What mathematical knowledge improves high school teaching?

Yvonne Lai

University of Nebraska-Lincoln

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CONSIDER
THE FOLLOWING …

- Compute (-1) x (-1).

1

- Solve 6x + 5 = 10.

x = 5/6

- Can the data in this table be modeled by a linear relationship?

Yes:
y=2x+5

x	y
1	7
3	11
4	13

Why does (-1) x (-1) = 1 ?

What are some different explanations and/or metaphors you could use to help a student make sense of this?

Why do we call "x" a variable in equations like $6x + 5 = 10$ when it stands for just one number?

How would you respond?

-1 x -1 prompt: Adapted from COACTIV released items.
Variable prompt: Adapted from Chazan (1993), f(x)=g(x)? An approach to modeling with algebra. *For the Learning of Mathematics 13*(3), 22-26.

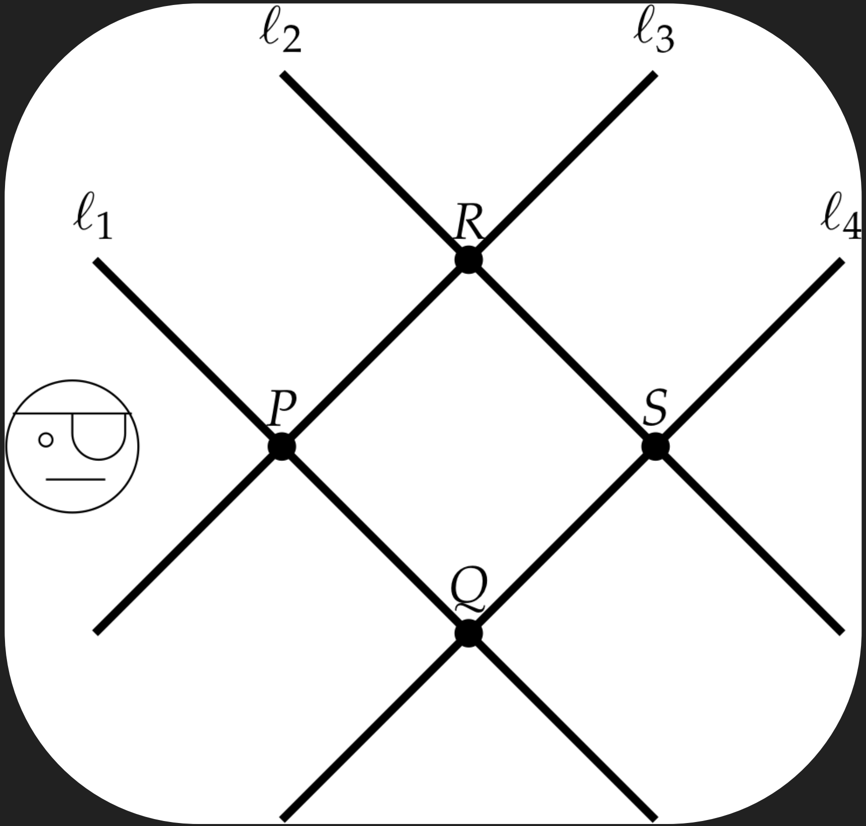
KATE
I thought about how
y changes when x changes.
(11-13)/(3 - 4) = 2.
y = 2x + something.

x	y
1	7
3	11
4	13

JANE
I got yes, y=2x+5.
I changed m and b until the pattern worked.

What are different, correct, and complete solutions to the table problem, consistent with these students' thinking?

What would you do or say to push Kate and Jane's thinking?



What conjectures could this diagram lead to?

What would you say or do to help your students say their ideas precisely?

How would you help students build on each others’ work?

How can prospective high school teachers learn the mathematics needed to teach well?

What mathematical knowledge improves high school teaching?

Do prospective high school teachers actually have opportunities to learn this math?

How can we ensure and improve these opportunities?

Mathematical knowledge for teaching (MKT)

MKT is … the mathematical knowledge entailed in recurrent teaching practices, such as:

- Listening to and building on student thinking
- Selecting strategic examples
- Giving accessible yet precise explanations

Ball, Thames, & Phelps, 2008;
Thompson & Thompson, 1996

MKT includes …

- Content knowledge (straight up math)
- Pedagogical content knowledge (can think of this as "applied" math, where the application is mathematics teaching)

cf. Bass (2005);
Lai & Howell (2016)

MKT impacts teaching

- Teachers' MKT predicts student achievement outcomes.

  GAST project (HS Geometry): Mohr-Schroeder, Ronau, Peters, Lee, & Bush, 2017
  COACTIV project (HS Algebra, Geometry): Baumert et al., 2010
  LMT and replications (Elementary): Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2011

- Teachers' <u>pedagogical content knowledge</u> explains student outcomes better than teachers' <u>content knowledge</u> does.
- Teachers' MKT (esp. PCK) impacts teaching actions, which explain student achievement outcomes

  GAST
  COACTIV
  MQI project: Gates Foundation, 2011; Hill et al., (2008)

How can prospective high school teachers learn the mathematics needed to teach well?

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US teachers' content knowledge (CK)

TEDS-M project reported on teachers' CK along two anchor points:

Could solve simple equations and evaluate algebraic expressions. Difficulty describing general patterns, relating equivalent representations of concepts.

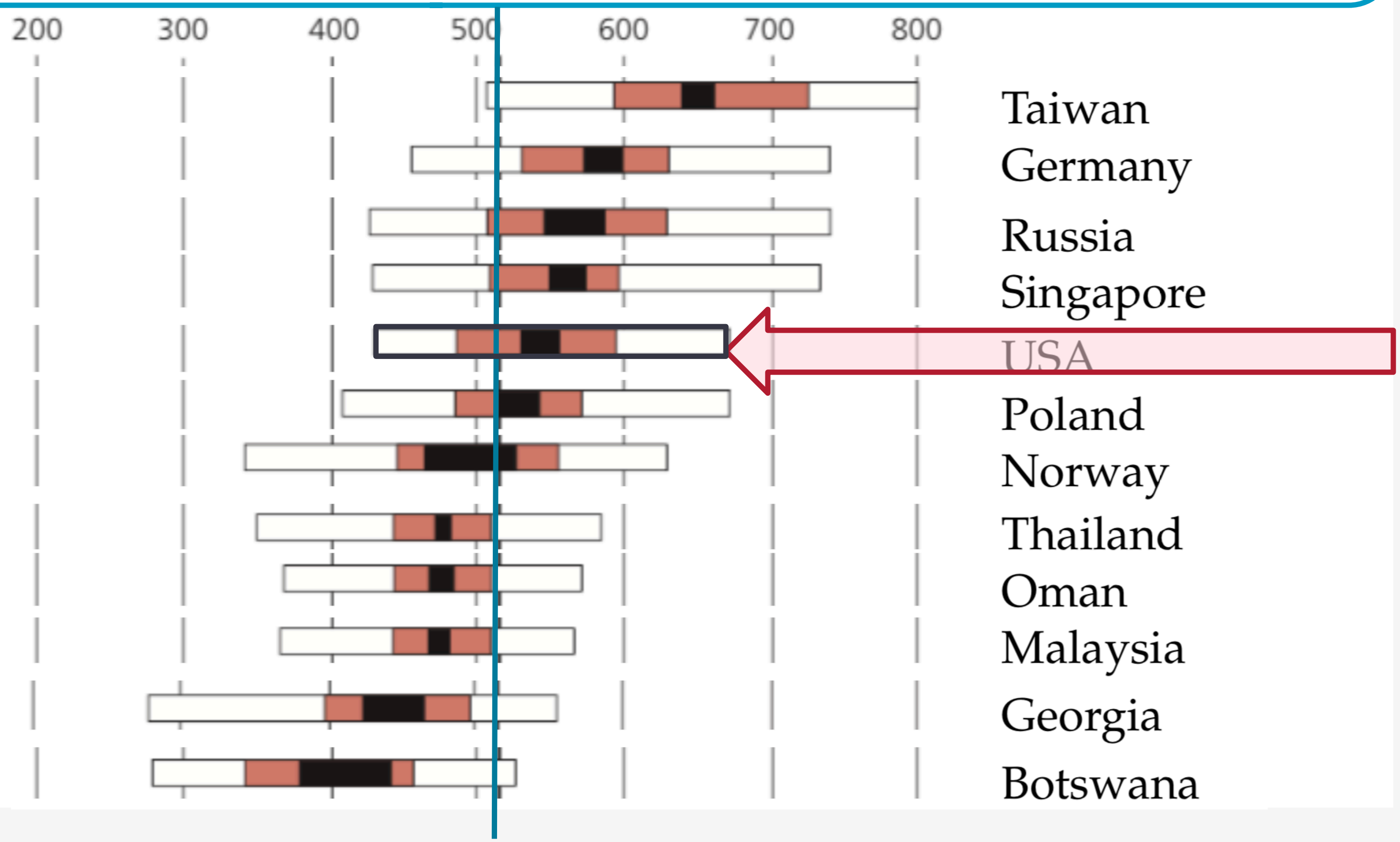
Could read, analyze, and apply abstract definitions and notation, and make short proofs. Unlikely to solve problems stated in purely abstract terms, construct more complex proofs.

CK of population of future upper secondary teachers (up to Grade 11/12)

US teachers' pedagogical content knowledge

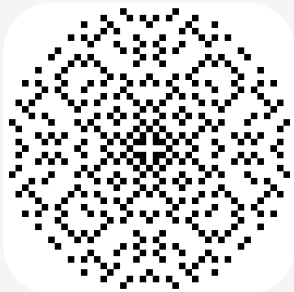
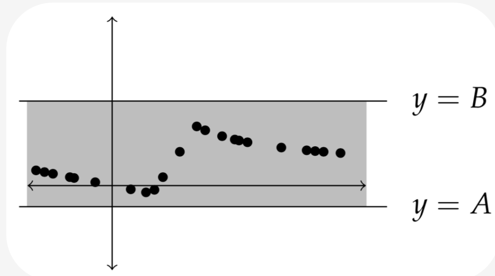
TEDS-M reported on teachers' PCK along one anchor point:

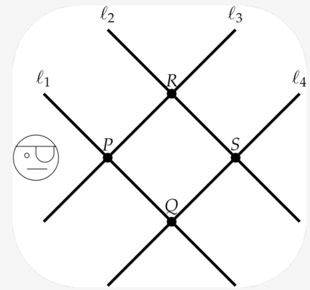
- Could identify prerequisite concepts for some secondary topics, evaluate students' mathematical work for short explanations or single-step proofs.
- Difficulty with evaluating student work in more complex mathematical situations.

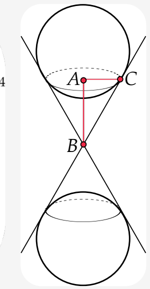


How teachers experience undergrad math

- US high school teachers take a lot of math courses!





- But many high school teachers do not believe that their advanced math courses are relevant to teaching.

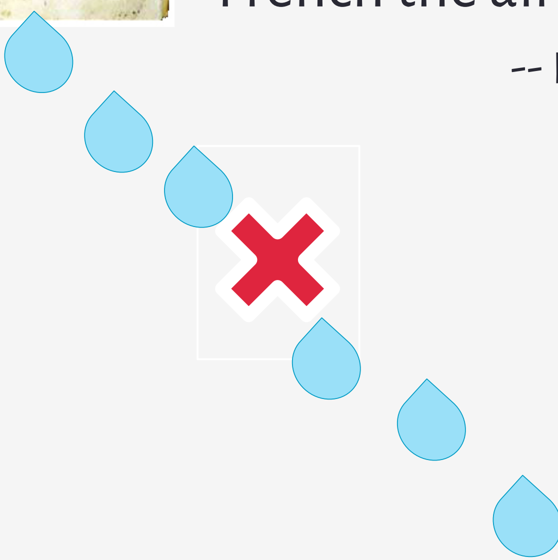
Zazkis & Leikin, 2010; Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman et al., 2018

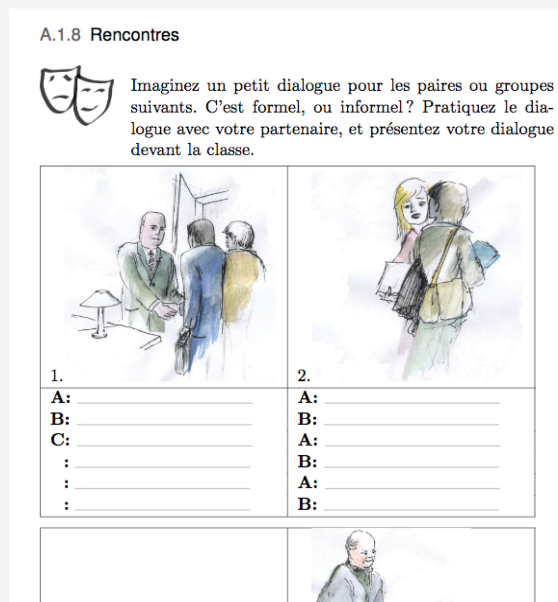
- And they may have a point …

Trickle-down theory doesn't work

"If we want to produce good French teachers ... Should we require them to learn Latin in college but not French? After all, Latin is the mother language of French and is linguistically more complex ... mastering a more complex language ... could enhance their understanding of the French the already know."

-- Hung-Hsi Wu, The Mis-Education of Mathematics Teachers (AMS Notices, March 2011)




A.1.8 Rencontres

Imaginez un petit dialogue pour les paires ou groupes suivants. C'est formel, ou informel ? Pratiquez le dialogue avec votre partenaire, et présentez votre dialogue devant la classe.

1.	2.
A: ____	A: ____
B: ____	B: ____
C: ____	A: ____
: ____	B: ____
: ____	A: ____
: ____	B: ____

We allow symbols to take an oversize presence

Baldinger & Lai, 2019

The product of three consecutive natural numbers is a multiple of 6. **Why?**

A multiple of 6 must have factors of 3 and 2. If you have three consecutive numbers, one will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

- $n \times (n+1) \times (n+2) = (n^2 + n) \times (n+2)$
- $= n^3 + n^2 + 2n$ [illegible]
- Cancelling [illegible] gives $1 + 1 + 2 + 2 = 6$.

Suppose these explanations are produced by high school students in a unit on proof. Which are valid? Which are not?

Suppose your university professor shows you these explanations in a unit on proof. Which are valid? Which are not?

We allow symbols to take an oversize presence

Baldinger & Lai, 2019

Teacher B02:

- "It's very much playing the game of how your professor likes proofs."

Teacher A06:

- "I think it's valid. But a college professor would call it invalid."
- "[This proof] seems like what they are looking for, I get that impression from courses I took… The algebra is important in college."

B02:
Verbal **VALID** ✔
Verbal **NOT Valid** ✖

A06:	
Verbal **VALID** ✔	Algebraic **NOT Valid** ✖
Verbal **NOT Valid** ✖	Algebraic **VALID** ✔

Knowing theorems and doing examples are not mathematical discovery

Ahrens & Lai, in progress

- "In my undergraduate education, I took courses that contained the content of this course. The way I learned it was, "This is what it is." We were given all theorems and asked to look at them, prove them, and then use them. There was rarely any conjecturing as to what might be true or not true."
- "As I look back at my undergraduate career, very little of it was spent doing these types of problems and because of that many of my first years of teaching were very traditional and close minded."
- "While completing my undergrad, I often times would teach in the way I was taught. Example were provided, students would practice, and then an assessment was given."

-- Interviews from EMU project

Do prospective high school teachers have opportunities to learn MKT?

Maybe.

- Comparatively, the US is doing okay, but there is room to improve, especially if we want to engage more high school students in mathematical discovery.

There is work for us!

- Undergraduate math may foster neither mathematical discovery nor norms for good explanation.
- Many teachers do not believe that advanced mathematics coursework is relevant to teaching … And they may be right.
- If someone doesn't believe that something is useful, they aren't likely to use it.

How can prospective high school teachers learn the mathematics needed to teach well?

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Do prospective high school teachers have opportunities to learn MKT?

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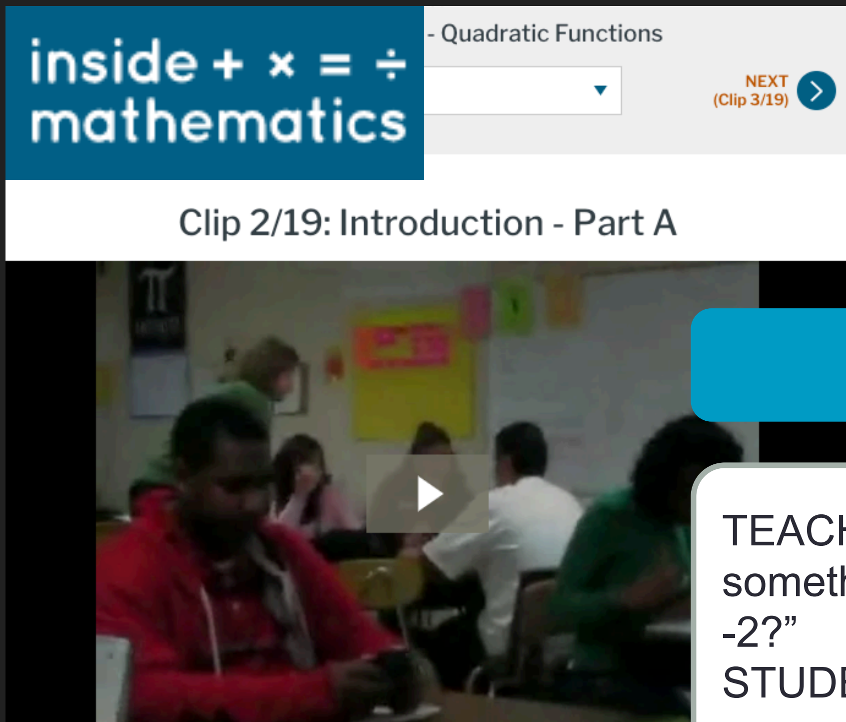
Showcase mathematical teaching practices

Give teachers an opportunity to see and use mathematical knowledge in teaching.

Example: Explaining mathematical ideas (MODULES^2)

Lai, Hart, & Patterson (ongoing)



TEACHER: “When finding an x-intercept, why do we start by putting in y = 0?”
STUDENT: “Because we want to solve for x.”

How would you respond?

TEACHER: “But how do we know that y = 0 and not something else? Why don’t we put in 10 for y, or 7, or -2?”
STUDENTS: “Because 0 is the easiest thing?” “Because you want to cancel it out.”
TEACHER: “Terence, what did you say, too?”
TERENCE: “ ’Cuz 0 is where the line crosses.”
TEACHER: “Because we are looking for the x-intercept, we are only moving in the x-direction only, we are only left and right, not up and down.”

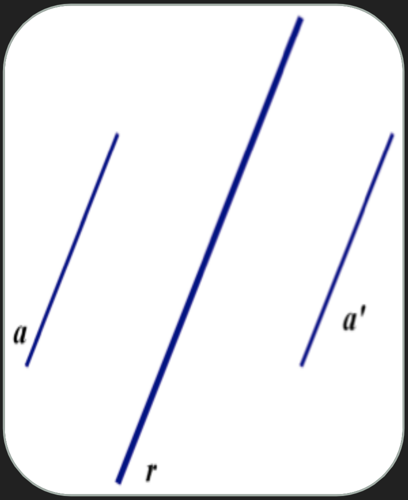
How did the teacher use the definition of graph and x-intercept to hear student contributions and help students understand?

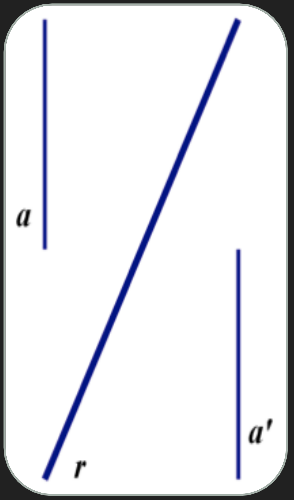
Example: Building on student work (MODULES²)

Alibegovic & Lischka (ongoing)

Definition. A ***reflection*** about line L is a transformation of the plane that, for every point P in the plane:

- $P' = P$ (if P is on L)
- L is the perpendicular bisector of PP' (if P is not on L).





- How would you respond to these students?
- Record a video of yourself where you use both students' work to help students understand how methods of constructing an image can be explained in terms of the definition of reflection.

Treat mathematics teaching as a legitimate application of mathematics

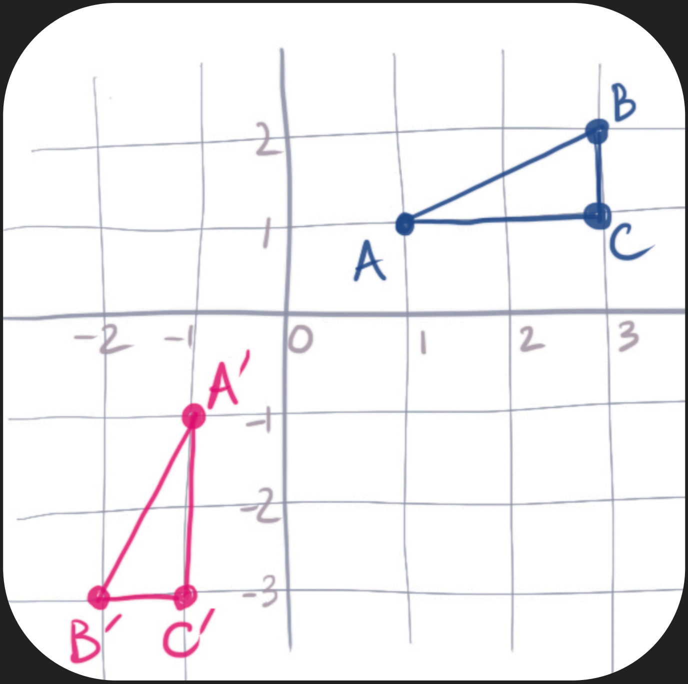
Motivate mathematics with an example from teaching practice

After building up the mathematics, step into to future teaching

Example of building up (META Math)

Suppose you are teaching high school geometry.

- What is a sequence of rotations and reflections that would map ABC to A’B’C’?
- Using language a high school student can understand, explain how to “undo” this sequence.
- Using vocabulary and concepts related to group structure, explain how you know that you could also map A’B’C’ to ABC using a single element or the group of rotations and reflections that preserve the origin.
- Does “order matter” when composing these elements?



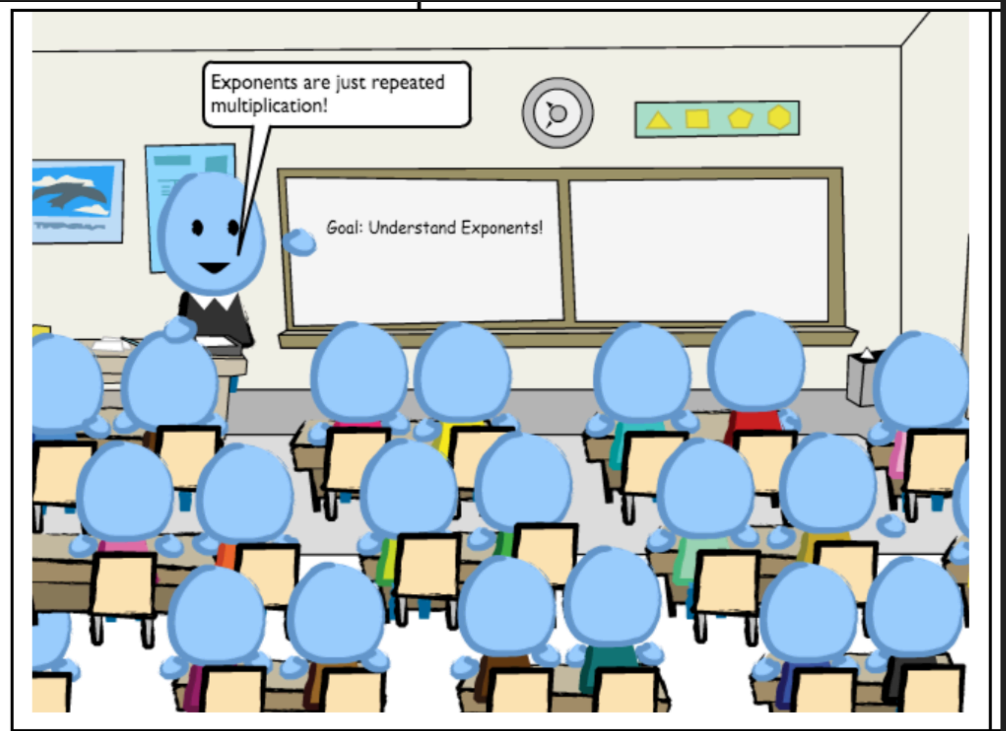
META math leadership: Ensley, Burroughs, Alvarez, Neudauer, Tanton

Example of stepping into teaching (ULTRA)

(During real analysis course)

How would you evaluate the pedagogical quality of this explanation?

If your evaluation depends on the context of the statement, provide some sense of why and when your evaluation might change.



ULTRA leadership: Wasserman, Weber, Mejia-Ramos, Fukawa-Connelly

Example of stepping into teaching (MODULES²)

Lai, Hart, & Patterson, ongoing

If $5^{\pi} = 5^{3}\ 5^{0.1}\ 5^{0.04}\ 5^{0.001}\ 5^{0.0005}$ ... ", the exponents keep on getting smaller and smaller.

You're multiplying by smaller and smaller numbers that get closer and closer to 0.

So 5^{π} should be really tiny, close to 0."

What issues or ideas come up for you when thinking about this student's comment?

Example of stepping into teaching (MODULES²)

Lai, Hart, & Patterson, ongoing

$1^0, (\frac{1}{2})^0, (\frac{1}{3})^0, (\frac{1}{4})^0, (\frac{1}{5})^0, (\frac{1}{6})^0, \ldots$

Based on this sequence,
$0^0 = 1$

$0^1, 0^{1/2}, 0^{1/3}, 0^{1/4}, 0^{1/5}, 0^{1/6}, \ldots$

Based on this sequence,
$0^0 = 0$

?!!!
???!

What happened?!

- We have sequences $a_n^{x_n}$ where $a_n \rightarrow 0$ and $x_n \rightarrow 0$
- ... but $a_n^{x_n}$ converged to different values!

Discuss the meaning of the limit notation. How could you explain these ideas to an algebra class?

Remember that context matters

The naked math task is not equivalent to the embedded math task!

Verbal explanation	Teaching Context	University Context	Total
Valid	**16**	**5**	21
Not Valid	**1**	**11**	12
Other	0	1	1
Total	17	17	34

Baldinger & Lai, 2019

- What people emphasize and think about depends on the expectations of their context.
- We need to shift how our students understand "university math".

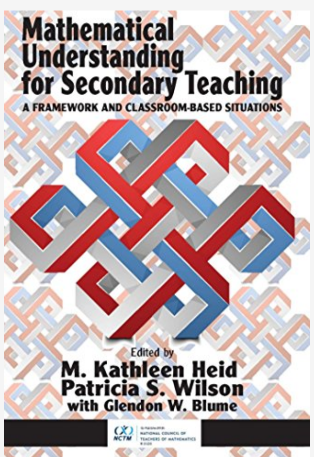
Go meta. Expand your imagery of teaching.

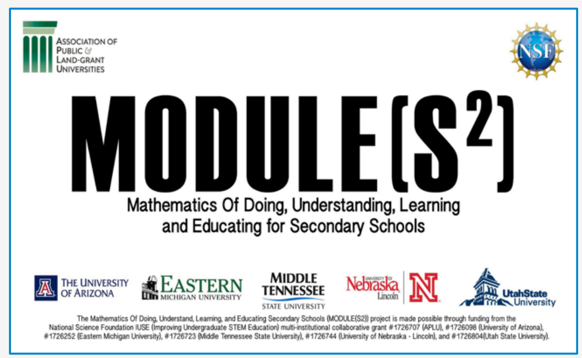
- Find and watch examples of teaching where ideas build on students' thinking. (TIMSS, Inside Mathematics, …)
- Ask questions about what you see. Imagine alternative scenarios and commitments.
- Discuss with prospective teachers.
  - What is a good example? How could "goodness" depend on context? (e.g., intro, review, digging deeper)
  - What are different ways to pose a conjecture? How would these differences shape the way students see the math?
  - What would be an effective warm-up for working on a proof? How do you create a warm-up that makes a proof more accessible but doesn't give away the good stuff?

MAKING PROGRESS

Hope for the future

- Lesson from elementary teacher education: To improve, we need institutional support and resources, designed by mathematicians and mathematics educators together.
- There is hope for the future of secondary education, because we are creating resources now.
- There are increasingly more resources that showcase mathematical teaching practices in ways that teachers find compelling.





META Math

MAA
MATHEMATICAL ASSOCIATION OF AMERICA

The work in front of us

- Expand our vision of what it means to teach, and what it means to teach teachers
- Treat mathematics teaching as a legitimate application of mathematics.
- Marshal resources for teaching MKT
- Build community of mathematics faculty who teach teachers

(plug: join the SIGMAA-MKT!)

THANK YOU

Yvonne Lai
yvonnexlai@unl.edu
University of Nebraska-Lincoln
Founding Chair, SIGMAA-MKT

Credits

PSL(2,Z): (cc) David Dumas https://homepages.math.uic.edu/~ddumas/slview/

Stars: (cc) Jordan Hackworth, https://flickr.com/photos/jordanhackworth/4891600614

Flint news collage: compiled by Gail Burrill, used in presentation at JMM 2019

-1 x -1 prompt: Adapted from COACTIV released items.

Variable prompt: Adapted from Chazan (1993), f(x)=g(x)? An approach to modeling with algebra. For the Learning of Mathematics 13(3), 22-26.

Kate/Jane task: Adapted from the Allen Minicase of the ETS Minicases project, by Howell, Lai, and Nabors-Olah. (c) ETS, used with permission.

Ovid's Metamorphosis: Public domain

French textbook snapshot: From Liberte, by Gretchen Angelo, 2003, p. 17, from the UMN Open Textbook Library

Inside Mathematics logo/video: Screenshots taken from https://www.insidemathematics.org/classroom-videos/public-lessons/9th-11th-grade-math-quadratic-functions/problem-1-part-a

META math examples from the META Math project, used with permission

ULTRA examples from ULTRA project, used with permission

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Relevant Links

Curriculum links:

- http://sigmaa.maa.org/mkt/curricula.html

Research links:

- (MQI report [among other instruments]) https://k12education.gatesfoundation.org/resource/gathering-feedback-on-teaching-combining-high-quality-observations-with-student-surveys-and-achievement-gains-3/
- (TEDS-M report) https://www.iea.nl/studies/iea/teds-m