EXCEPTIONAL SET ESTIMATES FOR ORTHOGONAL AND RADIAL PROJECTIONS IN $\mathbb{R}^n$

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ABSTRACT. We give different proofs of classic Falconer-type and Kaufman-type exceptional estimates for orthogonal projections using the high-low method. With the new techniques, we resolve Liu's conjecture on radial projections: given a Borel set $A \subset \mathbb{R}^n$, we have

$$\dim(\{x \in \mathbb{R}^n \setminus A \mid \dim(\pi_x(A)) < \dim A\}) \leq \lceil \dim A \rceil.$$

CONTENTS

1. INTRODUCTION

In this paper, we study the orthogonal and radial projections in $\mathbb{R}^n$.

Let $G(n,m)$ be the set of m-dimensional subspaces in $\mathbb{R}^n$, which is also known as the Grassmannian. For $V \in G(n,m)$, define $\pi_V : \mathbb{R}^n \to V$ to be the orthogonal projection onto V. Given $x \in \mathbb{R}^n$, define $\pi_x : \mathbb{R}^n \setminus \{x\} \to \mathbb{S}^{n-1}$ to be the radial projection centered at x:

$$\pi_x(y) = \frac{y-x}{|y-x|}.$$

We first discuss some background of the projection theory. We use $\dim X$ to denote the Hausdorff dimension of the set X. Projection theory dates back to Marstrand [8], who showed that if A is a Borel set in $\mathbb{R}^2$, then the projection of A onto almost every line through the origin has Hausdorff dimension $\min\{1, \dim A\}$. This was generalized to higher dimensions by Mattila [9], who showed that if A is

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a Borel set in $\mathbb{R}^n$, then the projection of A onto almost every k-plane through the origin has Hausdorff dimension $\min\{k, \dim A\}$. It turns out that one can obtain some finer results which are known as the exceptional set estimates. The exceptional set estimates give a bound on the set of directions where the projection is small. Two classic types of exceptional set estimates (see Proposition 1, 2) were obtained by Falconer [1] and Kaufman [5] using s-energy and Fourier analysis.

In this paper, we provide a approach to these two theorems from another perspective, by using incidence geometry and high-low method. The connection between projection theory and incidence geometry is well known now, and dates back to Wolff [13]. This connection is used in a fair amount of recent work in projection theory, but this paper revisits these two classical theorems of projection theory from the incidence geometry angle. We believe that our proofs have the same fundamental idea as the original proofs by Falconer and Kaufman, but writing the proofs using incidence geometry and high-low method makes these ideas a little more flexible. Actually, our approach can also be applied to resolve a conjecture of Liu on radial projections(see [6] Conjecture 1.2).

We first state the following two classic results of Falconer and Kaufman. We also recommend [10] Theorem 5.10 for the classic proofs.

Proposition 1 (Falconer-type)**.** *Suppose $A \subset \mathbb{R}^n$ is a Borel set of Hausdorff dimension α. For $0 \le s < \min\{m, \alpha\}$, define the exceptional set*

$$E_s(A) = \{V \in G(n,m) \mid \dim(\pi_V(A)) < s\}.$$

Then we have

$$\dim(E_s(A)) \le \max\{m(n-m) + s - \alpha, 0\}.$$

Proposition 2 (Kaufman-type)**.** *Suppose $A \subset \mathbb{R}^n$ is a Borel set of Hausdorff dimension α. For $0 \le s < \min\{m, \alpha\}$, define the exceptional set*

$$E_s(A) = \{V \in G(n,m) \mid \dim(\pi_V(A)) < s\}.$$

Then we have

$$\dim(E_s(A)) \le m(n-m-1) + s$$

Let us turn to the radial projections. We first state our two theorems.

Theorem 1. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\alpha = \dim A \in (k, k+1]$ for some $k \in \{1, \dots, n-1\}$. Fix $0 < s < k$ and let*

$$E_s(A) := \{y \in \mathbb{R}^n \setminus A \mid \dim(\pi_y(A)) < s\}.$$

Then,

$$\dim(E_s(A)) \le \max\{k + s - \alpha, 0\}.$$

Theorem 2. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\alpha = \dim A \in (k-1, k]$ for some $k \in \{1, \dots, n-1\}$. Define the exceptional set*

$$E(A) := \{x \in \mathbb{R}^n \setminus A \mid \dim(\pi_x(A)) < \alpha\}.$$

Then we have

$$\dim(E(A)) \le k.$$

Theorem 2 is sharp. If we let A be an α-dimensional subset of $\mathbb{R}^k$, we see that $E(A) = \mathbb{R}^k \setminus A$ which has dimension k.

We remark that Theorem 1 is a conjecture made by Lund, Pham and Thu ([7] Conjecture 1.2); Theorem 2 is Liu's conjecture ([6] Conjecture 1.2).

Recently, Orponen and Shmerkin [12] proved the $n = 2$ case for both Theorem 1 and Theorem 2. Their proof of Theorem 1 (when $n = 2$) is based on a Furstenberg-type estimate due to Fu and Ren [3]. Then by a swapping trick, they are able to prove Theorem 2 (when $n = 2$). In this paper, we prove the Theorems for all dimensions. We remark that the upper bound in Theorem 1 is a Falconer-type bound, compared with Proposition 1. In the later sections, we will see many similarities between the proofs of Proposition 1 and Theorem 1.

Let us also talk about the relations between orthogonal projection and radial projection. For an $(n-1)$-plane V, we can view the orthogonal projection π_V as a radial projection $\pi_{x(V)}$ whose projection center $x(V)$ lies on the infinite hyperplane. Therefore, we see that Theorem 1 with $k = n-1$ implies Proposition 1 with $m = 1$. However, Proposition 1 only implies a weaker result than Theorem 1, which is of form

$$\dim(E_s(A) \cap \Pi) \leq \max\{k + s - \alpha, 0\}.$$

Here, Π is an $(n-1)$-plane. Such comparison between orthogonal projection and radial projection was discussed in Orponen and Shmerkin's paper (see [12] (1.4)). For more backgrounds on the radial projections, we refer to Orponen's paper [11].

We briefly discuss about the strategies in the proofs. We mainly use the high-low method and the double counting technique. For instance, the proof of Proposition 3 uses the high-low method. Then the proof of Proposition 4 uses both of the tricks. Theorem 1 is a result of Proposition 3 and Proposition 4. Finally, the proof of Theorem 2 is a combination of Theorem 1 and a trick of Orponen and Shmerkin [12].

We talk about the structure of the paper. In Section 2, we prove Proposition 1. In Section 3, we prove Proposition 2. In Section 4, we prove Theorem 1. In Section 5, we prove Theorem 2.

1.1. **Some notations.** We will frequently use the following definitions.

Definition 1. *For a number $\delta > 0$ and any set X (in a metric space), we use $|X|_\delta$ to denote the maximal number of δ-separated points in X.*

The simplest definition of a (δ, s)-set is seen on $\mathbb{R}^n$:

Definition 2. *Let $\delta, s > 0$, and let $A \subset \mathbb{R}^n$ be a finite δ-separated set. We say A is a (δ, s)-set if it satisfies the following estimate:*

$$\#(A \cap B_r(x)) \lesssim (r/\delta)^s$$

for any $x \in \mathbb{R}^n$ and $r \geq \delta$.

Remark 1. *Throughout the rest of this paper, I will use $\#E$ to denote the cardinality of a set E and $|\cdot|$ to denote the measure of a region.*

These types of sets are vastly useful in helping us reduce discrete sums, find upper bounds for cardinalities of sets or measures of regions, etc. For instance, consider the following lemma:

Lemma 1. *Let $\delta, s > 0$ and let $B \subset \mathbb{R}^n$ be any set with $\mathcal{H}^s_\infty(B) =: \kappa > 0$. Then, there exists a (δ, s)-set $P \subset B$ with $\#P \gtrsim \kappa\delta^{-s}$.*

Proof. See [2] Lemma 3.13. $\square$

In this paper, we will focus on the planks T of dimensions

$$\underbrace{\delta \times \delta \times \cdots \times \delta}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$$

which are contained in $B^n(0,2)$, so we want a similar condition for when a collection of planks is a (δ, s)-set. Recall that $G(n,m)$ is the Grassmannian, and we also use $A(n,m)$ to denote the set of m-planes in $\mathbb{R}^n$ that intersect with $B^n(0,1)$. Therefore, $A(n,m)$ is a bounded subset of $AG(n,m)$ (the set of all m-dimensional spaces in $\mathbb{R}^n$). We can view $A(n,m)$ as a manifold, and $G(n,m)$ is a submanifold of $A(n,m)$. Let $A(n,m)$ be equipped with the metric d given by

$$d(V_1, V_2) = \|\pi_{V_1} - \pi_{V_2}\| + |a_1 - a_2|,$$

where $a_i = V_i \cap V_i^{\perp}$ (see also [12] Definition 2.2).

There is a natural correspondence between such planks and the δ-balls in $A(n,n-m)$. Given such a δ-thick plank T, we let $V_T \in A(n,n-m)$ be the central $(n-m)$-plane of T. We let T correspond to $B(V_T, \delta)$ $(\subset A(n,n-m))$. We can see that if T, T' are essentially the same (in the sense that $C^{-1}T \subset T' \subset CT$), then $V_T \in B(V_{T'}, C')$, and vice versa. Here the constant C, C' depend on each other. In particular, if $\{T\}$ is a set of essentially distinct planks, then $\{V_T\}$ are $\sim \delta$-separated points in $A(n,n-m)$. We call these planks $(n,n-m,\delta)$-planks. When the ambient dimension n and the codimension of the planks m are fixed, we just simply call such planks δ-planks.

Definition 3. *Let $\mathbb{T}$ be a collection of δ-planks in $\mathbb{R}^n$. We say $\mathbb{T}$ is a (δ, s)-set if*

(1) $\mathbb{T}$ are essentially distinct, and

(2) for each $r \geq \delta$ and any ball $B_r \subset A(n,n-m)$, we have

$$\#\{T \in \mathbb{T} \mid V_T \in B_r \neq \emptyset\} \lesssim (r/\delta)^s.$$

Remark 2. Actually, the condition (2) in the definition above can be replaced by

$$\#\{T \in \mathbb{T} : T \subset P_r\} \lesssim (r/\delta)^s, \quad \text{for any } (n, n-m, r\delta)\text{-plank } P_r,$$

which we will use often.

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2. Falconer-type estimate

In this section of the paper, we use the high-low method to prove Falconer's Bound in $\mathbb{R}^n$. We first discretize the problem.

2.1. δ-discretization.

We will prove the following δ-discretized version of Proposition 1:

Theorem 3. *Fix an integer $m \in [1,n]$, a number $a \in (0,n]$ and $0 < s < \min\{m, a\}$. For each $\varepsilon > 0$, there exists $C_{s,\varepsilon}$ so that the following holds. Let $\delta > 0$. Let $H \subset B^n(0,1)$ be a (δ, a)-set with $\#H \gtrsim (\log \delta^{-1})^{-2}\delta^{-a}$. Let $\mathcal{V}$ be a δ-separated subset of $G(n,m)$ such that $\mathcal{V}$ is a (δ, t)-set and $\#\mathcal{V} \gtrsim (\log \delta^{-1})^{-2}\delta^{-t}$ for some $t > 0$. Assume for each $V \in \mathcal{V}$, we have a collection of $(n, n-m, \delta)$-planks $\mathbb{T}_V$ orthogonal to V. $\mathbb{T}_V$ satisfies the s-dimensional condition:*

(1) $\#\mathbb{T}_V \lesssim \delta^{-s}$, *and*
(2) $\#\{T \in \mathbb{T}_V : T \subset P_r\} \lesssim (r/\delta)^s$, *for any* $(n, n-m, r)$*-plank* P_r *with* $\delta \leq r \leq 1$.

We also assume that each δ*-ball contained in* H *intersects* $\gtrsim (\log \delta^{-1})^{-2} \#\mathcal{V}$ *many planks from* $\bigcup_{V \in \mathcal{V}} \mathbb{T}_V$. *Then*

$$\delta^{-t} \leq C_{s,\varepsilon} \delta^{-m(n-m)-s+a-\varepsilon}.$$

We will first show that Theorem 3 implies Proposition 1, and then prove Theorem 3. Before starting the proof, we state a very useful lemma. We use the following notation. Fix a dimension m. For any $\delta = 2^{-k}$ ($k \in \mathbb{N}^+$), let $\mathcal{D}_\delta$ denote the lattice of δ-cubes in $[0,1]^m$.

Lemma 2. *Suppose* $X \subset [0,1]^m$ *with* $\dim X < s$. *Then for any* $\varepsilon > 0$, *there exist dyadic cubes* $\mathcal{C}_{2^{-k}} \subset \mathcal{D}_{2^{-k}}$ $(k > 0)$ *so that*

(1) $X \subset \bigcup_{k>0} \bigcup_{D \in \mathcal{C}_{2^{-k}}} D$,
(2) $\sum_{k>0} \sum_{D \in \mathcal{C}_{2^{-k}}} r(D)^s \leq \varepsilon$,
(3) $\mathcal{C}_{2^{-k}}$ *satisfies the* s*-dimensional condition: For* $l < k$ *and any* $D \in \mathcal{D}_{2^{-l}}$, *we have* $\#\{D' \in \mathcal{C}_{2^{-k}} : D' \subset D\} \leq 2^{(k-l)s}$.

Proof. See [4] Lemma 2. □

Remark 3. Besides $[0,1]^m$, this Lemma also works for other compact metric spaces, for example $\mathbb{S}^n$ and $G(n,m)$, which we will use throughout the rest of the paper.

Proof of Proposition 1 assuming Theorem 3. Suppose $A \subset \mathbb{R}^n$ is a Borel set. We may assume $A \subset B^n(0,1)$. Define the exceptional set

$$E_s(A) := \{V \in G(n,m) : \dim(\pi_V(A)) < s\}.$$

Recall that the definition of the t-dimensional Hausdorff content is given by

$$\mathcal{H}^t_\infty(B) := \inf\left\{\sum_i r(B_i)^t : B \subset \bigcup_i B_i\right\}.$$

A property for the Hausdorff content is that

$$\dim(B) = \sup\{t : \mathcal{H}^t_\infty(B) > 0\}.$$

We choose a, t such that $\mathcal{H}^a_\infty(A) > 0, \mathcal{H}^t_\infty(E_s(A)) > 0$. We only need to prove

$$a \leq m(n-m) + s - t,$$

since then we can send $a \to \dim(A)$ and $t \to \dim(E_s(A))$. As a and t are fixed, we may assume $\mathcal{H}^a_\infty(A), \mathcal{H}^t_\infty(E_s(A)) \sim 1$ are constants.

Fix a $V \in E_s(A)$. By definition, we have $\dim(\pi_V(A)) < s$. We also fix a small number $\epsilon_\circ$ which we will later send to 0. By Lemma 2, we can find a covering of $\pi_V(A)$ by disks $\mathbb{D}_V = \{D\}$, each of which has radius 2^{-j} for some integer $j > |\log_2 \epsilon_\circ|$. We define $\mathbb{D}_{V,j} := \{D \in \mathbb{D}_V : r(D) = 2^{-j}\}$. Lemma 2 yields the following properties:

$$\sum_{D \in \mathbb{D}_V} r(D)^s < 1, \tag{1}$$

and for each j and r-ball $B_r \subset V$, we have

$$\#\{D \in \mathbb{D}_{V,j} : D \subset B_r\} \lesssim \left(\frac{r}{2^{-j}}\right)^s. \tag{2}$$

After finding such a $\mathbb{D}_V$, we can define a set of δ-planks by lifting $\mathbb{D}_V$. More precisely, we define the plank sets $\mathbb{T}_{V,j} := \{\pi_V^{-1}(D) : D \in \mathbb{D}_{V,j}\} \cap B^n(0,2)$, $\mathbb{T}_V = \bigcup_j \mathbb{T}_{V,j}$. Each plank in $\mathbb{T}_{V,j}$ has dimensions

$$\underbrace{2^{-j} \times 2^{-j} \times \cdots \times 2^{-j}}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$$

such that the $1\times\cdots\times1$ 'side' is orthogonal to V. One easily sees that $A \subset \bigcup_{T\in\mathbb{T}_V} T$. By pigeonholing, there exists $j(V)$ such that

$$\mathcal{H}^a_\infty(A \cap (\cup_{T\in\mathbb{T}_{V,j(V)}} T)) \geq \frac{1}{10j(V)^2}\mathcal{H}^a_\infty(A). \tag{3}$$

For each $j > |\log_2 \epsilon_\circ|$, define $E_{s,j}(A) := \{V \in E_s(A) : j(V) = j\}$. Then we obtain a partition of $E_s(A)$:

$$E_s(A) = \bigsqcup_j E_{s,j}(A).$$

By pigeonholing again, there exists j such that

$$\mathcal{H}^t_\infty(E_{s,j}(A)) \geq \frac{1}{10j^2}\mathcal{H}^t_\infty(E_s(A)) \sim \frac{1}{10j^2}. \tag{4}$$

In the rest of the proof, we fix this j. We also set $\delta = 2^{-j}$. By Lemma 1, there exists a (δ, t)-set $\mathcal{V} \subset E_{s,j}(A)$ with cardinality $\#\mathcal{V} \gtrsim (\log\delta^{-1})^{-2}\delta^{-t}$.

Next, we consider the set $S := \{(x, V) \in A \times \mathcal{V} : x \in \cup_{T\in\mathbb{T}_{V,j}} T\}$. We also use μ to denote the counting measure on $\mathcal{V}$. Define the sections of S:

$$S_x = \{V : (x, V) \in S\}, \quad S_V := \{x : (x, V) \in S\}.$$

By (3) and Fubini, we have

$$(\mathcal{H}^a_\infty \times \mu)(S) \geq \frac{1}{10j^2}\mathcal{H}^a_\infty(A)\mu(\mathcal{V}). \tag{5}$$

This implies

$$(\mathcal{H}^a_\infty \times \mu)\bigg(\Big\{(x, V) \in S : \mu(S_x) \geq \frac{1}{20j^2}\mu(\mathcal{V})\Big\}\bigg) \geq \frac{1}{20j^2}\mathcal{H}^a_\infty(A)\mu(\mathcal{V}), \tag{6}$$

since

$$(\mathcal{H}^a_\infty \times \mu)\bigg(\Big\{(x, V) \in S : \mu(S_x) \leq \frac{1}{20j^2}\mu(\mathcal{V})\Big\}\bigg) \leq \frac{1}{20j^2}\mathcal{H}^a_\infty(A)\mu(\mathcal{V}). \tag{7}$$

By (6), we have

$$\mathcal{H}^a_\infty\bigg(\Big\{x \in A : \mu(S_x) \geq \frac{1}{20j^2}\mu(\mathcal{V})\Big\}\bigg) \geq \frac{1}{20j^2}\mathcal{H}^a_\infty(A) \sim \frac{1}{20j^2}. \tag{8}$$

We are ready to apply Theorem 3. Recall $\delta = 2^{-j}$ and $\#\mathcal{V} \gtrsim (\log\delta^{-1})^{-2}\delta^{-t}$. By (8) and Lemma 1, we can find a δ-separated subset of $\{x \in A : \#S_x \geq \frac{1}{20j^2}\#\mathcal{V}\}$ with cardinality $\gtrsim (\log\delta^{-1})^{-2}\delta^{-a}$. We denote this set by H. For $x \in H$, we see that there are $\gtrsim (\log\delta^{-1})^{-2}\#\mathcal{V}$ many planks from $\cup_{V\in\mathcal{V}}\mathbb{T}_{V,j}$ that intersect x. We can now apply Theorem 3 to obtain

$$\delta^{-a-t} \leq C_{s,\varepsilon}\delta^{-m(n-m)-s-\varepsilon}.$$

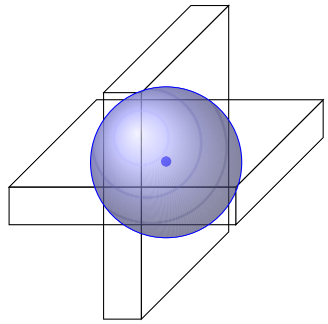

FIGURE 1. Dual Slabs

Letting $\epsilon_\circ \to 0$ (and hence $\delta \to 0$) and then $\varepsilon \to 0$, we obtain

$$a + t \leq m(n-m) + s.$$

□

2.2. **Discretized Falconer-type estimate.** In this subsection, we prove Theorem 3.

Proof of Theorem 3. For each $V \in \mathcal{V}$, let S_V be a

$$\underbrace{\delta^{-1} \times \delta^{-1} \times \cdots \times \delta^{-1}}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$$

slab centered at the origin such that the $1 \times 1 \times \cdots \times 1$ 'side' is orthogonal to V. Then, these slabs are dual to $\mathbb{T}_V$.

For all $T \in \mathbb{T}_V$, choose a bump function ψ_T such that $\psi_T \geq 1$ on T, ψ_T decays rapidly outside of T, and supp $\widehat{\psi}_T \subset S_V$.

Define

$$f_V = \sum_{T \in \mathbb{T}_V} \psi_T \quad \text{and} \quad f = \sum_{V \in \mathcal{V}} f_V.$$

Then, by definition, $f(x) \gtrsim (\log \delta^{-1})^{-2} \# \mathcal{V} \gtrsim (\log \delta^{-1})^{-4} \delta^{-t}$. So,

$$\delta^{n-a-2t} \lessapprox \# H (\# \mathcal{V})^2 \lessapprox \int_H |f|^2. \tag{9}$$

Here, $\lessapprox$ means $\lesssim (\log \delta^{-1})^{O(1)}$.

We are going to find an upper bound of $\int_H |f|^2$ using the high-low method. Let K be a large number to be determined later (we will actually choose $K \sim (\log \delta^{-1})^{O(1)}$). Let $\eta_{low}(\xi)$ be a smooth bump function on $B^n(0, (K\delta)^{-1})$ and $\eta_{high}(\xi) = 1 - \eta_{low}(\xi)$. We have the following high-low decomposition for f:

$$f = f_{low} + f_{high},$$

where $\widehat{f}_{low} = \eta_{low} \widehat{f}$ and $\widehat{f}_{high} = \eta_{high} \widehat{f}$. See Figure 1 for a diagram of the high part and low part and the dual slabs.

For $x \in H$, we have

$$(\log \delta^{-1})^{-2} \# \mathcal{V} \lesssim f(x) \leq |f_{high}(x)| + |f_{low}(x)|. \tag{10}$$

We will show that the high part dominates for $x \in H$, i.e., $|f_{high}(x)| \gtrsim (\log \delta^{-1})^{-2} \# \mathcal{V}$. It suffices to show

$$|f_{low}(x)| \le C^{-1} (\log \delta^{-1})^{-2} \# \mathcal{V}. \tag{11}$$

Recall that $f_{low} = \sum_{V \in \mathcal{V}} f_V * \eta_{low}^{\vee}$. Since η_{low} is a bump function on $B^n(0, (K\delta)^{-1})$, we see that $\eta_{low}^{\vee}$ is an L^1-normalized bump function essentially supported in $B^n(0, K\delta)$. Let $\chi(x)$ be a positive function $= 1$ on $B^n(0, K\delta)$ and decays rapidly outside $B^n(0, K\delta)$. We have

$$|\eta_{low}^{\vee}| \lesssim \frac{1}{|B^n(0, K\delta)|} \chi.$$

Therefore,

$$|f_{low}(x)| \lesssim \sum_{V \in \mathcal{V}} \sum_{T \in \mathbb{T}_V} \psi_T * \frac{1}{|B^n(0, K\delta)|} \chi(x) \lesssim \sum_{V \in \mathcal{V}} \sum_{T \in \mathbb{T}_V} K^{-m} \chi_{T_K}(x). \tag{12}$$

Here, each T_K is a $(n, n-m, K\delta)$-plank which is the K-thickening of the $\delta \times \cdots \times \delta$ 'side' of T, and χ_{T_K} is a bump function $= 1$ on T_K and decays rapidly outside T_K. We just ignore the rapidly decaying tail and think of each χ_{T_K} as a indicator of T_K. For a fixed $V \in \mathcal{V}$, we note that $\{T : T \in \mathbb{T}_V\}$ are orthogonal to V. Therefore, if we let $P_{K\delta}$ be an $(n, n-m, 100K\delta)$-plank orthogonal to V and contains x, then by Remark 2,

$$\sum_{T \in \mathbb{T}_V} \chi_{T_K}(x) \lesssim \#\{T \in \mathbb{T}_V : T \subset P_{K\delta}\} \lesssim K^s,$$

where the last inequality is by the s-dimensional condition of $\mathbb{T}_V$. Plugging this back into (12), we obtain

$$|f_{low}(x)| \lesssim K^{s-m} \# \mathcal{V}.$$

Noting that $s < m$, we may choose $K \sim (\log \delta^{-1})^{O(1)}$ large enough so that (11) holds.

Now, we have

$$\delta^{n-a-2t} \lessapprox \int |f_{high}|^2 = \int \left| \sum_{V \in \mathcal{V}} \widehat{f_V} \eta_{high} \right|^2$$

We use the following lemma to estimate the overlap of $\{\mathrm{supp}(\widehat{f_V} \eta_{high})\}_{V \in \mathcal{V}}$, or more precisely $\{S_V \setminus B^n(0, (K\delta)^{-1})\}_{V \in \mathcal{V}}$.

Lemma 3. *$\{S_V \setminus B^n(0, (K\delta)^{-1})\}_{V \in \mathcal{V}}$ is $\lesssim K^{O(1)} \delta^{-\dim(G(n-1,m-1))}$-overlapping.*

Proof. Let $\boldsymbol{\xi}_0 = (0, \ldots, 0, (K\delta)^{-1})$. We just need to show that the number of planks S_V that pass through 0 and $\boldsymbol{\xi}_0$ is $\lesssim K^{O(1)} \delta^{-\dim(G(n-1,m-1))}$. Since we allow $K^{O(1)}$-loss, we may assume $K = 100$ (it will be clear from our proof that such assumption is allowable).

Consider the $\widetilde{G} = \{W \in G(n, m) : 0, \boldsymbol{\xi}_0 \in W\}$ which is a submanifold of $G(n, m)$. Note that each $W \in \widetilde{G}$ is orthogonal to the $(n-1)$-plane $\Pi = \{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) : \xi_n = \frac{1}{100} \delta^{-1}\}$ and contains $\boldsymbol{\xi}_0 \in \Pi$. We can project each $W \in \widetilde{G}$ to Π, which gives an $(m-1)$-dimensional plane W' pass through $\boldsymbol{\xi}_0$. Conversely, given any $(m-1)$-plane W' in Π that contains $\boldsymbol{\xi}_0$, the inverse image of W' under the projection is an m-plane in $\widetilde{G}$. Actually, this map gives rise to a homeomorphism $\widetilde{G} \simeq G(n-1, m-1)$. The only thing we need is $\dim \widetilde{G} = \dim(G(n-1, m-1))$.

Let us come back to the slabs. If S_V contains $0, \boldsymbol{\xi}_0$, then the central m-plane of S_V corresponds to a point in $N_{100\delta}\widetilde{G}(\subset G(n,m))$. Therefore, since $\{S_V\}$ are essentially distinct for all $V \in \mathcal{V}$, we see that the number of S_V that pass through 0 and $\boldsymbol{\xi}_0$ is $\lesssim |N_{100\delta}\widetilde{G}(\subset G(n,m))|_\delta \lesssim \delta^{-\dim(G(n-1,m-1))}$. □

We are now able to find an upper bound to the high part of the integral. We have

$$\delta^{n-a-2t} \lessapprox \int |f_{high}|^2 = \int |\widehat{f}_{high}|^2 \lessapprox_s \delta^{-\dim(G(n-1,m-1))} \sum_{V\in\mathcal{V}} \int |\eta_{high}\widehat{f}_V|^2$$

by Lemma 3. Since $|\eta_{high}| \lesssim 1$ and the planks in $\mathbb{T}_V$ (for a fixed V) are essentially disjoint, we have

$$\int |\eta_{high}\widehat{f}_V|^2 \lesssim \sum_{V\in\mathcal{V}} \int |f_V|^2 \lesssim \sum_{V\in\mathcal{V}} \sum_{T\in\mathbb{T}_V} \int |\psi_T|^2 \leq (\#\mathcal{V})(\#\mathbb{T}_V)\delta^m \lesssim \delta^{-t-s+m}$$

Combining everything and noting that $\dim(G(n-1,m-1)) = (m-1)(n-m)$, we have that

$$\delta^{-t} \lessapprox_{s,\varepsilon} \delta^{-m(n-m)-s+a-\varepsilon}.$$

□

3. Kaufman-type estimate

In this section of the paper, we prove Proposition 2. We first state a discretized version.

Theorem 4. *Fix an integer $m \in [1,n]$, $0 < s < m$, $t > m(n-m-1)$ and $0 < u \leq t - m(n-m-1)$. For sufficiently small $\varepsilon > 0$ (depending on s, t, and u), the following holds. Let $\delta > 0$. Let $H \subset B^n(0,1)$ be a (δ,u)-set with $\#H \gtrsim (\log\delta^{-1})^{-2}\delta^{-u}$ (we use $\#H$ to denote the number of δ-balls in H). Let $\mathcal{V}$ be a δ-separated subset of $G(n,m)$ such that $\mathcal{V}$ is a (δ,t)-set and $\#\mathcal{V} \gtrsim (\log\delta^{-1})^{-2}\delta^{-t}$ for some $t > 0$. Assume for each $V \in \mathcal{V}$, we have a collection of $(n, n-m, \delta)$-planks $\mathbb{T}_V$ orthogonal to V. $\mathbb{T}_V$ satisfies the s-dimensional condition:*

1. *$\#\mathbb{T}_V \lesssim \delta^{-s}$,*
2. *$\#\{T \in \mathbb{T}_V : T \subset P_r\} \lesssim (r/\delta)^s$, for any P_r being a $(n, n-m, r)$-plank $(\delta \leq r \leq 1)$.*

We also assume that each δ-ball contained in H intersects $\gtrsim (\log\delta^{-1})^{-2}\#\mathcal{V}$ many planks from $\cup_{V\in\mathcal{V}}\mathbb{T}_V$. Then

$$\delta^{-u} \lessapprox_\varepsilon \delta^{-s-\varepsilon}.$$

We first prove Proposition 2 assuming Theorem 4. This will follow from the same scheme as we did for the proof of Proposition 1 assuming Theorem 3.

3.1. δ-discretization.

Proof of Proposition 2 assuming Theorem 4. Suppose $A \subset \mathbb{R}^n$ is a Borel set. We may assume $A \subset B^n(0,1)$. Define the exceptional set

$$E_s(A) := \{V \in G(n,m) : \dim(\pi_V(A)) < s\}.$$

Using the same argument as in the previous section, choose a, t such that $\mathcal{H}^a_\infty(A) > 0, \mathcal{H}^t_\infty(E_s) > 0$. We only need to prove

$$t \leq m(n-m-1) + s,$$

since then we can send $a \to \dim(A)$ and $t \to \dim(E_s(A))$. As a and t are fixed, we may assume $\mathcal{H}^a_\infty(A), \mathcal{H}^t_\infty(E_s(A)) \sim 1$ are constants.

Fix a $V \in E_s(A)$. By definition, we have $\dim(\pi_V(A)) < s$. We also fix a small number $\epsilon_\circ$ which we will later send to 0. By Lemma 2, we can find a covering of $\pi_V(A)$ by disks $\mathbb{D}_V = \{D\}$, each of which has radius 2^{-j} for some integer $j > |\log_2 \epsilon_\circ|$. We define $\mathbb{D}_{V,j} := \{D \in \mathbb{D}_V : r(D) = 2^{-j}\}$. Lemma 2 yields the following properties:

$$\sum_{D \in \mathbb{D}_V} r(D)^s < 1, \tag{13}$$

and for each j and r-ball $B_r \subset V$, we have

$$\#\{D \in \mathbb{D}_{V,j} : D \subset B_r\} \lesssim \left(\frac{r}{2^{-j}}\right)^s. \tag{14}$$

For each $V \in E_s(A)$, we can find such a $\mathbb{D}_V$. We also define the plank sets $\mathbb{T}_{V,j} := \{\pi_V^{-1}(D) : D \in \mathbb{D}_{V,j}\} \cap B^n(0,1)$, $\mathbb{T}_V = \bigcup_j \mathbb{T}_{V,j}$. Each plank in $\mathbb{T}_{V,j}$ has dimensions

$$\underbrace{2^{-j} \times 2^{-j} \times \cdots \times 2^{-j}}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$$

such that the $1 \times \cdots \times 1$ 'side' is orthogonal to V. One easily sees that $A \subset \bigcup_{T \in \mathbb{T}_V} T$. By pigeonholing, there exists $j(V)$ such that

$$\mathcal{H}^a_\infty(A \cap (\cup_{T \in \mathbb{T}_{V,j(V)}} T)) \geq \frac{1}{10 j(V)^2} \mathcal{H}^a_\infty(A). \tag{15}$$

For each $j > |\log_2 \epsilon_\circ|$, define $E_{s,j}(A) := \{V \in E_s(A) : j(V) = j\}$. Then we obtain a partition of $E_s(A)$:

$$E_s(A) = \bigsqcup_j E_{s,j}(A).$$

By pigeonholing again, there exists j such that

$$\mathcal{H}^t_\infty(E_{s,j}(A)) \geq \frac{1}{10j^2} \mathcal{H}^t_\infty(E_s(A)) \sim \frac{1}{10j^2}. \tag{16}$$

In the rest of the proof, we fix this j. We also set $\delta = 2^{-j}$. By Lemma 1, there exists a (δ, t)-set $\mathcal{V} \subset E_{s,j}(A)$ with cardinality $\#\mathcal{V} \gtrsim (\log \delta^{-1})^{-2} \delta^{-t}$.

Next, we consider the set $S := \{(x, V) \in A \times \mathcal{V} : x \in \cup_{T \in \mathbb{T}_{V,j}} T\}$. We also use μ to denote the counting measure on $\mathcal{V}$. Define the sections of S:

$$S_x = \{V : (x, V) \in S\}, \quad S_V := \{x : (x, V) \in S\}.$$

By (15) and Fubini, we have

$$(\mathcal{H}^a_\infty \times \mu)(S) \geq \frac{1}{10j^2} \mathcal{H}^a_\infty(A) \mu(\mathcal{V}). \tag{17}$$

This implies

$$(\mathcal{H}^a_\infty \times \mu)\Big(\Big\{(x, V) \in S : \mu(S_x) \geq \frac{1}{20j^2} \mu(\mathcal{V})\Big\}\Big) \geq \frac{1}{20j^2} \mathcal{H}^a_\infty(A) \mu(\mathcal{V}). \tag{18}$$

By (18), we have

$$\mathcal{H}^a_\infty\Big(\Big\{x \in A : \mu(S_x) \geq \frac{1}{20j^2} \mu(\mathcal{V})\Big\}\Big) \geq \frac{1}{20j^2} \mathcal{H}^a_\infty(A) \sim \frac{1}{20j^2}. \tag{19}$$

We are ready to apply Theorem 4. Recall $\delta = 2^{-j}$ and $\#\mathcal{V} \gtrsim (\log\delta^{-1})^{-2}\delta^{-t}$. We may assume $t > m(n-m-1)$, otherwise we are done. Set

$$u = \min\{t - m(n-m-1), a\} - \varepsilon.$$

By (19), we can find a (δ, u)-subset of $\{x \in A : \#S_x \geq \frac{1}{20j^2}\#\mathcal{V}\}$ with cardinality $\gtrsim (\log\delta^{-1})^{-2}\delta^{-u}$. We denote this set by H. For each $x \in H$, we see that there are $\gtrsim (\log\delta^{-1})^{-2}\#\mathcal{V}$ many planks from $\cup_{V\in\mathcal{V}}\mathbb{T}_{V,j}$ that intersect x. We can now apply Theorem 4 to obtain

$$\delta^{-u} \lesssim \delta^{-s-\varepsilon}.$$

Noting that $s < a$ and letting $\epsilon_\circ \to 0$ (and hence $\delta \to 0$) and then $\varepsilon \to 0$, we obtain $t \leq m(n-m-1) + s$. $\square$

3.2. **Discretized Kaufman-type estimate.**

Proof of Theorem 4. Set $\mathbb{T} := \bigcup_{V\in\mathcal{V}}\mathbb{T}_V$. For each $x \in H$, let $\mathbb{T}_x$ be the planks in $\mathbb{T}$ that intersect x. By assumption, we have $\#\mathbb{T}_x \gtrsim |\log\delta|^{-4}\delta^{-t}$. Also, $\mathbb{T}_x$ inherits the t-dimensional condition from $\mathcal{V}$. To see this, note that each $T \in \mathbb{T}_x$ is a $(n, n-m, \delta)$-plank passing through x and is orthogonal to some $V \in \mathcal{V}$. Such T corresponds to some $V_T \in G_x(n, n-m)$. Here, $G_x(n, n-m)$ is the set of $(n-m)$-planes passing through x. We may regard $G_x(n, n-m)$ as the Grassmannian $G(n, n-m)$. Since $\mathcal{V}$ is a (δ, t)-set, we see that $\{V_T : T \in \mathbb{T}_x\}$ is a (δ, t)-set in $G(n, n-m)$.

The theorem will be proved by comparing the upper and lower bound of $\#\mathbb{T}$. We easily see the upper bound

$$\#\mathbb{T} \lesssim \#\mathcal{V}\delta^{-s} \lesssim \delta^{-t-s}. \tag{20}$$

For the lower bound, we first choose a $\delta|\log\delta|^{O(1)}$-separated subset $H' \subset H$ with $\#H' \gtrsim |\log\delta|^{-O(1)}\delta^{-u}$. we have

$$\#\mathbb{T} = \#\left(\bigcup_{x\in H'}\mathbb{T}_x\right) \geq \#\left(\bigcup_{x\in H'}\left(\mathbb{T}_x \setminus \bigcup_{y\in H'\setminus\{x\}}\mathbb{T}_y\right)\right) \tag{21}$$

$$= \sum_{x\in H'}\#\left(\mathbb{T}_x \setminus \bigcup_{y\in H'\setminus\{x\}}\mathbb{T}_y\right) \tag{22}$$

$$= \sum_{x\in H'}\left(\#\mathbb{T}_x - \sum_{y\in H'\setminus\{x\}}\#\left(\mathbb{T}_x\cap\mathbb{T}_y\right)\right). \tag{23}$$

We show that

$$\#\mathbb{T}_x - \sum_{y\in H'\setminus\{x\}}\#\left(\mathbb{T}_x\cap\mathbb{T}_y\right) \geq \frac{1}{2}\#\mathbb{T}_x.$$

For fixed x, and $y \in H'\setminus\{x\}$, we want to find an upper bound for $\#(\mathbb{T}_x\cap\mathbb{T}_y)$. First, we consider the set of $(n-m)$-planes that pass through x and y:

$$G_{x,y}(n, n-m) := \{V \in G_x(n, n-m) : y \in V\}.$$

By the discussion in the proof of Lemma 3, We have

$$G_{x,y}(n, n-m) \simeq G_x(n-1, n-m-1)(\subset G_x(n, n-m)).$$

Since $\mathbb{T}_x\cap\mathbb{T}_y$ consists of δ-planks passing through x, y, we see that

$$\{V_T : T \in \mathbb{T}_x\cap\mathbb{T}_y\} \subset N_{\frac{\delta}{|x-y|}}\big(G_x(n-1, n-m-1)\big)(\subset G(n, n-m)).$$

Noting that $\dim(G_x(n-1,n-m-1)) = m(n-m-1)$, we can cover $N_{\frac{\delta}{|x-y|}}\big(G_x(n-1,n-m-1)\big)$ by $\sim \left(\frac{\delta}{|x-y|}\right)^{-m(n-m-1)}$ many $\frac{\delta}{|x-y|}$-balls in $B_{\frac{\delta}{|x-y|}} \subset G_x(n,n-m)$. By the (δ,t) property of $\mathbb{T}_x$, we have

$$\#\{V_T \in B_{\frac{\delta}{|x-y|}} : T \in \mathbb{T}_x\} \lesssim |x-y|^{-t}.$$

Therefore,

$$\#(\mathbb{T}_x \cap \mathbb{T}_y) \lesssim \left(\frac{\delta}{|x-y|}\right)^{-m(n-m-1)} |x-y|^{-t}.$$

So, we have

$$\begin{aligned}\sum_{y\in H'\setminus\{x\}} \#(\mathbb{T}_x \cap \mathbb{T}_y) &\lesssim \sum_{y\in H'\setminus\{x\}} \left(\frac{\delta}{|x-y|}\right)^{-m(n-m-1)} |x-y|^{-t} \\ &= \sum_{\delta|\log\delta|^{O(1)} \le d \le 1} \ \sum_{y\in H', |x-y|\sim d} \left(\frac{\delta}{d}\right)^{-m(n-m-1)} d^{-t}.\end{aligned}$$

Here the summation over d is over dyadic numbers. Since $\#(H' \cap B_d(x)) \lesssim (\frac{d}{\delta})^u$, the expression above is bounded by

$$\begin{aligned}&\lesssim \sum_{\delta|\log\delta|^{O(1)} \le d \le 1} \left(\frac{d}{\delta}\right)^{u} \left(\frac{d}{\delta}\right)^{m(n-m-1)} d^{-t} \\ &= \delta^{-t} \sum_{\delta|\log\delta|^{O(1)} \le d \le 1} \left(\frac{d}{\delta}\right)^{u+m(n-m-1)-t} \\ &\lesssim \delta^{-t}|\log\delta|^{O(1)(u+m(n-m-1)-t)}.\end{aligned}$$

Since $u+m(n-m-1)-t < 0$, by choosing the constant $O(1)$ big enough, we have

$$\sum_{y\in H'\setminus\{x\}} \#(\mathbb{T}_x \cap \mathbb{T}_y) \le C^{-1}|\log\delta|^{-4}\delta^{-t} \le \frac{1}{2}\#\mathbb{T}_x.$$

As a result, we have

$$\#\mathbb{T} \ge |\log\delta|^{-O(1)}\delta^{-u-t}.$$

Compared with the upper bound of $\#\mathbb{T}$ in (20), we finish the proof. $\square$

4. Falconer-type estimates for radial projections

In this section of the paper, we prove Theorem 1.

We introduce some notations. Fix $0 \le \sigma$, $\delta > 0$. For a bounded set $E \subset \mathbb{R}^n$, define

$$\mathcal{H}_{\delta,\infty}(E) := \inf\left\{\sum_j r(D_j)^s : E \subset \cup_j D_j\right\},$$

where the infimum runs over the coverings of E by a lattice of dyadic cubes $\{D_j\}$ with length $\ge \delta$, and $r(D)$ denotes the length of the cube. We state three useful lemmas about $\mathcal{H}^s_{\delta,\infty}$. Recall that $\mathcal{D}_\delta$ denotes the lattice of δ-cubes in $[0,1]^m$.

Lemma 4. *Suppose $X \subset [0,1]^m$. Then there exist dyadic cubes*

$$\mathcal{C} = \bigsqcup_{k=0}^{\log_2 \delta^{-1}} \mathcal{C}_{2^{-k}}$$

(with $\mathcal{C}_{2^{-k}} \subset \mathcal{D}_{2^{-k}}$) that cover X and

(1) $\sum_{D\in\mathcal{C}} r(D)^s = \mathcal{H}^s_{\delta,\infty}(X)$,

(2) $\mathcal{C}_{2^{-k}}$ satisfies the s-dimensional condition: For $l < k$ and any $D \in \mathcal{D}_{2^{-l}}$, we have $\#\{D' \in \mathcal{C}_{2^{-k}} : D' \subset D\} \le 2^{(k-l)s}$. In particular, $\mathcal{H}^s_{2^{-k},\infty}(\cup_{D\in\mathcal{C}_{2^{-k}}} D) = \#\mathcal{C}_{2^{-k}} 2^{-ks}$.

Proof. This lemma looks like Lemma 2, but it is much easier since we only care about the scales $\ge \delta$. We just choose $\mathcal{C}$ to be the covering that attain the "inf" in the definition of $\mathcal{H}^s_{\delta,\infty}(X)$. It is not hard to check the two properties are satisfied. □

The next lemma is [2] Proposition A.1. Though it is stated for $\mathcal{H}^s_\infty$ there, the proof also works for $\mathcal{H}^s_{\delta,\infty}$.

Lemma 5. *Suppose $X \subset [0,1]^m$, with $\mathcal{H}^s_{\delta,\infty}(X) = \kappa > 0$. Then there exists a (δ,s)-subset of X with cardinality $\gtrsim \kappa\delta^{-s}$.*

We also have the following lemma saying that the lemma above can be reversed.

Lemma 6. *Suppose $X \subset [0,1]^m$ is a (δ,s)-set with $\#X \ge \kappa\delta^{-s}$. Then, $\mathcal{H}^s_{\delta,\infty}(X) \gtrsim \kappa$. In particular, by Lemma 5, this implies that for any $\delta \le \Delta \le 1$, X contains a subset X' which is a (Δ,s)-set and satisfies $\#X' \gtrsim \kappa\Delta^{-s}$; and also implies that for any $u \le s$, X contains a subset X' which is a (Δ,u)-set and satisfies $\#X' \gtrsim \kappa\Delta^{-u}$.*

Proof. Assuming our (δ,s)-set X satisfies $\#X \ge \kappa\delta^{-s}$, we are going to show $\mathcal{H}^s_{\delta,\infty}(X) \gtrsim \kappa$. Let $\mathcal{C}$ be the covering of X that attains "inf" in the definition of $\mathcal{H}^s_{\delta,\infty}(X)$. Also let $\mathcal{C}_\Delta \subset \mathcal{C}$ be the set of Δ-cubes. We write $X = \bigsqcup_\Delta X_\Delta$, where X_Δ is the points in X covered $\mathcal{C}_\Delta$. By the definition of (δ,s)-set, each Δ-cube contains $\lesssim (\frac{\Delta}{\delta})^s$ many points from X_Δ. We have $\#\mathcal{C}_\Delta \gtrsim (\frac{\delta}{\Delta})^s \#X_\Delta$. We see that

$$\mathcal{H}^s_{\delta,\infty}(X) = \sum_{\Delta\ge\delta} \Delta^s \#\mathcal{C}_\Delta \gtrsim \delta^s \#X = \kappa.$$

□

Remark 4. From now on, when X is a (δ,s)-set, we will treat the two conditions $\#X \ge \delta^{-s+\varepsilon}$, $\mathcal{H}^s_{\delta,\infty}(X) \ge \delta^\varepsilon$ as the same.

We recall Theorem 1 here.

Theorem 5. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\alpha = \dim A \in (k, k+1]$ for some $k \in \{1,\dots,n-1\}$. Fix $0 < s < k$ and let*

$$E_s(A) := \{y \in \mathbb{R}^n \setminus A \mid \dim(\pi_y(A)) < s\}.$$

Then,

$$\dim(E_s(A)) \le \max\{k + s - \alpha, 0\}.$$

We will actually prove the following δ-discretized version which is a generalization of [12] Proposition 4.2.

Theorem 6. *Let $0 < \sigma < k$, $a \in (k, k+1]$ for some $k \in \{1, \dots, n-1\}$ and $t > \max\{k + \sigma - a, 0\}$. Let $\eta > 0$ be any small number. Then for ε and δ small enough depending on η, σ, a, and t, we have the following result.*

Let $E, F \subset B^n(0,1)$ be a (δ, t)-set and a (δ, a)-set respectively, with $\mathcal{H}^t_{\delta,\infty}(E) \gtrsim \delta^\varepsilon$, $\mathcal{H}^a_{\delta,\infty}(F) \gtrsim \delta^\varepsilon$, and $\mathrm{dist}(E, F) \geq 1/2$. Then, there exists $y \in E$ such that for all $F' \subset F$ with $\#F' \geq \delta^\epsilon \#F$, we have

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F')) > \delta^\eta.$$

We first show that Theorem 6 implies Theorem 5.

Proof that Theorem 6 implies Theorem 5. By a standard reduction, we can find subsets $A_1, A_2 \subset A$ with $\mathrm{dist}(A_1, A_2) > c > 0$, and $\dim(A_1) = \dim(A_2) = \dim(A) = \alpha$. We only need to show for any ball $B_{c/2}$ of radius $c/2$, $E_s(A) \cap B_{c/2}$ has dimension $\leq \max\{k+s-a, 0\}$. We may assume $\mathrm{dist}(B_{c/2}, A_1) > c/2$. We show that set

$$E := E_s(A_1) \cap B_{c/2} = \{y \in B_{c/2} : \dim(\pi_y(A_1)) < s\}$$

has dimension $\leq \max\{k + s - \dim(A_1), 0\}$. We may assume $A_1, E' \subset B^n(0,1)$ and $\mathrm{dist}(A_1, E') \geq 1/2$.

We choose $t < \dim(E'), a < \dim(A_1)$. Then $\mathcal{H}^t_\infty(E'), \mathcal{H}^a_\infty(A_1) > 0$. We only need to prove $t \leq \max\{k + s - a, 0\}$, since then we can send $a \to \dim(A_1), t \to \dim(E')$. For the sake of contradiction, assume that $t > \max\{k + s - a, 0\}$. Thus, we can find $\sigma > s$ so that $t > \max\{k + \sigma - a, 0\}$. Set $\eta = \sigma - s > 0$.

Now we fix a, t, so we may assume $\mathcal{H}^t_\infty(E'), \mathcal{H}^a_\infty(A_1) \sim 1$ are constants. For any $y \in E'$, we have $\dim(\pi_y(A_1)) < s$. Since $0 = \mathcal{H}^s_\infty(\pi_y(A_1)) = \lim_{\delta\to 0} \mathcal{H}^s_{\delta,\infty}(\pi_y(A_1))$, we find a subset of E' such that we have $\mathcal{H}^t_\infty(E') \sim 1$ and for small enough δ: $\mathcal{H}^s_{\delta,\infty}(\pi_x(A_1)) \leq 1$ for all $y \in E'$.

Using Lemma 2 to $\pi_y(A_1)$, we obtain a set of dyadic caps $\mathcal{C} = \bigsqcup_j \mathcal{C}_{y,j}$ in $\mathbb{S}^{n-1}$ that cover $\pi_y(A_1)$. Here each $\mathcal{C}_{y,j}$ is a set of 2^{-j}-caps that satisfy the s-dimensional condition as $\dim(\pi_y(A)) < s$. Also, the radius of these caps is less than $\varepsilon_\circ$ which is any given small number.

By the s-dimensional condition of $\mathcal{C}_{y,j}$, we have

$$\mathcal{H}^s_{2^{-j},\infty}\left(\bigcup_{C \in \mathcal{C}_{y,j}} C\right) = \#\mathcal{C}_{y,j} 2^{-js}.$$

Therefore, we have

$$\mathcal{H}^\sigma_{2^{-j},\infty}\left(\bigcup_{C \in \mathcal{C}_{y,j}} C\right) \leq \#\mathcal{C}_{y,j} 2^{-j\sigma} = 2^{-j\eta} \mathcal{H}^s_{2^{-j},\infty}\left(\bigcup_{C \in \mathcal{C}_{y,j}} C\right) \leq 2^{-j\eta}. \tag{24}$$

For each cap $C \in \mathcal{C}$, consider $\pi_y^{-1}(C) \cap \{x \in \mathbb{R}^n : 1 - \frac{1}{100} \leq |x - y| \leq 1\}$ which is a tube. We obtain a collection of finitely overlapping tubes

$$\mathbb{T}_y = \bigsqcup_j \mathbb{T}_{y,j}$$

that cover A_1 (see Figure 2). Here, each tube has its coreline passing through y and at distance ~ 1 from y. The tubes in $\mathbb{T}_{y,j}$ have dimensions $2^{-j} \times \cdots \times 2^{-j} \times 1$.

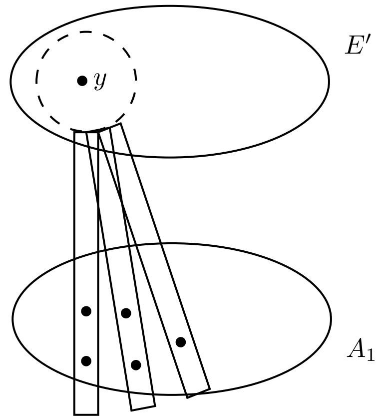

FIGURE 2. $\mathbb{T}_{y,j}$ in the radial projection

For any $y \in E'$, there exists a $j(y) \geq |\log_2 \varepsilon_\circ|$ such that

$$\mathcal{H}^a_\infty\left(A_1 \cap \bigcup_{T \in \mathbb{T}_{y,j(y)}} T\right) \geq \frac{1}{10j(y)^2}\mathcal{H}^a_\infty(A_1). \tag{25}$$

We have a partition $E' = \bigsqcup_j E'_j$ where $E'_j = \{y \in E' : j(y) = j\}$. We choose j such that $\mathcal{H}^t_\infty(E'_j) \gtrsim \frac{1}{j^2}$. We let $\delta = 2^{-j}$. Note that $\delta \leq \epsilon_0$ by assumption. By Lemma 1, there exists a subset $E'' \subset E'_j$ which is a (δ, t)-set and $\#E \gtrsim |\log\delta|^{-2}\delta^{-t}$. We use μ to denote the counting measure on E''.

Next, we consider the set $S = \{(y, x) \in E'' \times A_1 : x \in \bigcup_{T \in \mathbb{T}_{y,j}} T\}$. We also denote the y-section and x-section of S by S_y and S_x. By (25), we have $\mathcal{H}^a_\infty(S_y) \geq \frac{1}{10j(y)^2}\mathcal{H}^a_\infty(A_1)$, so we have

$$(\mathcal{H}^a_\infty \times \mu)(S) \geq \frac{1}{10j^2}\mathcal{H}^a_\infty(A_1)\mu(E''). \tag{26}$$

This implies

$$(\mathcal{H}^a_\infty \times \mu)\left(\left\{(y, x) \in S : \mu(S_x) \geq \frac{1}{20j^2}\mu(E'')\right\}\right) \geq \frac{1}{20j^2}\mathcal{H}^a_\infty(A_1)\mu(E''). \tag{27}$$

Therefore, we have

$$\mathcal{H}^a_\infty\left(\left\{x \in A_1 : \mu(S_x) \geq \frac{1}{20j^2}\mu(E'')\right\}\right) \geq \frac{1}{20j^2}\mathcal{H}^a_\infty(A_1) \sim |\log\delta|^{-2}. \tag{28}$$

By Lemma 1, we can find a subset F of $\left\{x \in A_1 : \mu(S_x) \geq \frac{1}{20j^2}\mu(E)\right\}$, so that F is a (δ, a)-set and $\#F \gtrsim |\log\delta|^{-2}\delta^{-a}$.

Hence,

(29)

$$|\log\delta|^{-2}\#F\#E \lesssim \#\left\{(y, x) \in E \times F : x \in \bigcup_{T \in \mathbb{T}_{y,j}} T\right\} = \sum_{y \in E} \#\left\{x \in F : x \in \bigcup_{T \in \mathbb{T}_{y,j}} T\right\}.$$

By pigeonholing, there exists a subset $E \subset E''$ with $E \gtrsim |\log\delta|^{-2}\#E'' \gtrsim \delta^{\varepsilon/2}\delta^{-t}$, so that for any $y \in E$:

$$\#\{x \in F : x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\} \gtrsim \delta^{\varepsilon}\#F.$$

We set $F_y := \{x \in F : x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\}$.

Now we use Theorem 6 to derive a contradiction. Since E is a (δ,t)-set with $\#E \gtrsim \delta^{\varepsilon/2}\delta^{-t}$, by Lemma 6, $\mathcal{H}^t_{\delta,\infty}(E) \gtrsim \delta^{\varepsilon}$. Similar reasoning shows $\mathcal{H}^a_{\delta,\infty}(F) \gtrsim \delta^{\varepsilon}$. Theorem 6 yields the existence of an $y \in E$ such that $\mathcal{H}^{\sigma}_{\delta,\infty}(\pi_y(F_y)) > \delta^{\eta}$. This contradicts (24). □

Before proving Theorem 6, we prove two propositions. Then we show Theorem 6 is a result of them. The first proposition is a quantitative version of Marstrand's projection theorem. The second proposition is a special case of Theorem 6 when $k = n-1$.

Proposition 3. *Set $d_{n,m} = m(n-m) = \dim(G(n,m))$. Let $0 < a < m$. Let $\eta > 0$ be any small number. Then for ε and δ small enough depending on η, a, we have the following result.*

Let $F \subset B^n(0,1)$ be a δ-separated set, where $\mathcal{H}^a_{\delta,\infty}(F) \gtrsim \delta^{\varepsilon}$. Let $G \subset G(n,m)$ be a δ-separated set, where $\mathcal{H}^{d_{n,m}}_{\delta,\infty}(G) \gtrsim \delta^{\varepsilon}$. Then, there exists $V \in G(n,m)$ such that for all $F' \subset F$ with $\#F' \geq \delta^{\epsilon}\#F$, we have

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F')) > \delta^{\eta}.$$

Here, π_V is the orthogonal projection onto V.

Proof. The main idea of the proof has appeared in the previous section when we proved a Falconer-type estimate.

Suppose the result is not true. By contradiction, for any $V \in G$, there exists $F_V \subset F$ with $\#F_V \geq \delta^{\varepsilon}\#F$ and

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F_V)) \leq \delta^{\eta}.$$

By the standard argument as in the previous proof, we can find a covering of F_V:

$$F_V \subset \bigsqcup_{\delta\leq\Delta\leq\eta} \mathbb{T}_{V,\Delta}.$$

Here, each $\mathbb{T}_{V,\Delta}$ consists of planks of dimensions $\underbrace{\Delta\times\Delta\times\cdots\times\Delta}_{m\text{ times}}\times\underbrace{1\times1\times\cdots\times1}_{n-m\text{ times}}$ that are orthogonal to V. Also, $\mathbb{T}_{V,\Delta}$ satisfies the a-dimensional spacing condition. Therefore,

$$\#\mathbb{T}_{V,\Delta} \leq \Delta^{-a}\mathcal{H}^a_{\delta,\infty}(\pi_V(F_V)) \leq \delta^{\eta}\Delta^{-a}. \tag{30}$$

By the standard pigeonhole argument, we can find a scale Δ, a $(\Delta, d_{n,m})$-subset $G' \subset G$ with $\#G' \gtrsim |\log\delta|^{-2}\Delta^{-d_{n,m}}$ and a (Δ,a)-subset $F' \subset F$ with $\#F' \gtrsim |\log\delta|^{-2}\Delta^{-a}$, so that

(1) for each $V \in G'$, $\#\left(F \cap \bigcup_{T\in\mathbb{T}_{V,\Delta}} T\right) \gtrsim \delta^{2\varepsilon}\#F$,
(2) and each $x \in F'$ is contained in $\gtrsim \delta^{2\varepsilon}\#G$ planks from $\bigcup_{V\in G'}\mathbb{T}_{V,\Delta}$.

Define $f = \sum_{V\in G'}\sum_{T\in\mathbb{T}_{V,\Delta}} \psi_T$ where ψ_T is a smooth bump function at T. We have

$$\delta^{O(\varepsilon)}\Delta^n \#F'(\#G')^2 \lesssim \int_{N_\Delta(F)} |f|^2 \lesssim \int |f_{high}|^2.$$

The last step is by a high-low argument and $a < m$. Noting (30), we have the following estimate for the high part

$$\int |f_{high}|^2 \lesssim \Delta^{-d_{n-1,m-1}}\Delta^m \#\Big(\bigcup_{V\in G'} \mathbb{T}_{V,\Delta}\Big) \lesssim \delta^\eta \Delta^{-d_{n-1,m-1}+m-a}\#G'.$$

Combining the estimates yields $\delta^{O(\varepsilon)} \lesssim \delta^\eta$, which is a contradiction if ε is much smaller depending on η. □

Proposition 4. *Let $0 < \sigma < n-1$, $a \in (n-1, n]$ and $t > \max\{n-1+\sigma-a, 0\}$. Let $\eta > 0$ be any small number. Then for ε and δ small enough depending on s, t, and η, we have the following result.*

Let $E, F \subset B^n(0,1)$ satisfy $\mathcal{H}^t_{\delta,\infty}(E) \gtrsim \delta^\varepsilon$, $\mathcal{H}^a_{\delta,\infty}(F) \gtrsim \delta^\varepsilon$, and $\mathrm{dist}(E,F) \geq 1/4$. Then, there exists $y \in E$ such that for all $F' \subset F$ with $\mathcal{H}^a_{\delta,\infty}(F') \geq \delta^{2\epsilon}$, we have

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F')) > \delta^\eta.$$

Proof. By Lemma 6, we can assume E is a (δ, t)-set with $\# \gtrsim \delta^{-t+\varepsilon}$ and F is a (δ, a)-set with $\# \gtrsim \delta^{-a+\varepsilon}$ by passing to a subset. Since $n-1+\sigma-a < \sigma$, it suffices to prove the proposition for $t < \sigma$. Assume for the sake of contradiction that for all $y \in E$ there exists $F_y \subset F$ with $\#F_y \geq \delta^\varepsilon \#F$ such that $\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y)) \leq \delta^\eta$. Then we do similar reduction to Proposition 4 using pigeonholing, and it can be reduced to the following lemma. Since we have done this kind of reduction many times, we omit it. To prove Proposition 4, it suffices to prove the following lemma. We will see that the result of the following lemma contradicts the condition $t > \max\{n-1+\sigma-a, 0\}$. □

Lemma 7. *Let $0 < t < \sigma < n-1$, $a \in (n-1, n]$. Let $0 < \delta \leq \Delta \leq \eta$, $\varepsilon > 0$, where δ, ε are small enough depending on η, t, σ, a. Let $E, F \subset B^n(0,1)$ be non-empty Δ-separated sets where*

1. *E is a (Δ, t)-set with cardinality $\#E \gtrsim \Delta^{-t}\delta^\varepsilon$,*
2. *F is a (Δ, a)-set with cardinality $\#F \gtrsim \Delta^{-a}\delta^\varepsilon$, and*
3. *$\mathrm{dist}(E, F) \geq \frac{1}{2}$.*

For all $y \in E$, we assume there exists a collection of Δ-tubes $\mathbb{T}_y$, such that

1. *each $T \in \mathbb{T}_y$ is of form $\pi_y^{-1}(C) \cap \{x \in \mathbb{R}^n : 1 - \frac{1}{100} \leq |x-y| \leq 1\}$ for some Δ-cap $C \subset \mathbb{S}^{n-1}$,*
2. *$\mathbb{T}_y$ is a (Δ, σ)-set of tubes with cardinality $\#\mathbb{T}_y \lesssim \delta^\eta \Delta^{-\sigma}$,*
3. *and for all $x \in F$, $\#\{y \in E : \exists T \in \mathbb{T}_y \text{ such that } x \in T\} \gtrsim \Delta^{-t}\delta^\varepsilon$.*

Then,

$$\delta^{O(\varepsilon)}\Delta^{-t} \lesssim \delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-(n-1)-\sigma+a},$$

which implies that $t \leq n-1+\sigma-a$ (if ε is very small depending on η, t, σ).

Proof. Fix a $y \in E$. For any $T \in \mathbb{T}_y$, choose a bump function ψ_T such that $\psi_T \geq 1$ on T, ψ_T decays rapidly outside of T, and supp $\widehat{\psi}_T$ is contained in the dual tube of T which is a $\Delta^{-1} \times \cdots \times \Delta^{-1} \times 1$-slab. Define

$$f_y = \sum_{T\in\mathbb{T}_y} \psi_T \quad\text{and}\quad f = \sum_{y\in E'} f_y.$$

Then, for $x \in N_\Delta(F)$, $f(x) \gtrsim \#\{y \in E : \exists T \in \mathbb{T}_y \text{ such that } x \in T\} \gtrsim \Delta^{-t}\delta^\varepsilon$ by assumption. Therefore,

$$\delta^{O(\varepsilon)}\Delta^{-2t-a+n} \lesssim \delta^{O(\varepsilon)}(\#E)^2(\#F)\Delta^n \lesssim \int_{N_\Delta(F)} |f|^2. \tag{31}$$

We will also use the same high-low argument as we did in the proof of Theorem 3. Let $\eta_{low}(\xi)$ be a smooth bump function on $B^n(0,(K\delta)^{-1})$ and $\eta_{high} = 1 - \eta_{low}$. We will choose $K \sim \delta^{-O(\varepsilon)}$. Define $f_{low} = \eta_{low}^\vee * f$ and $f_{high} = \eta_{high}^\vee * f$.

For $x \in N_\Delta(F)$, we have

$$\Delta^{-t}\delta^\varepsilon \lesssim f(x) \le |f_{low}(x)| + |f_{high}(x)|.$$

By the same argument as in the proof of Theorem 3, we have

$$|f_{low}(x)| \lesssim K^{\sigma-(n-1)}\#E << \Delta^{-t}\delta^\varepsilon,$$

if $K \sim \delta^{-O(\varepsilon)}$ is properly chosen. Therefore, we have $|f(x)| \lesssim |f_{high}(x)|$ on $N_\Delta(F)$.

We have

$$\int_{N_\Delta(F)} |f|^2 \lesssim \int |f_{high}|^2.$$

Here is where things become a little more different than the proof of Proposition 1. For each $T \in \bigcup_{y\in E}\mathbb{T}_y =: \mathbb{T}$, define

$$n_T := \#\{y \in E \mid T \in \mathbb{T}_y\}.$$

By looking at the overlaps of high parts of ψ_T in the frequency space and noting Lemma 3, we get that

$$\int |f_{high}|^2 = \int \Big|\sum_{T\in\mathbb{T}} n_T \cdot \psi_{T,high}\Big|^2 \lesssim \delta^{-O(\varepsilon)}\Delta^{-d_{n-1,n-2}} \sum_{T\in\mathbb{T}} n_T^2 \int |\psi_T|^2 \lesssim \delta^{-O(\varepsilon)}\Delta \sum_{T\in\mathbb{T}} n_T^2.$$

We now find an upper bound to $\sum_{T\in\mathbb{T}} n_T^2$.

$$\sum_{T\in\mathbb{T}} n_T^2 = \sum_{T\in\mathbb{T}} \#\{y,y' \in E \mid T \in \mathbb{T}_y \cap \mathbb{T}_{y'}\} = \sum_{y\in E}\sum_{y'\in E} \#\{T \in \mathbb{T} \mid T \in \mathbb{T}_y \cap \mathbb{T}_{y'}\}$$

Given that each $\mathbb{T}_y$ is a (Δ,σ)-set, the above expression is bounded by

$$\lesssim \sum_{y\in E}\sum_{y'\in E\setminus\{y\}} |y-y'|^{-\sigma} + \sum_{y\in E} \#\mathbb{T}_y.$$

The second term is bounded by

$$\sum_{y\in E} \#\mathbb{T}_y \lesssim \delta^\eta \Delta^{-t-\sigma}.$$

For the first term, we have

$$\begin{aligned}
&\lesssim \sum_{y\in E}\sum_{k=0}^{\log_2\Delta^{-1}}\sum_{|y-y'|\le 2^{-k}}\min\{|y-y'|^{-\sigma},\delta^{\eta}\Delta^{-\sigma}\}\\
&\lesssim \sum_{y\in E}\sum_{k=0}^{\log_2\Delta^{-1}}\#\{y'\in E\cap B^n(y,2^{-k})\}\min\{2^{k\sigma},\delta^{\eta}\Delta^{-\sigma}\}\\
&\lesssim \Delta^{-t}\sum_{k=0}^{\log_2\Delta^{-1}}(\Delta^{-1}2^{-k})^t\min\{2^{k\sigma},\delta^{\eta}\Delta^{-\sigma}\}\\
&= \Delta^{-t}\sum_{k=0}^{\log_2\Delta^{-1}}\Delta^{-t}\min\{2^{k(\sigma-t)},\delta^{\eta}\Delta^{-\sigma}2^{-kt}\}.
\end{aligned}$$

When $2^{k(\sigma-t)} = \delta^{\eta}\Delta^{-\sigma}2^{-kt}$ or equivalently $2^{k\sigma} = \delta^{\eta}\Delta^{-\sigma}$, the value of "min" dominates. The expression above is therefore bounded by $\delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-t-\sigma}$.

Combining all the estimates, we have

$$\sum_{T\in\mathbb{T}} n_T^2 \lesssim (\delta^{\frac{\sigma-t}{\sigma}\eta}+\delta^{\eta})\Delta^{-t-\sigma}.$$

Plugging into (31), we have

$$\delta^{O(\varepsilon)}\Delta^{-t}\lesssim \delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-(n-1)-\sigma+a}.$$

□

We now prove Theorem 6.

Proof of Theorem 6. We will show that the result hold for $\varepsilon \le \varepsilon_0(\eta,\sigma,a,t)$, $\delta \le \delta_0(\eta,\sigma,a,t)$, where $\varepsilon_0(\eta,\sigma,a,t),\delta_0(\eta,\sigma,a,t)$ depend on Proposition 3 and 4.

We will apply Proposition 4 with $n = k+1$. For our purpose, we determine the parameters of Proposition 4 in advance. For fixed η, we first choose small number ε' so that Proposition 4 holds for $\varepsilon = \varepsilon'$.

Let $\widetilde{G}$ be an open subset of $G(n,k+1)$ such that any $V\in\widetilde{G}$ satisfies

$$\mathrm{dist}(\pi_V(E),\pi_V(F))\ge \frac{1}{4}.$$

We choose G to be a δ-separated subset of $\widetilde{G}$ with $\mathcal{H}^{d_{n,m}}_{\delta,\infty}(G)\gtrsim 1$. By Proposition 3, if δ,ε are small enough depending on ε', a, there exists a subset $G_1\subset G$ with $\mathcal{H}^{d_{n,m}}_{\delta,\infty}(G_1)\gtrsim \mathcal{H}^{d_{n,m}}_{\delta,\infty}(G)-O(\delta^{\varepsilon'})$, so that for any $V\in G_1$ we have

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F'))>\delta^{\varepsilon'},\quad \text{for any } F'\subset F \text{ with } \#F'\ge \delta^{\varepsilon}\#F. \tag{32}$$

Similarly, there exists a subset $G_2\subset G$ with $\mathcal{H}^{d_{n,m}}_{\delta,\infty}(G_2)\gtrsim \mathcal{H}^{d_{n,m}}_{\delta,\infty}(G)-O(\delta^{\varepsilon'})$, so that for any $V\in G_2$ we have

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(E))>\delta^{\varepsilon'}. \tag{33}$$

Noting that $G_1\cap G_2\ne\emptyset$, we can find $V\in G_1\cap G_2$ so that (32) and (33) hold for this V.

Let F,E be sets in Theorem 6, so $\pi_V(F),\pi_V(E)$ are sets in $V=\mathbb{R}^{k+1}$. Note that $\mathcal{H}^t_{\delta,\infty}(\pi_V(E))\gtrsim\delta^{\varepsilon'}$, $\mathcal{H}^a_{\delta,\infty}(\pi_V(F))\gtrsim\delta^{\varepsilon'}$, and $\mathrm{dist}(\pi_V(E),\pi_V(F))\ge\frac{1}{2}$. We can apply Proposition 4 to $(\pi_V(F), \pi_V(E))$ to find a $\widetilde{y} \in \pi_V(E)$ such that: for all $\widetilde{F} \subset \pi_V(F)$ with $\mathcal{H}^a_{\delta,\infty}(\widetilde{F}) \geq \delta^{2\varepsilon'}$, we have

$$\mathcal{H}^\sigma_{\delta,\infty}\Big(\pi_{\widetilde{y}}(\widetilde{F})\Big) > \delta^\eta. \tag{34}$$

We use this property to finish the proof. We choose $y \in E$ so that $\pi_V(y) = \widetilde{y}$. We show that this y satisfies the requirement in Theorem 6. For any $F' \subset F$ with $\#F' \geq \delta^\varepsilon \#F$, by (32) we have $\mathcal{H}^a_{\delta,\infty}(\pi_V(F')) \geq \delta^{2\varepsilon'}$. We obtain (34) with $\widetilde{F} = \pi_V(F')$:

$$\mathcal{H}^\sigma_{\delta,\infty}\Big(\pi_{\widetilde{y}}\big(\pi_V(F')\big)\Big) > \delta^\eta.$$

Note that

$$\mathcal{H}^\sigma_{\delta,\infty}\Big(\pi_y(F')\Big) \geq \mathcal{H}^\sigma_{\delta,\infty}\Big(\pi_{\widetilde{y}}\big(\pi_V(F')\big)\Big),$$

as any covering of $\pi_y(F')$ naturally gives rise to a covering of $\pi_{\widetilde{y}}\big(\pi_V(F')\big)$. Therefore, we have

$$\mathcal{H}^\sigma_{\delta,\infty}\Big(\pi_y(F')\Big) > \delta^\eta.$$

□

5. Liu's conjecture on radial projections

In this section, we prove Liu's conjecture (Theorem 2). The idea is the same as in [12], but we still provide some details to clarify the numerology since we are in higher dimensions.

Theorem 7. *Given a Borel set $E \subset \mathbb{R}^n$, with $\dim E \in (k-1, k]$ for some $k \in \{1, \dots, n-1\}$, then*

$$\dim\{y \in \mathbb{R}^n \setminus E \mid \dim(\pi_x(E)) < \dim E\} \leq k.$$

It suffices to prove

Proposition 5. *Given a Borel set $E \subset \mathbb{R}^n$, with $\dim E \in (k-1, k]$ for some $k \in \{1, \dots, n-1\}$, and $\tau_0 > 0$ being a small number, then we have*

$$\dim\{x \in \mathbb{R}^n \setminus E \mid \dim(\pi_x(E)) \leq \dim E - \tau_0\} \leq k.$$

The proof is by contradiction to assume the set

$$F = \{x \in \mathbb{R}^n \setminus E \mid \dim(\pi_x(E)) \leq \dim E - \tau_0\} \tag{35}$$

satisfies $t = \dim F > k$. We will derive a contradiction through the following proposition and a standard reduction (see also [12] Proposition 4.8). Since we have done similar reductions many times, we omit it. So, it suffices to prove

Proposition 6. *Let $k \in \{1, \cdots, n-1\}$. Let $0 < s < k$, $t > k$ and $\tau_0 > 0$. For ε, δ small enough depending on s, t, τ_0, the following holds. Let $E, F \subset B^n(0,1)$ be (δ, s)-set and (δ, t)-set, with $\#E \gtrsim \delta^{-s+\varepsilon}$, $\#F \gtrsim \delta^{-t+\varepsilon}$, and $\mathrm{dist}(E, F) \geq 1/2$. Then there exists $x \in F$ such that*

$$|\pi_x(E')|_\delta \geq \delta^{-s+\tau_0}, \quad \text{for all } E' \subset E, \#E' \geq \delta^\varepsilon \#E. \tag{36}$$

Remark 5. (36) roughly says there exists $x \in F$ such that $\dim(\pi_x(E)) > \dim E - \tau_0$, contradicts the definition of F (35). Throughout the proof, we will use x to denote points in F and y to denote points in E.

We will discuss more about the proof in Appendix. Before that, we give an intuitive proof for Proposition 5. We will use Theorem 5.

An intuitive proof of Proposition 5. We just need to prove for $\dim E < k$. We set $s = \dim E$. By contradiction, we assume $t = \dim F > k$ (F is given by (35)). Also, by passing to a subset of F, we may assume $t \in (k, k+1)$. Now we let this F be the set A in Theorem 5. Since $s < k$, we have that the s-exceptional

$$E_s(F) = \{y \in \mathbb{R}^n \setminus F : \dim(\pi_y(F)) < s\}$$

has dimension $\leq k + s - t < s = \dim E$. Subtracting this small exceptional part from E, we may pass to a subset of E (still denoted by E) with the same dimension s and satisfying

$$\dim(\pi_y(F)) \geq s,$$

for any $y \in E$.

By δ-discretization, we may assume F is a t-dimensional set of points and E is an s-dimensional set of points. (Here, when we say F is a t-dimensional set, it means that F is a (δ, t)-set and $\#F \gtrsim \delta^{-t}$). For each $x \in F$ and $y \in E$, we connect them by a δ-tube. Let $\mathbb{T}$ be the set of δ-tubes produced in this way. We also identify comparable tubes. Roughly speaking, we define

$$\mathbb{T} := \{T : T \text{ connects some } x \in F, y \in E\}.$$

We also define $\mathbb{T}_x := \{T \in \mathbb{T} : x \in T\}$ for $x \in F$, and $\mathbb{T}^y := \{T \in \mathbb{T} : y \in T\}$ for $y \in E$. By definition, we have $\dim(\pi_x(E)) \leq s - \tau_0$ for $x \in F$. This condition morally says that $\mathbb{T}_x$ is an $(s-\tau_0)$-dimensional set. Since the tubes in $\mathbb{T}_x$ are finitely overlapping, we have

$$\delta^{-s} \leq \#E \lesssim \sum_{T \in \mathbb{T}_x} \#(T \cap E).$$

Since $\#\mathbb{T}_x \leq \delta^{-s+\tau_0}$, we may morally assume $\#(T_x \cap E) \gtrsim \delta^{-\tau_0/2}$ for any $T_x \in \mathbb{T}_x$. Morally, we may further assume for any $T \in \mathbb{T}$, we have $\#(T \cap E) \gtrsim \delta^{-\tau_0/2}$. The condition $\dim(\pi_y(F)) \geq s$ morally says that $\mathbb{T}^y$ is at least an s-dimensional set.

We consider the incidence between E and $\mathbb{T}$. We will derive a contradiction by comparing the upper and lower bounds of $I(E, \mathbb{T})$. First, we have

$$I(E, \mathbb{T}) = \sum_{T \in \mathbb{T}} \#(T \cap E) \gtrsim \#\mathbb{T}\delta^{-\tau_0/2}.$$

For the upper bound of the incidence, we have

$$\begin{aligned} I(E, \mathbb{T}) = \sum_{T \in T} \#(T \cap E) &\leq (\#\mathbb{T})^{1/2} \left(\sum_{T \in \mathbb{T}} \#(T \cap E)^2 \right)^{1/2} \\ &= (\#\mathbb{T})^{1/2} \left(\sum_{y, y' \in E} \#\{T \in \mathbb{T} : y, y' \in T\} \right)^{1/2} \\ &= (\#\mathbb{T})^{1/2} \left(\sum_{y \in E} \sum_{y' \in E} \#\{T \in \mathbb{T}^y : y' \in T\} \right)^{1/2}. \end{aligned}$$

By the s-dimensional condition for $\mathbb{T}^y$, we have

$$\#\{T \in \mathbb{T}^y : y' \in T\} \lesssim \left(\frac{\delta}{|y-y'|}\right)^s \#\mathbb{T}^y.$$

Therefore, we have

$$\begin{aligned} I(E,\mathbb{T}) &= (\#\mathbb{T})^{1/2}\left(\sum_{y\in E}\sum_{y'\in E\setminus\{y\}} \#\{T\in\mathbb{T}^y : y'\in T\} + \sum_{y\in E}\#\mathbb{T}^y\right)^{1/2} \\ &\lesssim (\#\mathbb{T})^{1/2}\left(\sum_{y\in E}\sum_{y'\in E\setminus\{y\}}\left(\frac{\delta}{|y-y'|}\right)^s \#\mathbb{T}^y + I(E,\mathbb{T})\right)^{1/2}. \end{aligned}$$

Using that E is an s-dimensional set, we have

$$\sum_{y'\in E\setminus\{y\}}(\frac{\delta}{|y-y'|})^s \lessapprox 1,$$

so we have

$$I(E,\mathbb{T}) \lessapprox (\#\mathbb{T})^{1/2} I(E,\mathbb{T})^{1/2}.$$

This means $I(E,\mathbb{T}) \lessapprox \#\mathbb{T}$, which contradicts the lower bound of $I(E,\mathbb{T})$.

□

Appendix A. Proof of Proposition 6

We provide more details for the proof of Proposition 6. We remark that the proof has the same idea as in [12]. We include here just for completeness.

We introduce some notations. Given set E, F and set of δ-tubes $\mathbb{T}$, we define

$$I(E,F,\mathbb{T}) := \#\{(y,x,T)\in E\times F\times\mathbb{T} : y,x\in T\}.$$

For $y\in E$, we define

$$\mathbb{T}^y := \{T\in\mathbb{T} : y\in T\}.$$

For $x\in F$, we define

$$\mathbb{T}_x := \{T\in\mathbb{T} : x\in T\}.$$

One easily sees that

$$I(E,F,\mathbb{T}) = \sum_{y\in E}\#\left(F\cap\bigcup_{T\in\mathbb{T}^y}T\right) = \sum_{x\in F}\#\left(E\cap\bigcup_{T\in\mathbb{T}_x}T\right).$$

In [12], Orponen and Shmerkin derive their Corollary 4.5 from Proposition 4.2. By the same argument, we can derive the following corollary from Theorem 6. We omit the proof.

Corollary 1. *Let $0\le\sigma\le s\le k$, $t\in(k,k+1]$, $\eta>0$ very small, and $s>\max\{k+\sigma-t,0\}$. Then, for sufficiently small ε,δ depending on s,σ,t,η, the following holds.*

Let $E,F\subset B^n(0,1)$ be (δ,s)-set and (δ,s)-set, with $\#E\gtrsim\delta^{-s+\varepsilon}$ and $\#F\gtrsim\delta^{-t+\varepsilon}$. Then, there exists a subset $E'\subset E$ with $\#E'\ge(1-\delta^\varepsilon)\#E$, and for every point $y\in E'$, there exist disjoint (possibly empty) families of δ-tubes $\mathbb{T}^y=\mathbb{T}^y_1\sqcup\cdots\sqcup\mathbb{T}^y_L$ (where $L=3\log(1/\delta)$), with the following properties:

(1) The tubes in $\mathbb{T}^y$ pass through y.

(2) *Each* $\mathbb{T}_j^y$ *can be writen as* $\mathbb{T}_j^y = \sqcup_i \mathbb{T}_{j,i}^y$*, where each* $\mathbb{T}_{j,i}^y$ *is a* (δ, σ)*-set with cardinality* $\gtrsim \delta^{-\sigma+\eta}$.
(3) $\#(T \cap F) \sim 2^j$*, for* $T \in \mathbb{T}_j^y$.
(4) $\mathbb{T}_j^y$ *is either empty, or* $\#(F \cap \bigcup_{T\in\mathbb{T}_j^y} T) \geq \delta^{2\varepsilon}\#F$ *in which case* $\#\mathbb{T}_j^y \geq \delta^{2\varepsilon}2^{-j}\#F$.
(5) $\#(F \cap \bigcup_{T\in\mathbb{T}^y} T) \geq (1-\delta^{\varepsilon})\#F$.

Let us return to the proof of Proposition 6. Since

$$s > \max\{k + s - t, 0\},$$

we apply Corollary 1 with $\sigma := s$. We find a set $E' \subset E$ with $\#E' \geq (1-\delta^{4\varepsilon})\#E$, and for all $y \in E'$ the tubes $\mathbb{T}^y = \mathbb{T}_1^y \sqcup \cdots \sqcup \mathbb{T}_L^y$ ($L = 3\log(1/\delta)$) satisfying the properties in Corollary 1.

Set $\mathbb{T}_j' = \cup_{y\in E'}\mathbb{T}_j^y$, $\mathbb{T} = \cup_j \mathbb{T}_j' = \cup_{y\in E'}\mathbb{T}^y$. Note that a $T \in \mathbb{T}$ may belong to both $\mathbb{T}^y$ and $\mathbb{T}^{y'}$ for different y, y'. By (5), we have

$$I(E', F, \mathbb{T}) = \sum_{y\in E'} \#\left(F \cap \bigcup_{T\in\mathbb{T}^y} T\right) \geq (1-\delta^{4\varepsilon})\#E'\#F. \tag{37}$$

Now, we make a counter assumption: (36) fails for all $x \in F$. Thus for every $x \in F$, there exists a subset $E_x' \subset E$ such that $\#E_x' \geq \delta^{\varepsilon}\#E$, and

$$|\pi_x(E_x')|_\delta < \delta^{-s+\tau_0}. \tag{38}$$

Since $\#E' \geq (1-\delta^{4\varepsilon})\#E$, we have $\#(E_x' \cap E') \gtrsim \delta^{\varepsilon}\#E$. We may assume $E_x' \subset E'$ by replacing E_x' with $E_x' \cap E'$. For each $x \in F$, we choose δ-tubes $\mathcal{T}_x$ passing through x so that $\mathcal{T}_x$ cover E_x' and

$$\#\mathcal{T}_x < \delta^{-s+\tau_0}. \tag{39}$$

We immediately have

$$I(E', F, \cup_x \mathcal{T}_x) = \sum_{x\in F} \#(E' \cap \mathcal{T}_x) \geq \delta^{\varepsilon}\#E'\#F. \tag{40}$$

The inequalities (37) and (40) together imply

$$I\left(E', F, \mathbb{T} \cap \bigcup_x \mathcal{T}_x\right) \geq (\delta^{\varepsilon} - \delta^{4\varepsilon})\#E'\#F. \tag{41}$$

By pigeonholing, there exists a j such that

$$I\left(E', F, \mathbb{T}_j' \cap \bigcup_x \mathcal{T}_x\right) \gtrsim \delta^{4\varepsilon}\#E'\#F. \tag{42}$$

We set $\mathbb{T}_j := \mathbb{T}_j' \cap \bigcup_x \mathcal{T}_x$.

Next, we introduce the high-tubes:

$$\mathbb{T}_j^{high} := \{T \in \mathbb{T}_j : \#(E' \cap T) \geq \delta^{-\tau_0/2}\}.$$

Also define $\mathbb{T}_j^{low} = \mathbb{T}_j \setminus \mathbb{T}_j^{high}$. We want to show that

$$I(E', F, \mathbb{T}_j^{high}) \gtrsim \delta^{\varepsilon}\#E'\#F. \tag{43}$$

To show this, note

$$I(E',F,\mathbb{T}_j^{low})=\sum_{x\in F}\#\left(E'\cap\bigcup_{T\in(\mathbb{T}_j^{low})_x}T\right)\lesssim\sum_{x\in F}\sum_{T\in(\mathbb{T}_j^{low})_x}\#(E'\cap T). \tag{44}$$

Note that $\#(E'\cap T)\le\delta^{-\tau_0/2}$ for $T\in\mathbb{T}_j^{low}$, $(\mathbb{T}_j^{low})_x\subset\mathcal{T}_x$ and (39). Therefore

$$I(E',F,\mathbb{T}_j^{low})\lesssim\#F\delta^{-s+\tau_0/2}\lesssim\delta^{\varepsilon}\#E'\#F,$$

if ε is small enough depending on τ_0. Combined with (42) yields (43).

Next, we show that there exists $E''\subset E'$ with $\#E''\gtrsim\delta^{\varepsilon}\#E'$, such that for $y\in E''$:

$$\#(\mathbb{T}_j^{high})^y\ge\delta^{4\varepsilon}\#\mathbb{T}_j^y.$$

Note that

$$\delta^{\varepsilon}\#E'\#F\lesssim I(E',F,\mathbb{T}_j^{high})=\sum_{y\in E'}I(F,(\mathbb{T}_j^{high})^y).$$

By pigeonholing, we can choose $E''\subset E'$ with $\#E''\ge\delta^{\varepsilon}\#E'$ and $I(F,(\mathbb{T}_j^{high})^y)\ge\delta^{\varepsilon}\#F$. Since $(\mathbb{T}_j^{high})^y\subset\mathbb{T}_j^y$ and each $T\in\mathbb{T}_j^y$ satisfies $\#(F\cap T)\sim 2^j$, we have $I(F,(\mathbb{T}_j^{high})^y)\sim 2^j\#(\mathbb{T}_j^{high})^y$ which implies $\#(\mathbb{T}_j^{high})^y\gtrsim\delta^{\varepsilon}2^{-j}\#F\gtrsim\delta^{\varepsilon}\#\mathbb{T}_j^y$.

Noting that $\mathbb{T}_j^y=\sqcup_i\mathbb{T}_{j,i}^y$ where each $\mathbb{T}_{j,i}^y$ is a (δ,s)-set with cardinality $\gtrsim\delta^{-s+\eta}$, by the same trick as in (21), we have

$$\#\mathbb{T}_j^{high}\ge\#\left(\bigcup_{y\in E''}(\mathbb{T}_j^{high})^y\right)\gtrsim\delta^{O(\eta+\varepsilon)}\sum_{y\in E''}\#(\mathbb{T}_j^{high})^y\gtrsim\delta^{O(\eta+\varepsilon)}\sum_{y\in E''}\#\mathbb{T}_j^y.$$

Combined with (4),

$$\#\mathbb{T}_j^{high}\gtrsim\delta^{O(\eta+\varepsilon)}\delta^{-s}2^{-j}\#F. \tag{45}$$

Finally, we estimate $I(E',\mathbb{T}_j^{high})$. We easily have the lower bound

$$I(E',\mathbb{T}_j^{high})\gtrsim\#\mathbb{T}_j^{high}\delta^{-\tau_0/2}. \tag{46}$$

We have the upper bound

$$\begin{aligned}I(E',\mathbb{T}_j^{high})&\le(\#\mathbb{T}_j^{high})^{1/2}\left(\sum_{T\in\mathbb{T}_j^{high}}(\#E'\cap T)^2\right)^{1/2}\\&\le(\#\mathbb{T}_j^{high})^{1/2}\left(\sum_{y\ne y'\in E'}\#\{T\in\mathbb{T}_j^y:y'\in T\}+I(E',\mathbb{T}_j^{high})\right)^{1/2}.\end{aligned}$$

By (2), $\#\{T\in\mathbb{T}_j^y:y'\in T\}\lesssim\delta^{-O(\eta)}(\delta/|y-y'|)^s\#\mathbb{T}_j^y$. We see that

$$\sum_{y\ne y'\in E'}\#\{T\in\mathbb{T}_j^y:y'\in T\}\lesssim\delta^{-O(\eta)}\sum_{y\ne y'\in E'}\left(\frac{\delta}{|y-y'|}\right)^s\#\mathbb{T}_j^y\lesssim\delta^{-O(\eta+\varepsilon)}\delta^{-s}2^{-j}\#F.$$

Plugging to the inequality above, we obtain

$$I(E',\mathbb{T}_j^{high})\lesssim\delta^{-O(\eta+\varepsilon)}\Big((\#\mathbb{T}_j^{high})^{1/2}(\delta^{-s}2^{-j}\#F)^{1/2}+\#\mathbb{T}_j^{high}\Big).$$

Comparing with (46), we obtain

$$\#\mathbb{T}_j^{high} \lesssim \delta^{-O(\eta+\varepsilon)+\tau_0/2}\delta^{-s}2^{-j}\#F,$$

which contradicts (45), since η, ε can be chosen much smaller than τ_0.

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