CONVOLUTION-EXACT PERVERSE SHEAVES ON THE AFFINE FLAG VARIETY

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Abstract. An unpublished result by Mirković states that convolution-exact sheaves on the flag variety of a simple linear algebraic group G over $\overline{\mathbb{F}_p}$ are tilting, that is, they admit both standard and costandard filtrations. The analogous statement for convolution-exact sheaves on the affine flag variety is false, but Arkhipov and Bezrukavnikov noted that it is still not known whether the projections of such sheaves to a different category, which they called the Iwahori–Whittaker category, are tilting. We make partial progress toward this question by considering reductions to the combinatorics of the extended affine Weyl group of G. In particular, we demonstrate an obstruction to a direct generalization of Mirković's original proof, even in the $G = \mathrm{SL}_2$ case. We also investigate Wakimoto filtrations, as introduced by Arkhipov and Bezrukavnikov, and their variants.

1. Introduction

Let G be a simple linear algebraic group over the field $k = \overline{\mathbb{F}_p}$. Fix a Borel subgroup B of G, and let the ind-variety $\mathcal{F}\ell$ be the corresponding affine flag variety, as in [1]. Let $l \neq p$ be a prime, let $D \coloneqq D^b(\mathcal{F}\ell)$ be the bounded derived category of constructible l-adic sheaves on $\mathcal{F}\ell$ (see for instance [2, 4]), let $D_I \coloneqq D^b_I(\mathcal{F}\ell)$ be the Iwahori-equivariant bounded derived category, and let $\mathcal{P} \subseteq D$ and $\mathcal{P}_I \subseteq D_I$ be the (full) subcategories of perverse sheaves. Let $*$ denote the convolution operation, which gives the category D_I a monoidal structure, and also gives a right action of D_I on D.

Let $\check{G}$ be the Langlands dual group of G over the field $\overline{\mathbb{Q}_l}$, with category of representations $\mathrm{Rep}(\check{G})$. Much work has been done in describing various categories of l-adic sheaves over the affine Grassmannian and affine flag variety of G in terms of the group $\check{G}$. For instance, the geometric Satake isomorphism gives an equivalence of (Tannakian) categories between a certain category of equivariant l-adic perverse sheaves over the affine Grassmannian of G and $\mathrm{Rep}(\check{G})$ [7]. Further work has been done in describing categories of l-adic sheaves on $\mathcal{F}\ell$ in terms of $\check{G}$ [1, 3].

Let W_f be the (finite) Weyl group associated to G, let Λ be the coweight lattice of G, and let $W = W_f \rtimes \Lambda$ be the extended affine Weyl group. Let $\ell \colon W \to \mathbb{Z}_{\geq 0}$ be the length function on W (which extends the length function on the (non-extended) affine Weyl group), and let ${}^fW \subseteq W$ be the set of minimal-length representatives of right cosets of W_f in W. Recall that the Schubert cells on $\mathcal{F}\ell$ are parameterized by W. For $w \in W$, let $j_w \colon \mathcal{F}\ell_w \hookrightarrow \mathcal{F}\ell$ denote the embedding of the Schubert cell corresponding to w. Furthermore, let $j_{w!} \coloneqq j_{w!}(\overline{\mathbb{Q}_l}[\ell(w)])$ and $j_{w*} \coloneqq j_{w*}(\overline{\mathbb{Q}_l}[\ell(w)])$ be the standard and costandard objects of $\mathcal{P}_I$, respectively.

An object $\mathcal{F} \in \mathcal{P}_I$ is said to be *convolution-exact* if $\mathcal{F} * X \in \mathcal{P}_I$ for all $X \in \mathcal{P}_I$. As mentioned in [1, Remark 7], we have the following unpublished result by Mirković:

Theorem 1.1 (Mirković). *A convolution-exact sheaf on the finite-dimensional flag variety G/B is tilting, i.e., it has a filtration with standard subquotients and a filtration with costandard subquotients.*

Here, the standard and costandard objects over G/B are defined analogously to those over $\mathcal{F}\ell$. But in [1, Remark 7], it is also mentioned that it is false in general that convolution-exact objects of $\mathcal{P}_I$ are tilting, with certain "central sheaves" serving as counterexamples. However, Arkhipov and Bezrukavnikov later note that the validity of the following weaker statement, which involves another abelian category $\mathcal{P}_{\mathcal{IW}}$ they refer to as the *Iwahori–Whittaker category*, and an exact "projection" functor $Av_\Psi \colon \mathcal{P}_I \to \mathcal{P}_{\mathcal{IW}}$, is unknown:

Question 1.2 ([1, Remark 11]). *If $\mathcal{F} \in \mathcal{P}_I$ is convolution-exact, then $Av_\Psi(\mathcal{F})$ is a tilting object of the Iwahori–Whittaker category.*

Note that $Av_\Psi(\mathcal{F}) \in \mathcal{P}_{\mathcal{IW}}$ is tilting if $\mathcal{F} \in \mathcal{P}_I$ is. Extending the argument for Theorem 1.1 given in [1, Remark 7] allows us to prove the following special case of Question 1.2. First, we set some more notation: following [1], for any $\mathcal{F} \in D_I$, we define

$$W^*_{\mathcal{F}} \coloneqq \{w \in W \mid j^*_w(\mathcal{F}) \neq 0\} \qquad \text{and} \qquad W^!_{\mathcal{F}} \coloneqq \{w \in W \mid j^!_w(\mathcal{F}) \neq 0\}.$$

Moreover, for any subset $S \subseteq W$, we define the "downward closure"

$$\overline{S} \coloneqq \{w \in W \mid w \leq s \text{ for some } s \in S \text{ under the (strong) Bruhat order}\}.$$

Finally, for $w_1, w_2 \in W$, we say that w_1 is a *prefix* of w_2 if $\ell(w_1) + \ell(w_1^{-1}w_2) = \ell(w_2)$. Prefixes in W admit a visual interpretation in terms of "shortest paths" on alcoves, in much the same way as prefixes in the (non-extended) affine Weyl group; this is discussed further in Section 2.1.

Theorem 1.3. *Suppose $\mathcal{F} \in \mathcal{P}_I$ is convolution-exact. If there exists an element $w_1 \in W$ that has each element of $(\overline{W^*_{\mathcal{F}}})^{-1}$ as a prefix, then $\mathcal{F}$ has a costandard filtration. Similarly, if there exists $w_2 \in W$ that has each element of $(\overline{W^!_{\mathcal{F}}})^{-1}$ as a prefix, then $\mathcal{F}$ has a standard filtration. In particular, if both such $w_1, w_2 \in W$ exist, then both $\mathcal{F} \in \mathcal{P}_I$ and $Av_\Psi(\mathcal{F}) \in \mathcal{P}_{\mathcal{IW}}$ are tilting.*

In the above result, $(\overline{W^*_{\mathcal{F}}})^{-1}$ denotes the set $\{w^{-1} \mid w \in \overline{W^*_{\mathcal{F}}}\}$, and similarly for $(\overline{W^!_{\mathcal{F}}})^{-1}$.

For instance, if all elements of $W^*_{\mathcal{F}}$ and $W^!_{\mathcal{F}}$ had length 0, then the result would apply. But unfortunately, as we will see, for most finite sets $S \subseteq W$ (let alone sets S of the form $\overline{T}^{-1}$) there do not exist a $w \in W$ having each element of S as a prefix, including many sets $S = (\overline{W^*_{\mathcal{F}}})^{-1}, (\overline{W^!_{\mathcal{F}}})^{-1}$ that arise naturally from convolution-exact sheaves over $\mathcal{F}\ell$. Moreover, as discussed in Remark 2, even if we only want to show $Av_\Psi(\mathcal{F})$ (not $\mathcal{F}$ itself) is tilting, we may at best only very slightly weaken the hypotheses of Theorem 1.3.

Explicitly, using computations in W, we are able to describe one natural condition on S which forces the existence of some $w \in W$ having each element of S as a prefix:

Proposition 1.4. *Let $S \subseteq W$ be a finite set such that $S \subseteq w_f \cdot {}^f W$ for some $w_f \in W_f$. Then there exists $w \in w_f \cdot {}^f W$ that has each element of S as a prefix.*

However, as alluded to above, it is rare that a set of the form $\overline{T}^{-1}$ satisfies the condition laid out in Proposition 1.4.

In another direction, we may use the so-called Wakimoto sheaves and their variants to prove various similarly combinatorially-flavored results; these ideas are outlined in Section 4.

The rest of the paper is organized as follows. Definitions, notation, and useful preliminary results are given in Section 2. We generalize the proof of Theorem 1.1 in Section 3, proving Theorem 1.3 and Proposition 1.4; we also discuss the obstructions to generalizing this result further. Then we consider Wakimoto filtrations in Section 4. We discuss possible directions for future research in Section 5.

Finally, as in [1], we remark that everything in this paper remains true if the underlying field $k = \overline{\mathbb{F}_p}$ is replaced by $k = \mathbb{C}$, the field of coefficients $\overline{\mathbb{Q}_l}$ is replaced by $\mathbb{C}$, and the category of l-adic constructible sheaves is replaced by the category of D-modules (as discussed for instance in [5]).

2. Preliminaries

2.1. **Root systems and alcoves.** Let $\Phi \subseteq V$ be the root system of G, where V is a real vector space with inner product $\langle \cdot, \cdot \rangle$ that is spanned by Φ; note Φ is irreducible because G is simple. Our choice of Borel subgroup B induces a system of positive roots Φ^+, with corresponding negative roots Φ^-. Let $\Lambda^+ \subseteq \Lambda$ be the corresponding set of dominant coweights.

Recall that $W = W_f \ltimes \Lambda$ acts on V on the left, where Λ acts by translations. We will often denote this action by $(w, v) \mapsto w \cdot v$. For clarity, for $\lambda \in \Lambda$ we write $t(\lambda) \in W$ to denote the corresponding element of W. We also let e denote the identity of W (note $e = t(0)$).

Let W_0 denote the non-extended affine Weyl group of W. We now recall basic facts about W_0 and W and their actions on the alcoves of Φ. First, W_0 is a normal subgroup of W, and acts freely and transitively on the set of alcoves of Φ. This action extends to a transitive action of W on the set of alcoves. Let $\mathcal{A}_0$ be the fundamental alcove, and let $\Omega \subseteq W$ be the stabilizer of $\mathcal{A}_0$. Then $W = W_0 \rtimes \Omega$. Moreover, for any $w \in W$, the length $\ell(w)$ equals the number of alcove boundaries separating $\mathcal{A}_0$ and $w \cdot \mathcal{A}_0$ (thus Ω is the set of elements of W with length 0). Furthermore, given $w_1, w_2 \in W$, we know that w_1 is a prefix of w_2 if and only if there exists a shortest path from the alcove $\mathcal{A}_0$ to the alcove $w_2 \cdot \mathcal{A}_0$ that passes through $w_1 \cdot \mathcal{A}_0$.

2.2. **Convolutions and W.** Under certain circumstances, convolutions involving $j_{w!}$ and j_{w*} for $w \in W$ can be related to multiplication in the group W. First, the unit object δ_e of the monoidal category D_I (under convolution) is given by $j_{e!} = j_{e*}$.

Lemma 2.1 ([1, Lemma 8])**.**

(1) *For $w_1, w_2 \in W$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, there are canonical isomorphisms $j_{w_1*} * j_{w_2*} = j_{w_1 w_2 *}$ and $j_{w_1!} * j_{w_2!} = j_{w_1 w_2 !}$.*
(2) *For $w \in W$, there are canonical isomorphisms*

$$j_{w*} * j_{w^{-1}!} = j_{w^{-1}!} * j_{w*} = \delta_e.$$

Convolution is not commutative in general; following [1, §3.6.3], we say a sheaf $\mathcal{F} \in \mathcal{P}_I$ is *central* if there exist canonical isomorphisms $\mathcal{F} * X = X * \mathcal{F}$ for all $X \in \mathcal{P}_I$.

A careful analysis of the proof of [1, Lemma 15] yields the following result. The sets $W^*_{\mathcal{F}}, W^!_{\mathcal{F}} \subseteq W$ are defined in Section 1.

Lemma 2.2 (cf. [1, Lemma 15])**.** *Let $\mathcal{F} \in D_I$. Then $W^*_{\mathcal{F}}$ and $W^!_{\mathcal{F}}$ are finite, and for all $w \in W$,*

$$W^!_{\mathcal{F} * j_{w*}} \subseteq \overline{W^!_{\mathcal{F}}} \cdot w \qquad \text{and} \qquad W^*_{\mathcal{F} * j_{w!}} \subseteq \overline{W^*_{\mathcal{F}}} \cdot w.$$

Remark 1. The $\overline{W^!_{\mathcal{F}}}$ and $\overline{W^*_{\mathcal{F}}}$ arise in the statement of Lemma 2.2 because the closure of $\mathcal{F}\ell_w$ includes all $\mathcal{F}\ell_v$ for $v \leq w$ in the strong Bruhat order. In general, we cannot replace $\overline{W^!_{\mathcal{F}}}$ with $W^!_{\mathcal{F}}$, nor $\overline{W^*_{\mathcal{F}}}$ with $W^*_{\mathcal{F}}$. For example, if $s \in W$ is a simple reflection and $\mathcal{F} = j_{s!}$, then $W^*_{\mathcal{F}} = \{s\}$ but $W^*_{\mathcal{F} * j_{s!}} = W^*_{j_{s!} * j_{s!}} = \{e, s\}$. Note $\overline{W^*_{\mathcal{F}}} = \{e, s\}$ in this case.

We now define more notation taken from [1]: for a triangulated category $\mathcal{C}$ (such as D_I) and a set $\mathcal{S}$ of objects of $\mathcal{C}$, we let $\langle \mathcal{S} \rangle$ denote the smallest subset of $\mathcal{C}$ containing $\mathcal{S} \cup \{0\}$ and closed under taking extensions (where C is an extension of A and B if there is a distinguished triangle $A \to C \to B \to A[1]$). We will occasionally refer to objects in $\langle \mathcal{S} \rangle$ as lying in the *span* of $\mathcal{S}$. This allows us to state the following result, which is a less general version of [1, Claim 1]. This result lets us describe certain (shifted) standard and costandard filtrations of $\mathcal{F}$ in terms of $W^*_{\mathcal{F}}$ and $W^!_{\mathcal{F}}$. First, let $(D^{\leq 0}, D^{\geq 0})$ denote the perverse t-structure on the derived category D, so that $\mathcal{P} = D^{\leq 0} \cap D^{\geq 0}$.

Lemma 2.3 (cf. [1, Claim 1])**.**

(1) *If $\mathcal{F} \in D^{\leq 0}$, then $\mathcal{F} \in \langle j_{w!}[d] \mid d \geq 0, w \in W^*_{\mathcal{F}} \rangle$. Similarly, if $\mathcal{F} \in D^{\geq 0}$, then $\mathcal{F} \in \langle j_{w*}[d] \mid d \leq 0, w \in W^!_{\mathcal{F}} \rangle$.*
(2) *If $\mathcal{F} \in D^{\geq 0}$ and $\mathcal{F} \in \langle j_{w*}[d] \mid d \geq 0, w \in W \rangle$, then $\mathcal{F} \in \mathcal{P}$ and $\mathcal{F} \in \langle j_{w*}, w \in W^!_{\mathcal{F}} \rangle$. Similarly, if $\mathcal{F} \in D^{\leq 0}$ and $\mathcal{F} \in \langle j_{w!}[d] \mid d \leq 0, w \in W \rangle$, then $F \in \mathcal{P}$ and $\mathcal{F} \in \langle j_{w!}, w \in W^*_{\mathcal{F}} \rangle$.*

A related result is given by the following lemma.

Lemma 2.4 ([1, Proof of Sublemma 2])**.** *For all $w_1, w_2 \in W$,*

$$j_{w_1*} * j_{w_2*} \in \langle j_{w*}[d] \mid d \geq 0, w \in W \rangle \qquad \text{and} \qquad j_{w_1!} * j_{w_2!} \in \langle j_{w!}[d] \mid d \leq 0, w \in W \rangle.$$

2.3. **The Iwahori–Whittaker category.** We now discuss the Iwahori–Whittaker category $\mathcal{P}_{\mathcal{IW}}$ mentioned in Section 1, which is abelian and can be thought of as a geometric counterpart of the realization of the anti-spherical module M_{asp} in terms of the Whittaker model. The full subcategories $D_{\mathcal{IW}} \subseteq D$ and $\mathcal{P}_{\mathcal{IW}} \subseteq \mathcal{P}$ are defined in full in [1, §1.6], but for our purposes, we will only need that the standard and costandard objects of $\mathcal{P}_{\mathcal{IW}}$ are given by sheaves Δ_w and ∇_w for $w \in {}^fW$, respectively, along with the following results. The functor $Av_\Psi \colon D_I \to D_{\mathcal{IW}}$, defined via $\mathcal{F} \mapsto \Delta_e * \mathcal{F}$, restricts to an exact functor $\mathcal{P}_I \to \mathcal{P}_{\mathcal{IW}}$. Moreover:

Lemma 2.5 ([1, Lemma 4])**.**

(1) $\Delta_e = \nabla_e$.
(2) *For $w = w_f w'$, where $w_f \in W_f$ and $w' \in {}^fW$,*

$$Av_\Psi(j_{w!}) = \Delta_{w'} \qquad \textit{and} \qquad Av_\Psi(j_{w*}) = \nabla_{w'}.$$

We also note that as mentioned in [1, §2.0.2], the objects of $\mathcal{P}_I$ are given by

$$\langle j_{w!}[d] \mid w \in W, d \geq 0 \rangle \cap \langle j_{w*}[d] \mid w \in W, d \leq 0 \rangle,$$

and the objects of $\mathcal{P}_{\mathcal{IW}}$ are given by

$$\langle \Delta_w[d] \mid w \in {}^fW, d \geq 0 \rangle \cap \langle \nabla_w[d] \mid w \in {}^fW, d \leq 0 \rangle.$$

2.4. **A proof of Theorem 1.1.** We now quickly reproduce the proof of Mirković's result, Theorem 1.1, that is given in [1, Remark 7].

Proof of Theorem 1.1. Recall that the standard and costandard perverse sheaves over the finite-dimensional flag variety G/B are defined analogously to those over $\mathcal{F}\ell$; we also denote them by $j_{w!}$ and j_{w*}, respectively, but these are only given for w in the finite Weyl group W_f. Let w_0 denote the longest element of W_f. Now suppose $\mathcal{F} \in \mathcal{P}_I$ is convolution-exact: then $\mathcal{F} * j_{w_0!}$ is perverse, so it lies in $\langle j_{w!}[d] \mid w \in W_f, d \geq 0 \rangle$. By standard facts about perverse sheaves, this implies that $\mathcal{F} * j_{w_0!} * j_{w_0*} = \mathcal{F}$ lies in $\langle j_{w!} * j_{w_0*}[d] \mid w \in W_f, d \geq 0 \rangle$, where $j_{w!} * j_{w_0*} = j_{ww_0*}[d]$ for all $w \in W_f$ (because $\ell(w_0) = \ell(w^{-1}) + \ell(ww_0)$). It follows that a costandard filtration exists. The proof that a standard filtration exists is similar. $\square$

Note that the above argument hinges on the existence of the longest element w_0 in W_f; the lack of a "longest element" in W is what makes extending this argument to the infinite-dimensional setting difficult.

2.5. **The central sheaves Z_λ.** Finally, we introduce certain convolution-exact sheaves over the affine flag variety $\mathcal{F}\ell$ that are not tilting. Let $\lambda \in \Lambda^+$ be a dominant weight of the Langlands dual $\check{G}$. In [1, §3.2], Arkhipov and Bezrukavnikov construct a representation $V_\lambda \in \mathrm{Rep}(\check{G})$ with highest weight λ, to which they apply the geometric Satake isomorphism and a functor constructed by Gaitsgory [4] to obtain a central sheaf $Z_\lambda \in \mathcal{P}_I$ such that $W^*_{Z_\lambda}$ and $W^!_{Z_\lambda}$ both contain the weights of V_λ that occur with nonzero multiplicity (i.e., if $\mu \in \Lambda$ is a weight of V_λ with nonzero multiplicity, then $t(\mu) \in W^*_{Z_\lambda}, W^!_{Z_\lambda}$). For $\lambda \neq 0$, the sheaves Z_λ are convolution-exact but not tilting [1, Remark 7], though the projection $Av_\Psi(Z_\lambda) \in \mathcal{P}_{\mathcal{IW}}$ is tilting [1, Remark 10].

3. Generalizing the proof of Mirković's result

In this section, we try to generalize the argument for Mirković's result regarding convolution-exact sheaves over G/B, as presented in Section 2.4, to convolution-exact sheaves over $\mathcal{F}\ell$. Theorem 1.3 is the result of this attempt at a generalization.

Proof of Theorem 1.3. Let $\mathcal{F} \in \mathcal{P}_I$ be convolution-exact. Suppose first that there exists $w_1 \in W$ that has each element of $(\overline{W^*_{\mathcal{F}}})^{-1}$ as a prefix. Then by Lemma 2.2, $W^*_{\mathcal{F}*j_{w_1!}} \subseteq \overline{W^*_{\mathcal{F}}} \cdot w_1$, so by Lemma 2.3, because $\mathcal{F} * j_{w_1!}$ is perverse (since $\mathcal{F}$ is convolution-exact),

$$\mathcal{F} * j_{w_1!} \in \langle j_{vw_1!}[d] \mid v \in \overline{W^*_{\mathcal{F}}}, d \geq 0 \rangle.$$

It then follows by a standard fact about perverse sheaves that

$$\mathcal{F} = \mathcal{F} * j_{w_1!} * j_{w_1^{-1}*} \in \langle j_{vw_1!} * j_{w_1^{-1}*}[d] \mid v \in \overline{W^*_{\mathcal{F}}}, d \geq 0 \rangle.$$

Fix $v \in \overline{W^*_{\mathcal{F}}}$ and $d \geq 0$. We know v^{-1} is a prefix of w_1, so $\ell(w_1^{-1}) = \ell(w_1^{-1}v^{-1}) + \ell(v)$, and we can write by Lemma 2.1

$$j_{vw_1!} * j_{w_1^{-1}*}[d] = j_{vw_1!} * j_{w_1^{-1}v^{-1}*} * j_{v*}[d] = j_{v*}[d].$$

Thus $\mathcal{F} \in \langle j_{v*}[d] \mid v \in W, d \geq 0 \rangle$, which is enough to imply that $\mathcal{F}$ has a filtration with costandard subquotients, say by Lemma 2.3.

The proof of the second statement is similar; one writes $\mathcal{F} = \mathcal{F} * j_{w_2*} * j_{w_2^{-1}!}$. $\square$

Remark 2. As mentioned in Section 1, even if we only want to show $Av_\Psi(\mathcal{F})$ is tilting, by applying more facts about Av_Ψ we may at best only slightly weaken the hypotheses of Theorem 1.3; and by considering the alcoves, one easily sees that the resulting statement is really not much of an improvement on the sets $W^*_{\mathcal{F}}, W^!_{\mathcal{F}}$ we can handle. Explicitly, take the proof of the first statement: a priori, given any $w_1 \in W$, it seems most general to use the full powers of Lemmas 2.4 and 2.5 to try to write each convolution $j_{vw_1!} * j_{w_1^{-1}*}[d]$ in the form $j_{w_f!} * j_{u_1*} * j_{u_2*}[d]$ for $w_f \in W_f$ and $u_1, u_2 \in W$, so that because $\Delta_e * -$ is a triangulated functor, $Av_\Psi(\mathcal{F}) = \Delta_e * \mathcal{F}$ would lie in the span of sheaves of the form

$$\begin{aligned} \Delta_e * j_{w_f!} * j_{u_1*} * j_{u_2*}[d] &= \Delta_e * j_{u_1*} * j_{u_2*}[d] \\ &\in \langle \Delta_e * j_{w*}[d] \mid d \geq 0, w \in W \rangle \\ &= \langle \nabla_w[d] \mid d \geq 0, w \in {}^fW \rangle, \end{aligned}$$

which would imply $Av_\Psi(\mathcal{F})$ has a costandard filtration. The most reasonable way to accomplish this is to try to write $vw_1 = w_f x$, where $w_f \in W_f$ and $\ell(vw_1) = \ell(w_f) + \ell(x)$, and $w_1^{-1} = yz$, where $\ell(w_1^{-1}) = \ell(y) + \ell(z)$ and x^{-1} is a prefix of y; this would allow us to write

$$j_{vw_1!} * j_{w_1^{-1}*[d]} = j_{w_f!} * j_{x!} * j_{y*} * j_{z*}[d] = j_{w_f!} * j_{xy*} * j_{z*}[d].$$

However, if such w_f, x, y, z exist, then x^{-1} is a prefix of w_1^{-1}, and we can assume without loss of generality that $y = w_1^{-1}$ and $z = e$ (or alternatively that $y = x^{-1}$ and $z = xw_1^{-1}$). After rearranging, we find that it is equivalent to require the existence of some $w_f \in W_f$ such that $v^{-1}w_f$ is a prefix of w_1 and $\ell(vw_1) = \ell(w_f) + \ell(w_f^{-1}vw_1)$. But even if we ignore the length condition, we will see that most finite sets S do not satisfy the resulting desired property: there exists $w \in W$ such that for all $v \in S$, there exists $w_v \in W_f$ for which $v^{-1}w_v$ is a prefix of w. Moreover, this property is only slightly more general than the original property: in the alcove picture (as described in Section 2.1), multiplying v^{-1} on the right by various elements of W_f can only yield other alcoves close to the alcove of v^{-1}, and all these alcoves are the prefixes of similar $w \in S$. (However, if $v^{-1}w_f$ were replaced by $w_f v^{-1}$, and we ignored the length condition, then we would be done by Proposition 1.4.)

But to be able to apply Theorem 1.3 to any particular $\mathcal{F}$, we need to first give a more explicit condition under which $w_1 \in W$ or $w_2 \in W$ exists. To do this, we work in terms of W itself, proving Proposition 1.4, which gives a natural broad class of finite subsets $S \subseteq W$ for which there exists a $w \in W$ having each element as a prefix. First, we prove a preliminary computational result.

Lemma 3.1. *Let $\lambda \in \Lambda$, and let $\Phi^+_{\geq 0} := \{\alpha \in \Phi^+ \mid \langle \lambda, \alpha \rangle \geq 0\}$ and $\Phi^+_{<0} := \{\alpha \in \Phi^+ \mid \langle \lambda, \alpha \rangle < 0\}$. Then $P := \Phi^+_{\geq 0} \cup (-\Phi^+_{<0})$ is a system of positive roots for Φ, so there exists a unique $w_f \in W_f$ such that $w_f \cdot P = \Phi^+$. Moreover, $w_f t(\lambda)$ is the (unique) minimal-length representative of the right coset $W_f t(\lambda) \subseteq W$, and has length*

$$\sum_{\alpha \in \Phi^+} \langle w_f \cdot \lambda, \alpha \rangle - \ell(w_f).$$

Proof. It is straightforward to check that $\Phi^+_{\geq 0}$ and $-\Phi^+_{<0}$ are disjoint and that P is a system of positive roots for Φ. Thus such a w_f exists uniquely, and moreover has length $\ell(w_f) = |\Phi^+_{<0}|$.

Now for any $w \in W_f$, it is standard (see for instance [6, Chapter 2]) that the length $\ell(wt(\lambda))$ equals the sum

$$\sum_{\alpha \in \Phi^+} |\langle \lambda, \alpha \rangle + \chi(w \cdot \alpha)| = \sum_{\alpha \in \Phi^+_{\geq 0}} (\langle \lambda, \alpha \rangle + \chi(w \cdot a)) - \sum_{\alpha \in \Phi^+_{<0}} (\langle \lambda, \alpha \rangle + \chi(w \cdot \alpha)),$$

where $\chi\colon \Phi \to \{0, 1\}$ is the indicator function of Φ^-. This can be rewritten as

$$|(w \cdot \Phi^+_{\geq 0}) \cap \Phi^-| - |(w \cdot \Phi^+_{<0}) \cap \Phi^-| + \sum_{\alpha \in P} \langle \lambda, \alpha \rangle.$$

For all $w \in W_f$, this quantity is at least as large as $-|\Phi^+_{<0}| + \sum_{\alpha \in P} \langle \lambda, \alpha \rangle$. Equality is achieved if and only if $(w \cdot \Phi^+_{\geq 0}) \subseteq \Phi^+$ and $(w \cdot (-\Phi^+_{<0})) \subseteq \Phi^+$, that is, if and only if $w \cdot P \subseteq \Phi^+$. It follows that for w ranging in W_f, the length $\ell(wt(\lambda))$ is uniquely minimized when $w = w_f$, and this minimum length equals

$$-|\Phi^+_{<0}| + \sum_{\alpha \in w_f^{-1} \Phi^+} \langle \lambda, \alpha \rangle = -\ell(w_f) + \sum_{\alpha \in \Phi^+} \langle w_f \cdot \lambda, \alpha \rangle,$$

as claimed. $\square$

We now use Lemma 3.1 to prove Proposition 1.4.

Proof of Proposition 1.4. Note that we may assume without loss of generality that $w_f = 1$. Write S as $\{w_i t(\lambda_i)\}$, where $w_i \in W_f$ and $\lambda_i \in \Lambda$ for all i. There exists some $\lambda \in \Lambda^+$ such $\lambda - (w_i \cdot \lambda_i) \in \Lambda^+$ for all i; we claim that setting $w := t(\lambda)$ suffices. It is clear that $t(\lambda) \in {}^f W$, say from Lemma 3.1. It remains to show that

$$\ell(t(\lambda)) = \ell(w_i t(\lambda_i)) + \ell(t(-\lambda_i) w_i^{-1} t(\lambda))$$

for all i. Fix i; then using the fact that $\lambda - (w_i \cdot \lambda_i) \in \Lambda^+$ and $\lambda \in \Lambda^+$, we compute

$$\begin{aligned}\ell(t(-\lambda_i)w_i^{-1}t(\lambda)) &= \ell(w_i^{-1}t((-w_i \cdot \lambda_i) + \lambda)) \\ &= \ell(w_i^{-1}) + \ell(t(\lambda - (w_i \cdot \lambda_i))) \\ &= \ell(w_i) + \sum_{\alpha \in \Phi^+} \langle \lambda - (w_i \cdot \lambda_i), \alpha \rangle \\ &= \ell(w_i) + \ell(t(\lambda)) - \sum_{\alpha \in \Phi^+} \langle w_i \cdot \lambda_i, \alpha \rangle.\end{aligned}$$

The result then follows from the fact that $w_i t(\lambda_i) \in {}^f W$ and Lemma 3.1. □

Combining Theorem 1.3 with Proposition 1.4 allows us to affirmatively answer Question 1.2 for some very elementary convolution-exact sheaves $\mathcal{F}$, for instance, those such that $W^!_{\mathcal{F}}, W^*_{\mathcal{F}} \subseteq \Omega$, as mentioned in Section 1. But the scope of these results does not include many of the more sophisticated sheaves we care about. For instance, suppose $\mathcal{F}$ is given by the central sheaf Z_λ for some $\lambda \in \Lambda^+$. Then, as described in Section 2.5, the set $W^*_{Z_\lambda}$ contains the weights of V_λ; in particular, $W^*_{Z_\lambda}$ contains the W_f-orbit of λ. It is then easy to see that in most cases, there is no element of W that contains each element of $(W^*_{Z_\lambda})^{-1}$ as a prefix (visually, one may consider the alcoves corresponding to elements of the W_f-orbit of λ). Note that such examples arise even in the $G = \mathrm{SL}_2$ case; writing $V = \mathbb{R}$, $\Phi = \{2, -2\}$, and $\Lambda^+ = \mathbb{Z}_{\geq 0}$, we notice that the argument fails for $\mathcal{F} = Z_\lambda$ for $\lambda \geq 1$.

4. Wakimoto filtrations

First we define the *Wakimoto sheaves* $J_w \in D_I$ for $w \in W$, following [1]. To define these, we first set $J_{t(\lambda)} \coloneqq j_{t(\lambda)*}$ for $\lambda \in \Lambda^+$ and $J_{t(\lambda)} = j_{t(\lambda)!}$ for $\lambda \in -\Lambda^+$ and $J_{w_f} = j_{w_f *}$ for $w_f \in W_f$; it can then be shown that these definitions may be (uniquely) extended to all $w \in W$ in such a way that $J_{t(\lambda)w} = J_{t(\lambda)} * J_w$ for all $\lambda \in \Lambda$ and $w \in W$. Moreover, we have the following.

Lemma 4.1 ([1, Theorem 5a and Proposition 5a])**.** *The sheaves $J_w \in D_I$ lie in the subcategory $\mathcal{P}_I$. Moreover, any convolution-exact object $\mathcal{F} \in \mathcal{P}_I$ has a filtration with subquotients of the form J_w.*

In fact, the proof of Lemma 4.1 still holds if one makes small modifications to the definition of J_w, for instance, if we instead require $J_{wt(\lambda)} = J_w * J_{t(\lambda)}$, or if we set $J_{t(\lambda)} \coloneqq j_{t(\lambda)!}$ for $\lambda \in \Lambda^+$ and $J_{t(\lambda)} \coloneqq j_{t(\lambda)*}$ for $\lambda \in -\Lambda^+$, or if we set $J_{w_f} = j_{w_f !}$, or in general any combination of these changes.

For example, suppose we require $J_{wt(\lambda)} = J_w * J_{t(\lambda)}$ (and for ease of notation, we replace J_w with $\widetilde{J}_w$). Then given a convolution-exact $\mathcal{F}$, we immediately see (using results in Section 2) that if $\mathcal{F}$ has a filtration with subquotients of the form $\widetilde{J}_w$ where $w \in w_f \Lambda^+$, then $\Delta_e * \mathcal{F}$ has a costandard filtration. We may find other similar results by applying different variants on the definitions, but it seems difficult in general to rework these results into statements that do not explicitly resort to the existence of a specific type of Wakimoto-like filtration.

5. Future work

While a direct attempt at generalizing the proof of Theorem 1.1 did not answer Question 1.2, it is conceivable that a different combinatorial reduction may suffice. For example, as noted in Remark 2, if the group elements somehow showed up in a different order in the proof of Theorem 1.3, it is possible that one would be able to write down a result with a much farther reach.

Alternatively, from a more sheaf-theoretic point of view, we may consider the so-called stalks and costalks of the sheaves involved. Explicitly, for $w \in W$, the *stalk* of a sheaf $\mathcal{F}$ at w may be defined as (a shift of) $j_x^*(\mathcal{F})$, where $x \in \mathcal{F}\ell_w$ and $j_x \colon x \to \mathcal{F}\ell$, with *costalks* defined with $j_x^!$ instead of j_x^* (cf. [1, §4]). In fact, it is possible to rewrite the proof of Theorem 1.1 given in Section 2.4 in these terms, so it is possible that similar considerations would allow one to make additional progress on Question 1.2.

Another possible perspective is given by the *Radon transform* on certain perverse sheaves over $\mathcal{F}\ell$, as detailed in [8]; this operation may be thought of as an affine analog of the functor $\mathcal{F} \mapsto \mathcal{F} * j_{w_0!}$ in the case of the (finite-dimensional) flag variety, which was employed in the proof of Theorem 1.1 given in Section 2.4.

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