ON THE ANALYTIC LANGLANDS CORRRESPONDENCE FOR PGL_2 IN GENUS 0 WITH WILD RAMIFICATION

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Contents

1. Introduction

1.1. **Analytic Langlands correspondence.** In [1, 3, 2], an analytic version of the Langlands correspondence was formulated for curves over local fields. The general setup, which we recount just for completeness, is as follows. Let X be a smooth projective irreducible curve over a local field F, let G be a connected simple algebraic group over F, and let B be its Borel subgroup. Let S be a finite set of F-points in X. By $\mathrm{Bun}_G(X, S)$ we denote the the algebraic stack of G-bundles $\mathcal{E}$ on X with reduction to B on $\mathcal{E}|_S$. On the spectral side, one considers a Hilbert space $\mathcal{H}$ of, roughly, square-integrable half-densities on the open dense substack of stable bundles in $\mathrm{Bun}_G(X, S)$; in [3], a commutative algebra of Hecke operators were constructed, initially only on a dense subspace of $\mathcal{H}$, but are conjectured to extend by continuity to compact normal operators on $\mathcal{H}$. On the "arithmetic" side, at least in the case $F = \mathbb{C}$, it is conjectured that the joint spectrum of Hecke operators should correspond to the set of ${}^LG$-opers with real monodromy, where ${}^LG$ is the Langlands dual group of G.

In [2], this recipe was implemented for $G = \mathrm{PGL}_2$, $X = \mathbb{P}^1$, and S a set of distinct F-points $t_0, \dots, t_{m+1}$ in X, where $m \geq 1$ (a necessary condition). Let us suppose $F = \mathbb{C}$, which will be relevant to us, but most of the discussion holds for a general local field, archimedean or not. In this case, a G-bundle with parabolic reduction is simply a rank 2 vector bundle, up to tensoring by line bundles, with distinguished dimension 1 subspaces in the fibers above the marked points $t_0, \dots, t_{m+1}$. Such bundles are called *quasiparabolic bundles*. In this case, the moduli stack of stable quasiparabolic bundles is known to be a smooth, quasiprojective variety, and is the union of two connected components, bundles of degree 0 and 1, respectively. There are isomorphisms identifying the two components, given by Hecke modification at any of the marked points;

Date: August 2, 2023.

so it suffices to consider the degree 0 component Bun_G^0. This space could be parametrized birationally by $\mathbb{P}^{m-1}$ ([2], lemma 3.1): by fixing the lines above $t_0 = 0$ and $t_{m+1} = \infty$, a generic quasiparabolic bundle is uniquely given by m complex numbers, each specifying the line above $t_1, \ldots, t_m$, up to simultaneous scaling. Therefore, $\mathcal{H} = L^2(\mathrm{Bun}_G^0) = L^2(\mathbb{P}_{\mathbb{C}}^{m-1})$ is simply the space of square-integrable half-densities on $\mathbb{P}^{m-1}$ (sections of $|\mathcal{K}|$, where $\mathcal{K} = \mathcal{O}(-m)$ is the canonical bundle). An element $\psi \in \mathcal{H}$ can therefore be realized as a complex-valued function $\psi(y_1, \ldots, y_m)$ on $\mathbb{C}^m \backslash \{0\}$, such that $\psi(z\mathbf{y}) = |z|^{-m}\psi(\mathbf{y})$ for any $z \in \mathbb{C}^\times$.

Under this parametrization, the Hecke operators take the following explicit form. For each $x \in \mathbb{C}\backslash\{t_i\}$, the Hecke operator H_x is given by

$$(H_x\psi)(y_1, \ldots, y_m) = \left(\prod_{i=0}^{m} |t_i - x|\right) \cdot \int_{\mathbb{C}} \psi\left(\frac{t_1 s - x y_1}{s - y_1}, \cdots, \frac{t_m s - x y_m}{s - y_m}\right) \frac{|s|^{m-2} ds d\bar{s}}{\prod_{i=1}^{m} |s - y_i|^2}. \tag{1.1}$$

It is shown in [2], section 3 that H_x indeed extend to compact, self-adjoint, mutually commuting operators on $\mathcal{H}$, with zero common kernel. Importantly, this relies on the fact that H_x is given by integrating certain unitary operators $U_{s,x}$ over $s \in \mathbb{C}$.

On the other hand, there are certain commuting global holomorphic differential operators on $\mathcal{H}$, in this case Gaudin operators G_i $(0 \le i \le m)$, which also act on $\mathcal{H}$. The key insight is that although these G_i are unbounded operators, they commute with H_x in a certain well-defined sense, so that we get a good spectral problem for both Hecke and differential operators (since Hecke operators are compact self-adjoint). In this case (PGL_2 and $\mathbb{P}^1$ over $\mathbb{C}$), the joint eigenvalues $\beta_k(x)$ (real-valued and continuous in x, labeled by $k \in \mathbb{N}$) satisfy a differential equation ([2], corollary 4.14):

$$\left(\partial_x^2 + \frac{1}{4}\sum_{i=0}^{m} \frac{1}{(x - t_i)^2} - \sum_{i=0}^{m} \frac{\mu_{i,k}}{x - t_i}\right) \beta_k(x) = 0, \tag{1.2}$$

which is an $\mathrm{SL}_2(\mathbb{C})$-oper (i.e. no ∂_x term); SL_2 is Langlands dual to PGL_2. Here $\mu_{i,k} \in \mathbb{C}$ are eigenvalues of G_i on the eigenfunction ψ_k corresponding to β_k (in particular it is shown that the joint spectrum of H_x is simple). Moreover, the monodromy representation of such a differential equation (where $\mu_{i,k}$ are now variables in $\mathbb{C}$) lands in $\mathrm{SL}_2(\mathbb{R})$ up to conjugation if (and, partially, only if) they come from a joint eigenfunction of Hecke operators ([2], theorem 4.15), thus establishing analytic Langlands correspondence.

1.2. **Summary of our paper.** In this paper we investigate what happens when we collide several points among t_i, i.e. when S is no longer a reduced divisor. For example, suppose we merge only t_0 and t_1. One obvious way of obtaining a limit of Hecke operators is to simply set $t_0 = t_1$ in eq. (1.1); this corresponds to choosing two lines in the fiber of the quasiparabolic bundle above the closed point $t_0 = t_1$. However, the resulting Hecke operators will have no eigenvectors.

Instead, we should make $t_0 = t_1$ a non-reduced point, in this case a $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-point. A generic line in its fiber is given by $(1, u_0 + u_1\varepsilon)$, so that in eq. (1.1) one should change variables y_0, y_1 by $u_0 = y_0$, $u_1 = \frac{y_1 - y_0}{t_1 - t_0}$. In order to have a well-defined limit as $t_1 \to t_0$, we should also use a twisted version of Hecke operators, whose twisting parameters are sent to infinity in an appropriate way.

We carry this out in section 2, giving us limits of Hecke operators H_x that are again given by integrating some unitary representation $U_{s,x}$ over $s \in \mathbb{C}$ (proposition 2.4). Moreover, we show that these H_x extend to bounded, compact, self-adjoint, and mutually commuting operators on $\mathcal{H}$, with zero common kernel, and therefore they have a joint discrete spectrum (corollary 2.10). In other words, we recover the main properties of Hecke operators required for establishing analytic Langlands correspondence in our case.

In section 3 we consider limits of differential operators G_i. To get a well-defined limit, we should also use their twisted counterparts, with twisting parameters sent to infinity. We show that just like the original tamely ramified case, the limits of Hecke operators satisfy a differential equation together with limits of Gaudin operators (theorem 3.1), and their joint eigenvalues $\beta_k(x)$ satisfy a differential equation similar to eq. (1.2). However, the important difference is that this equation will no longer have regular singularities at t_i, but also irregular singularities at the merged points (wild ramification). So the condition of real monodromy is not enough, and there should be a condition on the Stokes data or asymptotic expansion of solutions at irregular singularities. This is currently under investigation.

2. Limits of Hecke operators

2.1. **Twisted Hecke operators.** Let $t_0, \dots, t_{m+1} \in \mathbb{P}^1_{\mathbb{C}}$ be closed points. Without loss of generality, let us fix $t_{m+1} = \infty$. Let $x \in \mathbb{P}^1_{\mathbb{C}}$, $x \neq t_i, \infty$. Let $\lambda = (\lambda_0, \dots, \lambda_{m+1})$ be twisting parameters, which are purely imaginary.

For any purely imaginary number c, let $\mathcal{H}^c = L^2(\mathbb{P}^{m-1}_{\mathbb{C}}, |\mathcal{K}|^{1+c})$, whose elements we view as complex-valued functions $\psi(y_1, \dots, y_m)$ on $\mathbb{C}^m \backslash \{0\}$, homogeneous of degree $-m(1+c)$. They may also be viewed as functions $\psi(y_0, y_1, \dots, y_m)$ which are both translation-invariant and homogeneous of degree $-m(1+c)$, where geometrically y_i parametrize the quasiparabolic lines above t_i; this interpretation has more symmetry and makes formulas nicer. Let $\mathcal{H} := \mathcal{H}^0$.

The *twisted Hecke operators* H_x^λ are given by

$$(H_x^\lambda \psi)(y_0, \dots, y_m) = \left(\prod_{i=0}^m |t_i - x| \right) \cdot \int_{\mathbb{C}} \psi \left(\frac{t_0 - x}{s - y_0}, \cdots, \frac{t_m - x}{s - y_m} \right) \frac{ds d\bar{s}}{\prod_{i=0}^m |s - y_i|^{2(1+\lambda_i)}}. \tag{2.1}$$

It is easy to check that H_x^λ is a linear map which maps functions homogeneous of degree $-m(1+\lambda_{m+1})$ to functions homogeneous of degree $-m(1 + \frac{2}{m}(\sum_{i=0}^m \lambda_i) - \lambda_{m+1})$. We will limit ourselves to the case when $\lambda_{m+1} = 0$ and $\sum_{i=0}^m \lambda_i = 0$, so that functions in the domain and codomain of H_x^λ have the same homogeneity degree $-m$.

If we omit the constant term $\prod |t_i - x|$ in eq. (2.1), the formula gives so-called *modified* Hecke operators, denoted by $\mathbb{H}_x^\lambda$.

2.2. **The formula.** Suppose we wish to merge points $t_0, \dots, t_n$, where $n \leq m$. For simplicity, we let the other points remain distinct, but one can merge more than one group of points by the same procedure. For $0 \leq i \leq n$, take twisting parameters

$$\lambda_i = \frac{a}{\prod\limits_{\substack{0 \leq k \leq n \\ k \neq i}} (t_i - t_k)}, \tag{2.2}$$

and set the rest to 0, where a is imaginary. In the limiting process, we will make $t_i - t_{i-1}$ $(1 \leq i \leq n)$ all equal, real numbers δ, as we take the limit $\delta \to 0$.

We will reparametrize $(y_0, \dots, y_n)$ by new variables $(u_0, \dots, u_n)$, where

$$u_i = \sum_{0 \leq j \leq i} \frac{y_j}{\prod\limits_{\substack{0 \leq k \leq i \\ k \neq j}} (t_j - t_k)}. \tag{2.3}$$

In fact, define variables $u_{i,j}$, $0 \leq j \leq i \leq n$ recursively, as follows: $u_{i,0} = y_i$, $u_{i,j} = \frac{u_{i,j-1} - u_{i-1,j-1}}{t_i - t_{i-j}}$. Then it is easy to see $u_i = u_{i,i}$. We also let $u_i = y_i$ for $n+1 \leq i \leq m$ for simplicity. Note that now an element $\psi = \psi(u_0, \dots, u_m) \in \mathcal{H}$ will still be homogeneous of degree $-m$, but translation invariant only in the variables $u_0, u_{n+1}, \dots, u_m$ while $u_1, \dots, u_n$ remain fixed.

Definition 2.1. Consider the field $\mathbb{C}(s, x, t_0, u_{i,j})$ generated formally by these symbols. Define a derivation ∂ on this field, defined by $\partial s = \partial x = 0$, $\partial t_0 = 1$, and $\partial u_{i,j} = (j+1) u_{i+1,j+1}$.

Proposition 2.2. *The limit $\mathbb{H}_x$ of the modified Hecke operator $\mathbb{H}_x^\lambda$, as $\delta \to 0$, is given by*

$$\begin{aligned}
&(\mathbb{H}_x \psi)(u_0, \dots, u_m) \\
&= \int_{\mathbb{C}} \psi \left(\frac{t_0 - x}{s - u_0}, \partial \left(\frac{t_0 - x}{s - u_0} \right), \dots, \frac{1}{n!} \partial^n \left(\frac{t_0 - x}{s - u_0} \right), \frac{t_{n+1} - x}{s - u_{n+1}}, \dots \right) \frac{\exp(-\frac{2a}{n!} \mathrm{Re}\, \partial^n \log(s - u_0)) ds d\bar{s}}{|s - u_0|^{2n+2} \prod_{k=n+1}^m |s - u_k|^2}.
\end{aligned}$$

Proof. Let us show that the limit of the term $|s - u_0|^{2\lambda_0} \cdots |s - u_n|^{2\lambda_n}$ is $\exp(\frac{2a}{n!} \mathrm{Re}\, \partial^n \log(s - u_0))$. Use induction on n. The base case $n = 0$ is clear. In general, we have for $0 < i < n$,

$$\lambda_i = \frac{a}{\prod_{0 \leq k \neq i \leq n} (t_i - t_k)} = \frac{1}{t_n - t_0} \left(\frac{a}{\prod_{0 < k \neq i \leq n} (t_i - t_k)} - \frac{a}{\prod_{0 \leq k \neq i < n} (t_i - t_k)} \right),$$

so

$$\prod_{i=0}^{n}|s-u_i|^{2\lambda_i}=\left(\frac{\prod_{i=1}^{n}|s-u_i|^{2\lambda_{i,[1,n]}}}{\prod_{i=0}^{n-1}|s-u_i|^{2\lambda_{i,[0,n-1]}}}\right)^{\frac{1}{t_n-t_0}}, \tag{2.4}$$

where $\lambda_{i,[0,n-1]}=\frac{a}{\prod_{0\le k\ne i\le n-1}(t_i-t_k)}$ and $\lambda_{i,[1,n]}=\frac{a}{\prod_{1\le k\ne i\le n}(t_i-t_k)}$. By induction hypothesis, the limit of the RHS of eq. (2.4) as $\delta\to 0$ is

$$\lim_{\delta\to 0}\exp\left(\frac{2a}{(n-1)!}\operatorname{Re}\partial^{n-1}\frac{1}{n\delta}(\log(s-u_{1,0})-\log(s-u_{0,0}))\right)=\exp\left(\frac{2a}{n!}\operatorname{Re}\partial^{n}\log(s-u_0)\right),$$

by using $u_{1,0}=u_{0,0}+\delta u_{1,1}$.

Let us also consider the terms $\frac{t_i-x}{s-u_i}$. Use induction on n again. The base case $n=0$ is clear. The induction step is given by

$$\lim_{\delta\to 0}\frac{1}{(n-1)!}\partial^{n-1}\frac{1}{n\delta}\left(\frac{t_1-x}{s-u_{1,0}}-\frac{t_0-x}{s-u_{0,0}}\right)=\frac{1}{n!}\partial^{n}\left(\frac{t_0-x}{s-u_0}\right),$$

where we used $t_1=t_0+\delta$ and $u_{1,0}=u_{0,0}+\delta u_{1,1}$. □

2.3. **Non-reduced point with parabolic structure.** Write $\mathbb{C}[\varepsilon]=\mathbb{C}[\varepsilon]/(\varepsilon^{n+1})$. As mentioned in the introduction, let us consider a $\mathbb{C}[\varepsilon]$-point t_0 on $\mathbb{P}^1$ with parabolic structure, i.e. there is a chosen rank-1 free $\mathbb{C}[\varepsilon]$-submodule of $\mathbb{C}[\varepsilon]^{\oplus 2}$, the fiber of the quasiparabolic bundle $\mathcal{O}^{\oplus 2}$ above t_0. Generically, say it is the line spanned by $(1,\sum_{k=0}^{n}u_k\varepsilon^k)$.

Let $x\ne t_0$ be a closed point, and s a line in the fiber above x. After Hecke modification at (x,s) (and rewriting in terms of the original parametrization, see [2], sections 3.1, 3.2), the line $(1,\sum_{k=0}^{n}u_k\varepsilon^k)$ becomes

$$\left(\sum_{k=0}^{n}u_k\varepsilon^k-s,t_0-x+\varepsilon\right).$$

This is the same line as $(1,-\sum_{k=0}^{n}\frac{1}{k!}\partial^k(\frac{t_0-x}{s-u_0})\varepsilon^k)$, by part (a) of the following:

Proposition 2.3. *We have the following identities:*

(a) $t_0-x+\varepsilon=\left(\sum_{k=0}^{n}\frac{1}{k!}\partial^k(\frac{t_0-x}{s-u_0})\varepsilon^k\right)\left(s-\sum_{k=0}^{n}u_k\varepsilon^k\right)$;

(b) $\frac{1}{n!}\partial^n(\log(s-u_0))=[\varepsilon^n]\log(s-\sum_{k=0}^{n}u_k\varepsilon^k)$.

Proof. For any $X\in\mathbb{C}(s,x,t_0,u_{i,j})$, consider its Taylor series $T(X)=\sum_{k=0}^{n}\frac{1}{k!}\partial^k(X)\varepsilon^k$. It is easily checked by direct calculation that $T(X_1X_2)=T(X_1)T(X_2)$ and $T(\log(C-X))=\log(T(C-X))$, for any C such that $\partial C=0$. Part (a) is simply

$$T(t_0-x)=T\left(\frac{t_0-x}{s-u_0}\right)T(s-u_0).$$

Part (b) follows from $T(\log(s-u_0))=\log(T(s-u_0))$. □

2.4. **The unitary representation.** Let $H_x=(\prod_{i=0}^{m}|t_i-x|)\,\mathbb{H}_x$. We will now show that $H_x=\int_{\mathbb{C}}U_{s,x}d\nu(s)$, where $U_{s,x}$ are certain unitary operators on $\mathcal{H}=L^2(\mathbb{P}^{m-1}_{\mathbb{C}})$ and ν is some measure on $\mathbb{C}$.

To do this, we first get rid of translation and dilation invariance. Suppose for simplicity $n\le m-2$ so that we do not have to fix the glued point, though in general we still expect the conclusions of this subsection to hold. Set the unglued points $u_{m-1}=t_{m-1}=0$ and $u_m=t_m=1$. A short computation gives that the resulting modified Hecke operator is

$$(\mathbb{H}_x\psi)(u_0,\dots,u_{m-2})=\int_{\mathbb{C}}\psi\left(\frac{s(s-1)}{s-x}\left(\frac{x}{s}+\frac{t_0-x}{s-u_0},\partial\left(\frac{t_0-x}{s-u_0}\right),\dots,\frac{1}{n!}\partial^n\left(\frac{t_0-x}{s-u_0}\right),\frac{x}{s}+\frac{t_{n+1}-x}{s-u_{n+1}},\dots\right)\right)$$
$$\cdot\frac{|s(s-1)|^{m-2}\exp(-\frac{2a}{n!}\operatorname{Re}\partial^n\log(s-u_0))dsd\overline{s}}{|s-x|^m|s-u_0|^{2n+2}\prod_{k=n+1}^{m-2}|s-u_k|^2}.$$

The unitary operators $U_{s,x}$ will be given by the action of a group element

$$g_{s,x}=(g_{s,x,0},g_{s,x,n+1},\dots,g_{s,x,m-2})\in\mathrm{PGL}_2(\mathbb{C}[\varepsilon])\times\mathrm{PGL}_2(\mathbb{C})^{m-n-2}.$$

For the coordinates $u_{n+1},\dots,u_{m-2}$ parametrizing closed points, each individual action of $\mathrm{PGL}_2(\mathbb{C})$ on $L^2(\mathbb{C})$ is just the principal series representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = \frac{|ad-bc|}{|cz+d|^2} f\left(\frac{az+b}{cz+d}\right).$$

We now describe the action of $\mathrm{PGL}_2(\mathbb{C}[\varepsilon])$ for the coordinates $u_0,\dots,u_n$ parametrizing the fiber above the non-reduced point. The group $\mathrm{PGL}_2(\mathbb{C}[\varepsilon])$ acts naturally on $\mathbb{P}^1(\mathbb{C}[\varepsilon])$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}(z) = \frac{az+b}{cz+d}.$$

Suppose we identify $z = u_0 + \cdots + u_n\varepsilon^n$ with $(u_0,\dots,u_n)\in\mathbb{C}^{n+1}$, with the usual measure. Let us define a unitary representation of $\mathrm{PGL}_2(\mathbb{C}[\varepsilon])$ on $L^2(\mathbb{C}^{n+1})$, by

$$gf(z) = f(g^{-1}z)\cdot\left|\frac{\det g_0}{(c_0u_0+d_0)^2}\right|^{n+1}\cdot\exp(-2a\mathrm{Re}\left([\varepsilon^n]\log(cz+d)\right)),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ is the constant part of g.

Then, using proposition 2.3, it is easy to check the following:

Proposition 2.4. *We have $H_x = \int_{\mathbb{C}} U_{s,x}d\nu(s)$, where $U_{s,x}$ is the unitary operator given by the action of the group element $g_{s,x} = (g_{s,x,0}, g_{s,x,n+1},\dots,g_{s,x,m-2})$, where*

$$g_{s,x,0} = \begin{pmatrix} -(s-1)x & (t_0+\varepsilon)s(s-1) \\ -(s-x) & s(s-x) \end{pmatrix},\quad g_{s,x,k} = \begin{pmatrix} -(s-1)x & t_ks(s-1) \\ -(s-x) & s(s-x) \end{pmatrix}$$

for $n+1\le k\le m-2$, and $\nu(s) = |\frac{x(x-1)}{s(s-1)(s-x)}|dsd\overline{s}$. □

2.5. **Boundedness.** Initially, the Hecke operators are only partially defined. Let $V\subset\mathcal{H}$ be the (dense) subset of *continuous* functions ψ, translation-invariant and homogeneous of degree $-m$. Let $U\subset\mathbb{C}^{m+1}$ be the subset of points where no two coordinates are equal to each other.

Proposition 2.5. *For $\psi\in V$, the integral $(\mathbb{H}_x\psi)(u_0,\dots,u_m)$ converges and is continuous on U, and can be extended to an element of $\mathcal{H}$.*

Proof. We have to first show the integral converges, i.e. to check the behavior of the formula in proposition 2.2 at $s = u_0, u_{n+1},\dots,u_m,\infty$. Let us use translation invariance to set the last coordinate $u_m = 0$, and also without loss of generality set $t_m = 0$. We obtain

$$\mathbb{H}_x\psi(u_0,\dots,u_{m-1}) = \int_{\mathbb{C}}\psi\left(\frac{t_0s-xu_0}{s-u_0}, \partial\left(\frac{t_0-x}{s-u_0}\right),\dots,\frac{1}{n!}\partial^n\left(\frac{t_0-x}{s-u_0}\right),\frac{t_{n+1}s-xu_{n+1}}{s-u_{n+1}},\dots\right)$$
$$\cdot\frac{|s|^{m-2}\exp(-\frac{2a}{n!}\mathrm{Re}\,\partial^n\log(s-u_0))dsd\overline{s}}{|s-u_0|^{2n+2}\prod_{k=n+1}^{m-1}|s-u_k|^2}.$$

From this, it is clear that as $s\to\infty$, $\mathbb{H}_x\psi(u_0,\dots,u_{m-1})$ decays as $|s|^{-m-2}$, hence integrable. To check the behavior as $s\to u_0$, we use homogeneity and scale all arguments up by $(s-u_0)^{n+1}$; then there will be an additional $|s-u_0|^{(n+1)m}$ term in the measure, so that as $s\to u_0$ the integral behaves as $|s-u_0|^{(n+1)m-(2n+2)}$ which is also integrable. A similar calculation addresses the behaviors at $s = u_{n+1},\dots,u_m$.

Continuity of $\mathbb{H}_x\psi$ in U follows from continuity of ψ. Finally, $\mathbb{H}_x\psi$ is L^2-integrable by Cauchy-Schwarz and the fact that $\|H_x\|\le\int_{\mathbb{C}}|\frac{x(x-1)}{s(s-1)(s-x)}|dsd\overline{s}<\infty$, which is a consequence of proposition 2.4. □

Proposition 2.6. *The Hecke operators H_x extend to bounded, self-adjoint, mutually commuting operators on $\mathcal{H}$, for $x\neq t_i,\infty$.*

Proof. Boundedness follows from the previous proposition and $\|H_x\|<\infty$.

It is easy to check that $g_{s,x}^{-1} = g_{\sigma(s),x}$, where $\sigma(s) = \frac{x(s-1)}{s-x}$. This implies $U_{s,x}^* = U_{\sigma(s),x}$. Also, the measure $d\nu(s)$ is invariant under the involution $s\mapsto\sigma(s)$. This implies that H_x are self-adjoint.

Let x_1,x_2 be two distinct points distinct from t_i,∞. The fact that operators H_{x_1},H_{x_2} commute is a consequence of the general fact that Hecke modifications at distinct points $(x_1,s_1),(x_2,s_2)$ commute.

Concretely, it can also be checked directly using proposition 2.4; it can be reduced to the routine calculation that $d\nu_{x_1}(s_1)d\nu_{x_2}(s_2') = d\nu_{x_1}(s_1')d\nu_{x_2}(s_2)$, where

$$d\nu_{x_i}(s) = \left|\frac{x_i(x_i-1)}{s(s-1)(s-x_i)}\right| ds d\overline{s}$$

and $s_1' = \frac{s_2-1}{s_2-x_2} \cdot \frac{x_1 s_2 - x_2 s_1}{s_2 - s_1}$ and symmetric for s_2'. Here, s_1' is the coordinate of the parabolic line s_1 after Hecke modification at (x_2, s_2), and vice versa. □

2.6. **Compactness.**

Proposition 2.7. *The Hecke operators H_x are compact and norm-continuous in x, for $x \neq t_i, \infty$.*

Proof. Using proposition 2.4, the exact same argument as in ([2], proposition 3.13) goes through, except that we have to show that the rational map $\phi_N : \mathbb{A}^N_{\mathbb{C}} \mapsto \mathbf{G}_{n,m} = \mathrm{PGL}_2(\mathbb{C}[\varepsilon]/(\varepsilon^{n+1})) \times \mathrm{PGL}_2(\mathbb{C})^{m-n-2}$, given by $(s_1, \dots, s_N) \mapsto g_{s_1,x} \cdots g_{s_N,x}$, where, say, $N = 4m$, satisfies that the preimage of a measure zero set is measure zero. We supply a proof of this below. Denote $G = \mathrm{PGL}_2$.

Step 1: We show that for any $x \neq t \in \mathbb{C}$, the elements

$$g(s) = g_{t,x}(s) = \begin{pmatrix} -(s-1)x & ts(s-1) \\ -(s-x) & s(s-x) \end{pmatrix} \in G(\mathbb{C})$$

generate a dense subgroup of $G(\mathbb{C})$, as s ranges in $\mathbb{C}\backslash\{0,1,x\}$. As $g(s)^{-1} = g(\frac{x(s-1)}{s-x})$, this set is closed under inverses and contains the identity. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ be the Lie algebra of $G(\mathbb{C})$, and let H be the closure of the subgroup that these elements generate. Then H is a Lie group, so that we may consider its Lie algebra $\mathfrak{h}$. It suffices to show that $\mathfrak{h} = \mathfrak{g}$. By definition, $\mathfrak{h}$ contains the elements

$$g(s)^{-1}g'(s) = \frac{1}{s(s-1)(s-x)} \begin{pmatrix} \frac{sx(t-1)}{t-x} - \frac{1}{2}(s^2+x) & \frac{s^2 t(1-x)}{t-x} \\ \frac{x(x-1)}{t-x} & \frac{1}{2}(s^2+x) - \frac{sx(t-1)}{t-x} \end{pmatrix},$$

which linearly spans the 3-dimensional space $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Step 2: Denote $\mathbb{C}[\varepsilon] = \mathbb{C}[\varepsilon]/(\varepsilon^{n+1})$. We show that the elements $g(s) = g_{t+\varepsilon,x}(s)$ generate a dense subgroup of $G(\mathbb{C}[\varepsilon])$. Let H be the closure of the subgroup they generate, and let $\mathfrak{h}$ be its Lie algebra, which lies in $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}[\varepsilon])$. It suffices to show $\mathfrak{h} = \mathfrak{g}$. We know $\mathfrak{h}$ contains the elements $A(s) = s(s-1)(s-x)g(s)^{-1}g'(s) = A_0(s) + A_1(s)\varepsilon + \dots$, where

$$A_0(s) = \begin{pmatrix} \frac{sx(t-1)}{t-x} - \frac{1}{2}(s^2+x) & \frac{s^2 t(1-x)}{t-x} \\ \frac{x(x-1)}{t-x} & \frac{1}{2}(s^2+x) - \frac{sx(t-1)}{t-x} \end{pmatrix}, \quad A_1(s) = \frac{x(1-x)}{(t-x)^2}\begin{pmatrix} s & -s^2 \\ 1 & -s \end{pmatrix}.$$

Let us first produce an element $X \in \mathfrak{h}$ whose constant term is 0. Suppose we write $A_0(s) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then

$$A_1(s) = \begin{pmatrix} \frac{1-x}{(t-x)(t-1)}a + \frac{1}{2t(t-1)}b + \frac{1}{2(1-t)}c & -\frac{x}{t(t-x)}b \\ -\frac{1}{t-x}c & -(\frac{1-x}{(t-x)(t-1)}a + \frac{1}{2t(t-1)}b + \frac{1}{2(1-t)}c) \end{pmatrix}.$$

It is not hard to verify that the commutator $[A(s_1+1) - A(s_1), A(s_2+1) - A(s_2)]$ is given by

$$\frac{4(s_1-s_2)t(t-1)x(x-1)}{(t-x)^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{4(s_1-s_2)x(x-1)(2tx-t-x)}{(x-t)^3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\varepsilon + \dots$$

which is linearly independent from the elements of the above form. Thus some linear combination of them would give $X \in \mathfrak{h}$ whose constant term is 0 and ε term is nonzero.

Now that we have found one element $X \in \mathfrak{h}$, $X = B_1\varepsilon + \dots$ with $B_1 \neq 0$, consider its commutator with all the elements $A(s) = A_0 + A_1\varepsilon + \dots$. Since $[A(s), X] = [A_0, B_1]\varepsilon + \dots$, and $\mathfrak{sl}_2(\mathbb{C})$ is simple, by Step 1, we may now generate all elements of form $B_1\varepsilon + \dots$, where $B_1 \in \mathfrak{sl}_2(\mathbb{C})$. Now, taking commutators once again, we can generate all elements of form $B_2\varepsilon^2 + \dots$, where $B_2 \in \mathfrak{sl}_2(\mathbb{C})$, and so on. Thus we have shown that $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}[\varepsilon]) = \mathfrak{g}$ as desired.

Step 3: We show that the elements $g_{s,x} = (g_{t_0+\varepsilon,x}(s), g_{t_{n+1},x}(s), \dots, g_{t_{m-2},x}(s))$, where $s \in \mathbb{C}\backslash\{0,1,x\}$, generate a dense subgroup of $\mathbf{G}_{n,m} = G(\mathbb{C}[\varepsilon]) \times G(\mathbb{C})^{m-n-2}$. Use induction on $m-n$. The induction basis $m = n+2$ is already shown in Step 2. For the induction step, let H be the closure of the subgroup that $g_{s,x}$ generate. By induction hypothesis, H surjects onto $\mathbf{G}_{n,m-1} = G(\mathbb{C}[\varepsilon]) \times G(\mathbb{C})^{m-n-3}$ (the first $m-n$

factors) and $G(\mathbb{C})$ (the last factor). By lemma 2.8 below, it follows that $H \subset \mathbf{G}_{n,m-1} \times G(\mathbb{C})$ is the preimage of the graph of some smooth map $f : \mathbf{G}_{n,m-1} \to G(\mathbb{C})/L$, where $L \lhd G(\mathbb{C})$ is the kernel of $H \to \mathbf{G}_{n,m-1}$. Since $G(\mathbb{C})$ is simple, $L = 1$ or $L = G(\mathbb{C})$. In the latter case $H = \mathbf{G}_{n,m}$ and we are done; in the former case, since the center of $G(\mathbb{C})$ is trivial, f is given by projection of $\mathbf{G}_{n,m-1}$ onto one of its factors, composed with a map to $G(\mathbb{C})$. Such a map would send $g_{t_i,x}(s)$ (for some $n+1 \le i \le m-2$) or $g_{t_0+\varepsilon,x}(s)$ to $g_{t_{m-2},x}(s)$. If this is a map $G(\mathbb{C}) \to G(\mathbb{C})$, then it must be an automorphism, hence given by conjugation. But since the points $t_0, t_{n+1}, \ldots, t_{m-2}$ are all distinct, the numbers $\frac{\mathrm{tr}^2}{\det}$ of $g_{t_i,x}(s)$ are all distinct, which is a contradiction. If this is a map $G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C})$, by lemma 2.9 below, we get a contradiction for the same reason.

Step 4: We show the rational map $\phi_N : \mathbb{A}^N_{\mathbb{C}} \to \mathbf{G}_{n,m}$ is dominant, where $N = 4m$. Define ϕ_k similarly. Let U_k be the Zariski-closure of the image of ϕ_k, then it is a closed irreducible set in $\mathbf{G}_{n,m}$ of dimension at most k. Since $g_{s,x}g_{\sigma(s),x} = 1$, where $\sigma(s) = \frac{x(s-1)}{s-x}$, we have a chain $U_0 \subset U_2 \subset U_4 \subset \ldots$, and let $2k$ be the smallest index such that $U_{2k} = U_{2k+2}$. Then $U_{2k} = U_{2k+2} = \ldots$, so $U_{2k} \supset H$, so by Step 3, $U_{2k} = \mathbf{G}_{n,m}$. Since $G_{n,m}$ has dimension $3(m-1) < 4m$, $U_{4m} = \mathbf{G}_{n,m}$ as desired. □

In the proof above, we made use of the following two lemmas about Lie groups:

Lemma 2.8. *Let G_1, G_2 be Lie groups, and K a closed subgroup of $G_1 \times G_2$ that surjects onto both G_1 and G_2. Let $L \subset G_2$ be the kernel of $K \hookrightarrow G_1 \times G_2 \to G_1$. Then $L \lhd G_2$, and K is the preimage in $G_1 \times G_2$ of the graph of a smooth homomorphism $f : G_1 \to G_2/L$.*

Proof. In the case $L = 1$, $K \to G_1$ is an isomorphism, so f is given by its inverse composed with the map $K \hookrightarrow G_1 \times G_2 \to G_2$. In general, suppose $(1, \ell) \in L \subset K$. For any $g_2 \in G_2$, there exists $(g_1, g_2) \in K$, so $(g_1, g_2)^{-1}(1, \ell)(g_1, g_2) = (1, g_2^{-1}\ell g_2) \in K$, so L is normal. Let K' be the image of K in $G_1 \times (G_2/L)$. Then we may apply the $L = 1$ case to K', and K is the preimage in $G_1 \times G_2$ of the graph of a $f : G_1 \to G_2/L$. □

Lemma 2.9. *The surjective Lie group homomorphisms $f : G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C})$ are all of the form $\psi \circ \pi$, where $\pi : G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C})$ is projection to constant term, and $\psi \in \mathrm{Aut}(G(\mathbb{C}))$.*

Proof. Pass to Lie algebra homomorphism $df : \mathfrak{sl}_2(\mathbb{C}[\varepsilon]) \to \mathfrak{sl}_2(\mathbb{C})$. The restriction of df on $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{sl}_2(\mathbb{C}[\varepsilon])$ is an inner automorphism, since $\mathfrak{sl}_2(\mathbb{C})$ is simple and df is surjective. Since every element in $\mathfrak{sl}_2(\mathbb{C}[\varepsilon])$ with zero constant term is ad-nilpotent, we conclude that they lie in the kernel of df. So f is an automorphism precomposed with projection as well. □

2.7. **Spectral decomposition.** By the spectral theorem for commuting compact self-adjoint operators, we conclude the following.

Corollary 2.10. *There is an orthogonal decomposition $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, where $\mathcal{H}_k$ are finite dimensional joint eigenspaces: for any $\psi_k \in \mathcal{H}_k$, $H_x\psi_k = \beta_k(x)\psi_k$ where $\beta_k(x)$ are real-valued and continuous in x.*

Proof. We expect the operators $\frac{H_x}{2|x|\log|x|}$ to strongly converge to the identity, as $x \to \infty$; consequently the operators H_x have trivial common kernel, so all $\mathcal{H}_k$ are finite dimensional. Continuity of $\beta_k(x)$ follows from norm-continuity of H_x. □

3. Limits of differential operators

3.1. **Twisted Gaudin operators.** In ([2], section 4), a system of commuting, second-order differential operators G_i on variables $y_0, \ldots, y_m$, the *Gaudin operators*, were considered, which act on $\mathcal{H}$ as well. For us, we will use a twisted version, given by

$$\widehat{G}_i = \sum_{0 \le j \ne i \le m} \frac{1}{t_i - t_j} \left(-(y_i - y_j)^2 \partial_i \partial_j + (y_i - y_j)((1+\lambda_j)\partial_i - (1+\lambda_i)\partial_j)\right), \tag{3.1}$$

where $\lambda_0, \ldots, \lambda_m$ are twisting parameters. The usual G_i are given by setting twisting parameters to 0 and adding the constant $\frac{1}{2}\sum_{j\ne i} \frac{1}{t_i - t_j}$.

3.2. **Differential equation for Hecke operators.** Let us again glue $t_0, \dots, t_n$, among the $m+1$ points $t_0, \dots, t_m$. We wish to take the limit of the twisted Gaudin operators eq. (3.1), and show that they satisfy a differential equation together with limits of Hecke operators.

Take the same twisting parameters (eq. (2.2)), and reparametrize $y_0, \dots, y_m$ with $u_0, \dots, u_m$ in the same way (eq. (2.3)), as we take the same limiting procedure $\delta \to 0$.

Theorem 3.1. *For any $\psi = \psi(u_0, \dots, u_m) \in \mathcal{H}$, smooth on U with compact support modulo translation and dilation, the map $x \mapsto \mathbb{H}_x\psi$ is smooth for $x \neq t_i, \infty$, and satisfies*

$$\left(\partial_x^2 + \left(\frac{n+1}{x-t_0} + \frac{a}{(x-t_0)^{n+1}} + \frac{1}{x-t_{n+1}} + \cdots + \frac{1}{x-t_m}\right)\partial_x\right)\mathbb{H}_x\psi = \mathbb{H}_x(\widehat{G}\psi), \tag{3.2}$$

where $\widehat{G}$ is the limit of $\sum_{i=0}^m \frac{\widehat{G}_i}{x-t_i}$ as $\delta \to 0$.

At this point, the proof of theorem 3.1 is only on the formal (algebraic) level, i.e. the two sides are equal when when viewed as elements of the dual space. But we expect this issue to be solved by some analytic tricks.

Example 3.2. Consider the simplest nontrivial case, which is $n = 1$, $m = 2$. In this case the right hand side of eq. (3.2) is

$$\widehat{G} = \frac{\widehat{G}_{00}}{(x-t_0)^2} + \frac{\widehat{G}_{02}}{(x-t_0)(x-t_2)} + \frac{\widehat{G}_{002}}{(x-t_0)^2(x-t_2)},$$

where

$$\begin{cases} \widehat{G}_{00} = u_1^2\partial_1^2 + u_1(2\partial_1 - a\partial_0) \\ \widehat{G}_{02} = -(u_2-u_0)^2\partial_0\partial_2 + u_1(u_2-u_0)\partial_1\partial_2 + u_1(\partial_1 - a\partial_2) - (u_2-u_0)(\partial_0 - 2\partial_2) \\ \widehat{G}_{002} = -(u_2-u_0)^2\partial_1\partial_2 + (u_2-u_0)(a\partial_2 - \partial_1). \end{cases}$$

We are still attempting to find a simple, closed-form formula for $\widehat{G}$.

3.3. **Proof of theorem 3.1.**

3.3.1. *Preparations.* We first recall a combinatorial identity used in the proof of theorem 3.1.

Lemma 3.3. *Let d be a nonnegative integer. Let $p, q_1, \dots, q_d, a$ be nonnegative integers. Then*

$$\sum_{i\in\mathbb{Z}}(-1)^i i^a\binom{p}{i}\binom{i}{q_1}\cdots\binom{i}{q_d} = \begin{cases} 0, & \text{if } a < p - q_1 - \cdots - q_d; \\ (-1)^p\frac{p!}{q_1!\dots q_d!} & \text{if } a = p - q_1 - \cdots - q_d. \end{cases}$$

Proof. Consider the generating function

$$\begin{aligned} F(X, Y_1, \dots, Y_d) &= \sum_{p,q_1,\dots,q_d\in\mathbb{Z}_{\geq 0}}\left(\sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i i^a\binom{p}{i}\binom{i}{q_1}\cdots\binom{i}{q_d}\right)X^pY_1^{q_1}\dots Y_d^{q_d} \\ &= \sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i i^a\left(\sum_{p\in\mathbb{Z}_{\geq 0}}\binom{p}{i}X^p\right)\prod_{j=1}^d\left(\sum_{q_j\in\mathbb{Z}_{\geq 0}}\binom{i}{q_j}Y_j^{q_j}\right) \\ &= \sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i i^a\frac{X^i}{(1-X)^{i+1}}\prod_{j=1}^d(1+Y_j)^i \\ &= \frac{1}{1-X}\sum_{i\in\mathbb{Z}_{\geq 0}} i^a\left(\frac{X(1+Y_1)\dots(1+Y_d)}{X-1}\right)^i. \end{aligned}$$

It is well-known that $(\sum_{i\geq 0} i^aT^i)(1-T)^{a+1} = \sum_{i=0}^a i!\left\{{a \atop i}\right\}T^i(1-T)^{a-i}$ is a polynomial in T of degree a, where $\left\{{a \atop i}\right\}$ are Stirling numbers of the second kind. Here, let us take $T = \frac{X(1+Y_1)\dots(1+Y_d)}{X-1}$. Then if we define the polynomial

$$P = \sum_{i=0}^a i!\left\{{a \atop i}\right\}(-1)^i(X(1+Y_1)\dots(1+Y_d))^i(X(1+Y_1)\dots(1+Y_d) - X + 1)^{a-i},$$

we then have

$$F = P\cdot(1-(X(1+Y_1)\dots(1+Y_d)-X)+(X(1+Y_1)\dots(1+Y_d)-X)^2+\dots)^{a+1}. \tag{3.3}$$

We want to extract the coefficient of $X^pY_1^{q_1}\dots Y_d^{q_d}$. Let us view the variable X as in degree 1, and each Y_i as in degree -1. Then it is clear that the expression on the right hand side has degree at most a. So, if $a<p-q_1-\dots-q_d$, we have $[X^pY_1^{q_1}\dots Y_d^{q_d}]F=0$.

Let us now compute the coefficient when $a=p-q_1-\dots-q_d$. Since the latter expression in eq. (3.3) has negative degree, we have to take the leading term $(-1)^a a!X^a$ in P. In the rest, we need $[Y_1^{q_1}\dots Y_d^{q_d}]$ in

$$\frac{1}{(1+Y_1+\dots+Y_d)^{a+1}}=\sum_i\binom{a+i}{a}(-1)^i(Y_1+\dots+Y_d)^i,$$

which is $(-1)^{p-a}\binom{p}{a}\binom{p-a}{q_1,\dots,q_d}$. So, combined together, the desired coefficient is $(-1)^p\frac{p!}{q_1!\dots q_d!}$. □

The idea of the proof of theorem 3.1 is to first use an integration-by-parts formula, then directly compare the coefficients of each term on both sides. For simplicity in the proof, we introduce the following notations.

Definition 3.4. Let

$$d\mu(s)=\frac{\exp(-\frac{2a}{n!}\mathrm{Re}\,\partial^n\log(s-u_0))dsd\overline{s}}{|s-u_0|^{2(n+1)}|s-u_{n+1}|^2\cdots|s-u_m|^2},$$

$$c_k=\begin{cases}\frac{1}{k!}\partial^k(\frac{-1}{s-u_0}) & \text{if } 0\le k\le n\\ -\frac{1}{s-u_k} & \text{if } n+1\le k\le m,\end{cases}$$

and let

$$v_k=\begin{cases}\frac{1}{k!}\partial^k(\frac{t_0-x}{s-u_0}) & \text{if } 0\le k\le n\\ \frac{t_k-x}{s-u_k} & \text{if } n+1\le k\le m\end{cases}$$

be the variables after coordinate change.

Then $c_k=\partial_x v_k$, and they satisfy

$$c_k=\frac{v_k}{x-t_0}+\dots+\frac{v_0}{(x-t_0)^{k+1}},\quad v_k=(x-t_0)c_k-c_{k-1}$$

for $0\le k\le n$, and $c_k=\frac{v_k}{x-t_k}$ otherwise. Then the (modified) Hecke operators are given by

$$(\mathbb{H}_x\psi)(u_0,\dots,u_m)=\int_{\mathbb{C}}\psi(v_0,\dots,v_m)d\mu(s).$$

We have the integration-by-parts formula:

$$-\int_{\mathbb{C}}\partial_s\psi d\mu(s)=\int_{\mathbb{C}}((n+1)c_0+ac_n+c_{n+1}+\dots+c_m)\psi d\mu(s), \tag{3.4}$$

where the ac_n term comes from the fact that $\frac{\partial}{\partial s}\mathrm{Re}\,\partial^n(\log(s-u_0))=\frac12\partial^n(\frac{1}{s-u_0})$, from Cauchy-Riemann equations.

Now assume $\psi=\psi(v_0,\dots,v_m)$. Then

$$\begin{aligned}-\frac{\partial}{\partial s}\psi&=\left(\frac{\partial}{\partial u_0}+\frac{\partial}{\partial u_{n+1}}+\dots+\frac{\partial}{\partial u_m}\right)\psi\\&=\sum_{k=0}^n\frac{1}{k!}\partial^k\left(\frac{t_0-x}{(s-u_0)^2}\right)\psi_k+\sum_{k=n+1}^m\frac{t_k-x}{(s-u_k)^2}\psi_k.\end{aligned}$$

We have $\frac{1}{k!}\partial^k\frac{1}{(s-u_0)^2}=\sum_{i=0}^k c_ic_{k-i}$, and therefore

$$\frac{1}{k!}\partial^k\frac{t_0-x}{(s-u_0)^2}=(t_0-x)\sum_{i=0}^k c_ic_{k-i}+\sum_{i=0}^{k-1}c_ic_{k-1-i}.$$

So,

$$-\frac{\partial}{\partial s}\psi=\sum_{k=0}^n\left((t_0-x)\sum_{i=0}^k c_ic_{k-i}+\sum_{i=0}^{k-1}c_ic_{k-1-i}\right)\psi_k+\sum_{k=n+1}^m(t_k-x)c_k^2\psi_k. \tag{3.5}$$

Putting eq. (3.4) and eq. (3.5) together gives the integration-by-parts formula which we use in the proof.

3.3.2. *The quadratic part.* Let $\psi(u_0,\dots,u_m)$ be smooth with compact support in U. Let ψ_i,ψ_{ij} be the first and second derivatives of ψ, evaluated at the new variables v_i. The following calculations are purely formal (since we don't know $\mathbb{H}_x\psi$ is differentiable in x yet).

Let us expand the left hand side of eq. (3.2). From here on, everything is inside $\int_{\mathbb{C}} \bullet d\mu(s)$. First, because

$$\partial_x^2\psi(u_0,\dots,u_m) = \int_{\mathbb{C}} \sum_{0\le i,j\le m} c_ic_j\psi_{ij}d\mu(s),$$

we put in c_pc_q for the coefficient of ψ_{pq}. Then add the contribution of the integration-by-parts formula, applied to each ψ_i in

$$((n+1)c_0 + ac_n + c_{n+1} + \cdots + c_m)\left(\frac{\psi_0}{x-t_0} + \cdots + \frac{\psi_n}{(x-t_0)^{n+1}} + \sum_{k=n+1}^{m}\frac{\psi_k}{x-t_k}\right).$$

This gives that the coefficient of ψ_{pq} (we temporarily view ψ_{pq} and ψ_{qp} as distinct) is

$$\begin{cases} c_pc_q + \frac{1}{(x-t_0)^{q+1}}\left(\sum_{i=0}^{p-1}c_ic_{p-1-i} + (t_0-x)\sum_{i=0}^{p}c_ic_{p-i}\right) & \text{if } 0\le p,q\le n\\ c_pc_q + \frac{1}{x-t_q}\left(\sum_{i=0}^{p-1}c_ic_{p-1-i} + (t_0-x)\sum_{i=0}^{p}c_ic_{p-i}\right) & \text{if } 0\le p\le n<q\\ c_pc_q + \frac{1}{(x-t_0)^{q+1}}(t_p-x)c_p^2 & \text{if } 0\le q\le n<p\\ c_pc_q + \frac{1}{x-t_q}(t_p-x)c_p^2 & \text{if } n+1\le p,q\le m. \end{cases}$$

Now, let us express c_i in terms of v_i. We have, for $0\le p\le n$,

$$\sum_{i=0}^{p-1}c_ic_{p-1-i} + (t_0-x)\sum_{i=0}^{p}c_ic_{p-i} = -\sum_{i=0}^{p}c_iv_{p-i} = -\sum_{i+j\le p}\frac{v_iv_j}{(x-t_0)^{p-i-j+1}}.$$

The coefficient of $v_kv_\ell\psi_{pq}$ (as above, treat v_kv_ℓ and $v_\ell v_k$ differently) is (here 1_P is the indicator function):

$$\begin{cases} \frac{1}{(x-t_0)^{p+q-k-\ell+2}}(1_{(k\le p)\wedge(\ell\le q)} - 1_{(k+\ell\le p)}) & \text{if } 0\le p\le q\le n\\ \frac{1}{(x-t_q)(x-t_0)^{p+1}}(1_{(k\le p)\wedge(\ell=q)}(x-t_0)^k - 1_{(k+\ell\le p)}(x-t_0)^{k+\ell}) & \text{if } 0\le p\le n<q\\ \frac{1}{(x-t_p)(x-t_0)^{q+1}}(1_{(k=p)\wedge(\ell\le q)}(x-t_0)^\ell - 1_{(k=\ell=p)}) & \text{if } 0\le q\le n<p\\ \frac{1}{(x-t_p)(x-t_q)}(1_{(k=p)\wedge(\ell=q)} - 1_{(k=\ell=p)}) & \text{if } n+1\le p\le q\le m. \end{cases}$$

For comparison, let us take the limit of the right hand side of eq. (3.2),

$$\sum_{0\le i\le m}\frac{\widehat{G}_i}{x-t_i} = \frac{1}{2}\sum_{i\ne j}\frac{D_{ij}}{t_i-t_j}\left(\frac{1}{x-t_i} - \frac{1}{x-t_j}\right).$$

As a reminder, the u_i are related to y_i by

$$y_i = \sum_{0\le j\le i}u_j\prod_{0\le k\le j-1}(t_i-t_k), \quad \frac{\partial}{\partial y_i} = \sum_{i\le j\le n}\frac{\partial}{\partial u_j}\prod_{0\le k\le j, k\ne i}\frac{1}{t_i-t_k}$$

for $0\le i\le n$, and $u_i = y_i$ for $n+1\le i\le m$.

Consider the quadratic term $\sum_{i\ne j}\frac{(y_i-y_j)^2\partial_i\partial_j}{t_i-t_j}(\frac{1}{x-t_i} - \frac{1}{x-t_j})$. Suppose we sum over $0\le i\ne j\le n$ for now. In the limit, taking $\varepsilon = t_i - t_{i-1}$, the coefficient of $u_ku_\ell\partial_{u_p}\partial_{u_q}$ (in that order) is

$$\delta^{k+\ell-p-q-1}\frac{k!\ell!}{p!q!}\sum_{i\ne j}\frac{(-1)^{p+q+i+j}\binom{p}{i}\binom{q}{j}}{i-j}\left(\frac{1}{x-t_0-i\varepsilon} - \frac{1}{x-t_0-j\varepsilon}\right)\left(\binom{i}{k}\binom{i}{\ell} + \binom{j}{k}\binom{j}{\ell} - 2\binom{i}{k}\binom{j}{\ell}\right).$$

Use the expansion

$$\frac{1}{x-t_0-i\delta} = \frac{1}{x-t_0}\left(1 + \frac{i\delta}{x-t_0} + \cdots + \left(\frac{i\delta}{x-t_0}\right)^{p+q-k-\ell+1}\right) + (\text{multiple of } \delta^{p+q-k-\ell+2}),$$

it suffices to show the following combinatorial identity: for $r = 1, \dots, p+q-k-\ell$,

$$\sum_{i,j}(-1)^{i+j}(i^{r-1}+i^{r-2}j+\cdots+j^{r-1})\binom{p}{i}\binom{q}{j}\left(\binom{i}{k}-\binom{j}{k}\right)\left(\binom{i}{\ell}-\binom{j}{\ell}\right)=0,$$

and to evaluate this at $r = p+q-k-\ell+1$. In fact, every term is zero: for $a+b \le p+q-k-\ell-1$,

$$\sum_{i,j}(-1)^{i+j}i^a j^b\binom{p}{i}\binom{q}{j}\binom{i}{k}\binom{i}{\ell}=0.$$

Since $a+b \le p+q-k-\ell-1$, either $a \le p-k-\ell-1$ or $b \le q-1$. This then follows from lemma 3.3. The result should be

$$[u_k u_\ell \partial_p \partial_q] = -\frac{1}{2}(1_{(k+\ell\le p)}+1_{(k+\ell\le q)}-1_{(k\le p)\wedge(\ell\le q)}-1_{(k\le q)\wedge(\ell\le p)})$$

which, when we allow switching k, ℓ and p, q, is exactly the same as the left hand side.

The cases where one or two of i, j are larger than n do not involve any new ideas. This shows that the quadratic parts of both sides of eq. (3.2) are indeed equal.

3.3.3. *The linear part.* In the remaining, "linear" part of the left hand side, the coefficient of ψ_j is $(\frac{n+1}{x-t_0}+\frac{a}{(x-t_0)^{n+1}}+\sum_{k=n+1}^m\frac{1}{x-t_k})c_j$ minus

$$\begin{cases}\frac{1}{(x-t_0)^{j+1}}((n+1)c_0+ac_n+\sum_{k=n+1}^m c_k) & \text{if } 0\le j\le n\\ \frac{1}{x-t_j}((n+1)c_0+ac_n+\sum_{k=n+1}^m c_k) & \text{if } n+1\le j\le m.\end{cases}$$

The coefficient of $v_i\psi_j$ is then

$$\begin{cases}(\frac{n+1}{x-t_0}+\frac{a}{(x-t_0)^{n+1}}+\sum_{k=n+1}^m\frac{1}{x-t_k})\frac{1_{(i\le j)}}{(x-t_0)^{j-i+1}}-\frac{1}{(x-t_0)^{j+1}}(\frac{(n+1)1_{(i=0)}}{x-t_0}+\frac{a\cdot 1_{(i\le n)}}{(x-t_0)^{n-i+1}}+\frac{1_{(i\ge n+1)}}{x-t_i}) & \text{if } 0\le j\le n\\ (\frac{n+1}{x-t_0}+\frac{a}{(x-t_0)^{n+1}}+\sum_{k=n+1}^m\frac{1}{x-t_k})\frac{1_{i=j}}{x-t_j}-\frac{1}{x-t_j}(\frac{(n+1)1_{(i=0)}}{x-t_0}+\frac{a\cdot 1_{(i\le n)}}{(x-t_0)^{n-i+1}}+\frac{1_{(i\ge n+1)}}{x-t_i}) & \text{if } n+1\le j\le m.\end{cases}$$

Let's consider the linear part of the right hand side, $\sum_{i\ne j}\frac{(y_i-y_j)((1+\lambda_j)\partial_i-(1+\lambda_i)\partial_j)}{t_i-t_j}(\frac{1}{x-t_i}-\frac{1}{x-t_j})$. The coefficient of $u_k\partial_{u_\ell}$, from the contribution of $0\le i,j\le n$, is

$$\delta^{k-\ell-1}(-1)^\ell\frac{k!}{\ell!}\sum_{i\ne j}\frac{(-1)^{i+j}(\binom{i}{k}-\binom{j}{k})}{(x-t_0-i\varepsilon)(i-j)}\left(\binom{\ell}{i}\left((-1)^j+\frac{a}{n!}\binom{n}{j}(-1)^n\right)-\binom{\ell}{j}\left((-1)^i+\frac{a}{n!}\binom{n}{i}(-1)^n\right)\right).$$

So we have to show

$$\sum_{0\le i,j\le n}\frac{(i^r-j^r)(-1)^{i+j}(\binom{i}{k}-\binom{j}{k})}{i-j}\left(\binom{\ell}{i}\left((-1)^j+\frac{a}{n!}\binom{n}{j}(-1)^n\right)-\binom{\ell}{j}\left((-1)^i+\frac{a}{n!}\binom{n}{i}(-1)^n\right)\right)=0$$

for $r\le \ell-k$, and evaluate this at $r=\ell-k+1$. Applying lemma 3.3, the result is $\frac{2(n+1)}{(x-t_0)^{\ell-k+2}}$ when $k\ne 0$, and 0 when $k=0$. If we count in the contributions of $0\le i\le n<j$ or $0\le j\le n<i$, we get exactly the same answer as the above formula for the left hand side. Similar routine calculations show that the linear parts of eq. (3.2) are equal; this completes the proof.

Acknowledgments. This research is part of the SPUR summer program. The author is grateful to his mentor, Daniil Klyuev, for assisting him and providing many helpful insights. The author also thanks Prof. Pavel Etingof for suggesting this project, Prof. David Jerison for organizing the SPUR program, and Prof. Etingof and Prof. Jerison for many discussions and helpful suggestions.

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