SIZE OF THE LARGEST SUM-FREE SUBSET OF $[n]^3$

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Abstract. We prove a conjecture of Cameron and Aydinian [2] in three dimensions, showing the density of the largest sum-free subset of $[n]^3$ is $(10+\sqrt{15})/20$ as $n \to \infty$. We also give an alternate proof of the result in two dimensions, which was first shown by Elsholtz and Rackham [3].

1. Introduction

1.1. **History.** A set $S \subset \mathbb{Z}^d$ is *sum-free* if there are no solutions to the equation $x + y = z$ with $x, y, z \in S$. Let $[n] := \{1, \dots, n\}$, and let $S_d^{(n)}$ be the largest sum-free subset of $[n]^d$. We would like to determine its density asymptoticall, defined as the following:

$$c_d := \limsup_{n\to\infty} \frac{\left|S_d^{(n)}\right|}{n^d}.$$

This question for $d = 2$ was originally posed by Aydinian [2, Problem 10] at the 19th British Combinatorial Conference. In an unpublished note [1], Cameron asked the same question for a general d. It was also presented in a collection of open problems maintained by Green [4, Problem 6].

Observe that $\{v \in [n]^d : \mathbf{1}^\intercal v \in [tn, 2tn)\}$ is a sum-free subset of $[n]^d$ for any $t \geq 0$, where $\mathbf{1}$ denotes the all-ones vector in $\mathbb{R}^d$. For each d, we can optimize t to get a lower bound on c_d. It is conjectured that these lower bounds are sharp, i.e. $S_d^{(n)}$ is of this form.

For $d = 1$, this construction is optimized at $t = 1/2$, showing $c_1 \geq 1/2$. The matching upper bound follows a simple argument involving pairing elements that sums to the maximum of the set.

For $d = 2$, the construction is optimized at $t = 4/5$, showing $c_2 \geq 3/5$. The matching upper bound was recently given by Elsholtz and Rackham [3]. The proof involves doing case work on the shape of the upper convex hull of a sum-free set. Liu, Wang, Wilkes, and Yang [5] showed stability of this optimum, i.e. any near-maximal sum-free subset of $[n]^2$ must be close to this construction.

For $d = 3$, the construction is optimized at the following, showing $c_3 \geq (10+\sqrt{15})/20 = 0.693\dots$

$$S_3^* := \{v \in [n]^3 : \mathbf{1}^\intercal v \in [un, 2un)\} \quad \text{where} \quad u := \frac{15-\sqrt{15}}{10} = 1.112\dots. \tag{1.1}$$

The best upper bound $c_3 \leq 2/e = 0.735\dots$ is due to Cameron [1]. The argument involves pairing points that sum up to the point in the set with the largest product of coordinates. To obtain a sharp upper bound, a natural approach is to adapt the proof of the $d = 2$ case in [3]. However, this seems intractable, as the geometry of the upper convex hull becomes too complex in three dimensions.

1.2. **Main Results.** In this paper, we prove the conjecture for $d = 3$, showing $c_3 = (10+\sqrt{15})/20$.

Theorem 1.1. *The size of the largest sum-free set $S_3^{(n)} \subset [n]^3$ satisfies*

$$\left|S_3^{(n)}\right| = \left(\frac{10+\sqrt{15}}{20}\right)n^3 + O\left(n^{5/2}\right).$$

We also reprove the result for $d = 2$ given by [3], showing $c_2 = 3/5$.

Theorem 1.2. *The size of the largest sum-free set $S_2^{(n)} \subset [n]^2$ satisfies*

$$\left|S_2^{(n)}\right| = \frac{3}{5}n^2 + O(n).$$

1.3. **Proof Sketch.** We now sketch the proof of the upper bound of Theorem 1.1. For the purposes of this sketch, all inequalities and equalities hold up to lower order terms, which we omit.

The main idea is to average one dimensional inequalities on triples of lines with suitable weights to get a sharp upper bound. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection that forgets the last coordinate.

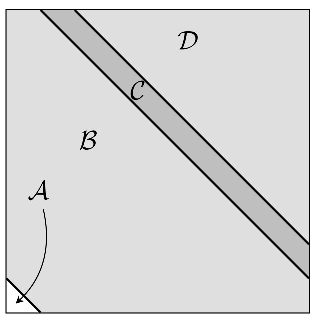
For any sum-free set $S \subset [n]^3$ and $v \in [n]^2$, let $\lambda(v) = |S \cap \pi^{-1}(v)|$ be the number of points of S lying on the segment with base v. We show in Lemma 2.1 that for any $x, y, z \in [n]^2$ with $x + y = z$,

$$\lambda(x) + \lambda(y) + \lambda(z) \leq 2n. \tag{1.2}$$

In addition, we have trivial constraints $\lambda(v) \leq n$ for all $v \in [n]^2$. Maximizing $|S|$ corresponds to maximizing $\sum_{v \in [n]^2} \lambda(v)$. The key point is that the linear program relaxation suffices to show the sharp upper bound. By duality, we want some weight function w on these constraints such that

(1) the constraints in the support of w attain equality when $S = S_3^*$ is conjectured optimizer,
(2) for each $v \in [n]^2$, the sum of the weights of all the constraints that contain v is roughly 1.

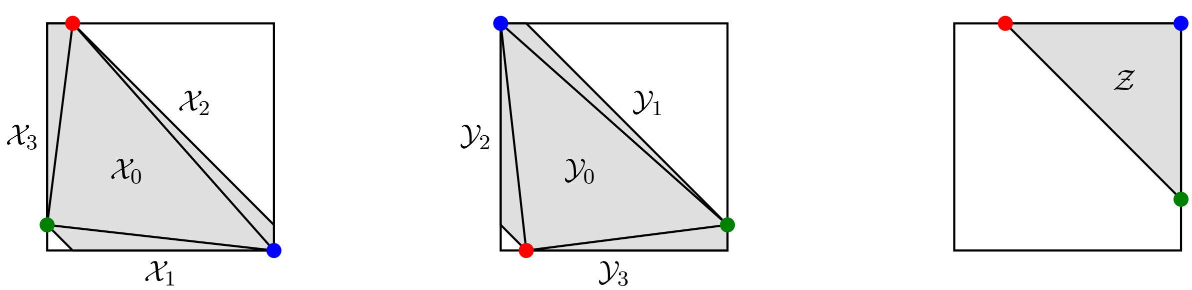
If we look at the conjectured optimizer S_3^* under the projection π, the following four regions appear:



$$\begin{aligned}
\mathcal{A} &= \{v \in [n]^2 : \mathbf{1}^\intercal v \in [0, (u-1)n)\}, \\
\mathcal{B} &= \{v \in [n]^2 : \mathbf{1}^\intercal v \in [(u-1)n, un)\}, \\
\mathcal{C} &= \{v \in [n]^2 : \mathbf{1}^\intercal v \in [un, (2u-1)n)\}, \\
\mathcal{D} &= \{v \in [n]^2 : \mathbf{1}^\intercal v \in [(2u-1)n, 2n]\}.
\end{aligned} \tag{1.3}$$

We can see that the set S_3^* contains none of $\pi^{-1}(\mathcal{A})$, all of $\pi^{-1}(\mathcal{C})$, and a $2/3$-fraction of $\pi^{-1}(\mathcal{B} \cup \mathcal{D})$. In Lemma 2.3, we check that S_3^* attains equality in (1.2) when $(x, y, z) \in (\mathcal{A} \times \mathcal{C} \times \mathcal{C}) \cup (\mathcal{B} \times \mathcal{B} \times \mathcal{D})$ and $x + y = z$. This motivates us to construct the weight function w as a sum of two weight functions, corresponding to this partition. Here, we will roughly explain the more challenging construction on $\{(x, y, z) \in \mathcal{B} \times \mathcal{B} \times \mathcal{D} : x + y = z\}$, as it is sharper and explains how the optimizer u arises.

Let $\mathcal{Z}$ be the triangular region in $[0, n]^2$ defined by $\mathcal{D}$. Partition the hexagonal region in $[0, n]^2$ defined by $\mathcal{B}$ into triangles in two ways, namely $\bigcup_{i=0}^3 \mathcal{X}_i$ and $\bigcup_{i=0}^3 \mathcal{Y}_i$, as shown in the figure below.



We see that the vertices of $\mathcal{X}_i$ and $\mathcal{Y}_i$ can be matched together so that they add up to the vertices of $\mathcal{Z}$. In the figure, we color-coded this for $\mathcal{X}_0$ and $\mathcal{Y}_0$. Once we match the vertices, we can match the entire triangles by taking convex combinations of the vertex maps, e.g. barycentric coordinates. More precisely, in Claim 2.6, we will construct invertible affine transformations $X_i : \mathcal{Z} \to \mathcal{X}_i$ and $Y_i : \mathcal{Z} \to \mathcal{Y}_i$ such that $X_i(z) + Y_i(z) = z$ for all $z \in \mathcal{Z}$. Ignoring integrality issues, we set weight

$$w(X_i(z), Y_i(z), z) = \frac{\text{Area}(\mathcal{X}_i)}{2\text{Area}(\mathcal{Z})}$$

for each $i \in \{0,1,2,3\}$ and $z \in \mathcal{Z}$, and set the weight to be 0 elsewhere. As $u = (15-\sqrt{15})/10$, we see that the area of the hexagon is double that of the triangle, i.e. $|\mathcal{B}| = 2|\mathcal{D}|$, so after a discretization procedure, the weight assignment above will satisfy (1) and (2) up to some lower order error.

We can similarly construct a weight function on $\{(x,y,z) \in \mathcal{A}\times\mathcal{C}\times\mathcal{C} : x+y=z\}$. Moreover, we see that $|\mathcal{C}| \geq 2|\mathcal{A}|$, so certain points $v \in \mathcal{C}$ will be given total weight less than 1. We can then give an appropriate weight to the trivial constraint $\lambda(v) \leq n$ to reach a total weight of roughly 1 at v.

1.4. **Notation.** In the paper, all logs are natural and all constants in asymptotic notations are universal. For any $v \in \mathbb{R}^d$, $r \in \mathbb{R}$, and $A \subset \mathbb{R}^d$, let $v + A := \{v + a : a \in A\}$ and $rA := \{ra : a \in A\}$.

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2. Proof of the Three Dimensional Result

In this section, we will prove our main result Theorem 1.1 carefully, building off the proof sketch.

2.1. **The Three Lines Lemma.** First, we prove a key lemma used for both Theorem 1.1 and Theorem 1.2. For any $v \in [n]^2$, let $\lambda(v) := |S \cap \pi^{-1}(v)|$ and $\lambda^*(v) := |S_3^* \cap \pi^{-1}(v)|$. For any $p, q \in \mathbb{R}^d$, let $\ell(p,q) := \{tp + (1-t)q : t \in [0,1]\}$ be the segment between them. Let $\mathbf{0}$ be the origin.

Lemma 2.1. *For any $x, y, z \in [n]^d$ with $x + y = z$, a vector $w \in \mathbb{R}^d_{\geq 0}$, and a sum-free set $S \subset [n]^d$,*

$$|S \cap \ell(x, x+w)| + |S \cap \ell(y, y+w)| + |S \cap \ell(z, z+w)| \leq 2\Big|[n]^d \cap \ell(\mathbf{0}, w)\Big| + O(1). \tag{2.1}$$

Proof. Observe that if $v \in [0,n]^d$ and $z + v \in [0,n]^d$, then $x + v \in [0,n]^d$ and $y + v \in [0,n]^d$. Also, we note that (2.1) clearly holds if $S \cap \ell(z, z+w) = \emptyset$.

Otherwise, let $m \in [0,1]$ be the largest number such that $z + mw \in S$. The observation above implies that $x + mw \in [0,n]^d$ and $y + mw \in [0,n]^d$. Note that $(x+tw)+(y+(m-t)w) = z+mw \in S$ for any $t \in [0,m]$. As $z+mw \in [n]^d$ and S is sum-free, $x+tw \in [n]^d$ if and only if $y+(m-t)w \in [n]^d$, and S contains at most one of them. The three segments (and so the lattice points on them) are translations of $\ell(\mathbf{0}, w)$. This allows us to bound the left hand side of (2.1) above by

$$\big|[n]^d \cap \ell(x, x+w)\big| + \big|[n]^d \cap \ell(y+mw, y+w)\big| + \big|[n]^d \cap \ell(z, z+mw)\big| \leq 2\big|[n]^d \cap \ell(\mathbf{0}, w)\big| + O(1). \qquad \square$$

Corollary 2.2. *For any $x, y, z \in [n]^2$ with $x + y = z$, and sum-free set $S \subset [n]^3$,*

$$\lambda(x) + \lambda(y) + \lambda(z) \leq 2n + O(1). \tag{2.2}$$

Proof. Apply Lemma 2.1 to $(x_1, x_2, 1), (y_1, y_2, 1), (z_1, z_2, 2)$ and $w = (0, 0, n-2)$, with $O(1)$ error. $\square$

Let us partition $[n]^2 = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$ as in (1.3). We will now show that S_3^* attains equality in (2.2) on triples of lines whose base points are in the set

$$\mathcal{E}_3 := \{(x,y,z) \in (\mathcal{A}\times\mathcal{C}\times\mathcal{C}) \cup (\mathcal{B}\times\mathcal{B}\times\mathcal{D}) : x + y = z\}. \tag{2.3}$$

Lemma 2.3. *For any $(x,y,z) \in \mathcal{E}_3$,*

$$\lambda^*(x) + \lambda^*(y) + \lambda^*(z) \geq 2n - O(1). \tag{2.4}$$

Proof. If $(x,y,z) \in \mathcal{A}\times\mathcal{C}\times\mathcal{C}$, then S_3^* contains none of the segment $\pi^{-1}(x) \cap [n]^3$ and all points on the segments $\pi^{-1}(y) \cap [n]^3$ and $\pi^{-1}(z) \cap [n]^3$, so (2.4) holds. If $(x,y,z) \in \mathcal{B}\times\mathcal{B}\times\mathcal{D}$ instead, then

$$\begin{aligned}
\lambda^*(x) + \lambda^*(y) + \lambda^*(z) &\geq (n - \lceil un\rceil + x_1 + x_2 + 1) + (n - \lceil un\rceil + y_1 + y_2 + 1) + (2\lfloor un\rfloor - z_1 - z_2 - 1) \\
&= 2n - 2\lceil un\rceil + 2\lfloor un\rfloor + 1 - \mathbf{1}^\intercal(x + y - z) \\
&= 2n - O(1)
\end{aligned}$$

$\square$

2.2. **Reduction to the Construction of a Weight Function.** In this section, we will reduce Theorem 1.1 to the existence of a weight function as in Lemma 2.5. First, we check the lower bound.

Lemma 2.4. *S_3^* defined in (1.1) is a sum-free of size $(10+\sqrt{15})n^3/20 + O(n^2)$.*

Proof. For any $x, y \in S_3^*$, $\mathbf{1}^\intercal(x+y) \geq 2un$, so $x + y \notin S_3^*$. Hence, S_3^* is sum-free. By a volume computation and a standard geometry of numbers argument, $|S_3^*| = (10+\sqrt{15})n^3/20 + O(n^2)$. □

To prove Theorem 1.1, we will sum inequalities (2.2) with weights given by the following lemma.

Lemma 2.5. *There exist weight functions $w : \mathcal{E}_3 \to \mathbb{R}_{\geq 0}$ and $\widetilde{w} : \mathcal{C} \to \mathbb{R}_{\geq 0}$ such that $\sum_{e\in\mathcal{E}_3} w(e) = O(n^2)$, and the following holds for all but at most $O(n^{3/2})$-many points $v \in [n]^2$:*

$$\widetilde{w}(v) + \sum_{e\in\mathcal{E}_3 : v\in e} w(e) = 1 + O\left(n^{-1/2}\right),$$

where we let $\widetilde{w}(v) = 0$ for $v \notin \mathcal{C}$.

We will defer the proof of this lemma to the next section. Here, we see how it implies Theorem 1.1.

Proof of Theorem 1.1. Let w and $\widetilde{w}$ be weight functions from Lemma 2.5. We compute

$$\begin{aligned} Q &:= \sum_{v\in\mathcal{C}} \widetilde{w}(v)\lambda(v) + \sum_{(x,y,z)\in\mathcal{E}_3} w(x,y,z)(\lambda(x)+\lambda(y)+\lambda(z)) \\ &= \sum_{v\in[n]^2} \left(1 + O\left(n^{-1/2}\right)\right)\lambda(v) + O\left(n^{5/2}\right) \\ &= |S| + O\left(n^{5/2}\right). \end{aligned} \tag{2.5}$$

Replacing S by S_3^* and λ by λ^*, we obtain

$$Q^* := \sum_{v\in\mathcal{C}} \widetilde{w}(v)\lambda^*(v) + \sum_{(x,y,z)\in\mathcal{E}_3} w(x,y,z)(\lambda^*(x)+\lambda^*(y)+\lambda^*(z)) = |S_3^*| + O\left(n^{5/2}\right). \tag{2.6}$$

By Corollary 2.2 and Lemma 2.3, for any $(x, y, z) \in \mathcal{E}_3$,

$$\lambda(x) + \lambda(y) + \lambda(z) \leq \lambda^*(x) + \lambda^*(y) + \lambda^*(z) + O(1).$$

Now, we will combine this with the trivial bound $\lambda(v) \leq n = \lambda^*(v)$ for all $v \in \mathcal{C}$ to get

$$\begin{aligned} Q &:= \sum_{v\in\mathcal{C}} \widetilde{w}(v)\lambda(v) + \sum_{(x,y,z)\in\mathcal{E}_3} w(x,y,z)(\lambda(x)+\lambda(y)+\lambda(z)) \\ &\leq \sum_{v\in\mathcal{C}} \widetilde{w}(v)\lambda^*(v) + \sum_{(x,y,z)\in\mathcal{E}_3} w(x,y,z)(\lambda^*(x)+\lambda^*(y)+\lambda^*(z)+O(1)) \\ &= Q^* + O\left(n^{5/2}\right). \end{aligned}$$

By (2.5) and (2.6), this gives the matching upper bound to Lemma 2.4, and Theorem 1.1 follows. □

2.3. **Constructing the Weight Function.** In this section, we prove Lemma 2.5 by constructing a continuous analog and applying a discretizing procedure. Clearly, adding Claim 2.6 and Claim 2.7 gives Lemma 2.5. This is motivated by the obvious partition of $\mathcal{E}_3$ into $\mathcal{A}\times\mathcal{C}\times\mathcal{C}$ and $\mathcal{B}\times\mathcal{B}\times\mathcal{D}$.

A standard geometry of numbers argument allows us to interchange between volumes of polytopal regions in $\mathbb{R}^d$ and the number of lattice points contained in the region up to $O(n^{d-1})$ error.

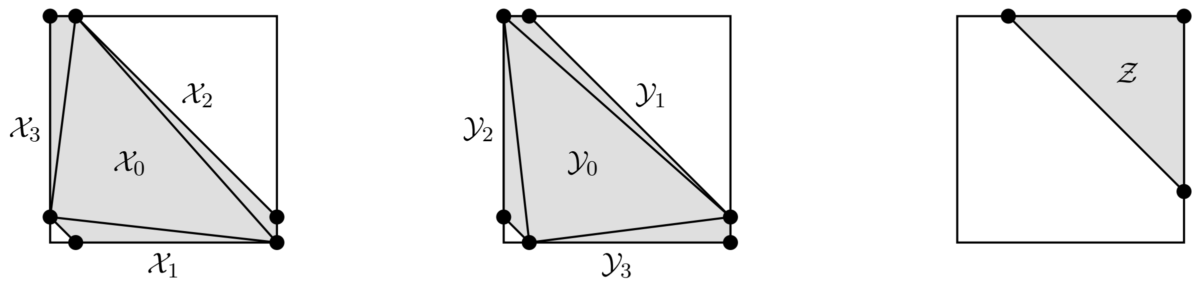
Claim 2.6. *There exists a weight function $w : \{(x, y, z) \in \mathcal{B}\times\mathcal{B}\times\mathcal{D} : x + y = z\} \to \mathbb{R}_{\geq 0}$ such that $\sum_e w(e) = O(n^2)$, and the following holds for all but at most $O(n^{3/2})$-many points $v \in \mathcal{B}\cup\mathcal{D}$:*

$$\sum_{e:v\in e} w(e) = 1 + O\left(n^{-1/2}\right). \tag{2.7}$$

Proof. Define the following regions in $[0,n]^2$: let $\mathcal{Z} = n \cdot \mathrm{conv}\{(1, 2u-2), (2u-2, 1), (1, 1)\}$ and

$$\begin{aligned}
\mathcal{X}_0 &= n \cdot \mathrm{conv}\{(0, u-1), (u-1, 1), (1, 0)\}, & \mathcal{Y}_0 &= n \cdot \mathrm{conv}\{(1, u-1), (u-1, 0), (0, 1)\},\\
\mathcal{X}_1 &= n \cdot \mathrm{conv}\{(0, u-1), (u-1, 0), (1, 0)\}, & \mathcal{Y}_1 &= n \cdot \mathrm{conv}\{(1, u-1), (u-1, 1), (0, 1)\},\\
\mathcal{X}_2 &= n \cdot \mathrm{conv}\{(1, u-1), (u-1, 1), (1, 0)\}, & \mathcal{Y}_2 &= n \cdot \mathrm{conv}\{(0, u-1), (u-1, 0), (0, 1)\},\\
\mathcal{X}_3 &= n \cdot \mathrm{conv}\{(0, u-1), (u-1, 1), (1, 0)\}, & \mathcal{Y}_3 &= n \cdot \mathrm{conv}\{(1, u-1), (u-1, 0), (0, 1)\},
\end{aligned}$$

where conv denotes the convex hull. Note, $\left(\bigcup_{i=0}^3 \mathcal{X}_i\right) \cap [n]^2 = \left(\bigcup_{i=0}^3 \mathcal{Y}_i\right) \cap [n]^2 = \mathcal{B}$ and $\mathcal{Z} \cap [n]^2 = \mathcal{D}$. We can compute $\mathrm{Area}(\mathcal{X}_i) = \mathrm{Area}(\mathcal{Y}_i)$ for $i \in \{0,1,2,3\}$ and $\sum_{i=0}^3 \mathrm{Area}(\mathcal{X}_i) = 2\,\mathrm{Area}(\mathcal{Z})$.



We now construct invertible affine transformations $X_i : \mathcal{Z} \to \mathcal{X}_i$ and $Y_i : \mathcal{Z} \to \mathcal{Y}_i$ for $i \in \{0,1,2,3\}$. For any $z \in \mathcal{Z}$, let $\mu_j \geq 0$ be the unique numbers such that $z = \mu_1(1, 2u-2) + \mu_2(2u-2, 1) + \mu_3(1,1)$ and $\mu_1 + \mu_2 + \mu_3 = n$. This amounts to a convex combination. Define X_i and Y_i by

$$\begin{aligned}
X_0(z) &= \mu_1(0, u-1) + \mu_2(u-1, 1) + \mu_3(1, 0), & Y_0(z) &= \mu_1(1, u-1) + \mu_2(u-1, 0) + \mu_3(0, 1),\\
X_1(z) &= \mu_1(0, u-1) + \mu_2(u-1, 0) + \mu_3(1, 0), & Y_1(z) &= \mu_1(1, u-1) + \mu_2(u-1, 1) + \mu_3(0, 1),\\
X_2(z) &= \mu_1(1, u-1) + \mu_2(u-1, 1) + \mu_3(1, 0), & Y_2(z) &= \mu_1(0, u-1) + \mu_2(u-1, 0) + \mu_3(0, 1),\\
X_3(z) &= \mu_1(0, u-1) + \mu_2(u-1, 1) + \mu_3(1, 0), & Y_3(z) &= \mu_1(1, u-1) + \mu_2(u-1, 0) + \mu_3(0, 1).
\end{aligned}$$

Observe that $X_i(z) + Y_i(z) = z$ for all $z \in \mathcal{Z}$. Let $B(p, r) := \{q \in \mathbb{R}^2 : \|p - q\|_\infty \leq r\}$ denote the L^∞-ball and let $\mathbf{1}\{\cdot\}$ denote the indicator function. We can now construct the weight function w as follows: for $(x, y, z) \in \mathcal{B} \times \mathcal{B} \times \mathcal{D}$ with $x + y = z$, define the weight function

$$w(x, y, z) := \frac{1}{2n} \sum_{i=0}^{3} \frac{\mathrm{Area}(\mathcal{X}_i)}{2\,\mathrm{Area}(\mathcal{Z})} \mathbf{1}\big\{X_i(z) \in B\big(x, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Y_i(z) \in B\big(y, \sqrt{n}\big)\big\}.$$

It suffices to check that w satisfies the conditions. We first check the total weight condition:

$$\sum_{\substack{(x,y,z) \in \mathcal{B}\times\mathcal{B}\times\mathcal{D} \\ x+y=z}} w(x, y, z) \leq \frac{1}{4n} \sum_{i=0}^{3} \frac{\mathrm{Area}(\mathcal{X}_i)}{\mathrm{Area}(\mathcal{Z})} \sum_{z \in \mathcal{D}} \sum_{x \in \mathcal{B}} \mathbf{1}\big\{x \in B\big(X_i(z), \sqrt{n}\big)\big\} = O\big(n^2\big).$$

We will now show (2.7) for most points in $\mathcal{D}$. For any $z \in \mathcal{D}$, we compute

$$\sum_{\substack{(x,y) \in \mathcal{B}\times\mathcal{B} \\ x+y=z}} w(x, y, z) = \frac{1}{4n} \sum_{i=0}^{3} \frac{\mathrm{Area}(\mathcal{X}_i)}{\mathrm{Area}(\mathcal{Z})} \sum_{\substack{(x,y) \in \mathcal{B}\times\mathcal{B} \\ x+y=z}} \mathbf{1}\big\{x \in B(X_i(z), \sqrt{n})\big\} \cdot \mathbf{1}\big\{y \in B(Y_i(z), \sqrt{n})\big\}.$$

For all $z \in \mathcal{D}$ such that $X_i(z)$ and $Y_i(z)$ are at least $2\sqrt{n}$ away from $\partial\mathcal{X}_i$ and $\partial\mathcal{Y}_i$ in L^∞-distance, respectively, observe that $x + y = z = X_i(z) + Y_i(z)$ implies $X_i(z) - x = y - Y_i(z)$, so we bound

$$\sum_{\substack{(x,y) \in \mathcal{B}\times\mathcal{B} \\ x+y=z}} \mathbf{1}\big\{x \in B(X_i(z), \sqrt{n})\big\} \cdot \mathbf{1}\big\{y \in B(Y_i(z), \sqrt{n})\big\} = \big|B\big(X_i(z), \sqrt{n}\big) \cap [n]^2\big| = 2n + O\Big(n^{1/2}\Big).$$

The boundary regions have $O(n^{3/2})$-many points, so for all but $O(n^{3/2})$-many $z \in \mathcal{D}$, we conclude

$$\sum_{\substack{(x,y)\in\mathcal{B}\times\mathcal{B}\\ x+y=z}} w(x,y,z) = \frac{1}{4n}\sum_{i=0}^{3}\frac{\mathrm{Area}(\mathcal{X}_i)}{\mathrm{Area}(\mathcal{Z})}\left(2n+O\left(n^{1/2}\right)\right) = 1+O\left(n^{-1/2}\right).$$

We will now show (2.7) for most points in $\mathcal{B}$. Without loss of generality, take any $x \in \mathcal{X}_i$. If x is at least $2\sqrt{n}$ away from the boundary $\partial\mathcal{X}_i$ in L^∞-distance, then

$$\sum_{\substack{(y,z)\in\mathcal{B}\times\mathcal{D}\\ x+y=z}} w(x,y,z) = \frac{1}{4n}\cdot\frac{\mathrm{Area}(\mathcal{X}_i)}{\mathrm{Area}(\mathcal{Z})}\sum_{\substack{(y,z)\in\mathcal{B}\times\mathcal{D}\\ x+y=z}} \mathbf{1}\left\{X_i(z)\in B(x,\sqrt{n})\right\}\cdot\mathbf{1}\left\{Y_i(z)\in B(y,\sqrt{n})\right\}.$$

If $X_i^{-1}(B(x,\sqrt{n})) \subset \mathcal{Z}$ and $Y_i\left(X_i^{-1}(B(x,\sqrt{n}))\right) \subset \mathcal{Y}_i$, i.e. images of x under the maps are $\Omega(\sqrt{n})$-away from the boundaries $\partial\mathcal{Y}_i$ and $\partial\mathcal{Z}$, then $X_i(z) \in B(x,\sqrt{n})$ implies $Y_i(z) \in B(y,\sqrt{n})$, and

$$\sum_{\substack{(y,z)\in\mathcal{B}\times\mathcal{D}\\ x+y=z}} w(x,y,z) = \frac{1}{4n}\cdot\frac{\mathrm{Area}(\mathcal{X}_i)}{\mathrm{Area}(\mathcal{Z})}\cdot\left|X_i^{-1}\left(B\left(x,\sqrt{n}\right)\right)\cap[n]^2\right|.$$

Since X_i is invertible linear transformation

$$\left|X_i^{-1}\left(B\left(x,\sqrt{n}\right)\right)\cap[n]^2\right| = \mathrm{Area}\left(X_i^{-1}\left(B\left(x,\sqrt{n}\right)\right)\right)+O\left(n^{1/2}\right) = \frac{\mathrm{Area}(\mathcal{Z})}{\mathrm{Area}(\mathcal{X}_i)}\left(2n+O\left(n^{1/2}\right)\right).$$

The boundary regions have $O(n^{3/2})$-many points, so for all but $O(n^{3/2})$-many $x \in \mathcal{B}$, we conclude

$$\sum_{\substack{(y,z)\in\mathcal{B}\times\mathcal{D}\\ x+y=z}} w(x,y,z) = \frac{1}{2}+O\left(n^{-1/2}\right).$$

By symmetry, the same holds for $y \in \mathcal{B}$. Adding the two equations together yields (2.7) on $\mathcal{B}$. □

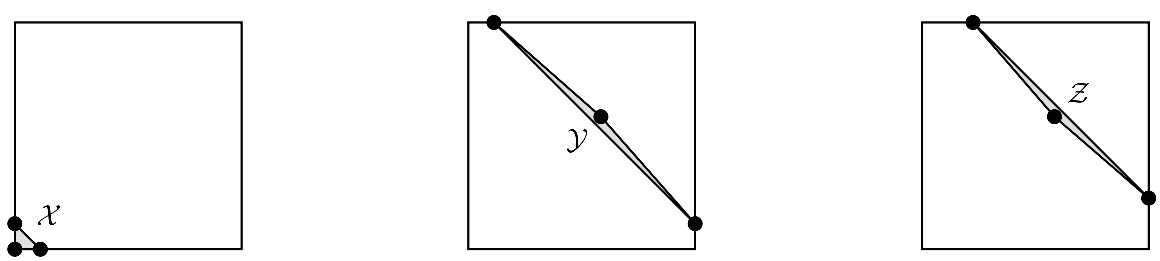
Claim 2.7. *There exist functions $w : \{(x,y,z)\in\mathcal{A}\times\mathcal{C}\times\mathcal{C} : x+y=z\} \to \mathbb{R}_{\geq 0}$ and $\widetilde{w} : \mathcal{C} \to \mathbb{R}_{\geq 0}$ such that $\sum_e w(e) = O(n^2)$ and the following holds for all but at most $O\left(n^{3/2}\right)$-many points $v \in \mathcal{A}\cup\mathcal{C}$:*

$$\widetilde{w}(v) + \sum_{e:v\in e} w(e) = 1+O\left(n^{-1/2}\right), \tag{2.8}$$

where we let $\widetilde{w}(v) = 0$ for $v \notin \mathcal{C}$.

Proof. Define the following regions in $[0,n]^2$, so that $\mathcal{A} = \mathcal{X}\cap[n]^2$, $\mathcal{Y}\cap[n]^2 \subset \mathcal{C}$, and $\mathcal{Z}\cap[n]^2 \subset \mathcal{C}$:

$$\begin{aligned}
\mathcal{X} &= n\cdot\mathrm{conv}\{(0,0),(u-1,0),(0,u-1)\},\\
\mathcal{Y} &= n\cdot\mathrm{conv}\left\{\left(\frac{3u-1}{4},\frac{3u-1}{4}\right),(u-1,1),(1,u-1)\right\},\\
\mathcal{Z} &= n\cdot\mathrm{conv}\left\{\left(\frac{3u-1}{4},\frac{3u-1}{4}\right),(2u-2,1),(1,2u-2)\right\}.
\end{aligned}$$



As before, we construct invertible affine transformations $Y : \mathcal{X} \to \mathcal{Y}$ and $Z : \mathcal{X} \to \mathcal{Z}$. For $x \in \mathcal{X}$, let $\mu_j \geq 0$ be the such that $x = \mu_1(0,0) + \mu_2(u-1,0) + \mu_3(0,u-1)$ and $\mu_1 + \mu_2 + \mu_3 = n$. Define

$$Y(x) = \mu_1\left(\frac{3u-1}{4}, \frac{3u-1}{4}\right) + \mu_2(u-1,1) + \mu_3(1,u-1),$$
$$Z(x) = \mu_1\left(\frac{3u-1}{4}, \frac{3u-1}{4}\right) + \mu_2(2u-2,1) + \mu_3(1,2u-2).$$

Observe that $x + Y(x) = Z(x)$ for all $x \in \mathcal{X}$. For $(x,y,z) \in \mathcal{A} \times \mathcal{C} \times \mathcal{C}$ with $x + y = z$, define weight

$$w(x,y,z) := \frac{1}{2n}\mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\}.$$

It suffices to check that w satisfies the conditions. We first check the total weight condition:

$$\sum_{\substack{(x,y,z)\in\mathcal{A}\times\mathcal{C}\times\mathcal{C}\\ x+y=z}} w(x,y,z) \leq \frac{1}{2n}\sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{C}} \mathbf{1}\big\{y \in B\big(Y(x), \sqrt{n}\big)\big\} = O\big(n^2\big).$$

We will now show (2.8) for most points in $\mathcal{X}$. For any $x \in \mathcal{X}$, we compute

$$\sum_{\substack{(y,z)\in\mathcal{C}\times\mathcal{C}\\ x+y=z}} w(x,y,z) = \frac{1}{2n}\sum_{\substack{(y,z)\in\mathcal{C}\times\mathcal{C}\\ x+y=z}} \mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\}.$$

For all $x \in \mathcal{D}$ such that $Y(x)$ and $Z(x)$ are at least $2\sqrt{n}$ away from $\partial\mathcal{Y}$ and $\partial\mathcal{Z}$ in L^∞-distance, respectively, observe that $Z(x) - Y(x) = x = z - y$ implies $Z(x) - z = Y(x) - y$, so we bound

$$\sum_{\substack{(y,z)\in\mathcal{C}\times\mathcal{C}\\ x+y=z}} \mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\} = \big|B\big(Z(x), \sqrt{n}\big) \cap [n]^2\big| = 2n + O\Big(n^{1/2}\Big).$$

The boundary regions have $O(n^{3/2})$-many points, so (2.8) holds for all but $O(n^{3/2})$-many $x \in \mathcal{X}$. We will now show (2.8) on $\mathcal{C}$. For any $y \in \mathcal{Y} \cap [n]^2$, since Y is an invertible affine transformation

$$\begin{aligned}\sum_{\substack{(x,z)\in\mathcal{A}\times\mathcal{C}\\ x+y=z}} w(x,y,z) &= \frac{1}{2n}\sum_{\substack{(x,z)\in\mathcal{A}\times\mathcal{C}\\ x+y=z}} \mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\}\\ &\leq \frac{1}{2n}\sum_{x\in[n]^2} \mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\}\\ &= \frac{\text{Area}(\mathcal{X})}{\text{Area}(\mathcal{Y})} + O\Big(n^{-1/2}\Big).\end{aligned}$$

For any $z \in \mathcal{Z} \cap [n]^2$, since Z is an invertible affine transformation

$$\begin{aligned}\sum_{\substack{(x,y)\in\mathcal{A}\times\mathcal{C}\\ x+y=z}} w(x,y,z) &= \frac{1}{2n}\sum_{\substack{(x,z)\in\mathcal{A}\times\mathcal{C}\\ x+y=z}} \mathbf{1}\big\{Y(x) \in B\big(y, \sqrt{n}\big)\big\} \cdot \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\}\\ &\leq \frac{1}{2n}\sum_{x\in[n]^2} \mathbf{1}\big\{Z(x) \in B\big(z, \sqrt{n}\big)\big\}\\ &= \frac{\text{Area}(\mathcal{X})}{\text{Area}(\mathcal{Z})} + O\Big(n^{-1/2}\Big).\end{aligned}$$

It follows a simple computation that $\text{Area}(\mathcal{X}) < \text{Area}(\mathcal{Z}) < \text{Area}(\mathcal{Y})$. Therefore, to show (2.8) on $\mathcal{C}$, it now suffices to set $\widetilde{w}(v) = \max\big(0, 1 - \sum_{e:v\in e} w(e)\big) \geq 0$ for every $v \in \mathcal{C}$. $\square$

3. Reproof of the Two Dimensional Result

3.1. **Overview.** In this section, we will reprove Theorem 1.2, first shown in [3]. For completeness, we first check the lower bound construction $S_2^* := \{v \in [n]^2 : \mathbf{1}^\intercal v \in [4n/5, 8n/5)\}$.

Lemma 3.1. *$S_2^* \subset [n]^2$ is a sum-free set of size $3n^2/5 + O(n)$.*

Proof. For any $x, y \in S_2^*$, $\mathbf{1}^\intercal(x+y) \geq 8n/5$, so $x + y \notin S_2^*$. Hence, S_3^* is sum-free. By an area computation and a standard geometry of numbers argument, $|S_2^*| = 3n^2/5 + O(n)$. □

For the upper bound, the same argument as Theorem 1.1 does not apply immediately: the analog of regions $\mathcal{B}$ and $\mathcal{D}$ in (1.3) are intervals $[0, 4/5]$ and $[3/5, 1]$. They intersect, so we cannot hope for something like Lemma 2.5. This is unsurprising since the upper bound we seek is $3/5$, and combining Lemma 2.1 on triples of lines cannot give any upper bound better than $2/3$.

Instead of lines in the axial direction, we will consider segments in the $\mathbf{1}$-direction, and derive an analog of Corollary 2.2 showing S_2^* is optimal on those segments. Combined with an analog of Lemma 2.5, we will derive an stability statement in a region around S_2^*. Finally, we will do some numerical computations to rule out cases where S contains any point not in this stability region.

3.2. **Stability of S_2^*.** The goal of this section is to deduce the following stability result.

Lemma 3.2. *Fix any $\alpha, \beta \geq 0$ such that $3\alpha + \beta \leq 4/5$. Then, $|S| \leq 3n^2/5 + O(n)$ for any sum-free*

$$S \subset R(\alpha, \beta) := \{v \in [n]^2 : \mathbf{1}^\intercal v \in [(4/5 - \alpha)n, (8/5 + \beta)n)\}. \tag{3.1}$$

Hence, S_2^* is the largest sum-free subset of $R(\alpha, \beta)$. Let $[2n]/2 = \{k/2 : k \in [2n]\}$ be the set of half-integers. We will prove the lemma by combining triples of segments along $\mathbf{1}$ intersecting

$$\mathcal{E}_2 := \{(x, y, z) \in ([2n]/2)^2 \times ([2n]/2)^2 \times ([2n]/2)^2 : x + y = z, \mathbf{1}^\intercal x = \mathbf{1}^\intercal y = \lfloor 4n/5 \rfloor\}.$$

For each triple in $\mathcal{E}_2$, the following analog of Corollary 2.2 holds.

Lemma 3.3. *For any $(x, y, z) \in \mathcal{E}_2$ and $a, b, c \geq 0$, if $x - a\mathbf{1}, y - b\mathbf{1}, z + c\mathbf{1} \in [0, n]^2$, then*

$$\ell(x - a\mathbf{1}, x + (b+c)\mathbf{1}), \ell(y - b\mathbf{1}, y + (a+c)\mathbf{1}), \ell(z - (a+b)\mathbf{1}, z + c\mathbf{1}) \subset [0, n]^2. \tag{3.2}$$

Moreover, for any sum-free set $S \subset [n]^2$,

$$|S \cap \ell(x - a\mathbf{1}, x + (b+c)\mathbf{1})| + |S \cap \ell(y - b\mathbf{1}, y + (a+c)\mathbf{1})| + |S \cap \ell(z - (a+b)\mathbf{1}, z + c\mathbf{1})| \tag{3.3}$$

is at most $2(a + b + c) + O(1)$, and equality is attained when $S = S_2^*$.

Proof. Define partial order $\preceq$ on $\mathbb{R}^2$ where $p \preceq q$ if $p_1 \leq q_1$ and $p_2 \leq q_2$. Since $x + y = z$

$$\mathbf{0} \preceq x - a\mathbf{1} \preceq x + (b+c)\mathbf{1} = (z + c\mathbf{1}) - (y - b\mathbf{1}) \preceq n\mathbf{1} - \mathbf{0} = n\mathbf{1}.$$

Hence, $x + (b+c)\mathbf{1} \in [0, n]^2$, and similarly $y + (a+c)\mathbf{1} \in [0, n]^2$ by symmetry. Now

$$\mathbf{0} = \mathbf{0} + \mathbf{0} \preceq (x - a\mathbf{1}) + (y - b\mathbf{1}) = z - (a+b)\mathbf{1} \preceq z \preceq n\mathbf{1}$$

so $z - (a+b)\mathbf{1} \in [0, n]^2$. Then, (3.2) follows the convexity of $[0, n]^2$. Now, we apply Lemma 2.1 to the segments in (3.2). The endpoints of the segments being half-integral gives $O(1)$ error, so

$$(3.3) \leq 2\left|[n]^2 \cap \ell(\mathbf{0}, (a+b+c)\mathbf{1})\right| + O(1) = 2(a+b+c) + O(1)$$

When $S = S_2^*$, its boundary intersects the segments in (3.2) at x, y, z respectively, so

$$\begin{aligned}
(3.3) &= \left|[n]^2 \cap \ell(x, x + (b+c)\mathbf{1})\right| + \left|[n]^2 \cap \ell(y, y + (a+c)\mathbf{1})\right| + \left|[n]^2 \cap \ell(z - (a+b)\mathbf{1}, z)\right| \\
&= \left|[n]^2 \cap \ell(\mathbf{0}, (b+c)\mathbf{1})\right| + \left|[n]^2 \cap \ell(\mathbf{0}, (a+c)\mathbf{1})\right| + \left|[n]^2 \cap \ell(\mathbf{0}, (a+b)\mathbf{1})\right| + O(1) \\
&= 2(a+b+c) + O(1)
\end{aligned}$$

□

To combine the inequality on (3.3), we construct a weight function on $\mathcal{E}_2$. The idea is similar to Lemma 2.5, but do it explicitly in the discrete setting, thereby obtaining a sharper error term.

Lemma 3.4. *There exists weight function $w : \mathcal{E}_2 \to \mathbb{R}_{\geq 0}$ such that $\sum_{e \in \mathcal{E}_2} w(e) = O(n)$, and the following holds for all but at most $O(1)$-many $v \in ([2n]/2)^2$ satisfying $\mathbf{1}^\intercal v \in \{\lfloor 4n/5 \rfloor, 2\lfloor 4n/5 \rfloor\}$:*

$$\sum_{e \in \mathcal{E}_2 : v \in e} w(e) = 1. \tag{3.4}$$

Proof. Let $m = \lfloor n/10 \rfloor$ and $r = \lfloor 4n/5 \rfloor - 8\lfloor n/10 \rfloor$, so that $8m + r = \lfloor 4n/5 \rfloor$. Define the weight function w to be 1 on the following triples for all $k \in \{-2m, -2m+1, \cdots, 2m\}$, and 0 otherwise:

$$\left(\left(7m + r + \frac{k}{2}, m - \frac{k}{2}\right), \left(m + \frac{k}{2}, 7m + r - \frac{k}{2}\right), \left(8m + r + \frac{2k}{2}, 8m + r - \frac{2k}{2}\right)\right),$$

$$\left(\left(5m + r + \frac{k+1}{2}, 3m - \frac{k+1}{2}\right), \left(3m + \frac{k}{2}, 5m + r - \frac{k}{2}\right), \left(8m + r + \frac{2k+1}{2}, 8m + r - \frac{2k+1}{2}\right)\right).$$

Then, for all but at most $O(1)$-many $v \in ([2n]/2)^2$ with $\mathbf{1}^\intercal v \in \{\lfloor 4n/5 \rfloor, 2\lfloor 4n/5 \rfloor\}$, (3.4) holds as only one summand is nonzero, and it is equal to 1. The total weight is $O(m)$ which is $O(n)$. ☐

We will now combine Lemma 3.3 and Lemma 3.4 to show Lemma 3.2.

Proof of Lemma 3.2. Similar to before, for all $v \in ([2n]/2)^2$ such that $\mathbf{1}^\intercal v = \lfloor 4n/5 \rfloor$, define

$$\lambda_-(v) = |S \cap \ell(v - n\alpha\mathbf{1}, v + n(\alpha+\beta)\mathbf{1})| \quad \text{and} \quad \lambda_-^*(v) = |S_2^* \cap \ell(v - n\alpha\mathbf{1}, v + n(\alpha+\beta)\mathbf{1})|,$$

and for all $v \in ([2n]/2)^2$ such that $\mathbf{1}^\intercal v = 2\lfloor 4n/5 \rfloor$, define

$$\lambda_+(v) = |S \cap \ell(v - 2n\alpha\mathbf{1}, v + n\beta\mathbf{1})| \quad \text{and} \quad \lambda_+^*(v) = |S_2^* \cap \ell(v - 2n\alpha\mathbf{1}, v + n\beta\mathbf{1})|.$$

For any $(x, y, z) \in \mathcal{E}_2$, let $a, b \in [0, n\alpha]$ and $c \in [0, n\beta]$ each be maximal such that $x - a\mathbf{1}, y - b\mathbf{1}, z + c\mathbf{1} \in [0, n]^2$. They exist since $x, y, z \in [0, n]^2$, so $a = b = c = 0$ works. By Lemma 3.3,

$$\lambda_-(x) + \lambda_-(y) + \lambda_+(z) \leq \lambda_-^*(x) + \lambda_-^*(y) + \lambda_+^*(z) + O(1). \tag{3.5}$$

As $3\alpha + \beta \leq 4/5$, the segments in $\lambda_-(v)$ and $\lambda_+(v)$ are pairwise disjoint and their union covers $R(\alpha, \beta)$. For $S \subset R(\alpha, \beta)$, we use (3.5) with weights from Lemma 3.4 to upper bound

$$\begin{aligned}
|S| &= \sum_{\substack{v \in ([2n]/2)^2 \\ \mathbf{1}^\intercal v = \lfloor 4n/5 \rfloor}} \lambda_-(v) + \sum_{\substack{v \in ([2n]/2)^2 \\ \mathbf{1}^\intercal v = 2\lfloor 4n/5 \rfloor}} \lambda_+(v) \\
&= O(n) + \sum_{(x,y,z) \in \mathcal{E}_2} w(x,y,z)(\lambda_-(x) + \lambda_-(y) + \lambda_+(z)) \\
&\leq O(n) + \sum_{(x,y,z) \in \mathcal{E}_2} w(x,y,z)\big(\lambda_-^*(x) + \lambda_-^*(y) + \lambda_+^*(z) + O(1)\big) \\
&= O(n) + \sum_{\substack{v \in ([2n]/2)^2 \\ \mathbf{1}^\intercal v = \lfloor 4n/5 \rfloor}} \lambda_-^*(v) + \sum_{\substack{v \in ([2n]/2)^2 \\ \mathbf{1}^\intercal v = 2\lfloor 4n/5 \rfloor}} \lambda_+^*(v) \\
&= |S_2^*| + O(n).
\end{aligned}$$

☐

3.3. **Numerical Computations.** To prove Theorem 1.2, we need to rule out instances where some sum-free $S \subset [n]^2$ contains points outside stability region $R(\alpha, \beta)$ for some α and β. The idea is that having any point at the corners near $\mathbf{0}$ or $\mathbf{1}$ imposes a lot of conditions, so it cuts out many elements from S. More precisely, we have the following two lemmas, corresponding to the two corners.

Lemma 3.5. *If sum-free $S \subset [n]^2$ contains some v with $\mathbf{1}^\intercal v \geq 17n/10$, then $|S| \leq 3n^2/5 + O(n)$.*

Lemma 3.6. *If sum-free $S \subset [n]^2$ contains some v with $\mathbf{1}^\intercal v \leq 17n/30$, then $|S| \leq 3n^2/5 + O(n)$.*

The proofs of Lemma 3.5 and Lemma 3.6 require some numerical computations. Here, we first show how to combine Lemma 3.1, Lemma 3.2, Lemma 3.5, and Lemma 3.6 to deduce Theorem 1.2.

Proof of Theorem 1.2. Recall that Lemma 3.1 gives the lower bound. For the upper bound, we fix any sum-free set $S \subset [n]^2$ and do case work on possible values of $\mathbf{1}^\intercal v$ for $v \in S$.

- If some $v \in S$ satisfies $\mathbf{1}^\intercal v \geq 17n/10$, Lemma 3.5 gives the upper bound $|S| \leq 3n^2/5 + O(n)$.
- If some $v \in S$ satisfies $\mathbf{1}^\intercal v \leq 17n/30$, Lemma 3.6 gives the upper bound $|S| \leq 3n^2/5 + O(n)$.
- Otherwise, $S \subset R(\alpha, \beta)$ for $\alpha = 7/30$ and $\beta = 1/10$, as defined in (3.1). We can check that $3\alpha + \beta = 4/5$, so Lemma 3.2 applies to show $|S| \leq 3n^2/5 + O(n)$. □

Now, it remains to show Lemma 3.5 and Lemma 3.6.

Proof of Lemma 3.5. We rescale $[n]^2$ to the unit square $[0,1]^2$. The neglected boundary effect gives $O(n)$ error by a standard geometry of numbers argument. Let (xn, yn) be the point in S that maximizes $r = xy$. If $\mathbf{1}^\intercal v \geq 17n/10$ for some $v \in S$, then $r \geq 7/10$. For every $p \in [0,x] \times [0,y]$, p and $v - p$ cannot both be in S as they sum to v, so we cut out an area of $r/2$. By maximality of r,

$$\frac{|S| - O(n)}{n^2} \leq r + \int_r^1 \frac{r}{t} dt - \frac{r}{2} = \frac{r}{2} - r\log r \leq \frac{7}{20} - \frac{7}{10}\log\left(\frac{7}{10}\right) = 0.599\ldots < 3/5,$$

where we note that $r/2 - r\log r$ is decreasing on $[7/10, 1]$, so we can bound it by its value at $7/10$. □

This proof is a modification of an argument due to Cameron [1]: upon obtaining the upper bound in r in the display, he obtained a global upper bound by maximizing it over r. The maximizer and maximum both turn out to be $1/\sqrt{e} = 0.606\ldots$. We now show Lemma 3.6.

Proof of Lemma 3.6. We rescale $[n]^2$ to the unit square $[0,1]^2$. The neglected boundary effect gives $O(n)$ error by a standard geometry of numbers argument. Let $v = (a,b)$ be the point in S/n with the smallest L^1-norm, so $a + b \leq 17/30$. Without loss of generality, assume $a \geq b$. Define the regions

$$U = \{p \in [0,1]^2 : \mathbf{1}^\intercal p \geq 17/10\} \quad \text{and} \quad L = \{p \in [0,1]^2 : \mathbf{1}^\intercal p \leq a + b\}.$$

By definition of v and Lemma 3.5, we can assume S/n is disjoint from U and L. Observe that for any $A, B \subset [0,1]^2$ such that $B \subset v + A$, each $p \in B$ and $v + p \in A$ cannot both be in S/n, so

$$\left|\frac{S}{n} \cap (A \cup B)\right| \leq \text{Area}(A) \cdot n^2 + O(n), \tag{3.6}$$

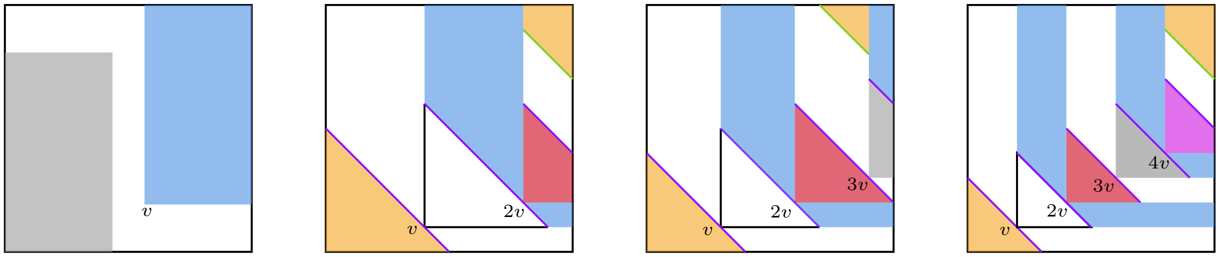
which we will refer to as "cutting out region B". For each positive integer t, define L-shaped regions

$$R_t := \left\{p \in [0,1]^2 : p - tv \notin [0,1]^2, p - (t-1)v \in [0,1]^2\right\}.$$

We see that $R_t - v \subset R_{t-1}$, allowing us to apply (3.6). Moreover, the area of these regions are

$$\text{Area}(R_t) = \begin{cases} a + b - (2t-1)ab & \text{if } 1 \leq t < \lceil 1/a \rceil \\ (1-(t-1)a)(1-(t-1)b) & \text{if } t = \lceil 1/a \rceil \\ 0 & \text{if } t > \lceil 1/a \rceil \end{cases}. \tag{3.7}$$

We casework on $\lceil 1/a \rceil$ and cut out an area of at least $2/5$ in all cases, i.e. $(|\overline{S}| + O(n))/n^2 \geq 2/5$. In the figures below, U and L are colored orange, and we cut them out trivially. The diagonal lines $x + y = t(a+b)$ for positive integer t are colored purple and $x + y = 17/10$ is colored green. Using (3.6), we will also cut out blue, red, and pink regions, with the latter two requiring more careful boundary analysis. Grey regions will also require boundary analysis, except we will not cut them out.



Case (1) *Case* (2) *Case* (3) *Case* (4)

(1) Suppose $a \geq 1/2$. Note that $R_2 \subset v + R_1$. By (3.6), we can cut out at least $\text{Area}(R_2)$, so
$$\frac{|\overline{S}| + O(n)}{n^2} \geq \text{Area}(R_2) = (1-a)(1-b),$$
which in this case is minimized at $(a, b) = (17/30, 0)$ with value $13/30 > 2/5$.

(2) Suppose $1/3 \leq a < 1/2$. Note that $R_2 \setminus (v + L) \subset v + (R_1 \setminus L)$ and $(v + L) \cap [0,1]^2 \subset v + L$, so we can cut out at least $\text{Area}(R_2)$ from the region $R_1 \cup R_2$. Also, we see that the red region $(2v + L) \cap [0,1]^2 \subset v + (v + L) \cap [0,1]^2$, so we also cut it out. It has area
$$\frac{1}{2}(a+b)^2 - \frac{1}{2}\max(0, 3a+b-1)^2 - \frac{1}{2}\max(0, 3b+a-1)^2 \geq \frac{1}{2}(a+b)^2 - \frac{1}{2}(3a+b-1)^2 - \frac{1}{2}\left(\frac{1}{30}\right)^2,$$
where we bound the third term below by checking that $3b + a - 1 \leq 1/30$, achieved at $(a, b) = (1/3, 7/30)$. We also exclude the orange region $U \cap ([2a, 1] \times [2b, 1])$ with area
$$\frac{1}{2}\left(2 - \frac{17}{10}\right)^2 - \frac{1}{2}\max\left(0, 2a - \frac{7}{10}\right)^2 - \frac{1}{2}\max\left(0, 2b - \frac{7}{10}\right)^2 \geq \frac{9}{200} - \frac{1}{2}\left(2a - \frac{7}{10}\right)^2,$$
where we see the third term is 0 as $b \leq 17/30 - a \leq 7/30$. Hence, in total we cut out
$$\begin{aligned}\frac{|\overline{S}| + O(n)}{n^2} &\geq \text{Area}(R_2) + \text{Area}\big((2v + L) \cap [0,1]^2\big) + \text{Area}(U \cap ([2a, 1] \times [2b, 1])) \\ &= a + b - 3ab + \frac{1}{2}(a+b)^2 - \frac{1}{2}(3a+b-1)^2 - \frac{1}{1800} + \frac{9}{200} - \frac{1}{2}\left(2a - \frac{7}{10}\right)^2,\end{aligned}$$
which in this case is minimized at $(a, b) = (1/3, 0)$ with value $0.432\ldots > 2/5$.

(3) Suppose $1/4 \leq a < 1/3$. As before, we cut out at least $\text{Area}(R_2)$ from the region $R_1 \cup R_2$. Now, the red region $(2v + L) \cap [0,1]^2 \subset v + (v + L) \cap [0,1]^2$ and contains disjoint translations of the grey region $(3v + L) \cap [0,1]^2$ and trapezoid $L \cap ([1-3a, 1-2a] \times [0,1])$. We compute
$$\begin{aligned}\text{Area}(L \cap ([1-3a, 1-2a] \times [0,1])) &= \frac{1}{2}\max(0, 4a+b-1)^2 - \frac{1}{2}\max(0, 3a+b-1)^2 \\ &\geq \frac{1}{2}(4a+b-1)^2 - \frac{1}{2}(3a+b-1)^2,\end{aligned} \tag{3.8}$$
where we use that $a \geq 1/4$. Also, this means the orange region $U \setminus R_4$ has area $(3a - 7/10)^2/2$. The blue region $R_4 \setminus (3v + L) \subset v + R_3 \setminus (2v + L)$, so we also cut it out. In total, we cut out
$$\begin{aligned}\frac{|\overline{S}| + O(n)}{n^2} &\geq \text{Area}(R_2) + \text{Area}((2v + L) \cap [0,1]^2) + \text{Area}(R_4 \setminus (3v + L)) + \text{Area}(U \setminus R_4) \\ &\geq \text{Area}(R_2) + \text{Area}(R_4) + \text{Area}(L \cap ([1-3a, 1-2a] \times [0,1])) + \text{Area}(U \setminus R_4) \\ &= a + b - 3ab + (1-3a)(1-3b) + \frac{1}{2}(4a+b-1)^2 - \frac{1}{2}(3a+b-1)^2 + \frac{1}{2}\left(3a - \frac{7}{10}\right)^2,\end{aligned}$$
which in this case is minimized at $a = b = 1/4$ with value $0.4075 > 2/5$.

(4) Suppose $1/5 \leq a < 1/4$. As before, we cut out at least $\text{Area}(R_2)$ from the region $R_1 \cup R_2$. Also, the argument (3.8) from case (3) on the grey and red regions still holds, so we can cut out at least $\text{Area}(R_4) + \text{Area}(L \cap ([1-3a, 1-2a] \times [0,1]))$ from $R_3 \cup R_4$. The pink region $(4v + L) \cap [0,1]^2 \subset v + (3v + L) \cap [0,1]^2$, so we also cut it out. Now, we do case on $a + b$.

(a) Suppose $a + b \geq 17/50$. The top orange region $U \setminus R_5$ overlaps with the pink region $(4v + L) \cap [0,1]^2$, so we can cut out the entire region R_5. Hence, in total, we cut out

$$\frac{|\overline{S}| + O(n)}{n^2} \geq \text{Area}(R_2) + \text{Area}(R_4 \cup R_5) + \text{Area}(L \cap ([1-3a, 1-2a] \times [0,1]))$$
$$\geq a + b - 3ab + (1-3a)(1-3b) + \frac{1}{2}(4a+b-1)^2 - \frac{1}{2}(3a+b-1)^2,$$

which in this case is minimized at $a = b = 5/21$ with value $0.404\ldots > 2/5$.

(b) Suppose $a + b < 17/50$. We need to consider the white parallelogram between lines $x + y = 5(a+b)$ and $x + y = 17/10$, and with $x \in [4a, 1]$. We can compute its area $(4a-1)(5(a+b)-17/10)$ and subtract it from the total we cut out in case (a) to obtain

$$\frac{|\overline{S}| + O(n)}{n^2} \geq a + b - 3ab + (1-3a)(1-3b) + \frac{1}{2}(4a+b-1)^2 - \frac{1}{2}(3a+b-1)^2 - (4a-1)\left(5(a+b) - \frac{17}{10}\right),$$

which in this case is minimized at $(a,b) = (1/4, 9/100)$ with value $0.446\ldots > 2/5$.

(5) Suppose $a < 1/5$. As $v \neq \mathbf{0}$, there exists a unique positive integer $k \geq 3$ such that either

$$\frac{1}{2k} \leq a < \frac{1}{2k-1} \quad \text{or} \quad \frac{1}{2k+1} \leq a < \frac{1}{2k}.$$

(a) For the first case, the non-empty regions are $\{R_t : t \in [2k]\}$. By (3.6), we include

$$\frac{|S| - O(n)}{n^2} \leq \sum_{t=1}^{k} \text{Area}(R_{2t-1}) = k(a + b - (2k-1)ab),$$

which under the (slightly relaxed) constraints $1/2k \leq a \leq 1/(2k-1)$ and $0 \leq b \leq a$ is maximized at $a = b = 1/(2k-1)$ with value $k/(2k-1) \leq 3/5$, as $k \geq 3$.

(b) For the second case, the non-empty regions are $\{R_t : t \in [2k+1]\}$. By (3.6), we cut out

$$\frac{|\overline{S}| + O(n)}{n^2} \geq \sum_{t=1}^{k} \text{Area}(R_{2t}) = k(a + b - (2k+1)ab),$$

which under the (slightly relaxed) constraints $1/(2k+1) \leq a \leq 1/2k$ and $0 \leq b \leq a$ is minimized at $a = b = 1/2k$ with value $(2k-1)/4k > 2/5$, as $k \geq 3$.

In all the cases above, we showed $\left(|\overline{S}| - O(n)\right)/n^2 \geq 2/5$, so $|S| \leq 3n^2/5 + O(n)$. $\square$

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