}{2} \frac{\sin(2\rho) - 2\rho}{\rho^{1+2/\ell}}$$

a bounded function. In [17, Theorem 2] a suitable local well-posedness/small data theory for such a nonlinearity is addressed when $\ell = 2$ and thus Theorem 5.1.1 follows from the same arguments in this chapter. For $\ell > 2$, one would need to develop a suitable well-posedness theory for (5.6.3). This presents some difficulties due the fractional power, $1 + 2/\ell$, in the nonlinearity.

One can also consider the $4d$ equivariant Yang-Mills system:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$
$$\partial_\beta F^{\alpha\beta} + [A_\beta, F^{\alpha\beta}] = 0, \quad \alpha, \beta = 0, \dots, 3$$

for the connection form A_α and the curvature $F_{\alpha\beta}$. After, making the equivariant ansatz:

$$A_\alpha^{ij} = (\delta_\alpha^i x^j - \delta_\alpha^j x^i)\frac{1 - \psi(t,r)}{r^2}$$

one obtains the following equation for ψ:

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r - \frac{2\psi(1-\psi^2)}{r^2} = 0$$

which can be written in the form

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \ell^2 \frac{f(\psi)}{r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1) \end{aligned} \tag{5.6.4}$$

for $f(\rho) = g(\rho)g'(\rho)$ and $g(\rho) = 1/2(1-\rho^2)$ and $\ell = 2$. This equation is of the same form as (5.6.2) with $\ell = 2$ and a more general metric g. The local well-posedness/small data scattering theory for (5.6.4) is addressed in [17, Theorem 2]. The proof and conclusions of Theorem 1.1 thus hold for solutions of this equation with suitable modifications as in the case of 1-equivariant wave maps to more general targets addressed above.

As we mentioned in the introduction, modulo a suitable local well-posedness/small data theory, one should be able to apply our methods to prove the analog of Theorem 5.1.3 for the odd higher equivariance classes, $\ell = 3, 5, 7, \dots,$. The reason is that if ℓ is odd, the linearized version of equation (5.6.2) is a $2\ell + 2$ dimensional free radial wave equation with $2\ell + 2 = 0$

mod 4 for ℓ odd, and in these dimensions Proposition 5.2.2 holds, see [18, Corollary 5].

However, as demonstrated in [18], Proposition 5.2.2 *fails* for $\ell = 2, 4, 6, \ldots$, since $2\ell+2 = 2$ mod 4 for ℓ even. Therefore it is impossible to prove Corollary 5.5.8 in these cases and our contradiction argument for the compactness of the error term $\vec{b}_n$ does not go through. So our method is not suited to prove the complete conclusions of Theorem 5.1.3 for either the even equivariance classes or the $4d$ Yang-Mills system, which corresponds roughly to the case $\ell = 2$. However, the rest of the argument preceding the proof of Proposition 5.5.1 should go through and in particular one should be able to deduce Proposition 5.5.7. This would allow one to conclude that the error terms $\vec{b}_n$ contain no profiles and converge to zero in a Strichartz norm adapted to the nonlinearity in (5.6.2). This is a slightly weaker result than showing that the $\vec{b}_n$'s vanish in the energy space, but on its own, it is already quite strong.

CHAPTER 6

CLASSIFICATION OF $2D$ EQUIVARIANT WAVE MAPS TO POSITIVELY CURVED TARGETS: PART II

6.1 Introduction

We continue our study of the equivariant wave maps problem from $1+2$ dimensional Minkowski space to 2–dimensional surfaces of revolution.

Recall that in spherical coordinates,

$$(\psi, \omega) \mapsto (\sin\psi\cos\omega, \sin\psi\sin\omega, \cos\psi),$$

on $\mathbb{S}^2$, the metric, g, is given by the matrix $g = \mathrm{diag}(1, \sin^2(\psi))$. In the case of 1-equivariant wave maps, we require our wave map, U, to have the form

$$U(t, r, \omega) = (\psi(t, r), \omega) \mapsto (\sin\psi(t, r)\cos\omega,\ \sin\psi(t, r)\sin\omega,\ \cos\psi(t, r)),$$

where (r, ω) are polar coordinates on $\mathbb{R}^2$. In this case, the Cauchy problem reduces to

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0 \\ &(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1). \end{aligned} \tag{6.1.1}$$

Wave maps exhibit a conserved energy, which in this equivariant setting is given by

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\,dr = \text{const.},$$

and they are invariant under the scaling

$$\vec{\psi}(t,r) := (\psi(t,r), \psi_t(t,r)) \mapsto (\psi(\lambda t, \lambda r), \lambda \psi_t(\lambda t \lambda r)).$$

The conserved energy is also invariant under this scaling which means that the Cauchy problem under consideration is energy critical.

We refer the reader to the previous chapter for a more detailed introduction and history of the equivariant wave maps problem.

As in Chapter 5, we note that any wave map $\vec{\psi}(t,r)$ with finite energy and continuous dependence on $t \in I$ satisfies $\psi(t,0) = m\pi$ and $\psi(t,\infty) = n\pi$ for all $t \in I$ for fixed integers m, n. This determines a disjoint set of energy classes

$$\mathcal{H}_{m,n} := \{(\psi_0, \psi_1) \,|\, \mathcal{E}(\psi_0, \psi_1) < \infty \quad \text{and} \quad \psi_0(0) = m\pi, \, \psi_0(\infty) = n\pi\}. \tag{6.1.2}$$

We will mainly consider the spaces $\mathcal{H}_{0,n}$ and we denote these by $\mathcal{H}_n := \mathcal{H}_{0,n}$. In this case we refer to n as the degree of the map. We also define $\mathcal{H} = \bigcup_{n \in \mathbb{Z}} \mathcal{H}_n$ to be the full energy space.

In our analysis, an important role is played by the unique (up to scaling) non-trivial harmonic map, $Q(r) = 2\arctan(r)$, given by stereographic projection. We note that Q solves

$$Q_{rr} + \frac{1}{r} Q_r = \frac{\sin(2Q)}{2r^2}. \tag{6.1.3}$$

Observe in addition that $(Q, 0) \in \mathcal{H}_1$ and in fact $(Q, 0)$ has minimal energy in $\mathcal{H}_1$ with $\mathcal{E}(Q) := \mathcal{E}(Q, 0) = 4$. Note the slight abuse of notation above in that we will denote the energy of the element $(Q, 0) \in \mathcal{H}_1$ by $\mathcal{E}(Q)$ rather than $\mathcal{E}(Q, 0)$.

Recall that in Chapter 5 we showed that for any data $\vec{\psi}(0)$ in the zero topological class,

$\mathcal{H}_0$, with energy $\mathcal{E}(\vec{\psi}) < 2\mathcal{E}(Q)$ there is a corresponding unique global wave map evolution $\vec{\psi}(t,r)$ that scatters to zero in the sense that the energy of $\vec{\psi}(t)$ on any arbitrary, but fixed compact region vanishes as $t \to \infty$, see Theorem 5.1.1. An equivalent way to view this scattering property is that there exists a decomposition

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + o_{\mathcal{H}}(1) \quad \text{as} \quad t \to \infty \tag{6.1.4}$$

where $\vec{\varphi}_L(t) \in \mathcal{H}_0$ solves the linearized version of (6.1.1):

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0 \tag{6.1.5}$$

This result was proved via the concentration-compactness/rigidity method which was developed by the Kenig and Merle in [36] and [37], and it provides a complete classification of all solutions in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$, namely, they all exist globally and scatter to zero. We note that this theorem is also a consequence of the work by Sterbenz and Tataru in [75] if one considers their results in the equivariant setting.

In the previous chapter we also study *degree one* wave maps, $\vec{\psi}(t) \in \mathcal{H}_1$, with energy $\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q)$ that blow up in finite time. Because we are working in the equivariant, energy critical setting, blow-up can only occur at the origin in $\mathbb{R}^2$ and in an energy concentration scenario. We show that if blow-up does occur, say at $t = 1$, then there exists a scaling parameter $\lambda(t) = o(1-t)$, a degree zero map $\vec{\varphi} \in \mathcal{H}_0$ and a decomposition

$$\vec{\psi}(t,r) = \vec{\varphi}(r) + (Q\,(r/\lambda(t))\,, 0) + o_{\mathcal{H}}(1) \quad \text{as} \quad t \to 1. \tag{6.1.6}$$

Here we complete our study of *degree one* solutions to (6.1.1), i.e., solutions that lie in $\mathcal{H}_1$, with energy below $3\mathcal{E}(Q)$, by providing a classification of such solutions with this energy constraint. Since the degree of the map is preserved for all time, scattering to zero is not possible for a degree one solution. However, we show that a decomposition of the form (6.1.6) holds in the global case. In particular we establish the following theorem:

Theorem 6.1.1 (Classification of solutions in $\mathcal{H}_1$ with energies below $3\mathcal{E}(Q)$)**.** *Let $\vec{\psi}(0) \in \mathcal{H}_1$ and denote by $\vec{\psi}(t) \in \mathcal{H}_1$ the corresponding wave map evolution. Suppose that $\vec{\psi}$ satisfies*

$$\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q).$$

Then, one of the following two scenarios occurs:

(1) ***Finite time blow-up:*** *The solution $\vec{\psi}(t)$ blows up in finite time, say at $t = 1$, and there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \eta$, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + (Q\,(\cdot/\lambda(t))\,, 0) + \vec{\epsilon}(t) \tag{6.1.7}$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

(2) ***Global Solution:*** *The solution $\vec{\psi}(t) \in \mathcal{H}_1$ exists globally in time and there exists a continuous function, $\lambda : [0, \infty) \to (0, \infty)$ with $\lambda(t) = o(t)$ as $t \to \infty$, a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linear wave equation* (6.1.5)*, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q\,(\cdot/\lambda(t))\,, 0) + \vec{\epsilon}(t) \tag{6.1.8}$$

such that $\vec{\epsilon}(t) \in \mathcal{H}_0$ and $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to \infty$.

Remark 21. One should note that the requirement $\lambda(t) = o(t)$ as $t \to \infty$ in part (2) above leaves open many possibilities for the asymptotic behavior of global degree one solutions to (6.1.1) with energy below $3\mathcal{E}(Q)$. If $\lambda(t) \to \lambda_0 \in (0, \infty)$ then our theorem says that the solution $\psi(t)$ asymptotically decouples into a soliton, Q_{λ_0}, plus a purely dispersive term, and one can call this *scattering to* Q_{λ_0}. If $\lambda(t) \to 0$ as $t \to \infty$ then this means that the solution is concentrating $\mathcal{E}(Q)$ worth of energy at the origin as $t \to \infty$ and we refer to this phenomenon as *infinite time blow-up*. Finally, if $\lambda(t) \to \infty$ as $t \to \infty$ then the solution can be thought of as concentrating $\mathcal{E}(Q)$ worth of energy at spacial infinity as $t \to \infty$ and we call this *infinite time flattening*.

We also would like to highlight the fact that global solutions of the type mentioned above, i.e., infinite time blow-up and flattening, have been constructed in the case of the $3d$ semi-linear focusing energy critical wave equation by Donninger and Krieger in [19]. No constructions of this type are known at this point for the energy critical wave maps studied here. In addition, a classification of all the possible dynamics for maps in $\mathcal{H}_1$ at energy levels $\geq 3\mathcal{E}(Q)$ remains open.

Remark 22. We emphasize that Chapter 5 goes hand-in-hand with this chapter. In fact, part (1) of Theorem 6.1.1 was established in Chapter 5. Therefore, in order to complete the proof of Theorem 6.1.1 we need to prove only part (2) and the rest of this chapter will be devoted to that goal. The broad outline of the proof of Theorem 6.1.1 (2) is similar in nature to the proof of part (1). With this is mind we will often refer the reader to the previous chapter where the details are nearly identical instead of repeating the same arguments here.

Remark 23. We remark that Theorem 6.1.1 is reminiscent of the recent works of Duyckaerts, Kenig, and Merle in [22, 21, 24, 23] for the energy critical semi-linear focusing wave equation in 3 spacial dimensions and again we refer the reader to the previous chapter for a more detailed description of the similarities and differences between these papers and this work.

Remark 24. Finally, we would like to note that the same observations in appendix of the previous chapter regarding 1-equivariant wave maps to more general targets, higher equivariance classes and the $4d$ equivariant Yang-Mills system hold in the context of the global statement in Theorem 6.1.1.

6.2 Preliminaries

For the reader's convenience, we recall a few facts and notations from [15] and the previous chapter that are used frequently in what follows. We define the 1-equivariant energy space to be

$$\mathcal{H} = \{\vec{U} \in \dot{H}^1 \times L^2(\mathbb{R}^2; \mathbb{S}^2) \,|\, U \circ \rho = \rho \circ U, \;\; \forall \rho \in SO(2)\}.$$

$\mathcal{H}$ is endowed with the norm

$$\mathcal{E}(\vec{U}(t)) = \|\vec{U}(t)\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^2;\mathbb{S}^2)} = \int_{\mathbb{R}^2} (|\partial_t U|_g^2 + |\nabla U|_g^2)\, dx. \tag{6.2.1}$$

As noted in the introduction, by our equivariance condition we can write $U(t, r, \omega) = (\psi(t, r), \omega)$ and the energy of a wave map becomes

$$\mathcal{E}(U, \partial_t U)(t) = \mathcal{E}(\psi, \psi_t)(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr = \text{const.} \tag{6.2.2}$$

We also define the localized energy as follows: Let $r_1, r_2 \in [0, \infty)$. Then

$$\mathcal{E}_{r_1}^{r_2}(\vec{\psi}(t)) := \int_{r_1}^{r_2} \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr.$$

Following Shatah and Struwe, [68], we set

$$G(\psi) := \int_0^\psi |\sin \rho| \; d\rho. \tag{6.2.3}$$

Observe that for any $(\psi, 0) \in \mathcal{H}$ and for any $r_1, r_2 \in [0, \infty)$ we have

$$\begin{aligned} |G(\psi(r_2)) - G(\psi(r_1))| &= \left| \int_{\psi(r_1)}^{\psi(r_2)} |\sin \rho| \; d\rho \right| \\ &= \left| \int_{r_1}^{r_2} |\sin(\psi(r))|\, \psi_r(r)\, dr \right| \le \frac{1}{2} \mathcal{E}_{r_1}^{r_2}(\psi, 0). \end{aligned} \tag{6.2.4}$$

We also recall from Chapter 5 the definition of the space $H \times L^2$.

$$\|(\psi_0, \psi_1)\|_{H\times L^2}^2 := \int_0^\infty \left(\psi_1^2 + (\psi_0)_r^2 + \frac{\psi_0^2}{r^2} \right) r\, dr. \tag{6.2.5}$$

We note that for degree zero maps $(\psi_0, \psi_1) \in \mathcal{H}_0$ the energy is comparable to the $H \times L^2$ norm provided the L^∞ norm of ψ_0 is uniformly bounded below π. This equivalence of norms is detailed in Lemma 5.2.1, see also [17, Lemma 2]. The space $H \times L^2$ is not defined for maps $(\psi_0, \psi_1) \in \mathcal{H}_1$, but one can instead consider the $H \times L^2$ norm of $(\psi_0 - Q_\lambda, 0)$ for $\lambda \in (0, \infty)$, and $Q_\lambda(r) = Q(r/\lambda)$. In fact, for maps $\vec{\psi} \in \mathcal{H}_1$ such that $\mathcal{E}(\vec{\psi}) - \mathcal{E}(Q)$ is small, one can choose $\lambda > 0$ so that

$$\|(\psi_0 - Q_\lambda, \psi_1)\|_{H\times L^2}^2 \simeq \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q).$$

This amounts to the coercivity of the energy near Q up to the scaling symmetry. For more details we refer the reader to [14, Proposition 4.3], Lemma 5.2.5, and [3].

6.2.1 Properties of global wave maps

We will need a few facts about global solutions to (6.1.1). The results in this section constitute slight refinements and a few consequences of the work of Shatah and Tahvildar-Zadeh in [71, Section 3.1] on global equivariant wave maps and originate in the work of Christodoulou and Tahvildar-Zadeh on spherically symmetric wave maps, see [12].

Proposition 6.2.1. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Let $0 < \lambda < 1$. Then we have*

$$\limsup_{t\to\infty} \mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) \to 0 \quad \textit{as} \quad A \to \infty. \tag{6.2.6}$$

In fact, we have

$$\mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) \to 0 \quad as \quad t,\, A \to \infty \quad for \quad A \le (1-\lambda)t. \tag{6.2.7}$$

We note that Proposition 6.2.1 is a refinement of [71, (3.4)], see also [12, Corollary 1] where the case of spherically symmetric wave maps is considered. To prove this result, we follow [12], [71], and [68] and introduce the following quantities:

$$\begin{aligned} e(t,r) &:= \psi_t^2(t,r) + \psi_r^2(t,r) + \frac{\sin^2(\psi(t,r))}{r^2} \\ m(t,r) &:= 2\psi_t(t,r)\psi_r(t,r). \end{aligned}$$

Observe that with this notation the energy identity becomes:

$$\partial_t e(t,r) = \frac{1}{r}\partial_r \left(r\, m(t,r)\right), \tag{6.2.8}$$

which we can conveniently rewrite as

$$\partial_t(re(t,r)) - \partial_r(r\, m(t,r)) = 0. \tag{6.2.9}$$

Using the notation in [12], we set

$$\begin{aligned} \alpha^2(t,r) &:= r\left(e(t,r) + m(t,r)\right) \\ \beta^2(t,r) &:= r(e(t,r) - m(t,r)) \end{aligned}$$

and we define null coordinates

$$u = t - r, \quad v = t + r.$$

Next, for $0 \leq \lambda < 1$ set

$$\mathscr{E}_\lambda(u) := \int_{\frac{1+\lambda}{1-\lambda}u}^{\infty} \alpha^2(u,v)\,dv \tag{6.2.10}$$

$$\mathscr{F}(u_0,u_1) := \lim_{v\to\infty} \int_{u_0}^{u_1} \beta^2(u,v)\,du. \tag{6.2.11}$$

Also, let $\mathscr{C}_\rho^{\pm}$ denote the interior of the forward (resp. backward) light-cone with vertex at $(t,r) = (\rho, 0)$ for $\rho > 0$ in (t,r) coordinates.

As in [71, Section 3.1], one can show that the integral in (6.2.10) and the limit in (6.2.11) exist for a wave map of finite energy, see also [12, Section 2] for the details of the argument for the spherically symmetric case.

By integrating the energy identity (6.2.9) over the region $(\mathscr{C}_{u_0}^{+} \backslash \mathscr{C}_{u_1}^{+}) \cap \mathscr{C}_{v_0}^{-}$, where $0 < u_0 < u_1 < v_0$, we obtain the identity

$$\int_{u_0}^{u_1} \beta^2(u,v)\,du = \int_{u_0}^{v_0} \alpha^2(u_0,v)\,dv - \int_{u_1}^{v_0} \alpha^2(u_1,v)\,dv.$$

Letting $v_0 \to \infty$ we see that

$$0 \leq \mathscr{F}(u_0,u_1) = \mathscr{E}_0(u_0) - \mathscr{E}_0(u_1), \tag{6.2.12}$$

which shows that $\mathscr{E}_0$ is decreasing. Next, note that

$$\mathscr{F}(u,u_2) = \mathscr{F}(u,u_1) + \mathscr{F}(u_1,u_2) \geq \mathscr{F}(u,u_1)$$

for $u_2 > u_1$, and thus $\mathscr{F}(u,u_1)$ is increasing in u_1. $\mathscr{F}(u,u_1)$ is also bounded above by $\mathscr{E}(u)$ so

$$\mathscr{F}(u) := \lim_{u_1\to\infty} \mathscr{F}(u,u_1)$$

exists and, as in [71], [12], we have

$$\mathscr{F}(u) \to 0 \quad \text{as} \quad u \to \infty. \tag{6.2.13}$$

Finally note that the argument in [12, Lemma 1] shows that for all $0 < \lambda < 1$ we have

$$\mathscr{E}_\lambda(u) \to 0 \quad \text{as} \quad u \to \infty, \tag{6.2.14}$$

which is stated in [71, (3.3)]. To deduce (6.2.14), follow the exact argument in [12, proof of Lemma 1] using the relevant multiplier inequalities for equivariant wave maps established in [68, proof of Lemma 8.2] in place of [12, equation (6)]. We can now prove Proposition 6.2.1.

Proof of Proposition 6.2.1. Fix $\lambda \in (0,1)$ and $\delta > 0$. Find A_0 and T_0 large enough so that

$$0 \le \mathscr{F}(A) \le \delta, \quad 0 \le \mathscr{E}_\lambda((1-\lambda)t) \le \delta$$

for all $A \ge A_0$ and $t \ge T_0$. In (u, v)–coordinates consider the points

$$\begin{aligned} &X_1 = ((1-\lambda)t, (1+\lambda)t), \quad X_2 = (A, 2t - A) \\ &X_3 = (A, \bar{v}), \quad X_4 = ((1-\lambda)t, \bar{v}) \end{aligned}$$

where $\bar{v}$ is very large. Integrating the energy identity (6.2.9) over the region Ω bounded by the line segments X_1X_2, X_2X_3, X_3X_4, X_4X_1 we obtain,

$$\mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) = -\int_{2t-A}^{\bar{v}} \alpha^2(A, v)\, dv + \int_A^{(1-\lambda)t} \beta^2(u, \bar{v})\, du + \int_{(1+\lambda)t}^{\bar{v}} \alpha^2((1-\lambda)t, v)\, dv.$$

Letting $\bar{v} \to \infty$ above and recalling that $\mathscr{F}(u, u_1)$ is increasing in u_1 we have

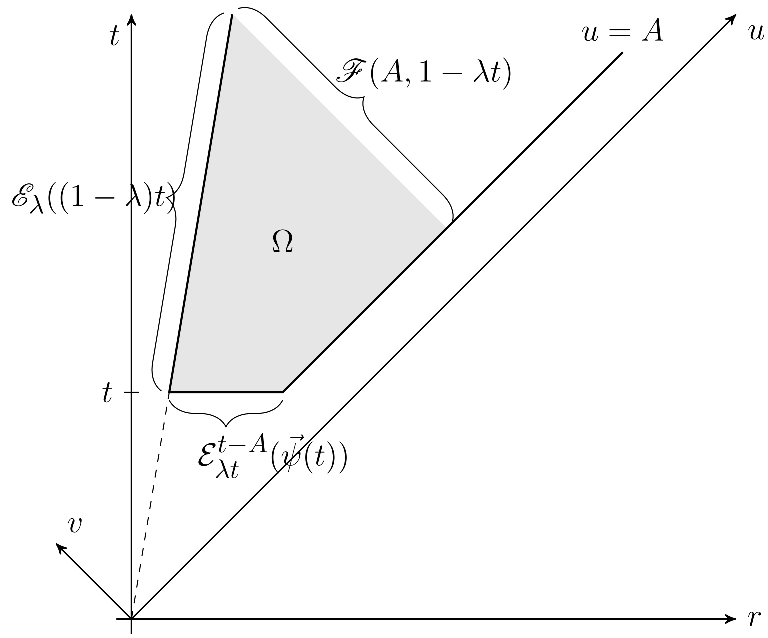

Figure 6.1: The quadrangle Ω over which the energy identity is integrated is the gray region above.

$$\begin{aligned} \mathcal{E}_{\lambda t}^{t-A}(\vec{\psi}(t)) &\leq \mathscr{E}_\lambda((1-\lambda)t) + \mathscr{F}(A, (1-\lambda)t) \\ &\leq \mathscr{E}_\lambda((1-\lambda)t) + \mathscr{F}(A). \end{aligned}$$

The proposition now follows from (6.2.14) and (6.2.13). □

We will also need the following corollaries of Proposition 6.2.1:

Corollary 6.2.2. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \int_A^T \int_0^{t-A} \psi_t^2(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad A \to \infty. \tag{6.2.15}$$

Proof. We will use the following virial identity for solutions to (6.1.1):

$$\partial_t(r^2 m) - \partial_r(r^2\psi_t^2 + r^2\psi_r^2 - \sin^2\psi) + 2r\psi_t^2 = 0. \tag{6.2.16}$$

Now, fix $\delta > 0$ so that $\delta < 1/3$ and find A_0, T_0 so that for all $A \geq A_0$ and $t \geq T_0$ we have

$$\mathcal{E}_{\delta t}^{t-A}(\vec{\psi}(t)) \leq \delta.$$

Then,

$$\int_0^{\delta t} e(t,r) r^2 \, dr \leq \mathcal{E}(\vec{\psi}(t)) \delta t$$

and as long as we ensure that $A \leq 1/3t$, we obtain

$$\int_{\delta t}^{2t/3} e(t,r) \, r^2 \, dr \leq \delta t.$$

This implies that

$$\int_0^{2t/3} e(t,r) \, r^2 \, dr \leq C\delta t, \quad \text{and} \quad \int_0^{2t/3} e(t,r) \, r^3 \, dr \leq C\delta t^2.$$

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\chi(x) = 1$ for $|x| \leq 1/3$, $\chi(x) = 0$ for $|x| \geq 2/3$ and $\chi'(x) \leq 0$. Then, using the virial identity (6.2.16) we have

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty m(t,r)\chi(r/t) \, r^2 \, dr &= \int_0^\infty \partial_t (r^2 m(t,r)) \, \chi(r/t) \, dr - \frac{2}{t^2} \int_0^\infty \psi_t \psi_r \, r^3 \chi'(r/t) \, dr \\
&= \int_0^\infty \partial_r (r^2(\psi_t^2 + \psi_r^2) - \sin^2(\psi))\chi(r/t) \, dr \\
&\quad - 2 \int_0^\infty \psi_t^2(t,r)\chi(r/t) \, r \, dr + O(\delta) \\
&= \frac{1}{t^2} \int_0^\infty (r^2(\psi_t^2 + \psi_r^2) - \sin^2(\psi))\chi'(r/t) \, r \, dr \\
&\quad - 2 \int_0^\infty \psi_t^2(t,r)\chi(r/t) \, r \, dr + O(\delta) \\
&= -2 \int_0^\infty \psi_t^2(t,r)\chi(r/t) \, r \, dr + O(\delta).
\end{aligned}$$

Integrating in t between 0 and T yields

$$\int_0^T \int_0^\infty \psi_t^2(t,r)\chi(r/t)\, r\, dr\, dt \le C\delta T$$

with an absolute constant $C > 0$. By the definition of $\chi(x)$ this implies

$$\int_0^T \int_0^{t/3} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta T.$$

Next, note that we have

$$\begin{aligned}\int_A^T \int_{t/3}^{t-A} \psi_t^2(t,r)\, r\, dr\, dt &\le \int_A^{T_0} \mathcal{E}(\vec{\psi})\, dt + \int_{T_0}^T \int_{t/3}^{t-A} e(t,r)\, r\, dr\, dt \\ &\le (T_0 - A)\mathcal{E}(\vec{\psi}) + (T - T_0)\delta.\end{aligned}$$

Therefore,

$$\frac{1}{T}\int_A^T \int_0^{t-A} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta + \frac{T_0}{T}\mathcal{E}(\vec{\psi})$$

Hence,

$$\limsup_{T\to\infty} \frac{1}{T}\int_A^T \int_0^{t-A} \psi_t^2(t,r)\, r\, dr\, dt \le C\delta$$

for all $A \ge A_0$, which proves (6.2.15). $\square$

Corollary 6.2.3. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a smooth global wave map. Recall that $\vec{\psi}(t) \in \mathcal{H}$ implies that there exists $k \in \mathbb{Z}$ such that for all t we have $\psi(t,\infty) = k\pi$. Then for any $\lambda > 0$ we have*

$$\|\psi(t) - \psi(t,\infty)\|_{L^\infty(r \ge \lambda t)} \to 0 \quad \textit{as} \quad t \to \infty. \tag{6.2.17}$$

Before proving Corollary 6.2.3, we can combine Proposition 6.2.1 and Corollary 6.2.3 to immediately deduce the following result.

Corollary 6.2.4. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a global wave map. Let $0 < \lambda < 1$. Then we have*

$$\limsup_{t \to \infty} \|\vec{\psi}(t) - (\psi(t, \infty), 0)\|^2_{H \times L^2(\lambda t \le r \le t - A)} \to 0 \quad as \quad A \to \infty. \tag{6.2.18}$$

Proof. Say $\vec{\psi}(t) \in \mathcal{H}_k$. Observe that Corollary 6.2.3 shows that for t_0 large enough we have, say,

$$|\psi(t, r) - k\pi| \le \frac{\pi}{100}$$

for all $t \ge t_0$ and $r \ge \lambda t$. This in turn implies that for $t \ge t_0$ we can find a $C > 0$ such that

$$|\psi(t, r) - k\pi|^2 \le C \sin^2(\psi(t, r)) \quad \forall\, t \ge t_0,\ r \ge \lambda t.$$

Now (6.2.18) follows directly from (6.2.6). $\square$

The first step in the proof of Corollary 6.2.3 is the following lemma:

Lemma 6.2.5. *Let $\vec{\psi}(t) \in \mathcal{H}$ be a smooth global wave map. Let $R > 0$ and suppose that the initial data $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_1$ satisfies $\operatorname{supp}(\partial_r \psi_0), \operatorname{supp}(\psi_1) \subset B(0, R)$. Then for any $t \ge 0$ and for any $A < t$ we have*

$$\|\psi(t) - \psi(t, \infty)\|_{L^\infty(r \ge t - A)} \le \sqrt{\mathcal{E}(\vec{\psi})} \sqrt{\frac{A + R}{t - A}}. \tag{6.2.19}$$

Proof. By the finite speed of propagation we note that for each $t \ge 0$ we have $\operatorname{supp}(\psi_r(t)) \subset$

$B(0, R+t)$. Hence, for all $t \geq 0$ we have

$$\begin{aligned}
|\psi(t,r) - \psi(t,\infty)| &\leq \int_r^\infty \left|\psi_r(t,r')\right| \, dr' \\
&\leq \left(\int_r^{R+t} \psi_r^2(t,r')\, r'\, dr'\right)^{\frac{1}{2}} \left(\int_r^{R+t} \frac{1}{r'}\, dr'\right)^{\frac{1}{2}} \\
&\leq \sqrt{\mathcal{E}(\vec{\psi})}\sqrt{\log\left(\frac{t+R}{r}\right)}.
\end{aligned}$$

Next observe that if $r \geq t - A$ then

$$\log\left(\frac{t+R}{r}\right) \leq \log\left(1 + \frac{A+R}{r}\right) \leq \log\left(1 + \frac{A+R}{t-A}\right) \leq \frac{A+R}{t-A}.$$

This proves (6.2.19). $\square$

Proof of Corollary 6.2.3. Say $\psi(t) \in \mathcal{H}_k$, that is $\psi(t,\infty) = k\pi$ for all t. First observe that by an approximation argument, it suffices to consider wave maps $\vec{\psi}(t) \in \mathcal{H}_k$ with initial data $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_k$ with

$$\operatorname{supp}(\partial_r \psi_0), \operatorname{supp}(\psi_1) \subset B(0,R)$$

for $R > 0$ arbitrary, but fixed. Now, let $t_n \to \infty$ be any sequence and set

$$A_n := \sqrt{t_n}.$$

Then, for each $r \geq \lambda t_n$ we have

$$|\psi(t_n, r) - k\pi| \leq \|\psi(t_n) - k\pi\|_{L^\infty(\lambda t_n \leq r \leq t_n - A_n)} + \|\psi(t_n) - k\pi\|_{L^\infty(r \geq t_n - A_n)}.$$

By Lemma 6.2.5 we know that

$$\|\psi(t_n) - k\pi\|_{L^\infty(r \geq t_n - A_n)} \leq \sqrt{\mathcal{E}(\psi)}\sqrt{\frac{\sqrt{t_n} + R}{t_n - \sqrt{t_n}}} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.2.20}$$

Hence it suffices to show that

$$\|\psi(t_n) - k\pi\|_{L^\infty(\lambda t_n \leq r \leq t_n - A_n)} \to 0 \quad \text{as} \quad n \to \infty.$$

To see this, first observe that (6.2.20) implies that

$$\psi(t_n, t_n - A_n) \to k\pi$$

as $n \to \infty$. Therefore it is enough to show that

$$\|\psi(t_n) - \psi(t_n, t_n - A_n)\|_{L^\infty(\lambda t_n \leq r \leq t_n - A_n)} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.2.21}$$

With G defined as in (6.2.3) we can combine (6.2.4) and Proposition 6.2.1 to deduce that for all $r \geq \lambda t_n$ we have

$$|G(\psi(t_n, r)) - G(\psi(t_n, t_n - A_n))| \leq \frac{1}{2}\mathcal{E}_{\lambda t_n}^{t_n - A_n}(\vec{\psi}(t_n)) \to 0.$$

as $n \to \infty$. This immediately implies (6.2.21) since G is a continuous, increasing function. $\square$

6.3 Profiles for global degree one solutions with energy below $3\mathcal{E}(Q)$

In this section we carry out the proof of Theorem 6.1.1 (2). We start by first deducing the conclusions along a sequence of times. To be specific, we establish the following proposition:

Proposition 6.3.1. *Let $\psi(t) \in \mathcal{H}_1$ be a global solution to* (6.1.1) *with*

$$\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q).$$

Then there exist a sequence of times $\tau_n \to \infty$, a sequence of scales $\lambda_n \ll \tau_n$, a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linear wave equation (6.1.5)*, and a decomposition*

$$\vec{\psi}(\tau_n) = \vec{\varphi}_L(\tau_n) + \left(Q\left(\cdot/\lambda_n\right), 0\right) + \vec{\epsilon}(\tau_n) \tag{6.3.1}$$

such that $\vec{\epsilon}(\tau_n) \in \mathcal{H}_0$ and $\vec{\epsilon}(\tau_n) \to 0$ in $H \times L^2$ as $n \to \infty$.

To prove Proposition 6.3.1 we proceed in several steps. We first construct the sequences τ_n and λ_n while identifying the large profile, $Q(\cdot/\lambda_n)$. Once we have done this, we extract the radiation term φ_L. In the last step, we prove strong convergence of the error

$$\vec{\epsilon}(\tau_n) := \vec{\psi}(\tau_n) - \vec{\varphi}_L(\tau_n) - \left(Q\left(\cdot/\lambda_n\right), 0\right) \to 0$$

in the space $H \times L^2$.

6.3.1 The harmonic map at $t = +\infty$

Here we prove the analog of Struwe's result [76, Theorem 2.1] for global wave maps of degree different than zero, i.e., $\psi(t) \in \mathcal{H} \backslash \mathcal{H}_0$ for all $t \in [0, \infty)$. This will allow us to identify the sequences τ_n, λ_n and the harmonic maps $Q(\cdot/\lambda_n)$ in the decomposition (6.3.1).

Theorem 6.3.2. *Let $\vec{\psi}(t) \in \mathcal{H} \backslash \mathcal{H}_0$ be a smooth, global solution to* (6.1.1)*. Then, there exists a sequence of times $t_n \to \infty$ and a sequence of scales $\lambda_n \ll t_n$ so that the following results hold: Let*

$$\vec{\psi}_n(t, r) := \Big(\psi(t_n + \lambda_n t, \lambda_n r), \lambda_n \dot{\psi}(t_n + \lambda_n t, \lambda_n r)\Big) \tag{6.3.2}$$

be the global wave map evolutions associated to the initial data

$$\vec{\psi}_n(r) := (\psi(t_n, \lambda_n r), \lambda_n \dot{\psi}(t_n, \lambda_n, r)).$$

Then, there exists $\lambda_0 > 0$ *so that*

$$\vec{\psi}_n \to (\pm Q(\cdot/\lambda_0), 0) \quad in \quad L^2_t([0,1); H^1 \times L^2)_{\mathrm{loc}}.$$

We begin with the following lemma, which follows from Corollary 6.2.2 and is the global-in-time version of Corollary 5.2.9 from the previous chapter. The statement and proof are also very similar to [24, Lemma 4.4] and [22, Corollary 5.3].

Lemma 6.3.3. *Let* $\vec{\psi}(t) \in \mathcal{H}$ *be a smooth global wave map. Let* $A : (0,\infty) \to (0,\infty)$ *be any increasing function such that* $A(t) \nearrow \infty$ *as* $t \to \infty$ *and* $A(t) \le t$ *for all* t*. Then, there exists a sequence of times* $t_n \to \infty$ *such that*

$$\lim_{n\to\infty} \sup_{\sigma>0} \frac{1}{\sigma} \int_{t_n}^{t_n+\sigma} \int_0^{t-A(t_n)} \dot{\psi}^2(t,r)\, r\, dr\, dt = 0. \tag{6.3.3}$$

Proof. The proof is analogous to the argument given in [22, Corollary 5.3]. We argue by contradiction. The existence of a sequence of times t_n satisfying (6.3.3) is equivalent to the statement

$$\begin{gathered}\forall A(t) \nearrow \infty \text{ with } A(t) \le t \text{ as } t \to \infty,\ \forall \delta > 0,\ \forall T_0 > 0,\ \exists \tau \ge T_0 \text{ so that}\\ \sup_{\sigma>0} \frac{1}{\sigma} \int_{\tau}^{\tau+\sigma} \int_0^{t-A(\tau)} \dot{\psi}^2(t,r)\, r\, dr\, dt \le \delta.\end{gathered}$$

So we assume that (6.3.3) fails. Then,

$$\exists A(t) \nearrow \infty \text{ with } A(t) \le t \text{ as } t \to \infty,\ \exists \delta > 0,\ \exists T_0 > 0,\ \forall \tau \ge T_0,\ \exists \sigma > 0 \text{ so that}$$
$$\frac{1}{\sigma}\int_\tau^{\tau+\sigma}\int_0^{t-A(\tau)} \dot\psi^2(t,r)\, r\, dr\, dt > \delta. \tag{6.3.4}$$

Now, by Corollary 6.2.2 we can find a large A_1 and a $T_1 = T_1(A_1) > T_0$ so that for all $T \ge T_1$ we have

$$\frac{1}{T}\int_{A_1}^{T}\int_0^{t-A_1} \dot\psi^2(t,r)\, r\, dr\, dt \le \delta/100. \tag{6.3.5}$$

Since $A(t) \nearrow \infty$ we can fix $T > T_1$ large enough so that $A(t) \ge A_1$ for all $t \ge T$. Define the set X as follows:

$$X := \left\{ \sigma > 0 \,:\, \frac{1}{\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot\psi^2(t,r)\, r\, dr\, dt \ge \delta \right\}.$$

Then X is nonempty by (6.3.4). Define $\rho := \sup X$. We claim that $\rho \le T$. To see this assume that there exists $\sigma \in X$ so that $\sigma \ge T$. Then we would have

$$T + \sigma \le 2\sigma.$$

This in turn implies, using (6.3.5), that

$$\frac{1}{2\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot\psi^2(t,r)\, r\, dr\, dt \le \frac{1}{T+\sigma}\int_{A_1}^{T+\sigma}\int_0^{t-A_1} \dot\psi^2(t,r)\, r\, dr\, dt \le \delta/100$$

where we have also used the fact that $A(T) \ge A_1$. This would mean that

$$\frac{1}{\sigma}\int_T^{T+\sigma}\int_0^{t-A(T)} \dot\psi^2(t,r)\, r\, dr\, dt \le \delta/50,$$

which is impossible since we assumed that $\sigma \in X$. Therefore $\rho \leq T$. Moreover, we know that

$$\int_T^{T+\rho} \int_0^{T-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt \geq \delta\rho. \tag{6.3.6}$$

Now, since $T + \rho > T > T_1 > T_0$ we know that there exists $\sigma > 0$ so that

$$\int_{T+\rho}^{T+\rho+\sigma} \int_0^{t-A(T+\rho)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta\sigma.$$

Since $A(t)$ is increasing, we have $A(T) \leq A(T+\rho)$ and hence the above implies that

$$\int_{T+\rho}^{T+\rho+\sigma} \int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta\sigma. \tag{6.3.7}$$

Summing (6.3.6) and (6.3.7) we get

$$\int_T^{T+\rho+\sigma} \int_0^{t-A(T)} \dot{\psi}^2(t,r)\, r\, dr\, dt > \delta(\sigma+\rho),$$

which means that $\rho + \sigma \in X$. But this contradicts that fact that $\rho = \sup X$. $\square$

The rest of the proof of Theorem 6.3.2 will follow the same general outline of [76, proof of Theorem 2.1]. Let $\vec{\psi}(t) \in \mathcal{H}_1$ be a smooth global wave map.

We begin by choosing a scaling parameter. Let $\delta_0 > 0$ be a small number, for example $\delta_0 = 1$ would work. For each $t \in (0, \infty)$ choose $\lambda(t)$ so that

$$\delta_0 \leq \mathcal{E}_0^{2\lambda(t)}(\vec{\psi}(t)) \leq 2\delta_0. \tag{6.3.8}$$

Then using the monotonicity of the energy on interior cones we know that for each $|\tau| \leq \lambda(t)$

we have

$$\mathcal{E}_0^{\lambda(t)}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{2\lambda(t)-|\tau|}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{2\lambda(t)}(\vec{\psi}(t)) \leq 2\delta_0. \tag{6.3.9}$$

Similarly, we have

$$\delta_0 \leq \mathcal{E}_0^{2\lambda(t)+|\tau|}(\vec{\psi}(t+\tau)) \leq \mathcal{E}_0^{3\lambda(t)}(\vec{\psi}(t+\tau)). \tag{6.3.10}$$

Lemma 6.3.4. *Let $\vec{\psi}(t) \in \mathcal{H}\backslash\mathcal{H}_0$ and $\lambda(t)$ be defined as above. Then we have $\lambda(t) \ll t$ as $t \to \infty$.*

Proof. Suppose $\vec{\psi} \in \mathcal{H}_k$ for $k \geq 1$. It suffices to show that for all $\lambda > 0$ we have $\lambda(t) \leq \lambda t$ for all t large enough. Fix $\lambda > 0$. By Corollary 6.2.3 we have

$$\|\psi(t) - k\pi\|_{L^\infty(r \geq \lambda t)} \to 0 \tag{6.3.11}$$

as $t \to \infty$. For the sake of finding a contradiction, suppose that there exists a sequence $t_n \to \infty$ with $\lambda(t_n) \geq \lambda t_n$ for all $n \in \mathbb{N}$. By (6.2.4) and (6.3.11) we would then have that

$$\mathcal{E}_0^{2\lambda(t_n)}(\vec{\psi}(t_n)) \geq \mathcal{E}_0^{\lambda t_n}(\vec{\psi}(t_n)) \geq 2G(\psi(t_n, \lambda t_n)) \to 2G(k\pi) \geq 4 > 2\delta_0,$$

which contradicts (6.3.8) as long as we ensure that $\delta_0 < 2$. $\square$

We can now complete the proof of Theorem 6.3.2.

Proof of Theorem 6.3.2. Let $\lambda(t)$ be defined as in (6.3.8). Choose another scaling parameter $A(t)$ so that $A(t) \to \infty$ and $\lambda(t) \leq A(t) \ll t$ for $t \to \infty$, for example one could take $A(t) := \max\{\tilde{\lambda}(t), t^{1/2}\}$ where $\tilde{\lambda}(t) := \sup_{0 \leq s \leq t} \lambda(s)$. By Lemma 6.3.3 we can find a sequence

$t_n \to \infty$ so that by setting $\lambda_n := \lambda(t_n)$ and $A_n := A(t_n)$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{t_n}^{t_n+\lambda_n} \int_0^{t-A_n} \dot{\psi}^2(t,r)\, r\, dr\, dt = 0.$$

Now define a sequence of global wave maps $\vec{\psi}_n(t) \in \mathcal{H}\backslash\mathcal{H}_0$ by

$$\vec{\psi}_n(t,r) := \Big(\psi(t_n + \lambda_n t, \lambda_n r), \lambda_n \dot{\psi}(t_n + \lambda_n t, \lambda_n r)\Big).$$

and write the full wave maps in coordinates on $\mathbb{S}^2$ as $U_n(t,r,\omega) := (\psi_n(t,r), \omega)$. Observe that we have

$$\int_0^1 \int_0^{r_n} \dot{\psi}_n^2(t,r)\, r\, dr\, dt \to 0 \quad \text{as} \quad n \to \infty \tag{6.3.12}$$

where $r_n := (t_n - A_n)/\lambda_n \to \infty$ as $n \to \infty$ by our choice of A_n. Also note that

$$\mathcal{E}(\vec{\psi}_n(t)) = \mathcal{E}(\vec{\psi}(t_n + \lambda_n t)) = \mathcal{E}(\vec{\psi}) = C.$$

This implies that the sequence $\vec{\psi}_n$ is uniformly bounded in $L^\infty_t(\dot{H}^1 \times L^2)$. Note that (6.2.4) implies that ψ_n is uniformly bounded in $L^\infty_t L^\infty_x$. Hence we can extract a further subsequence so that

$$\vec{\psi}_n \rightharpoonup \vec{\psi}_\infty \quad \text{weakly in} \quad L^2_t(H^1 \times L^2)_{\text{loc}}$$

and, in fact, locally uniformly on $[0,1) \times (0,\infty)$. By (6.3.12), the limit

$$\vec{\psi}_\infty(t,r) = (\psi_\infty(r), 0) \quad \forall (t,r) \in [0,1) \times (0,\infty)$$

and is thus a time-independent weak solution to (6.1.1) on $[0,1) \times (0,\infty)$. This means that the corresponding full, weak wave map $\tilde{U}_\infty(t,r,\omega) = U_\infty(r,\omega) := (\psi_\infty(r),\omega)$ is a time-independent weak solution to (1.1.8) on $[0,1)\times\mathbb{R}^2\setminus\{0\}$. By Hélein's theorem [32, Theorem 2],

$$U_\infty : \mathbb{R}^2\setminus\{0\} \to \mathbb{S}^2$$

is a smooth finite energy, co-rotational harmonic map. By Sacks-Uhlenbeck, [65], we can then extend U_∞ to a smooth finite energy, co-rotational harmonic map $U : \mathbb{R}^2 \to \mathbb{S}^2$. Writing $U(r,\omega) = (\psi_\infty(r),\omega)$, we have either $\psi_\infty \equiv 0$ or $\psi_\infty = \pm Q(\cdot/\lambda_0)$ for some $\lambda_0 > 0$.

Following Struwe, we can also establish strong local convergence

$$\vec{\psi}_n \to (\psi_\infty, 0) \quad \text{in} \quad L^2_t([0,1); H^1\times L^2)_{\text{loc}} \tag{6.3.13}$$

using the equation (1.1.8) and the local energy constraints from (6.3.9):

$$\mathcal{E}_0^1(\vec{\psi}_n(t)) \le 2\delta_0, \quad \mathcal{E}_0^1(\psi_\infty) \le 2\delta_0,$$

which hold uniformly in n for $|t|\le 1$. For the details of this argument we refer the reader to [76, Proof of Theorem 2.1 (ii)]. Finally we note that the strong local convergence in (6.3.13) shows that indeed $\psi_\infty \not\equiv 0$ since by (6.3.10) we have

$$\delta_0 \le \mathcal{E}_0^3(\vec{\psi}_n(t))$$

uniformly in n for each $|t|\le 1$. Therefore we can conclude that there exists $\lambda_0 > 0$ so that $\psi_\infty(r) = \pm Q(r/\lambda_0)$. □

As in the previous chapter, the following consequences of Theorem 6.3.2, which hold for global degree one wave maps with energy below $3\mathcal{E}(Q)$, will be essential in what follows.

Corollary 6.3.5. *Let $\psi(t) \in \mathcal{H}_1$ be a smooth global wave map such that $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$.*

Then we have

$$\psi_n - Q(\cdot/\lambda_0) \to 0 \quad as \quad n \to \infty \quad in \quad L^2_t([0,1); H)_{\text{loc}}, \tag{6.3.14}$$

with $\psi_n(t,r)$, $\{t_n\}$, $\{\lambda_n\}$, *and* λ_0 *as in Theorem 6.3.2.*

Corollary 6.3.5 is the global-in-time analog of Corollary 5.2.13. For the details, we refer the reader to the proof of Lemma 5.2.11, Lemma 5.2.12, and Corollary 5.2.13. At this point we note that we can, after a suitable rescaling, assume, without loss of generality, that λ_0 in Theorem 6.3.2, and Corollary 6.3.5, satisfies $\lambda_0 = 1$.

Arguing as in the proof of Proposition 5.5.4 we can also deduce the following consequence of Theorem 6.3.2.

Proposition 6.3.6. *Let* $\psi(t) \in \mathcal{H}_1$ *be a smooth global wave map such that* $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$. *Let* α_n *be any sequence such that* $\alpha_n \to \infty$. *Then, there exists a sequence of times* $\tau_n \to \infty$ *and a sequence of scales* $\lambda_n \ll \tau_n$ *with* $\alpha_n \lambda_n \ll \tau_n$, *so that*

(a) As $n \to \infty$ *we have*

$$\lim_{n \to \infty} \int_0^{\tau_n - A_n} \dot{\psi}^2(\tau_n, r)\, r\, dr \to 0, \tag{6.3.15}$$

where $A_n \to \infty$ *satisfies* $\lambda_n \leq A_n \ll \tau_n$.

(b) As $n \to \infty$ *we have*

$$\lim_{n \to \infty} \int_0^{\alpha_n \lambda_n} \left(\left| \psi_r(\tau_n, r) - \frac{Q_r(r/\lambda_n)}{\lambda_n} \right|^2 + \frac{|\psi(\tau_n, r) - Q(r/\lambda_n)|^2}{r^2} \right) r\, dr = 0. \tag{6.3.16}$$

Remark 25. Proposition 6.3.6 follows directly from Lemma 6.3.3, Corollary 6.3.5 and a diagonalization argument. As mentioned above, we refer the reader to Proposition 5.5.4, parts (a) and (b) for the details. Also note that $\tau_n \in [t_n, t_n + \lambda_n]$ where $t_n \to \infty$ is the sequence in Proposition 6.3.6. Finally $A_n := A(t_n)$ is the sequence that appears in the proof of Theorem 6.3.2.

As in the previous chapter we will also need the following simple consequence of Proposition 6.3.6.

Corollary 6.3.7. *Let α_n, λ_n, and τ_n be defined as in Proposition 6.3.6. Let $\beta_n \to \infty$ be any sequence such that $\beta_n < c_0\alpha_n$ for some $c_0 < 1$. Then, for every $0 < c_1 < C_2$ such that $C_2 c_0 < 1$ there exists $\tilde{\beta}_n$ with $c_1\beta_n \le \tilde{\beta}_n \le C_2\beta_n$ such that*

$$\psi(\tau_n, \tilde{\beta}_n\lambda_n) \to \pi \quad as \quad n \to \infty. \tag{6.3.17}$$

6.3.2 Extraction of the radiation term

In this subsection we construct what we will refer to as the radiation term, $\varphi_L(t) \in \mathcal{H}_0$ in the decomposition (6.3.1).

Proposition 6.3.8. *Let $\psi(t) \in \mathcal{H}_1$ be a global wave map with $\mathcal{E}(\vec{\psi}) = \mathcal{E}(Q) + \eta < 3\mathcal{E}(Q)$. Then there exists a solution $\varphi_L(t) \in \mathcal{H}_0$ to the linear wave equation* (6.1.5) *so that for all $A \ge 0$ we have*

$$\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}_L(t)\|_{H\times L^2(r\ge t-A)} \to 0 \quad as \quad t \to \infty. \tag{6.3.18}$$

Moreover, for n large enough we have

$$\mathcal{E}(\vec{\varphi}_L(\tau_n)) \le C < 2\mathcal{E}(Q). \tag{6.3.19}$$

Proof. To begin we pick any $\alpha_n \to \infty$ and find τ_n, λ_n as in Proposition 6.3.6. Now let

$\beta_n \to \infty$ be any other sequence such that $\beta_n \ll \alpha_n$. By Corollary 6.3.7 we can assume that

$$\psi(\tau_n, \beta_n\lambda_n) \to \pi \tag{6.3.20}$$

as $n \to \infty$. We make the following definition:

$$\phi_n^0(r) = \begin{cases} \pi - \frac{\pi - \psi(\tau_n, \beta_n\lambda_n)}{\beta_n\lambda_n} r & \text{if} \quad 0 \le r \le \beta_n\lambda_n \\ \psi(\tau_n, r) & \text{if} \quad \beta_n\lambda_n \le r < \infty \end{cases} \tag{6.3.21}$$

$$\phi_n^1(r) = \begin{cases} 0 & \text{if} \quad 0 \le r \le \beta_n\lambda_n \\ \dot{\psi}(\tau_n, r) & \text{if} \quad \beta_n\lambda_n \le r < \infty. \end{cases} \tag{6.3.22}$$

We claim that $\vec{\phi}_n := (\phi_n^0, \phi_0^1) \in \mathcal{H}_{1,1}$ and $\mathcal{E}(\vec{\phi}_n) \le C < 2\mathcal{E}(Q)$. Clearly $\phi_n^0(0) = \pi$ and $\phi_n^0(\infty) = \pi$. We claim that

$$\mathcal{E}_{\beta_n\lambda_n}^{\infty}(\vec{\phi}_n) = \mathcal{E}_{\beta_n\lambda_n}^{\infty}(\vec{\psi}(\tau_n)) \le \eta + o_n(1). \tag{6.3.23}$$

Indeed, since $\psi(\tau_n, \beta_n\lambda_n) \to \pi$ we have $G(\psi(\tau_n, \beta_n\lambda_n)) \to 2 = \frac{1}{2}\mathcal{E}(Q)$ as $n \to \infty$. Therefore, by (6.2.4) we have

$$\mathcal{E}_0^{\beta_n\lambda_n}(\psi(\tau_n), 0) \ge 2G(\psi(\tau_n, \beta_n\lambda_n)) \ge \mathcal{E}(Q) - o_n(1)$$

for large n which proves (6.3.23) since $\mathcal{E}_{\beta_n\lambda_n}^{\infty}(\vec{\psi}(\tau_n)) = \mathcal{E}_0^{\infty}(\vec{\psi}(\tau_n)) - \mathcal{E}_0^{\beta_n\lambda_n}(\vec{\psi}(\tau_n))$.

We can also directly compute $\mathcal{E}_0^{\beta_n\lambda_n}(\phi_n^0, 0)$. Indeed,

$$\mathcal{E}_0^{\beta_n\lambda_n}(\phi_n^0, 0) = \int_0^{\beta_n\lambda_n} \left(\frac{\pi - \psi(\tau_n, \beta_n\lambda_n)}{\beta_n\lambda_n}\right)^2 r\,dr + \int_0^{\beta_n\lambda_n} \frac{\sin^2\left(\frac{\pi-\psi(\tau_n,\beta_n\lambda_n)}{\beta_n\lambda_n} r\right)}{r}\,dr$$

$$\leq C\left|\pi - \psi(\tau_n, \beta_n\lambda_n)\right|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\mathcal{E}(\vec{\phi}_n) \leq \eta + o_n(1)$. This means that for large enough n we have the uniform estimates $\mathcal{E}(\vec{\phi}_n) \leq C < 2\mathcal{E}(Q)$. Therefore, by the degree 0 global existence and scattering result for energies below $2\mathcal{E}(Q)$, (which holds with exactly the same statement in $\mathcal{H}_{1,1}$ as in $\mathcal{H}_0 = \mathcal{H}_{0,0}$), we have that the wave map evolution $\vec{\phi}_n(t) \in \mathcal{H}_{1,1}$ with initial data $\vec{\phi}_n$ is global in time and scatters to π as $t \to \pm\infty$. The scattering statement means that for each n we can find initial data $\vec{\phi}_{n,L}$ so that the solution, $S(t)\vec{\phi}_{n,L}$, to the linear wave equation (6.1.5) satisfies

$$\|\vec{\phi}_n(t) - (\pi, 0) - S(t)\vec{\phi}_{n,L}\|_{H\times L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Abusing notation, we set

$$\vec{\phi}_{n,L}(t) := S(t - \tau_n)\vec{\phi}_{n,L}.$$

By the definition of $\vec{\phi}_n$ and the finite speed of propagation observe that we have

$$\phi_n(t - \tau_n, r) = \psi(t, r) \quad \forall r \geq t - \tau_n + \beta_n\lambda_n.$$

Therefore, for all fixed m we have

$$\|\vec{\psi}(t) - (\pi, 0) - \vec{\phi}_{m,L}(t)\|_{H\times L^2(r\geq t-\tau_m+\beta_m\lambda_m)} \to 0 \quad \text{as} \quad t \to \infty, \tag{6.3.24}$$

and, in particular

$$\|\vec{\phi}_n - (\pi, 0) - \vec{\phi}_{m,L}(\tau_n)\|_{H\times L^2(r\geq \tau_n - \tau_m + \beta_m\lambda_m)} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.25}$$

Now set $\vec{\varphi}_n = (\varphi_n^0, \varphi_n^1) := (\phi_n^0, \phi_n^1) - (\pi, 0) \in \mathcal{H}_0$. We have $\mathcal{E}(\vec{\varphi}_n) \leq C < 2\mathcal{E}(Q)$ by construction. Therefore the sequence $S(-\tau_n)\vec{\varphi}_n$ is uniformly bounded in $H \times L^2$. Let $\vec{\varphi}_L = (\varphi_L^0, \varphi_L^1) \in \mathcal{H}_0$ be the weak limit of $S(-\tau_n)\vec{\varphi}_n$ in $H \times L^2$ as $n \to \infty$, i.e.,

$$S(-\tau_n)\vec{\varphi}_n \rightharpoonup \vec{\varphi} \quad \text{weakly in} \quad H \times L^2$$

as $n \to \infty$. Denote by $\vec{\varphi}_L(t) := S(t)\vec{\varphi}_L$ the linear evolution of $\vec{\varphi}_L$ at time t. Following the construction in [1, Main Theorem] we have the following profile decomposition for $\vec{\varphi}_n$:

$$\vec{\varphi}_n(r) = \vec{\varphi}_L(\tau_n, r) + \sum_{j=2}^{k} \left(\varphi_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j),\, \frac{1}{\lambda_n^j}\dot{\varphi}_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j) \right) + \vec{\gamma}_n^k(r) \tag{6.3.26}$$

where if we label $\varphi_L =: \varphi_L^1$, $\tau_n =: t_n^1$, and $\lambda_n^1 = 1$ this is exactly a profile decomposition as in Corollary 5.2.15. Now observe that for each fixed m we can write

$$\begin{aligned}\vec{\varphi}_n(r) - \vec{\phi}_{m,L}(\tau_n, r) &= \vec{\varphi}_L(\tau_n, r) - \vec{\phi}_{m,L}(\tau_n, r)\\ &\quad + \sum_{j=2}^{k} \left(\varphi_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j),\, \frac{1}{\lambda_n^j}\dot{\varphi}_L^j(t_n^j/\lambda_n^j, r/\lambda_n^j) \right) + \vec{\gamma}_n^k(r)\end{aligned} \tag{6.3.27}$$

and (6.3.27) is still a profile decomposition in the sense of Corollary 5.2.15 for the sequence $\vec{\varphi}_n(r) - \vec{\phi}_{m,L}(\tau_n, r)$. Since the pseudo-orthogonality of the $H \times L^2$ norm is preserved after sharp cut-offs, see [18, Corollary 8] or Proposition 5.2.19, we then have

$$\|\vec{\varphi}_n - \vec{\phi}_{m,L}(\tau_n)\|^2_{H\times L^2(r\geq \tau_n-\tau_m+\beta_m\lambda_m)} = \|\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n)\|^2_{H\times L^2(r\geq \tau_n-\tau_m+\beta_m\lambda_m)}$$
$$+\sum_{j=2}^{k}\|\vec{\varphi}^j_L(t^j_n/\lambda^j_n)\|^2_{H\times L^2(r\geq \tau_n-\tau_m+\beta_m\lambda_m)} + \|\vec{\gamma}^k_j\|^2_{H\times L^2(r\geq \tau_n-\tau_m+\beta_m\lambda_m)} + o_n(1)$$

Note that (6.3.25) implies that the left-hand-side above tends to zero as $n\to\infty$. Therefore, since all of the terms on right-hand-side are non-negative we can deduce that

$$\|\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n)\|^2_{H\times L^2(r\geq \tau_n-\tau_m+\beta_m\lambda_m)} \to 0 \quad \text{as} \quad n\to\infty.$$

Since,

$$\vec{\varphi}_L(\tau_n) - \vec{\phi}_{m,L}(\tau_n) = S(\tau_n)(\vec{\varphi} - S(-\tau_m)\vec{\phi}_{m,L})$$

is a solution to the linear wave equation, we can use the monotonicity of the energy on exterior cones to deduce that

$$\|\vec{\varphi}_L(t) - \vec{\phi}_{m,L}(t)\|^2_{H\times L^2(r\geq t-\tau_m+\beta_m\lambda_m)} \to 0 \quad \text{as} \quad t\to\infty.$$

Combining the above with (6.3.24) we can conclude that

$$\|\vec{\psi}(t) - (\pi,0) - \vec{\varphi}_L(t)\|^2_{H\times L^2(r\geq t-\tau_m+\beta_m\lambda_m)} \to 0 \quad \text{as} \quad t\to\infty.$$

The above holds for each $m\in\mathbb{N}$ and for *any* sequence $\beta_m\to\infty$ with $\beta_m < c_0\alpha_m$. Taking $\beta_m \ll \alpha_m$ and recalling that $\tau_m\to\infty$ and λ_m are such that $\alpha_m\lambda_m \ll \tau_m$ we have that $\tau_m - \beta_m\lambda_m \to \infty$ as $m\to\infty$. Therefore, for any $A>0$ we can find m large enough so that $\tau_m - \beta_m\lambda_m > A$, which proves (6.3.18) in light of the above.

It remains to show (6.3.19). But this follows immediately from the decomposition (6.3.26) and the almost orthogonality of the nonlinear wave map energy for such a decomposition, see Lemma 5.2.16, since we know that the left-hand-side of (6.3.26) satisfies

$$\mathcal{E}(\vec{\varphi}_n) \leq C < 2\mathcal{E}(Q)$$

for large enough n. $\square$

Now that we have constructed the radiation term $\vec{\varphi}_L(t)$ we denote by $\varphi(t) \in \mathcal{H}_0$ the global wave map that scatters to the linear wave $\vec{\varphi}_L(t)$, i.e., $\vec{\varphi}(t) \in \mathcal{H}_0$ is the global solution to (6.1.1) such that

$$\|\vec{\varphi}(t) - \vec{\varphi}_L(t)\|_{H \times L^2} \to 0 \quad \text{as} \quad t \to \infty. \tag{6.3.28}$$

The existence of such a $\varphi(t) \in \mathcal{H}_0$ locally around $t = +\infty$ follows immediately from the existence of wave operators for the corresponding $4d$ semi-linear equation. The fact that $\varphi(t)$ is global-in-time follows from Theorem 5.1.1 since (6.3.19) and (6.3.28) together imply that $\mathcal{E}(\vec{\varphi}) < 2\mathcal{E}(Q)$.

We will need a few facts about the degree zero wave map $\vec{\varphi}(t)$ which we collect in the following lemma.

Lemma 6.3.9. *Let $\vec{\varphi}(t)$ be defined as above. Then we have*

$$\limsup_{t \to \infty} \|\vec{\varphi}(t)\|_{H \times L^2(|r-t| \geq A)} \to 0 \quad \textit{as} \quad A \to \infty, \tag{6.3.29}$$

$$\lim_{t \to \infty} \mathcal{E}_{t-A}^{\infty}(\vec{\varphi}(t)) \to \mathcal{E}(\vec{\varphi}) \quad \textit{as} \quad A \to \infty. \tag{6.3.30}$$

Proof. First we prove (6.3.29). We have

$$\|\vec{\varphi}(t)\|^2_{H\times L^2(|r-t|\geq A)} \leq \|\vec{\varphi}(t) - \vec{\varphi}_L(t)\|^2_{H\times L^2} + \|\varphi_L(t)\|^2_{H\times L^2(|r-t|\geq A)}.$$

By (6.3.28) the first term on the right-hand-side above tends to 0 as $t \to \infty$ so it suffices to show that

$$\limsup_{t\to\infty} \|\varphi_L(t)\|^2_{H\times L^2(|r-t|\geq A)} \to 0 \quad \text{as} \quad A \to \infty.$$

Since $\varphi_L(t)$ is a solution to (6.1.5) the above follows from [18, Theorem 4] by passing to the analogous statement for the corresponding $4d$ free wave $v_L(t)$ given by

$$rv_L(t,r) := \varphi_L(t,r).$$

To prove (6.3.30) we note that the limit as $t \to \infty$ exists for any fixed A due to the monotonicity of the energy on exterior cones. Next observe that we have

$$\lim_{t\to\infty} \mathcal{E}_0^{t-A}(\vec{\varphi}(t)) \leq \lim_{t\to\infty} \|\vec{\varphi}(t)\|^2_{H\times L^2(r\leq t-A)} \to 0 \quad \text{as} \quad A \to \infty \tag{6.3.31}$$

by (6.3.29) and then (6.3.30) follows immediately from the conservation of energy. □

Now, observe that we can combine Proposition 6.3.8 and (6.3.28) to conclude that for all $A \geq 0$ we have

$$\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}(t)\|_{H\times L^2(r\geq t-A)} \to 0 \quad \text{as} \quad t \to \infty. \tag{6.3.32}$$

With this in mind we define $a(t)$ as follows:

$$\vec{a}(t) := \vec{\psi}(t) - \vec{\varphi}(t) \tag{6.3.33}$$

and we aggregate some preliminary information about a in the following lemma:

Lemma 6.3.10. *Let $\vec{a}(t)$ be defined as in* (6.3.33)*. Then $\vec{a}(t) \in \mathcal{H}_1$ for all t. Moreover,*

- *for all $\lambda > 0$ we have*

$$\|\vec{a}(t) - (\pi, 0)\|_{H \times L^2(r \geq \lambda t)} \to 0 \quad as \quad t \to \infty, \tag{6.3.34}$$

- *the quantity $\mathcal{E}(\vec{a}(t))$ has a limit as $t \to \infty$ and*

$$\lim_{t \to \infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}). \tag{6.3.35}$$

Proof. By definition we have $a(t) \in \mathcal{H}_1$ for all t since

$$a(t, 0) = 0, \; a(t, \infty) = \pi.$$

To prove (6.3.34) observe that for every $A \leq (1 - \lambda)t$ we have

$$\begin{aligned} \|\vec{a}(t) - (\pi, 0)\|^2_{H \times L^2(r \geq \lambda t)} \leq \|\vec{\psi}(t) - (\pi, 0)\|^2_{H \times L^2(\lambda t \leq r \leq t - A)} \\ + \|\vec{\varphi}(t)\|^2_{H \times L^2(\lambda t \leq r \leq t - A)} \\ + \|\vec{a}(t) - (\pi, 0)\|^2_{H \times L^2(r \geq t - A)}. \end{aligned}$$

Then (6.3.34) follows by combining (6.3.32), (6.3.29), and (6.2.18). To prove (6.3.35) we first claim that

$$\lim_{A \to \infty} \lim_{t \to \infty} \mathcal{E}^{\infty}_{t-A}(\vec{\psi}(t)) = \mathcal{E}(\vec{\varphi}). \tag{6.3.36}$$

Indeed, we have

$$
\begin{aligned}
\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t)) &= \int_{t-A}^{\infty} [(\psi_t(t) - \varphi_t(t) + \varphi_t(t))^2 + (\psi_r(t) - \varphi_r(t) + \varphi_r(t))^2]\, r\, dr \\
&\quad + \int_{t-A}^{\infty} \frac{\sin^2(\psi(t) - \pi - \varphi(t) + \varphi(t))}{r}\, dr \\
&= \mathcal{E}_{t-A}^{\infty}(\vec{\varphi}(t)) + \|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}(t)\|^2_{\dot{H}^1 \times L^2(r \geq t-A)} \\
&\quad + O\left(\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}(t)\|_{\dot{H}^1 \times L^2(r \geq t-A)} \|\vec{\varphi}(t)\|_{\dot{H}^1 \times L^2(r \geq t-A)}\right) \\
&\quad + \int_{t-A}^{\infty} \frac{\sin^2(\psi(t) - \pi - \varphi(t) + \varphi(t)) - \sin^2(\varphi(t))}{r}\, dr \\
&= \mathcal{E}_{t-A}^{\infty}(\vec{\varphi}(t)) + O\left(\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}(t)\|^2_{H \times L^2(r \geq t-A)}\right) \\
&\quad + O\left(\sqrt{\mathcal{E}(\vec{\varphi})}\|\vec{\psi}(t) - (\pi, 0) - \vec{\varphi}(t)\|_{H \times L^2(r \geq t-A)}\right),
\end{aligned}
$$

which proves (6.3.36) in light of (6.3.30) and (6.3.32). In the third equality above we have used the simple trigonometric inequality:

$$
\left|\sin^2(x - y + y) - \sin^2(y)\right| \leq 2\,|\sin(y)|\,|x - y| + 2\,|x - y|^2 .
$$

Now, fix $\delta > 0$. By (6.3.29), (6.3.36), and (6.3.32) we can choose A, T_0 large enough so that for all $t \geq T_0$ we have

$$
\begin{aligned}
&\|\vec{\varphi}(t)\|_{H \times L^2(r \leq t-A)} \leq \delta, \\
&\left|\mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t)) - \mathcal{E}(\vec{\varphi})\right| \leq \delta, \\
&\|\vec{a}(t) - (\pi, 0)\|^2_{H \times L^2(r \geq t-A)} \leq \delta.
\end{aligned}
$$

Then for all $t \geq T_0$ and A as above we can argue as before to obtain

$$
\begin{aligned}
\mathcal{E}(\vec{a}(t)) &= \mathcal{E}_0^{t-A}(\vec{a}(t)) + O(\|\vec{a}(t) - (\pi,0)\|^2_{H \times L^2(r \geq t-A)}) \\
&= \mathcal{E}_0^{t-A}(\vec{\psi}(t)) + O\left(\sqrt{\mathcal{E}(\vec{\psi})}\|\vec{\varphi}(t)\|_{H \times L^2(r \leq t-A)}\right) \\
&\quad + O\left(\|\vec{\varphi}(t)\|^2_{H \times L^2(r \leq t-A)}\right) + O\left(\|\vec{a}(t) - (\pi,0)\|^2_{H \times L^2(r \geq t-A)}\right) \\
&= \mathcal{E}(\vec{\psi}) - \mathcal{E}_{t-A}^{\infty}(\vec{\psi}(t)) + O(\delta) \\
&= \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}) + O(\delta),
\end{aligned}
$$

which proves (6.3.35). $\square$

We will also need the following technical lemma in the next section.

Lemma 6.3.11. *For any sequence $\sigma_n > 0$ with $\lambda_n \ll \sigma_n \ll \tau_n$ we have*

$$
\lim_{n \to \infty} \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n + \sigma_n} \int_0^{\infty} \dot{a}^2(t,r)\, r\, dr\, dt = 0. \tag{6.3.37}
$$

Proof. Fix $0 < \lambda < 1$. For each n we have

$$
\begin{aligned}
\frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n + \sigma_n} \int_0^{\infty} \dot{a}^2(t,r)\, r\, dr\, dt \leq{}& \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n + \sigma_n} \int_0^{\lambda t} \dot{a}^2(t,r)\, r\, dr\, dt \\
&+ \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n + \sigma_n} \int_{\lambda t}^{\infty} \dot{a}^2(t,r)\, r\, dr\, dt.
\end{aligned}
$$

By (6.3.34) we can conclude that

$$
\lim_{n \to \infty} \sup_{t \geq \tau_n} \int_{\lambda t}^{\infty} \dot{a}^2(t,r)\, r\, dr = 0.
$$

Hence it suffices to show that

$$
\lim_{n \to \infty} \frac{1}{\sigma_n} \int_{\tau_n}^{\tau_n + \sigma_n} \int_0^{\lambda t} \dot{a}^2(t,r)\, r\, dr\, dt = 0.
$$

Observe that for every n we have

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t}\dot{a}^2(t,r)\,r\,dr\,dt \lesssim \frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t}\dot{\psi}^2(t,r)\,r\,dr\,dt \qquad (6.3.38)$$
$$+\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t}\dot{\varphi}^2(t,r)\,r\,dr\,dt.$$

We first estimate the first integral on the right-hand-side above. Let $A_n \to \infty$ be the sequence in Proposition 6.3.6, see also Remark 25, and let $t_n \to \infty$ be the sequence in Theorem 6.3.2. Recall that we have $\tau_n \in [t_n, t_n + \lambda_n]$ and $\lambda_n \le A_n \ll t_n$.

Observe that for n large enough we have that for each $t \in [\tau_n, \tau_n+\sigma_n]$ we have $\lambda t \le t - A_n$. Hence,

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t}\dot{\psi}^2(t,r)\,r\,dr\,dt \le \frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{t-A_n}\dot{\psi}^2(t,r)\,r\,dr\,dt.$$

Next, note that since $\lambda_n \ll \sigma_n$ we can ensure that for n large enough we have $\lambda_n+\sigma_n \le 2\sigma_n$. Therefore,

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{t-A_n}\dot{\psi}^2(t,r)\,r\,dr\,dt \le \frac{2}{\lambda_n+\sigma_n}\int_{t_n}^{t_n+\lambda_n+\sigma_n}\int_0^{t-A_n}\dot{\psi}^2(t,r)\,r\,dr\,dt \to 0$$

as $n \to \infty$ by Lemma 6.3.3.

Lastly we estimate the second integral on the righ-hand-side of (6.3.38). For each $A > 0$ we can choose n large enough so that $\lambda t \le t - A$ for each $t \in [\tau_n, \tau_n + \sigma_n]$. So we have

$$\frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{\lambda t}\dot{\varphi}^2(t,r)\,r\,dr\,dt \le \frac{1}{\sigma_n}\int_{\tau_n}^{\tau_n+\sigma_n}\int_0^{t-A}\dot{\varphi}^2(t,r)\,r\,dr\,dt.$$

Taking the limsup as $n \to \infty$ of both sides and then letting $A \to \infty$ on the right we have by (6.3.29) that the left-hand-side above tends to 0 as $n \to \infty$. This concludes the proof. $\square$

6.3.3 Compactness of the error

For the remainder of this section, we fix $\alpha_n \to \infty$ and find $\tau_n \to \infty$ and $\lambda_n \ll \tau_n$ as in Proposition 6.3.6. We define $\vec{b}_n = (b_{n,0}, b_{n,1}) \in \mathcal{H}_0$ as follows:

$$b_{n,0}(r) := a(\tau_n, r) - Q(r/\lambda_n), \tag{6.3.39}$$

$$b_{n,1}(r) := \dot{a}(\tau_n, r). \tag{6.3.40}$$

As in Section 5.5.3 of the previous chapter, our goal in this subsection is to show that $\vec{b}_n \to 0$ in the energy space. Indeed we prove the following result:

Proposition 6.3.12. *Define $\vec{b}_n \in \mathcal{H}_0$ as in* (6.3.39), (6.3.40). *Then,*

$$\|\vec{b}_n\|_{H\times L^2} \to 0 \quad as \quad n \to \infty. \tag{6.3.41}$$

Remark 26. In light of (6.3.28), it is clear that Proposition 6.3.12 implies Proposition 6.3.1.

We begin with the following consequences of the previous sections.

Lemma 6.3.13. *Let $\vec{b}_n \in \mathcal{H}_0$ be defined as above. Then we have*

(a) As $n \to \infty$ we have

$$\|b_{n,1}\|_{L^2} \to 0. \tag{6.3.42}$$

(b) As $n \to \infty$ we have

$$\|b_{n,0}\|_{H(r\le\alpha_n\lambda_n)} \to 0. \tag{6.3.43}$$

(c) For any fixed $\lambda > 0$ we have

$$\|b_{n,0}\|_{H(r \geq \lambda \tau_n)} \to 0 \quad as \quad n \to \infty. \tag{6.3.44}$$

(d) There exists a $C > 0$ so that

$$\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q) \tag{6.3.45}$$

for n large enough.

Proof. To prove (6.3.42) fix $0 < \lambda < 1$ and observe that we have

$$\begin{aligned}\int_0^\infty b_{n,1}^2(r)\, r\, dr \leq \int_0^{\lambda\tau_n} \dot{\psi}^2(\tau_n, r)\, r\, dr + \int_0^{\lambda\tau_n} \dot{\varphi}^2(\tau_n, r)\, r\, dr \\ + \int_{\lambda\tau_n}^\infty \dot{a}(\tau_n, r)^2\, r\, dr.\end{aligned}$$

Then (6.3.42) follows from (6.3.15), (6.3.29), and (6.3.34).

Next we prove (6.3.43). To see this, observe that for each n we have

$$\|b_{n,0}\|^2_{H(r \leq \alpha_n \lambda_n)} \leq \|\psi(\tau_n) - Q(\cdot/\lambda_n)\|^2_{H(r \leq \alpha_n\lambda_n)} + \|\varphi(\tau_n)\|^2_{H(r \leq \alpha_n \lambda_n)}.$$

The first term on the right-hand-side tends to zero as $n \to \infty$ by (6.3.16). To estimate the second term on the right-hand-side we note that for fixed $A > 0$ we can find n large enough so that $\alpha_n \lambda_n \leq \tau_n - A$ and so we have

$$\|\varphi(\tau_n)\|^2_{H(r \leq \alpha_n\lambda_n)} \leq \|\varphi(\tau_n)\|^2_{H(r \leq \tau_n - A)}.$$

Taking the limsup as $n \to \infty$ on both sides above and then taking $A \to \infty$ on the right and recalling (6.3.29) we see that the limit as $n \to \infty$ of the left-hand side above must be zero.

This proves (6.3.43).

To deduce (6.3.44) note that

$$\|b_{n,0}\|^2_{H(r\geq\lambda\tau_n)} \leq \|a(\tau_n)-\pi\|^2_{H(r\geq\lambda\tau_n)} + \|Q(\cdot/\lambda_n)-\pi\|^2_{H(r\geq\lambda\tau_n)}.$$

The first term on the right-hand-side above tends to zero as $n\to\infty$ by (6.3.34). The second term tends to zero since $\lambda\tau_n/\lambda_n\to\infty$ as $n\to\infty$.

Finally, we establish (6.3.45). First observe that for any fixed $\lambda>0$, (6.3.44) implies that

$$\begin{aligned}\mathcal{E}(\vec{b}_n) &= \mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n) + \mathcal{E}_{\lambda\tau_n}^{\infty}(\vec{b}_n)\\ &= \mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n) + o_n(1)\end{aligned}$$

as $n\to\infty$. So it suffices to control $\mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n)$. Next, observe that for n large enough, (6.3.31) gives that

$$\|\vec{\varphi}(\tau_n)\|_{H\times L^2(r\leq\lambda\tau_n)} \leq \|\vec{\varphi}(\tau_n)\|_{H\times L^2(r\leq\tau_n-A)}$$

and the right-hand side is small for n, A large. This means that the contribution of $\vec{\varphi}(\tau_n)$ is negligible on $r\leq\lambda\tau_n$, and thus

$$\mathcal{E}_0^{\lambda\tau_n}(\vec{b}_n) = \mathcal{E}_0^{\lambda\tau_n}(\vec{\psi}(\tau_n)-(Q(\cdot/\lambda_n),0)) + o_n(1).$$

Next, recall that Proposition 6.3.6 implies that

$$\mathcal{E}_0^{\alpha_n\lambda_n}(\vec{\psi}(\tau_n)-Q(\cdot/\lambda_n),0) = o_n(1), \tag{6.3.46}$$

which shows in particular that

$$\mathcal{E}^{\infty}_{\alpha_n\lambda_n}(\vec{\psi}(\tau_n)) \le \eta + o_n(1) \tag{6.3.47}$$

where $\eta := \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q) < 2\mathcal{E}(Q)$. Also, (6.3.46) means that it suffices to show that

$$\mathcal{E}^{\lambda\tau_n}_{\alpha_n\lambda_n}(\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)) \le C < 2\mathcal{E}(Q).$$

Note that since $\alpha_n \to \infty$ we have

$$\mathcal{E}^{\infty}_{\alpha_n\lambda_n}(Q(\cdot/\lambda_n)) = \mathcal{E}^{\infty}_{\alpha_n}(Q) = o_n(1).$$

Hence,

$$\mathcal{E}^{\lambda\tau_n}_{\alpha_n\lambda_n}(\vec{\psi}(\tau_n) - (Q(\cdot/\lambda_n), 0)) = \mathcal{E}^{\lambda\tau_n}_{\alpha_n\lambda_n}(\vec{\psi}(\tau_n)) + o_n(1) \le \eta + o_n(1),$$

which completes the proof. □

Next, we would like to show that the sequence $\vec{b}_n$ does not contain any nonzero profiles. This next result is the global-in-time analog of Proposition 5.5.7 and the proof is again, reminiscent of the the arguments given in [22, Section 5].

Denote by $\vec{b}_n(t) \in \mathcal{H}_0$ the wave map evolution with data $\vec{b}_n$. By (6.3.45) and Theorem 5.1.1 we know that $\vec{b}_n(t) \in \mathcal{H}_0$ is global in time and scatters to zero as $t \to \pm\infty$.

The statements of the following proposition and its corollary are identical to the corresponding statements, Proposition 5.5.7 and Corollary 5.5.8 in the finite time blow-up case.

Proposition 6.3.14. *Let $b_n \in \mathcal{H}_0$ and the corresponding global wave map evolution $\vec{b}_n(t) \in$*

$\mathcal{H}_0$ be defined as above. Then, there exists a decomposition

$$\vec{b}_n(t,r) = b_{n,L}(t,r) + \vec{\theta}_n(t,r) \tag{6.3.48}$$

where $\vec{b}_{n,L}$ satisfies the linear wave equation (6.1.5) with initial data $\vec{b}_{n,L}(0,r) := (b_{n,0}, 0)$. Moreover, $b_{n,L}$ and $\vec{\theta}_n$ satisfy

$$\left\| \frac{1}{r} b_{n,L} \right\|_{L^3_t(\mathbb{R}; L^6_x(\mathbb{R}^4))} \longrightarrow 0 \tag{6.3.49}$$

$$\|\vec{\theta}_n\|_{L^\infty_t(\mathbb{R}; H\times L^2)} + \left\| \frac{1}{r}\theta_n \right\|_{L^3_t(\mathbb{R}; L^6_x(\mathbb{R}^4))} \longrightarrow 0 \tag{6.3.50}$$

as $n \to \infty$.

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Before beginning the proof of Proposition 6.3.14 we use the conclusions of the proposition to deduce the following corollary which will be an essential ingredient in the proof of Proposition 6.3.12.

Corollary 6.3.15. *Let $\vec{b}_n(t)$ be defined as in Proposition 6.3.14. Suppose that there exists a constant δ_0 and a subsequence in n so that $\|b_{n,0}\|_H \geq \delta_0$. Then there exists $\alpha_0 > 0$ such that for all $t > 0$ and all n large enough along this subsequence we have*

$$\|\vec{b}_n(t)\|_{H\times L^2(r\geq t)} \geq \alpha_0 \delta_0. \tag{6.3.51}$$

Proof. First note that since $\vec{b}_{n,L}$ satisfies the linear wave equation (6.1.5) with initial data $\vec{b}_{n,L}(0) = (b_{n,0}, 0)$ we know by [18, Corollary 5] and Corollary 5.2.3, that there exists a constant $\beta_0 > 0$ so that for each $t \geq 0$ we have

$$\|\vec{b}_{n,L}(t)\|_{H\times L^2(r\geq t)} \geq \beta_0 \|b_{n,0}\|_H.$$

On the other hand, by Proposition 6.3.14 we know that

$$\|\vec{b}_n(t) - \vec{b}_{n,L}(t)\|_{H\times L^2(r\geq t)} \leq \|\vec{\theta}_n(t)\|_{H\times L^2} = o_n(1).$$

Putting these two facts together gives

$$\begin{aligned}\|\vec{b}_n(t)\|_{H\times L^2(r\geq t)} &\geq \|b_{n,L}(t)\|_{H\times L^2(r\geq t)} - o_n(1)\\ &\geq \beta_0\|b_{n,0}\|_H - o_n(1).\end{aligned}$$

This yields (6.3.51) by passing to a suitable subsequence and taking n large enough. $\square$

The proof of Proposition 6.3.14 is very similar to the proof of Proposition 5.5.7. Instead of going through the entire argument again here, we establish the main ingredients of the proof and we refer the reader to the previous chapter for the remainder of the argument.

Since $\vec{b}_n \in \mathcal{H}_0$ and $\mathcal{E}(\vec{b}_n) \leq C < 2\mathcal{E}(Q)$ we can, by Corollary 5.2.15, consider the following profile decomposition for $\vec{b}_n$:

$$b_{n,0}(r) = \sum_{j\leq k} \varphi_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) + \gamma_{n,0}^k(r), \tag{6.3.52}$$

$$b_{n,1}(r) = \sum_{j\leq k} \frac{1}{\lambda_n^j}\dot{\varphi}_L^j\left(\frac{-t_n^j}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right) + \gamma_{n,1}^k(r), \tag{6.3.53}$$

where each $\vec{\varphi}_L^j$ is a solution to (6.1.5) and where we have for each $j \neq k$:

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^k} + \frac{\left|t_n^j - t_n^k\right|}{\lambda_n^j} \to \infty \quad \text{as} \quad n \to \infty. \tag{6.3.54}$$

Moreover, if we denote by $\vec{\gamma}_{n,L}^k(t)$ the linear evolution of $\vec{\gamma}_n^k$, i.e., solution to (6.1.5), we have for $j \leq k$ that

$$\left(\gamma_{n,L}^{k}(\lambda_n^j t_n^j, \lambda_n^j \cdot), \lambda_n^j \dot{\gamma}_{n,L}^{k}(\lambda_n^j t_n^j, \lambda_n^j \cdot)\right) \rightharpoonup 0 \text{ in } H \times L^2 \quad \text{as} \quad n \to \infty \tag{6.3.55}$$

$$\limsup_{n\to\infty} \left\| \frac{1}{r} \gamma_{n,L}^{k} \right\|_{L_t^3 L_x^6(\mathbb{R}^4)} \to 0 \quad \text{as} \quad k \to \infty. \tag{6.3.56}$$

Finally we have the following Pythagorean expansions:

$$\|b_{n,0}\|_H^2 = \sum_{j \leq k} \left\| \varphi_L^j \left(\frac{-t_n^j}{\lambda_n^j} \right) \right\|_H^2 + \|\gamma_{n,0}^k\|_H^2 + o_n(1) \tag{6.3.57}$$

$$\|b_{n,1}\|_{L^2}^2 = \sum_{j \leq k} \left\| \dot{\varphi}_L^j \left(\frac{-t_n^j}{\lambda_n^j} \right) \right\|_{L^2}^2 + \|\gamma_{n,1}^k\|_{L^2}^2 + o_n(1). \tag{6.3.58}$$

As in the previous chapter, the proof of Proposition 6.3.14 will consist of a sequence of steps designed to show that each of the profiles φ_L^j must be identically zero. Arguing exactly as in Lemma 5.5.9 and Corollary 5.5.10 we can first deduce that the times t_n^j can be taken to be 0 for each n, j and that the the initial velocities $\dot{\varphi}_L^j(0)$ must all be identically zero as well. We summarize this conclusion in the following lemma:

Lemma 6.3.16. *In the decomposition* (6.3.52), (6.3.53) *we can assume, without loss of generality, that* $t_n^j = 0$ *for every* n *and for every* j*. In addition, we then have*

$$\dot{\varphi}_L^j(0, r) \equiv 0 \quad \textit{for every} \quad j.$$

The proof of Lemma 6.3.16 is identical to the proof of Lemma 5.5.9 and Corollary 5.5.10. We refer the reader to the previous chapter for the details.

By Lemma 6.3.16 we can rewrite our profile decomposition as follows:

$$b_{n,0}(r) = \sum_{j \le k} \varphi_L^j \left(0, r/\lambda_n^j\right) + \gamma_{n,0}^k(r) \tag{6.3.59}$$

$$b_{n,1}(r) = o_n(1) \text{ in } L^2 \text{ as } n \to \infty, \tag{6.3.60}$$

Note that in addition to the Pythagorean expansions in (6.3.57) we also have the following almost-orthogonality of the nonlinear wave map energy, which was established in Lemma 5.2.16:

$$\mathcal{E}(\vec{b}_n) = \sum_{j \le k} \mathcal{E}(\varphi_L^j(0), 0) + \mathcal{E}(\gamma_{n,0}^k, 0) + o_n(1). \tag{6.3.61}$$

Note that $\varphi^j, \gamma_{n,0}^k \in \mathcal{H}_0$ for every j, for every n, and for every k. Using the fact that $\mathcal{E}(\vec{b}_n) \le C < 2\mathcal{E}(Q)$, (6.3.61) and Theorem 5.1.1 imply that, for every j, the nonlinear wave map evolution of the data $(\varphi_L^j(0, r/\lambda_n^j), 0)$ given by

$$\vec{\varphi}_n^j(t,r) := \left(\varphi^j \left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right), \frac{1}{\lambda_n^j} \dot{\varphi}^j \left(\frac{t}{\lambda_n^j}, \frac{r}{\lambda_n^j}\right)\right) \tag{6.3.62}$$

is global in time and scatters as $t \to \pm\infty$. Moreover we have the following nonlinear profile decomposition guarranteed by Proposition 5.2.17:

$$\vec{b}_n(t,r) = \sum_{j \le k} \vec{\varphi}_n^j(t,r) + \vec{\gamma}_{n,L}^k(t,r) + \vec{\theta}_n^k(t,r) \tag{6.3.63}$$

where the $\vec{b}_n(t,r)$ are the global wave map evolutions of the data $\vec{b}_n$, $\vec{\gamma}_{n,L}^k(t,r)$ is the linear evolution of $(\gamma_n^k, 0)$, and the errors $\vec{\theta}_n^k$ satisfy

$$\limsup_{n \to \infty} \left(\|\vec{\theta}_n^k\|_{L_t^\infty(H \times L^2)} + \left\| \frac{1}{r} \theta_n^k \right\|_{L_t^3(\mathbb{R}; L_x^6(\mathbb{R}^4))} \right) \to 0 \quad \text{as} \quad k \to \infty. \tag{6.3.64}$$

Recall that we are trying to show that all of the profiles φ^j must be identically equal to zero. As in the previous chapter we can make the following crucial observations about the scales λ_n^j. Since we have concluded that we can assume that all of the times $t_n^j = 0$ for all n, j we first note that the orthogonality condition (6.3.54) implies that for $j \neq k$:

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} \to \infty \text{ as } n \to \infty.$$

Next, recall that by Lemma 6.3.13 we have

$$\|b_{n,0}\|_{H(r\leq\alpha_n\lambda_n)} \to 0 \text{ as } n \to \infty, \tag{6.3.65}$$

$$\|b_{n,0}\|_{H(r\geq\lambda\tau_n)} \to 0 \text{ as } n \to \infty, \ \forall\lambda > 0 \text{ fixed.} \tag{6.3.66}$$

Combining the above two facts with Proposition 5.2.19 we can conclude that for each λ_n^j corresponding to a nonzero profile φ^j we have

$$\lambda_n \ll \lambda_n^j \ll \tau_n \quad \text{as} \quad n \to \infty. \tag{6.3.67}$$

Now, let k_0 be the index corresponding to the first nonzero profile φ^{k_0}. We can assume, without loss of generality that $k_0 = 1$. By (6.3.65), (6.3.67) and [22, Appendix B] we can find a sequence $\tilde{\lambda}_n$ so that

$$\begin{aligned}
&\tilde{\lambda}_n \ll \alpha_n\lambda_n \\
&\lambda_n \ll \tilde{\lambda}_n \ll \lambda_n^1 \\
&\tilde{\lambda}_n \ll \lambda_n^j \text{ or } \lambda_n^j \ll \tilde{\lambda}_n \quad \forall j > 1.
\end{aligned}$$

Define

$$\beta_n = \frac{\tilde{\lambda}_n}{\lambda_n} \to \infty$$

and we note that $\beta_n \ll \alpha_n$ and $\tilde{\lambda}_n = \beta_n \lambda_n$. Therefore, up to replacing β_n by a sequence $\tilde{\beta}_n \simeq \beta_n$ and $\tilde{\lambda}_n$ by $\tilde{\tilde{\lambda}}_n := \tilde{\beta}_n \lambda_n$, we have by Corollary 6.3.7 and a slight abuse of notation that

$$\psi(\tau_n, \tilde{\lambda}_n) \to \pi \quad \text{as} \quad n \to \infty. \tag{6.3.68}$$

We define the set

$$\mathcal{J}_{\text{ext}} := \{ j \geq 1 \mid \tilde{\lambda}_n \ll \lambda_n^j \}.$$

Note that by construction $1 \in \mathcal{J}_{\text{ext}}$.

The above technical ingredients are necessary for the proof of the following lemma and its corollary. The analog in the finite-time blow-up case is Lemma 5.5.13.

Lemma 6.3.17. *Let φ^1, λ_n^1 be defined as above. Then for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1} \int_0^{\lambda_n^1} \int_{\varepsilon \lambda_n^1 + t}^{\infty} \left| \sum_{j \in \mathcal{J}_{ext}, j \leq k} \dot{\varphi}_n^j(t,r) + \dot{\gamma}_{n,L}^k(t,r) \right|^2 r\, dr\, dt = o_n^k \tag{6.3.69}$$

where $\lim_{k\to\infty} \limsup_{n\to\infty} o_n^k = 0$. Also, for all $j > 1$ and for all $\varepsilon > 0$ we have

$$\frac{1}{\lambda_n^1} \int_0^{\lambda_n^1} \int_{\varepsilon \lambda_n^1 + t}^{\infty} (\dot{\varphi}_n^j)^2(t,r)\, r\, dr dt \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.70}$$

Remark 27. We refer the reader to the proof Lemma 5.5.13 for the details of the proof of Lemma 6.3.17. The proof of (6.3.69) is nearly identical to the proof of (5.5.57) the one difference being that here we use Lemma 6.3.11 in place of the argument following equation (5.5.66). The proof of (6.3.70) is identical to the proof of (5.5.58) and we omit it here.

Note that (6.3.69) and (6.3.70) together directly imply the following result:

Corollary 6.3.18. *Let φ^1 be as in Lemma 6.3.17. Then for all $\varepsilon > 0$ we have*

$$\frac{1}{\lambda_n^1} \int_0^{\lambda_n^1} \int_{\varepsilon\lambda_n^1 + t}^{\infty} \left| \dot{\varphi}_n^1(t,r) + \dot{\gamma}_{n,L}^k(t,r) \right|^2 r\, dr\, dt = o_n^k \tag{6.3.71}$$

where $\lim_{k\to\infty} \limsup_{n\to\infty} o_n^k = 0$.

The proof of Proposition 6.3.14 now follows from the exact same argument as the proof Proposition 5.5.7. We refer the reader to the previous chapter for the details.

We can now complete the proof of Proposition 6.3.12.

Proof of Proposition 6.3.12. We argue by contradiction. Assume that Proposition 6.3.12 fails. Then, up to extracting a subsequence, we can find a $\delta_0 > 0$ so that

$$\|b_{n,0}\|_H \geq \delta_0 \tag{6.3.72}$$

for every n. Next, we rescale. Set

$$\mu_n := \frac{\lambda_n}{\tau_n}.$$

Since $\lambda_n \ll \tau_n$ as $n \to \infty$, our new scale $\mu_n \to 0$ as $n \to \infty$. We next define rescaled wave maps:

$$g_n(t,r) := \psi(\tau_n + \tau_n t, \tau_n r), \tag{6.3.73}$$

$$h_n(t,r) := \varphi(\tau_n + \tau_n t, \tau_n r). \tag{6.3.74}$$

Since $\vec{g}_n(t)$ and $\vec{h}_n(t)$ are defined by rescaling $\vec{\psi}$ and $\vec{\varphi}$ we have that $\vec{g}_n(t) \in \mathcal{H}_1$ is a global-in-time wave map and the wave map $\vec{\varphi}(t) \in \mathcal{H}_0$ is global-in-time and scatters to 0 as $t \to \pm\infty$. We then have

$$a(\tau_n + \tau_n t, \tau_n r) = g_n(t,r) - h_n(t,r).$$

Similarly, we define

$$\begin{aligned}\tilde{b}_{n,0}(r) &:= b_{n,0}(\tau_n r),\\ \tilde{b}_{n,1}(r) &:= \tau_n b_{n,1}(\tau_n r)\end{aligned}$$

and the corresponding rescaled wave map evolutions

$$\begin{aligned}\tilde{b}_n(t,r) &:= b_n(\tau_n t, \tau_n r),\\ \partial_t \tilde{b}_n(t,r) &:= \tau_n \dot{b}_n(\tau_n t, \tau_n r).\end{aligned}$$

After this rescaling, our decomposition becomes

$$g_n(0,r) = h_n(0,r) + Q\left(\frac{r}{\mu_n}\right) + \tilde{b}_{n,0}(r) \tag{6.3.75}$$

$$\dot{g}_n(0,r) = \dot{h}_n(0,r) + \tilde{b}_{n,1}(r). \tag{6.3.76}$$

We can rephrase (6.3.44) and (6.3.43) in terms of this rescaling and we obtain:

$$\forall \lambda > 0 \text{ fixed, } \|\tilde{b}_{n,0}\|_{H(r\geq\lambda)} \to 0 \text{ as } n \to \infty, \tag{6.3.77}$$

$$\|\tilde{b}_{n,0}\|_{H(r\leq\alpha_n\mu_n)} \to 0 \text{ as } n \to \infty. \tag{6.3.78}$$

Also, (6.3.29) implies that

$$\lim_{A\to\infty} \limsup_{n\to\infty} \|\vec{h}_n(0)\|_{H\times L^2(r\leq 1-A/\tau_n)} = 0, \tag{6.3.79}$$

$$\lim_{A\to\infty} \limsup_{n\to\infty} \|\vec{h}_n(0)\|_{H\times L^2(r\geq 1+A/\tau_n)} = 0. \tag{6.3.80}$$

Next, we claim that for every n a decomposition of the form (6.3.75) is preserved up to a small error if we replace the terms in (6.3.75) with their respective wave map evolutions at some future times to be defined precisely below.

By Corollary 6.3.7 we can choose a sequence $\gamma_n \to \infty$ with

$$\gamma_n \ll \alpha_n$$

so that

$$g_n(0, \gamma_n\mu_n) \to \pi \quad \text{as} \quad n \to \infty.$$

Define $\delta_n \to 0$ by

$$|g_n(0, \gamma_n\mu_n) - \pi| =: \delta_n \to 0.$$

Using (6.3.16) we define $\varepsilon_n \to 0$ by

$$\|\vec{g}_n(0) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\leq\alpha_n\mu_n)} =: \varepsilon_n \to 0.$$

Finally, choose $\beta_n \to \infty$ so that

$$\beta_n \leq \min\{\sqrt{\gamma_n}, \delta_n^{-1/2}, \varepsilon_n^{-1/2}\}$$
$$g_n(0, \beta_n\mu_n/2) \to \pi \quad \text{as} \quad n \to \infty. \tag{6.3.81}$$

As in the previous chapter, we make the following claims:

(*i*) As $n \to \infty$ we have

$$\|\vec{g}_n(\beta_n\mu_n/2) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\beta_n\mu_n)} \to 0. \tag{6.3.82}$$

(*ii*) For each n, on the interval $r \in [\beta_n\mu_n, \infty)$ we have

$$\begin{aligned}
&\vec{g}_n\left(\frac{\beta_n\mu_n}{2}, r\right) - (\pi, 0) = \vec{h}_n\left(\frac{\beta_n\mu_n}{2}, r\right) + \vec{\tilde{b}}_n\left(\frac{\beta_n\mu_n}{2}, r\right) \\
&\quad + \vec{\tilde{\theta}}_n\left(\frac{\beta_n\mu_n}{2}, r\right), \\
&\|\vec{\tilde{\theta}}_n\|_{L^\infty_t(H\times L^2)} \to 0.
\end{aligned} \tag{6.3.83}$$

We first prove (6.3.82). The proof is very similar to the corresponding argument in the finite-time blow-up case, see the proof of (5.5.94). We repeat the argument here for completeness.

First note that we have

$$\|\vec{g}_n(0) - (Q(\cdot/\mu_n), 0)\|_{H\times L^2(r\le\gamma_n\mu_n)} \le \varepsilon_n \to 0.$$

Unscale the above by setting $\tilde{g}_n(t, r) = g_n(\mu_n t, \mu_n r)$, which gives

$$\|(\tilde{g}_n(0), \partial_t\tilde{g}_n(0)) - (Q(\cdot), 0)\|_{H\times L^2(r\le\gamma_n)} \le \varepsilon_n \to 0.$$

Now using Corollary 5.2.6 and the finite speed of propagation we claim that we have

$$\|(\tilde{g}_n(\beta_n/2), \partial_t\tilde{g}_n(\beta_n/2)) - (Q(\cdot), 0)\|_{H\times L^2(r\le\beta_n)} = o_n(1). \tag{6.3.84}$$

To see this, we need to show that Corollary 5.2.6 applies. Indeed define

$$\hat{g}_{n,0}(r) := \begin{cases} \pi \quad \text{if} \quad r \geq 2\gamma_n \\ \pi + \frac{\pi - \tilde{g}_n(0,\gamma_n)}{\gamma_n}(r - 2\gamma_n) \quad \text{if} \quad \gamma_n \leq r \leq 2\gamma_n \\ \tilde{g}_n(0,r) \quad \text{if} \quad r \leq \gamma_n. \end{cases}$$

$$\hat{g}_{n,1}(r) = \begin{cases} \partial_t \tilde{g}_n(0,r) \quad \text{if} \quad r \leq \gamma_n \\ 0 \quad \text{if} \quad r \geq \gamma_n \end{cases}$$

Then, by construction we have $\vec{\hat{g}}_n \in \mathcal{H}_1$, and since

$$\|\vec{\hat{g}}_n - (\pi, 0)\|_{H \times L^2(\gamma_n \leq r \leq 2\gamma_n)} \leq C\delta_n$$

we then can conclude that

$$\begin{aligned} \|\vec{\hat{g}}_n - (Q,0)\|_{H \times L^2} &\leq \|\vec{\hat{g}}_n - (Q,0)\|_{H \times L^2(r \leq \gamma_n)} + \|\vec{\hat{g}}_n - (\pi,0)\|_{H \times L^2(\gamma_n \leq r \leq 2\gamma_n)} \\ &\quad + \|(\pi,0) - (Q,0)\|_{H \times L^2(r \geq \gamma_n)} \\ &\leq C(\varepsilon_n + \delta_n + \gamma_n^{-1}). \end{aligned}$$

Now, given our choice of β_n, (6.3.84) follows from Corollary 5.2.6 and the finite speed of propagation. Rescaling (6.3.84) we have

$$\|(g_n(\beta_n \mu_n/2), \partial_t g_n(\beta_n \mu_n/2)) - (Q(\cdot/\mu_n), 0)\|_{H \times L^2(r \leq \beta_n \mu_n)} \to 0.$$

This proves (6.3.82). Also note that by monotonicity of the energy on interior cones and the comparability of the energy and the $H \times L^2$ norm in $\mathcal{H}_0$, for small energies, we see that

(6.3.42) and (6.3.78) imply that

$$\|(\tilde{b}_n(\beta_n\mu_n/2), \partial_t\tilde{b}_n(\beta_n\mu_n/2))\|_{H\times L^2(r\leq\beta_n\mu_n)} \to 0. \tag{6.3.85}$$

Next we prove (6.3.83). First we define

$$\begin{aligned}
\tilde{g}_{n,0}(r) &= \begin{cases} \pi - \frac{\pi - g_n(0,\mu_n\beta_n/2)}{\frac{1}{2}\mu_n\beta_n} r & \text{if} \quad r \leq \beta_n\mu_n/2 \\ g_n(0,r) \quad \text{if} \quad r \geq \beta_n\mu_n/2 \end{cases} \\
\tilde{g}_{n,1}(r) &= \dot{g}_n(0,r).
\end{aligned}$$

Then, let $\chi \in C^\infty([0,\infty))$ be defined so that $\chi(r) \equiv 1$ on the interval $[2,\infty)$ and supp$\chi \subset [1,\infty)$. Define

$$\begin{aligned}
\vec{\breve{g}}_n(r) &:= \chi(4r/\beta_n\mu_n)(\vec{\tilde{g}}_n(r) - (\pi,0)) \\
\vec{\breve{b}}_n(r) &:= \chi(4r/\beta_n\mu_n)\vec{\tilde{b}}_n(r)
\end{aligned}$$

and observe that we have the following decomposition

$$\vec{\breve{g}}_n(r) = \vec{h}_n(0,r) + \vec{\breve{b}}_n(r) + o_n(1),$$

where the $o_n(1)$ is in the sense of $H \times L^2$ – here we also have used (6.3.79). Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of Corollary 5.2.15 because of Proposition 6.3.14 and [18, Lemma 11] or Lemma 5.2.20. We can then consider the nonlinear profiles. Note that by construction we have $\vec{\breve{g}}_n \in \mathcal{H}_0$ and as in the previous chapter, we can use (6.3.81) to show that $\mathcal{E}(\vec{\breve{g}}_n) \leq C < 2\mathcal{E}(Q)$ for large n. The corresponding wave map evolution $\vec{\breve{g}}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t \to \pm\infty$ by Theorem 5.1.1. We also need to check that $\mathcal{E}(\vec{\breve{b}}_n) \leq C < 2\mathcal{E}(Q)$. Note that by construction and the definition of $\tilde{b}_n$, we have

$$\begin{aligned}
\mathcal{E}(\vec{\breve{b}}_n) &\leq \mathcal{E}(\vec{\tilde{b}}_n) + O\left(\int_0^\infty \frac{4r^2}{\beta_{n,0}^2\mu_n^2}(\chi')^2(4r/\beta_n\mu_n)\frac{b_n^2((1-\tau_n)r)}{r}\,dr\right) \\
&\quad + \int_{\beta_n\mu_n/2}^{\beta_n\mu_n} \frac{\sin^2(\chi(4r/\beta_n\mu_n)b_{n,0}((1-\tau_n)r))}{r}\,dr \\
&\leq \mathcal{E}(\vec{\tilde{b}}_n) + O\left(\int_{\beta_n\lambda_n/2}^{\beta_n\lambda_n} \frac{b_{n,0}^2(r)}{r}\,dr\right) \\
&= \mathcal{E}(\vec{\tilde{b}}_n) + o_n(1) \leq C < 2\mathcal{E}(Q),
\end{aligned}$$

where the last line follows from (6.3.43) since $\beta_n \ll \alpha_n$.

Arguing as in the previous chapter, we can use Proposition 6.3.14, Proposition 5.2.17, and

Lemma 5.2.18 to obtain the following nonlinear profile decomposition

$$\begin{aligned}
&\vec{\breve{g}}_n(t,r) = \vec{h}_n(t,r) + \vec{\breve{b}}_n(t,r) + \vec{\breve{\theta}}_n(t,r), \\
&\|\vec{\breve{\theta}}_n\|_{L_t^\infty(H\times L^2)} \to 0.
\end{aligned}$$

Finally observe that by construction and the finite speed of propagation we have

$$\begin{aligned}
\vec{\breve{g}}_n(t,r) &= \vec{g}_n(t,r) - \pi, \\
\vec{\breve{b}}_n(t,r) &= \vec{\tilde{b}}_n(t,r).
\end{aligned}$$

for all $t \in \mathbb{R}$ and $r \in [\beta_n\mu_n/2 + |t|\,,\infty)$. Therefore, in particular we have

$$\vec{g}_n(\beta_n\mu_n/2, r) - (\pi, 0) = \vec{h}_n(\beta_n\mu_n/2, r) + \vec{\tilde{b}}_n(\beta_n\mu_n/2, r) + \vec{\breve{\theta}}_n(\beta_n\mu_n/2, r)$$

for all $r \in [\beta_n\mu_n, \infty)$ which proves (6.3.83).

We can combine (6.3.82), (6.3.83), (6.3.85), and (6.3.79) together with the monotonicity

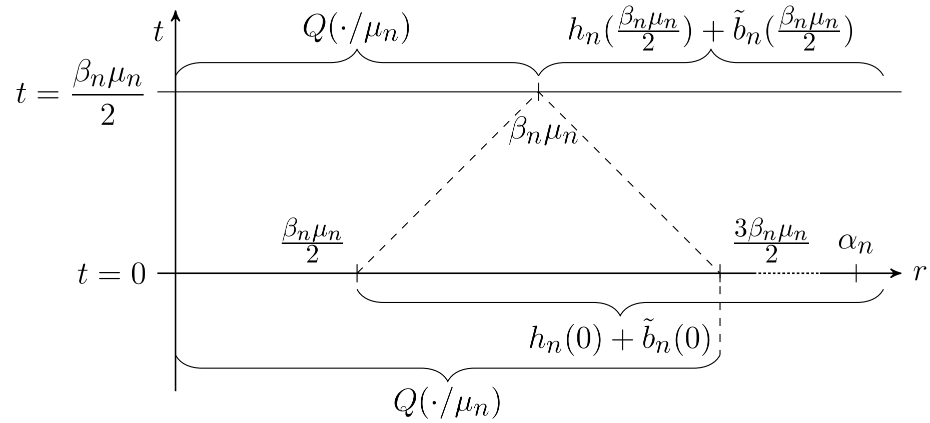

Figure 6.2: A schematic description of the evolution of the decomposition (6.3.75) from time $t = 0$ until time $t = \frac{\beta_n\mu_n}{2}$. At time $t = \frac{\beta_n\mu_n}{2}$ the decomposition (6.3.86) holds.

of the energy on interior cones and the fact that $\|Q(\cdot/\mu_n) - \pi\|_{H(r \geq \beta_n\mu_n)} = o_n(1)$, to obtain the decomposition

$$\vec{g}_n(\beta_n\mu_n/2, r) = (Q(r/\mu_n), 0) + \vec{h}_n(\beta_n\mu_n/2, r) \tag{6.3.86}$$
$$+ \vec{\tilde{b}}_n(\beta_n\mu_n/2, r) + \vec{\tilde{\theta}}_n(r),$$
$$\|\vec{\tilde{\theta}}_n\|_{H\times L^2} \to 0. \tag{6.3.87}$$

Now, let $s_n \to \infty$ be any sequence such that $s_n \geq \beta_n\mu_n/2$ for each n. The next step is to prove the following decomposition at time s_n:

$$\vec{g}_n(s_n, r) - (\pi, 0) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(s_n, r) + \vec{\zeta}_n(r) \quad \forall r \in [s_n, \infty), \tag{6.3.88}$$
$$\|\vec{\zeta}_n\|_{H\times L^2} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.89}$$

We proceed as in the proof of (6.3.83). By (6.3.82) we can argue as in Corollary 6.3.7 and find $\rho_n \to \infty$ with $\rho_n \ll \beta_n$ so that

$$g_n(\beta_n\mu_n/2,\ \rho_n\mu_n) \to \pi \quad \text{as} \quad n \to \infty. \tag{6.3.90}$$

Define

$$\hat{f}_{n,0}(r) = \begin{cases} \pi - \frac{\pi - g_n(\beta_n\mu_n/2, \rho_n\mu_n)}{\rho_n\mu_n} r & \text{if} \quad r \leq \rho_n\mu_n \\ g_n(\beta_n\mu_n/2, r) \quad \text{if} \quad r \geq \rho_n\mu_n \end{cases}$$

$$\hat{f}_{n,1}(r) = \dot{g}_n(\beta_n\mu_n/2, r).$$

Let $\chi \in C^\infty$ be as above and set

$$\vec{f}_n(r) := \chi(2r/\rho_n\mu_n)(\vec{\hat{f}}_n(r) - (\pi, 0)),$$
$$\vec{\hat{b}}_n(r) := \chi(2r/\rho_n\mu_n)\vec{\tilde{b}}_n(\beta_n\mu_n/2, r).$$

Observe that we have the following decomposition:

$$\vec{f}_n(r) = \vec{h}_n(\beta_n\mu_n/2, r) + \vec{\hat{b}}_n(r) + o_n(1).$$

where the $o_n(1)$ above is in the sense of $H \times L^2$. Moreover, the right-hand side above, without the $o_n(1)$ term, is a profile decomposition in the sense of Corollary 5.2.15 because of Proposition 6.3.14 and [18, Lemma 11] or Lemma 5.2.20. We can then consider the nonlinear profiles. Note that by construction we have $\vec{f}_n \in \mathcal{H}_0$ and, as usual, we can use (6.3.90) to show that $\mathcal{E}(\vec{f}_n) \leq C < 2\mathcal{E}(Q)$ for large n. The corresponding wave map evolution $\vec{f}_n(t) \in \mathcal{H}_0$ is thus global in time and scatters as $t \to \pm\infty$ by Theorem 5.1.1.

As in the proof of (6.3.83) it is also easy to show that $\mathcal{E}(\vec{\hat{b}}_n) \leq C < 2\mathcal{E}(Q)$ where here we use (6.3.85) instead of (6.3.43).

Again we can use Proposition 6.3.14, Proposition 5.2.17 and Lemma 5.2.18 to obtain the

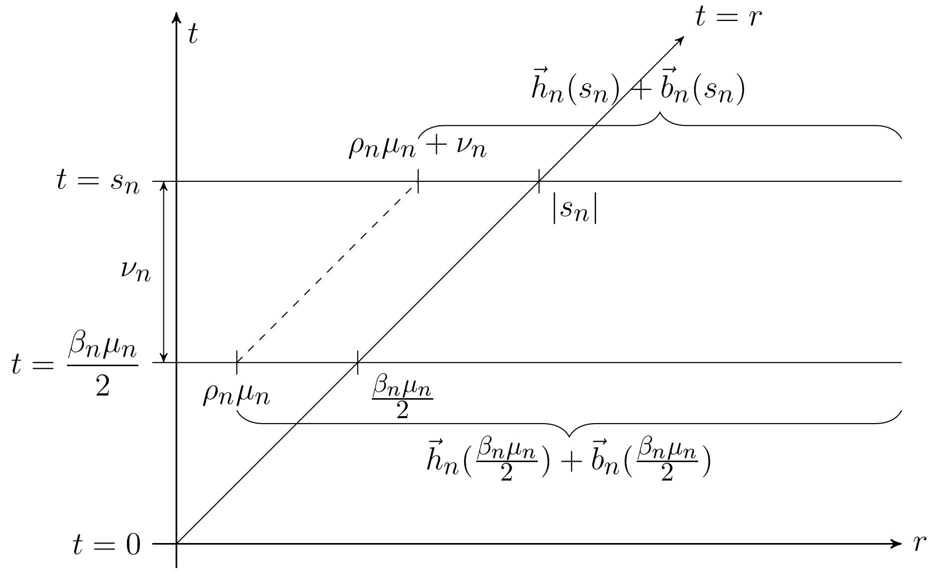

Figure 6.3: A schematic depiction of the evolution of the decomposition (6.3.86) up to time s_n. On the interval $[s_n, +\infty)$, the decomposition (6.3.88) holds.

following nonlinear profile decomposition

$$\vec{f}_n(t,r) = \vec{h}_n(\beta_n\mu_n/2 + t, r) + \tilde{\vec{b}}_n(t,r) + \tilde{\vec{\zeta}}_n(t,r),$$
$$\|\tilde{\vec{\zeta}}_n\|_{L_t^\infty(H\times L^2)} \to 0.$$

In particular, for

$$\nu_n := s_n - \beta_n\mu_n/2$$

we have

$$\vec{f}_n(\nu_n, r) = \vec{h}_n(s_n, r) + \tilde{\vec{b}}_n(\nu_n, r) + \tilde{\vec{\zeta}}_n(\nu_n, r).$$

By the finite speed of propagation we have that

$$\vec{f}_n(\nu_n, r) = \vec{g}_n(s_n, r) - (\pi, 0),$$
$$\tilde{\vec{b}}_n(\nu_n, r) = \tilde{\vec{b}}_n(s_n, r)$$

as long as $r \geq \rho_n\mu_n + \nu_n$. Using the fact that $\rho_n \ll \beta_n$ we have that $s_n \geq \rho_n\mu_n + \nu_n$ and hence,

$$\vec{g}_n(s_n, r) - (\pi, 0) = \vec{h}_n(s_n, r) + \vec{\tilde{b}}_n(s_n, r) + \vec{\tilde{\zeta}}_n(\nu_n, r) \quad \forall r \in [s_n, \infty).$$

Setting $\vec{\zeta}_n := \vec{\tilde{\zeta}}_n(\nu_n)$ we obtain (6.3.88) and (6.3.89). With this decomposition we can now complete the proof.

One the one hand observe that by rescaling, (6.3.34), and the fact that $2\tau_n s_n \geq \tau_n + \tau_n s_n$ for n large we have

$$\begin{aligned}
\|\vec{g}_n(s_n) - \vec{h}_n(s_n) - (\pi, 0)\|_{H\times L^2(r\geq s_n)} &= \|\vec{a}(\tau_n + \tau_n s_n, \tau_n\cdot) - (\pi, 0)\|_{H\times L^2(r\geq s_n)} \\
&= \|\vec{a}(\tau_n + \tau_n s_n) - (\pi, 0)\|_{H\times L^2(r\geq \tau_n s_n)} \\
&\leq \|\vec{a}(\tau_n + \tau_n s_n) - (\pi, 0)\|_{H\times L^2(r\geq \frac{1}{2}(\tau_n + \tau_n s_n))} \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Combining the above with the decomposition (6.3.88) and (6.3.89) we obtain that

$$\|\vec{\tilde{b}}_n(s_n)\|_{H\times L^2(r\geq s_n)} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.91}$$

On the other hand, combining our assumption (6.3.72) and Corollary 6.3.15 we know that there exists $\alpha_0 > 0$ so that

$$\|\vec{\tilde{b}}_n(s_n)\|_{H\times L^2(r\geq s_n)} = \|\vec{b}_n(\tau_n s_n)\|_{H\times L^2(r\geq \tau_n s_n)} \geq \alpha_0\delta_0.$$

But this contradicts (6.3.91). □

We can now complete the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $\vec{a}(t)$ be defined as in (6.3.33). Recall that by (6.3.35) we have

$$\lim_{t\to\infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(\vec{\varphi}). \tag{6.3.92}$$

By Proposition 6.3.1 we have found a sequence of times $\tau_n \to \infty$ so that

$$\mathcal{E}(\vec{a}(\tau_n)) \to \mathcal{E}(Q)$$

as $n \to \infty$. This then implies that

$$\lim_{t\to\infty} \mathcal{E}(\vec{a}(t)) = \mathcal{E}(Q).$$

We now use the variational characterization of Q to show that in fact $\|\dot{a}(t)\|_{L^2} \to 0$ as $t \to \infty$. To see this observe that since $a(t) \in \mathcal{H}_1$ we can deduce by (5.2.18) that

$$\mathcal{E}(Q) \leftarrow \mathcal{E}(a(t), \dot{a}(t)) \geq \int_0^\infty \dot{a}^2(t,r)\, r\, dr + \mathcal{E}(Q).$$

Next observe that the decomposition in Lemma 5.2.5 provides us with a function $\lambda : (0,\infty) \to (0,\infty)$ such that

$$\|a(t,\cdot) - Q(\cdot/\lambda(t))\|_H \leq \delta(\mathcal{E}(a(t),0) - \mathcal{E}(Q)) \to 0.$$

This also implies that

$$\mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)), 0)) \to 0 \tag{6.3.93}$$

as $t \to \infty$. Since $t \mapsto a(t)$ is continuous in H for $t \in [0,\infty)$ it follows from Lemma 5.2.5 that

$\lambda(t)$ is continuous on $[0, \infty)$. Therefore we have established that

$$\vec{\psi}(t) - \vec{\varphi}(t) - (Q(\cdot/\lambda(t)), 0) \to 0 \quad \text{in} \quad H \times L^2 \quad \text{as} \quad t \to \infty.$$

It remains to show that $\lambda(t) = o(t)$. This follows immediately from the asymptotic vanishing of $\nabla_{t,r} a(t)$ outside the light cone and from (6.3.93). To see this observe that by (6.3.34) with $\lambda = 1$ we have that $a(t, r) - (\pi, 0) = o(1)$ in $H \times L^2(r \geq t)$ as $t \to \infty$. Therefore we have

$$\mathcal{E}^{\infty}_{\frac{t}{\lambda(t)}}(Q) = \mathcal{E}^{\infty}_{t}(\pi - Q(\cdot/\lambda(t))) \leq \mathcal{E}(\vec{a}(t) - (Q(\cdot/\lambda(t)), 0)) + o(1) \to 0$$

as $t \to \infty$. But this then implies that $\frac{t}{\lambda(t)} \to \infty$ as $t \to \infty$. This completes the proof. $\square$

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