Study Guide for the Math 1210 Oral Final Exam [Spring 20]

Overview: *What to expect:*

- *several questions relating to problems on the "Unit-II Quest" and the "Integration Quest";* [1]
- *"skills questions" prompting you to solve or explain how to solve certain standard problems—these problems might involve short computations (< 1.5 minutes);*
- *"theorem/definition questions" relating to key definitions and theorems of the course—see below.*

For problems on the exam that require short calculations, you should have a notepad or notebook to write on. You are encouraged to share your thinking as you are writing, but you are not required to do so. Significant credit will be awarded for sharing thoughts that are relevant to solving a problem, even if your final solution is incorrect. We'll have video on during the exam so that you can show your work to the camera if necessary.

It's fine to have notes (on paper) and/or your book nearby but relying heavily on notes to solve a problem would result in a loss of some credit (e.g., an A response dropped to B). Also, we'll be sharing questions on screen through Zoom. There will be no need for you to interact with Zoom (your computer) during the exam session.

During the exam, instructors will not be confirming correctness or pointing out errors. As you progress through the exam, keep in mind what was mentioned above: ***Significant credit will be awarded for sharing thoughts that are relevant to solving a problem, even if your final solution is incorrect.****.*

This Guide: Below you will find descriptions of the skills, knowledge, and level of understanding you will need to earn a high score on the Oral Final Exam. Answers to sample questions are to be worked out by students collaboratively at our Class's Piazza site. Students need to practice finding solutions and explaining their thinking, not memorizing solutions.

Unit-I Skills

(1) **Pre-Calculus**

(a) interval notation

(b) Finding domains and ranges

(c) basic algebra reviewed; e.g., factorization of polynomials, rationalizing the numerator or denominator of a fraction (in a limit problem), and properties of exponents.

(d) Solving inequalities (suggestion: use a sign line)

(e) Properties of lines (slope, intercept, parallel, perpendicular); finding equations of a line (using the point-slope as well as slope intercept formulas).

(f) You should know formulas yielding the area/perimeter of circles, rectangles, and triangles; you should know the Pythagorean Theorem, the distance formula, and be able find an equation of a circle given its center and radius.

(g) You should also know the general quadratic formula. This formula states solutions (possibly complex-valued) of the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Q1 *Sample Oral Exam Questions*: (a) Where will a laser beam following the line through $(1, 2)$ and $(4, 3)$ strike the line $x = 7$?

(b) If $A(t) = 24 \cdot 2^{-t/8}$ is the amount of radioactive iodine, in mg, that a hospital will have in storage after t days, then how much will the hospital after 16 days?

(c) Find the domain of $f(x) = \left(\frac{9-x^2}{x-1}\right)^{3/4}$, expressing your answer in interval notation.

(2) **Limits**

(a) You should have a solid intuitive understanding of limits: $\lim_{x\to a} f(x) = L$ means that we can make $f(x)$ as close as desired to L for all choices of x sufficiently close to but not equal to a.

(b) You should be able to find limits visually/graphically, experimentally (plugging values of x close to a to try to find $\lim_{x\to a} f(x)$), and algebraically (with the assistance of limit laws).

(c) You should feel comfortable using limit laws/properties appearing in Theorem 1 on page 102 of the textbook. An especially useful law that follows from Theorem 1 is the "Polynomial Plug-In Law": If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

(d) The algebraic approach to finding $\lim_{x\to a} f(x)$ often requires simplifying f before evaluating the limit (e.g., factoring and canceling; multiplying by the conjugate over the conjugate; finding a common denominator)

(e) You should be prepared to analyze limits by considering one-sided limits (especially useful to study functions that have absolute values or are piecewise defined).

[1] [Quests given were essentially take-home, low-stakes quiz-tests (hence "qu-ests").]

Q2 *Sample Oral Exam Questions*: Find the following limits. If a limit does not exist, explain why.

(a) $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$ (b) $\lim_{x\to 2} \frac{x^2-4}{|x-2|}$ (c) $\lim_{t\to\infty} \frac{t^3+2t+2}{5-2t^2+4t^3}$ (d) $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{x-2}$.

For the function whose graph appears to the right, find or explain why the limit DNE.

(e) $\lim_{x\to -1} f(x)$

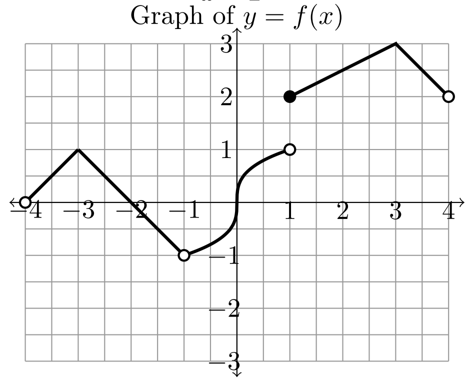
(f) $\lim_{x\to -4^+} f(x)$

(g) $\lim_{x\to 1} f(x)$

(h) $\lim_{x\to\infty} f\left(1+\frac{1}{x}\right)$

(i) $\lim_{x\to -3} f(f(x))$

Graph of $y = f(x)$



(3) **Continuity and Differentiability**

(a) Recall that calculus is a mathematical toolkit for analyzing functions. Limits are used to analyze and explore two fundamental properties of a function f:
- continuity at a number a in its domain—can we make $f(x)$ as close as desired to $f(a)$ for all choices of x sufficiently close to $f(a)$?
- differentiability of f at a number a in its domain—does the limit of the average rates of change $\frac{f(a+h)-f(a)}{h}$ of f over intervals with endpoints a and $a+h$ exist as h approaches 0 yielding an instantaneous rate of change of f at a?

(b) You should know the definitions of continuity at a and differentiability at a. You might be asked to state these during your oral exam.

The function f is *continuous* at a provided (i) $f(a)$ is defined, (ii) $\lim_{x\to a} f(x)$ exists, and (iii) $\lim_{x\to a} f(x) = f(a)$.

The *derivative of f at a*, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{provided the limit exists.}$$

When the derivative of f at a exists, we say f is *differentiable at a*.

(c) You should be able to use the definitions of continuity and differentiability to determine if a function f is continuous at a or differentiable at a.

Q3 *Sample Oral Exam Question*: Let

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases}$$

(a) Is f continuous at 1? Justify your answer using the definition of continuity of f at 1.

(b) Is f differentiable at 3? Justify your answer using the definition of differentiability of f at 3.

(d) You should feel comfortable analyzing continuity and differentiability graphically:
- Continuity at a means there is no hole or tear in the graph of f at a;
- Differentiability at a means that there is a tangent line with finite slope to the graph of f at a that can be used to well approximate values of f in an open interval containing a.

The graphical understanding of differentiability reveals that f is not differentiable at a if the graph of f has
- a corner or cusp at a,
- a vertical tangent at a,
- a discontinuity at a.

(e) As just indicated, understanding differentiability graphically yields:
- if f is differentiable at a, then f is continuous at a (which can be formally proved using the definitions of differentiability and continuity); however,
- continuity at a does not ensure differentiability at a. (Consider $f(x) = |x|$, which is continuous at 0, but $f'(0)$ does not exist.)

Q4 *Sample Oral Exam Question*: (a) Consider the function f whose graph appears at the top of the page on the right. What is the domain of f? What is the range of f? At what numbers in $(-4, 4)$ is f not continuous? At what numbers in $(-4, 4)$ is f not differentiable?

(f) You should see how the "Polynomial Plug-In Law" discussed in (2) above yields the theorem: *A polynomail function is everywhere continuous—continuous at each real number.* Other useful continuity theorems include those on Page 121 of the textbook as well as the Intermediate Value Theorem (page 123). *You should be able to state the Intermediate Value Theorem.*

(g) If you are having trouble solving an equation or showing that a function takes on a certain target value, *then the Intermediate Value Theorem is a tool you should consider using.*

Q5 *Sample Oral Exam Question*: (a) Is there a real number that is 1 more than its cube?

(4) **Differentiation Rules, Tangent Lines, and Rates of Change.**

(a) You should feel comfortable applying the following differentiation rules—these are all the rules for the course:

- Rules from Section 3.1:
  - (i) $\frac{d}{dx}[c] = 0$ (c, a constant)
  - (ii) $\frac{d}{dx}[x^n] = nx^{n-1}$ for every real number n
  - (iii) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
  - (iv) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$
  - (v) $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
- Product and Quotient Rules
  - (vi) Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$
  - (vii) Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{hi(x)}{lo(x)}\right] = \dfrac{lo(x)\frac{d}{dx}[hi(x)] - hi(x)\frac{d}{dx}[lo(x)]}{(lo(x))^2}$
- Chain Rule
  - (viii) General Chain Rule: $\frac{d}{dx}[g(f(x))] = g'(f(x))f'(x)$ or
  $$\frac{d}{dx}[g(f(x))] = g'(f(x))\frac{d}{dx}[f(x)].$$
  - (ix) Consequences of the Chain Rule
    - ∗ "General Power Rule" (or "The Power Rule for Functions"):
    $$\frac{d}{dx}\left[\Big(f(x)\Big)^n\right] = n\Big(f(x)\Big)^{n-1}\frac{d}{dx}[f(x)],$$
    which might also be written $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$, where $u = u(x)$ is a differentiable function of x.
    - ∗ "Exponential-Function Rule" (from Section 5.4): $\dfrac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}\dfrac{d}{dx}[f(x)]$.
    - ∗ "Logarithmic-Function Rule" (from Section 5.5): $\dfrac{d}{dx}[\ln(f(x))] = \dfrac{1}{f(x)}\dfrac{d}{dx}[f(x)]$

Q6 *Sample Oral Exam Questions*: (a)

If $f(x) = x^3 + 2x^4 + e^x + x^e + \ln 5$, then what is $f'(x)$?

(b) If $f(x) = \dfrac{e^{2x}}{x^2+1}$, what is $f'(x)$?

(c) If $f(x) = x^x\sqrt{x}$, what is $f'(x)$?

(d) if $f(x) = \left(e^{2x} + 7\right)^6$, then what is $f'(0)$?

(b) You should know that $f'(a)$ yields the slope of the line tangent to the graph of f at $(a, f(a))$.

Q7 *Sample Oral Exam Questions*: (a) Find an equation of the line tangent to the graph of $f(x) = \ln(2x+1)+3$ at $(0, 3)$.

(b) Is there a point on the graph of $f(x) = x^6 + 5x^3 + x^2 - 7x$ at which the tangent line to the graph has slope 1?

(c) You should know that $f'(a)$ may be interpreted as the instantaneous rate of change in f at a. For example, suppose that an object is traveling along a line and $s = s(t)$ yields its position/displacement in feet relative to a fixed point on the line at time t, where t is measured in seconds. Then $\frac{ds}{dt} = s'(t)$ is the (instantaneous) velocity of the object at time t in feet per second and $\frac{d^2s}{dt^2} = s''(t)$ is the acceleration of the object at time t in ft/sec^2.

Q8 *Sample Oral Exam Questions*: (a) The position of an object traveling along a line is given by $s(t) = t^3 + 2t + 1$, where s is measured in feet and t is measured in seconds. What is the object's velocity at $t = 1$? (Include units.) What is the object's acceleration at $t = 1$? (Include units). Is there a time t when the object's velocity is 29 ft/sec?

(b) Let $P(t) = 100e^{0.1t}$ be the population of bacteria in a culture at time t (measured in hours). What is the rate of change in the the culture's population at time $t = 0$? Include units.

(c) Boyle's Law states that when a sample of gas is compressed at constant temperature, the product of the pressure P and the volume V remains constant:

$$V = \frac{C}{P},$$

for some constant C. Find the rate of change of volume with respect to pressure.

Unit-I Definitions and Theorems

(1) You should be able to state the definition of continuity of f at a:

The function f is *continuous* at a provided (i) $f(a)$ is defined, (ii) $\lim_{x\to a} f(x)$ exists, and (iii) $\lim_{x\to a} f(x) = f(a)$.

(2) You should be able to state the definition of derivative.

The *derivative of f at x*, denoted $f'(x)$, is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limit exists.}$$

(3) You should know that differentiability at a implies continuity but not conversely.

(4) You should be able to state the Intermediate Value Theorem.

"If f is a continuous function on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that $f(c) = M$

Q9 *Sample Oral Exam Question*: (a) If $f'(5) = 3$, must $\lim_{x\to 5} f(x)$ exist? Explain.

Unit-II Skills

(1) Computation of derivatives of logarithmic and exponential functions—covered above.

(2) Determining appropriate units for derivatives.

Q10 *Sample Oral Exam Question*: (a) A conical pile of sand is growing larger—the radius r of its base is 3 ft and growing at the rate of 1/2 foot per minute. Its height h is 4 feet and growing at the rate of 1 ft per minute. Given that $V = \frac{1}{3}\pi r^2 h$ is the volume of the pile, find $\frac{dV}{dt}$. What are correct units for $\frac{dV}{dt}$?

(3) You should know and be able to apply the following.

Name	Formula
Pythagorean Theorem (right triangle with legs x, y, hypothenuse h)	$h^2 = x^2 + y^2$
Area enclosed by a circle of radius r	πr^2
Perimeter of a circle of radius r	$2\pi r$
Volume of a rectangular box, base xy, height h	xyh
Surface area of a rectangular box, base xy, height h, no top	$2xh + 2yh + xy$
Surface area of a rectangular box, base xy, height h, with top	$2xh + 2yh + 2xy$

Q11 *Sample Oral Exam Question*: (a) A circular oil slick currently has a radius of 100 meters and its radius is increasing at a rate of 5 meters per hour. What is the current rate of change of its area with respect to time? Include units.

(4) You should be able to find critical numbers of functions.

Q12 *Sample Oral Exam Question*: (a) Find the critical numbers, if any, of $f(x) = x^{2/3}$.

(b) The critical numbers of $f(x) = x + \dfrac{1}{x}$ are ____________________.

(5) You should be able to apply the First and Second-Derivative Tests to assess whether a critical number yields a relative maximum value of a function, a relative minimum value of a function, or neither a relative maximum nor relative minimum.

Q13 *Sample Oral Exam Question*: (a) Find and classify the critical numbers of $f(x) = x^3 - 3x$, where "classify" means determine whether the critical number yields a relative maximum value of f, a relative minimum value of f, or neither.

(6) You should be able to determine the (open) intervals on which a function is increasing and those on which it is decreasing using the f' sign line.

***Q14** Sample Oral Exam Question*: (a) On what interval(s) is the function $f(x) = 1 - (x-2)^2$ increasing? decreasing?

(b) True/False justify your answer: If $f'(x) > 0$ for every x in $(-\infty, \infty)$, then the graph of f has at most one x-intercept.

(7) You should be able to determine the (open) intervals on which the graph of a function is concave up and those on which it is concave down using the f'' sign line.

***Q15** Sample Oral Exam Question*: (a) On what intervals is the graph of the function $f(x) = x + \frac{1}{x}$ concave up? concave down?

(8) You should be able to find points of inflection on the graph of f.

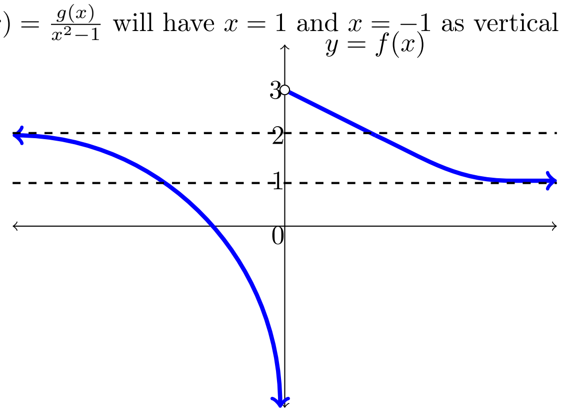
***Q16** Sample Oral Exam Question*: (a) Explain how to use an appropriate sign line to locate points of inflection on the graph of a function f.

(9) You should be able to find horizontal and vertical asymptotes of the graph of a function.

***Q17** Sample Oral Exam Question*: (a) What are the horizontal and vertical asymptotes (if any) of the graph of $f(x) = e^{-x} + \frac{1}{x}$?

(b) True/False justify your answer: Any function of the form $f(x) = \frac{g(x)}{x^2-1}$ will have $x = 1$ and $x = -1$ as vertical asymptotes.

(c) Let f be the function whose graph is at right.



Multiple-select problem. Let g be the function defined by $g(x) = f\left(\frac{1}{x}\right)$ for all $x \neq 0$. Let h be the function defined by $h(x) = f(e^{-x})$ for all x. Circle all statements that are correct.

(i) The graph of f has two horizontal asymptotes.
(ii) The graph of g has two horizontal asymptotes.
(iii) The graph of h has two horizontal asymptotes.

(10) You should be able to use the Closed Interval Method to find the absolute maximum value and absolute minimum value of a continuous function on a closed interval.

***Q18** Sample Oral Exam Question*: (a) Find the absolute maximum value and absolute minimum value of $f(x) = \ln(1+x^2)$ on $[-\sqrt{e-1}, \sqrt{e+1}]$.

(11) You should be able to sketch graphs or portions of graphs using Tan's "ninefold path" to sketching curves.

***Q19** Sample Oral Exam Question*: (a) Sketch the graph of function f such that

- f is continuous on $(-\infty, \infty)$,
- $f' > 0$ on $(-\infty, 0)$ and $f' < 0$ on $(0, \infty)$,
- $f'' > 0$ on $(-\infty, 0)$, and $f'' > 0$ on $(0, \infty)$.

(12) You should be able to use laws of exponents and laws of logarithms.

***Q20** Sample Oral Exam Question*: (a) Simplify $\frac{2^{1/2}2^{3/2}}{27^{-1/3}} + \ln\left(\frac{1}{e^3}\right)$.

(13) You should be able to solve equations involving exponential functions and be able to solve real-world problems involving exponential and logarithmic functions.

***Q21** Sample Oral Exam Question*: (a) The population of bacteria in a culture t hours after 1 p.m. is given by

$$P(t) = 1000e^{0.1t}.$$

When will the population reach 2000?

(14) Understand that the logarithmic function $\log_b$ $(b > 0,\ b \neq 1)$ is the inverse of the exponential function $f(x) = b^x$, which means that $\log_b(b^x) = x$ for all x and $b^{\log_b x} = x$ for $x > 0$.

***Q22** Sample Oral Exam Question*: (a) Multiple-Choice

$\ln(1) + e^{3\ln(2)} + \log_2(8) =$ (i) 8 (ii) 9 (iii) 10 (iv) 11 (v) 12.

(15) Be familiar with the graphs of exponential and logarithmic functions

***Q23** Sample Oral Exam Question*: (a) Sketch the graph of $g(x) = \ln x$.

(16) You should know the basic properties of exponential and logarithmic functions boxed and tabulated in the regular study guide for Exam 2.

***Q24** Sample Oral Exam Question*: (a) What is the domain of $f(x) = \ln(1-x)$? Explain your reasoning.

Key Theorems & Definitions, which you should be able to state

(1) A *critical number* (or *critical point*) of a function f is any number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

(2) A point $(a, f(a))$ on the graph of f at which the concavity of the graph changes is a *point of inflection*.

(3) The vertical line $x = a$ is a *vertical asymptote* of the graph of the function f provided that at least one of the one-sided limits $\lim_{x \longrightarrow a^+} f(x)$ and $\lim_{x \longrightarrow a^-} f(x)$ is ∞ or $-\infty$.

(4) The horizontal line $y = b$ is a *horizontal asymptote* of the graph of the function f if at least one of $\lim_{x \longrightarrow \infty} f(x)$ and $\lim_{x \longrightarrow -\infty} f(x)$ is the number b.

(5) The *First Derivative Test*:

- If c is a critical number at which f is continuous and the sign of f' changes from negative to the left of c to positive to the right, then f has a relative minimum at c.

  $\searrow$ $\nearrow$
  minimum

- If c is a critical number at which f is continuous and the sign of f' changes from positive to the left of c to negative to the right, then c has a relative maximum at c.

  maximum
  $\nearrow$ $\searrow$

- If c is a critical number at which f is continuous and the sign of f' does not change at c (that is, it's the same on either side of c), then f has neither a relative maximum nor a relative minimum at c.

(6) *The Second-Derivative Test*:
   (a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a relative minimum at c;
   (b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a relative maximum at c.

(7) **Extreme Value Theorem**: If a function f is continuous on a closed interval $[a, b]$, then f has an absolute maximum value and an absolute minimum value on $[a, b]$.

***Q25** Sample Oral Exam Questions*: (a) State the definition of horizontal asymptote and use it to find the horizontal asymptotes of the graph of $f(x) = \begin{cases} \frac{3x^2+1}{x^2+2} & \text{if } x < 0 \\ \frac{1}{x^2+2} & \text{if } x > 0. \end{cases}$

(b)

Dr. Calculo has applied the First Derivative Test to $f(x) = \begin{cases} 2x+3 & \text{if } x < 0 \\ -x^2 & \text{if } x \geq 0 \end{cases}$, noting, correctly, that $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$. Calculo concludes that f attains an absolute maximum at $x = 0$. Is Calculo correct? Explain.

An *A* student might answer "The First Derivative Test doesn't apply here because f is not continuous at .

(c) True/False justify your answer: If $f''(0) = 0$, then $(0, f(0))$ is a point of inflection on the graph of f.

(d) Multiple-Select Problem: Suppose that $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = 3$. Then which, if any, of the following must be true—select all that apply:

(i) f is continuous at 2;
(ii) f has neither a relative maximum nor a relative minimum at 2.
(iii) The line $x = 2$ is not a vertical asymptote of the graph of f.
(iv) $(2, f(2))$ cannot be a point of inflection on the graph of f.

Unit-III Skills

(1) You should be able to find "general antiderivatives"; equivalently, you should be able to find indefinite integrals.

If $f(x)$ is a function, all possible antiderivatives of $f(x)$ on an interval of interest are represented by the symbol $\int f(x)dx$. Under this notation, we might say that

$$\int f(x)dx = \text{ the family of all functions having derivative } f(x);$$

We might also say,

$$\int f(x)dx = \text{the general antiderivative of } f(x).$$

For example, $\int xdx = \frac{x^2}{2} + C$ means that all functions having derivative $f(x) = x$ on say, $(-\infty, \infty)$ are of the form $\frac{x^2}{2} + C$ where C is an arbitrary constant t called a constant of integration.

Remark: In general, finding antiderivatives is not as straightforward as finding derivatives. There are simple-looking functions such as $f(x) = e^{x^2}$ that do not have "nice antiderivatives".[2] On the final exam, if you are asked to find an antiderivative of a function f or to evaluate an indefinite integral $\int f\,dx$, then you will be dealing with a function f that *does* have a nice antiderivative, which you'll be able to find using the integration/antidifferentiation rules and techniques you have learned in this course.

(2) You should know the following integration rules.

Rule	Notation
Integral of a constant	$\int kdx = kx + C$
Constants slide through integrals	$\int kf(x)dx = k\int f(x)dx$
The integral of a sum is the sum of the integrals	$\int (f(x) + g(x))\,dx = \int f(x)dx + \int g(x)dx$
Integral of a power of x	$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$
Integral of $\frac{1}{x}$	$\int \frac{dx}{x} = \ln\|x\| + C$
Integral of an exponential	$\int e^x dx = e^x + C$

Q26 *Sample Oral Exam Questions*:

Evaluate the following indefinite integrals.

(a) $\int (3x^2 + 4)\,dx$ (b) $\int (3e^x + \sqrt{x})\,dx$ (c) $\int (3x+1)^2\,dx$ (d) $\displaystyle\int \frac{1+x}{x^2}\,dx$

(3) You should feel comfortable with the following natural application of antidifferentiation: the velocity of an object as a function of time $v(t)$ is an antiderivative of its acceleration $a(t)$ and the position of an object as a function of time $s(t)$ is an antiderivative of the velocity $v(t)$, that is,

$$v(t) = \int a(t)dt \quad \text{and} \quad s(t) = \int v(t)dt,$$

where we interpret the preceding equations to specify the forms of v and s, with constants of integration being determined by appropriate'initial conditions.

For example if the acceleration as a function of time is $a(t) = t^2 - 7$ then its velocity will have the form $v(t) = \int a(t)dt = \int \left(t^2 - 7\right)dt = \frac{t^3}{3} - 7t + C$ where the constant C is determined after an initial condition is specified for v.

Q27 *Sample Oral Exam Question* : (a) An object is thrown straight up. Assume that after its release, the only force it experiences is that due to gravity causing the object's acceleration at time t in seconds to be

$$a(t) = -9.8 \quad \text{meters/sec.}$$

Find the form of the object's position function.

(4) You should feel comfortable solving first-order initial-value problems.

Definition. An equation containing one or more derivatives of a function of interest is called a *differential equation*.

For instance, $f'(x) = \frac{1}{x} + 2x$ is a differential equation.

A *solution* to a differential equation is a function which satisfies the differential equation. For instance, $f(x) = \ln|x| + x^2$ is a solution of the differential equation $f'(x) = \frac{1}{x} + 2x$ and $f(x) = \ln|x| + x^2 + C$ is called a general

[2] Roughly speaking, "not nice" here means "can't be expressed as a finite algebraic combination of functions with which you are familiar".

solution of the equation.

Definition Let x_0 and y_0 be constants. A problem of the form

$$f'(x) = \text{ some function of } x, \qquad f(x_0) = y_0$$

is called a *first-order initial-value problem*. (The condition "$f(x_0) = y_0$" is called an *initial condition*.)

Example. Find a function f satisfying $f'(x) = \frac{1}{x} + 2x$ and $f(1) = 3$. The first step in the solution process is to integrate both sides of $f'(x) = \frac{1}{x} + 2x$ obtaining a general solution

$$f(x) = \ln|x| + x^2 + C.$$

The second step is to substitute $x = 1$ into both sides of the general solution and use $f(1) = 3$ to determine the constant C in the general solution: $3 = \ln|1| + 1^2 + C$, so that $C = 2$. Thus our Initial-Value Problem "IVP" has solution $f(x) = \ln|x| + x^2 + 2$.

***Q28** Sample Oral Exam Questions*: (a) Find f given that $f'(x) = \frac{1}{x^2} + x$ and $f(1) = 2$.

(b) Water is leaking from a container in such a way that the water level's height h in centimeters is changing at the rate

$$h'(t) = -\frac{1}{\sqrt{t+1}} \text{ cm/min} \quad 0 \leq t \leq 10$$

Assuming $h(0) = 10$, what is the water level 8 minutes later?

(5) You should be prepared to find indefinite integrals via the method of substitution, often called "u-substitution."

***Q29** Sample Oral Exam Questions*: (a) Find $\int \frac{\sqrt{\ln x}}{x}\, dx$

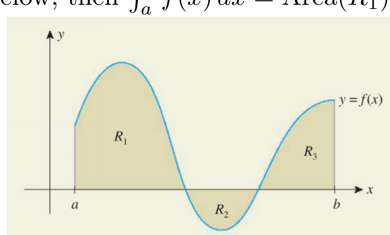
(b) Find $\int x^2 e^{x^3}\, dx$.

(6) You should be able to find a definite integral $\int_a^b f(x)\, dx$ using

- The area interpretation of definite integral: if $a < b$ and f is a nonnegative continuous function over $[a, b]$, then $\int_a^b f(x)\, dx$ is the area under the graph of f over $[a, b]$. More generally, if $a < b$, f is continuous on $[a, b]$, and the graph of f is pictured only over $[a, b]$, then

  $\int_a^b f(x)\, dx =$ area of region(s) under the graph of f and above the x axis minus the area of region(s) above the graph of f and under the x-axis.

  Thus if the graph of f is as pictured below, then $\int_a^b f(x)\, dx = \text{Area}(R_1) - \text{Area}(R_3) + \text{Area}(R_3)$.



- The Fundamental Theorem of Calculus: Let f be a continuous function on $[a, b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f; that is, $F'(x) = f(x)$.
- Properties of the definite integral:

  Some Properties of the definite integral: Suppose that $f(x), g(x)$ are functions defined on $[a, b]$ for which $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ exist. If c is any number between a and b and k any constant then

  (i) $\int_a^a f(x)dx = 0$

  (ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  (iii) $\int_a^b (f(x) + kg(x))\, dx = \int_a^b f(x)dx + k\int_a^b g(x)dx$

  (iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Q30 *Sample Oral Exam Questions*: Find

(a) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ (b) $\int_0^2(e^x+x)\,dx$ (c) $\int_0^1(x+3\sqrt{1-x^2})\,dx$ (d) $\int_0^1|x^2-1|\,dx$

(e) True/False justify your answer: If $a<b$ and if $\int_a^b f(x)\,dx>0$, then $f(x)>0$ for every x in (a,b).

(f) Assume f is integrable over $[-1,5]$, satisfying $\displaystyle\int_{-1}^{5} f(x)\,dx = -3$ and $\displaystyle\int_{1}^{5} f(x)\,dx = 7$. Then $\displaystyle\int_{1}^{-1} f(x)\,dx =$ ____________________________.

(7) You should feel comfortable evaluating definite integrals using the method of substitution: There are two natural ways to organize your work (see pages 443 and 444 of the textbook), which we will now illustrate. *Note well: you must organize your work in one of these ways. In particular, the original definite integral must be connected by a string of valid equalities (showing how the Fundamental Theorem of Calculus is being applied) to your final answer* When you use u-substitution to evaluate an integral, be sure to define u and compute du. Note this will yield partial credit (if you don't arrive at the correct answer).

For example, suppose that we want to find $\int_0^{\sqrt{2}} xe^{x^2}\,dx$.

(a) FIRST METHOD: you compute first the corresponding indefinite integral $\int xe^{x^2}\,dx$, for which we use the substitution $u=x^2$, $du=2xdx$ or $\frac{1}{2}\,du = x\,dx$. We have

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Thus, $F(x)=\frac{1}{2}e^{x^2}$ is an antiderivative of $f(x)=xe^{x^2}$ and the Fundamental Theorem of Calculus allows us to conclude that

$$\int_0^{\sqrt{2}} xe^{x^2}\,dx = \left(\frac{1}{2}e^{x^2}\right)\Big|_0^{\sqrt{2}} = \left(\frac{1}{2}e^2 - \frac{1}{2}e^0\right) = \frac{1}{2}e^2 - \frac{1}{2}.$$

(b) SECOND METHOD: we choose to work with the definite integral from the beginning, but now we need to keep track of the limits of integration. In order to find $\int_0^{\sqrt{2}} xe^{x^2}\,dx$ we use again the substitution $u=x^2$, $du=2xdx$ but now we keep track of the limits of integration. Since

$$\begin{cases} x=0 \implies u=0 \\ x=\sqrt{2} \implies u=2 \end{cases}$$

the definite integral becomes

$$\int_0^{\sqrt{2}} xe^{x^2}\,dx = \frac{1}{2}\int_0^2 e^u du = \frac{1}{2}\left(e^u\right)\Big|_0^2 = \frac{1}{2}\left(e^2-e^0\right) = \frac{1}{2}\left(e^2-1\right).$$

The advantage of this method is that after you find the integral in terms of u there is no need to use the substitution again in order to get an expression in terms of x since the new limits of integration take care of that. However, if you use this method be sure to change the limits of integration (and change them only once)!

Q31 *Sample Oral Exam Questions*: (a) Find $\int_0^1\sqrt{3x+1}\,dx$ (b) Find $\int_1^e \frac{\ln x}{x}\,dx$.

(8) You should know and be able to use the Net-Change formula, giving a different way of looking at the net change in a function over an interval:

The net change in a function f over an interval $[a,b]$ is given by

$$f(b)-f(a) = \int_a^b f'(x)\,dx,$$

provided that f' is continuous in $[a,b]$.

Q32 *Sample Oral Exam Question*: (a) The rate of change in the price of oil in dollars per barrel t months after January 1st, 2020 is approximated by

$$p'(t) = -8t,\ \ 0\le t\le 3,$$

According to this approximation, what was the net change in oil price this year from January 1st though the end of March?

(9) You should know how to find the average value over $[a, b]$ of an integrable function over $[a, b]$:

Suppose that $f(x)$ is a function integrable on $[a, b]$. Then its **average value over** $[a, b]$ is by definition

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Q33 *Sample Oral Exam Question*: (a) The temperature of a 3-centimeter-long horizontal metal wire x centimeters from its left endpoint is $T(x) = 1 + x^2$ degrees Celsius. Find the average temperature of the wire.

Unit-III Theorem

You might be asked to state the Fundamental Theorem of Calculus: Let f be a continuous function on $[a, b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f; that is, $F'(x) = f(x)$