Positivity of the mass using $p-$harmonic functions on asymptotically flat 3-manifolds

SPUR Final Paper, Summer 2021

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August 5, 2021

Abstract

The positive mass theorem states that for asymptotically flat spacetimes, the ADM mass, defined by Arnowitt-Deser-Misner, is nonnegative. This theorem has been given several proofs, notably by Schoen-Yau, Witten, Geroch, and others. We generalize an argument of Jang which proves the positivity of the ADM mass of an asymptotically flat spacelike hypersurface (M^3, g) with a single exterior region $M_{\text{ext}} \cong \mathbb{R}^3 \setminus B_1(0)$. We focus on a quantity asymptotic to the Hawking quasi-local mass defined on level sets of a p-harmonic function, for $1 < p < 3$, and demonstrate a monoticity formula similar to the Geroch monotonicity formula. For $p = 2$, we recover Jang's argument.

1. Introduction

In this paper we announce a proof of the positive mass theorem and sketch out in large our future argument. Let (M^3, g) be a complete Riemannian 3-manifold. An open submanifold $N \subseteq M$ is an *asymptotically flat end* of order τ if there exists a diffeomorphism $\Phi\colon N \to \mathbb{R}^3 \setminus B_1(0)$ for which the metric satisfies the decay conditions

$$|(\Phi_* g)_{ij} - \delta_{ij}| = o(r^{-\tau}); \quad |\partial_k (\Phi_* g)_{ij}| = o(r^{-\tau-1}); \quad |\partial_k \partial_l (\Phi_* g)_{ij}| = o(r^{-\tau-2}). \tag{1.1}$$

We say M is *asymptotically flat* of order τ if there exists a compact set $K \subset M$ such that $M \setminus K = N_1 \sqcup \cdots \sqcup N_k$ is the disjoint union of k asymptotically flat ends of order τ, and $R(g) \in L^1(M)$, where R is the scalar curvature of M. For simplicity, we shall consider asymptotically flat manifolds with a single end M_{end}. Given such a manifold the *ADM mass* is defined as

$$\mathfrak{m}(M, g) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} ((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i}) \nu^i \, \mathrm{d}S_r, \tag{1.2}$$

where S_r is the sphere of radius r in the coordinate system, ν is the normal vector to the sphere with respect to $\Phi_* g$, and $\mathrm{d}S_r$ is the volume form induced on the sphere by $\Phi_* g$. Bartnik [Ba] showed that for a manifold with a single end asymptotically flat of order $\tau \geq \frac{1}{2}$, this quantity does not depend on the choice of K or diffeomorphism Φ, and thus is a geometric invariant. By [HI, Lemma 4.1], there exists an *exterior region* $M_{\text{ext}} \supset M_{\text{end}}$ diffeomorphic to $\mathbb{R}^3$ minus a finite number of balls with disjoint closure, with ∂M_{ext} minimal. In the physical setting, M is taken to be spacelike hypersurface of a Lorentzian manifold $(\mathcal{N}^4, \widetilde{g})$, with timelike normal vector field ξ, obeying the *dominant energy condition* $\mu \geq |J|$, where $\mu = \mathcal{T}(\xi, \xi)$ is the *mass-energy density* and J, which satisfies $\widetilde{g}(J, \xi') = -\mathcal{T}(\xi, \xi')$ for arbitrary ξ', is the *momentum density* [CP]. Here, $\mathcal{T}$ is the *stress-energy tensor*, which satisfies the Einstein field equation $\mathrm{Ric}(\widetilde{g}) - \frac{1}{2}R(\widetilde{g}) = 8\pi\mathcal{T}$. In the case where M is a maximal surface this condition is equivalent to $R(g) \geq 0$. The main result is:

Theorem 1.1. *Let (M^3, g) be a complete, asymptotically flat, manifold with nonnegative scalar curvature and a single end contained in an exterior region with boundary homeomorphic to a 2-sphere. Then $\mathfrak{m}(M, g) \geq 0$.*

Outline of Proof. We follow the general idea developed in [Ja]. Suppose there exists positive solution f to the p-Laplace equation on M_{ext} for $1 \leq p \leq 3$, that is,

$$\Delta_p f \overset{\text{def}}{=} \mathrm{div}_g(|\nabla f|^{p-2}\nabla f) = 0, \tag{1.3}$$

satisfying the Dirichlet boundary condition $f \equiv t_0$ on ∂M_{ext} for some $t_0 > 0$ and $f = 0$ at infinity. Motivated by the fundamental solution $|x|^{\frac{p-n}{p-1}}$ on $\mathbb{R}^n$, we define $r = f^{\frac{p-1}{p-3}}$. We then consider level sets $\Sigma_t = r^{-1}(\{t\})$ and define for regular values of r

$$W(t) = \frac{1}{16\pi}\int_{\Sigma_t} t\left[\left(R_{\Sigma_t} - \frac{1}{2}H^2\right) + \frac{1}{2}\left(H - \frac{2|\nabla r|}{t}\right)^2\right] \mathrm{d}\sigma_t, \tag{1.4}$$

where R_{Σ_t} and H are the scalar and mean curvatures of Σ_t respectively, and $\mathrm{d}\sigma_t$ is the volume form on Σ_t. Note that when $\Sigma_t \cong \mathbb{S}^2$, the first term in the integrand is related to the *Hawking quasi-local mass* by

$$\mathfrak{m}_H(\Sigma_t) = \sqrt{\frac{\mathrm{area}(\Sigma_t)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2 \mathrm{d}\sigma_t\right) = \sqrt{\frac{\mathrm{area}(\Sigma_t)}{\mathrm{area}(\mathbb{S}^2_t)}} \cdot \frac{1}{16\pi}\int_{\Sigma_t} t\left(R_{\Sigma_t} - \frac{1}{2}H^2\right)\mathrm{d}\sigma_t, \tag{1.5}$$

where $\mathbb{S}^2_t \subset \mathbb{R}^3$ is the sphere of radius t. The second term in (1.4) can be shown to be negligible for large t. Since we expect the Hawking quasi-local mass for 2-spheres to approach the ADM mass, by choosing t_0 so that $\mathrm{area}(\Sigma_t)/\,\mathrm{area}(\mathbb{S}^2_t) \to 1$, we can obtain $\lim_{t\to\infty} W(t) = \mathfrak{m}(M, g)$. The argument proceeds by establishing $W(t_0) \geq 0$ and the monotonicity of W. On an interval of regular values $W'(t)$ is well-defined and is shown to be nonnegative. In general, however, there will be critical values for which W is ill-defined. To handle this, we would like to consider the modificaton

$$\widetilde{W}(t) = \left[W(t) - \frac{t}{4}\chi(\Sigma_t)\right] + \frac{t}{2}|\Gamma(\Sigma_t)|, \tag{1.6}$$

where $\chi(X)$ and $\Gamma(X)$ are, respectively, the Euler characteristic and the set of connected components of a topological space X, and $|S|$ is the cardinality of a set S. We verify that $\widetilde{W}(t_0) = W(t_0)$ and that $\lim_{t\to\infty}\widetilde{W}(t) = \lim_{t\to\infty} W(t)$, as near the horizon and near infinity Σ_t is homeomorphic to a

2-sphere. Now suppose that r has only isolated critical *values*. Then to examine the monotonicity of W or $\widetilde{W}$, one only needs to study the potential jump discontinuities occuring at each critical value. Suppose that the potential jump discontinuities of W only arise due to changes in the topology of Σ_t across critical values. Then the first term $[W(t) - \frac{t}{4}\chi(\Sigma_t)]$, which removes the explicit dependence of $W(t)$ on the topology of Σ_t, does not change across critical values. Thus we may calculate

$$\mathfrak{m} = W(t_0) + \int_{t_0}^{\infty} \Big[W'(t) + \Big(2|\Gamma(\Sigma_t)| - \chi(\Sigma_t)\Big)\Big]\,\mathrm{d}t + \sum_{t_i \in \mathrm{Crit}(r)} t_i \left[\lim_{t \to t_i^+} |\Gamma(\Sigma_t)| - \lim_{t \to t_i^-} |\Gamma(\Sigma_t)|\right]. \quad (1.7)$$

We have that $W(t_0) \geq 0$, and as $W'(t)$ is nonnegative and $2|\Gamma(X)| - \chi(X) \geq 0$ for orientable 2-surfaces, the second term is nonnegative as well. The last term will be 0 by the maximum principle. Roughly speaking, an additional component can never exist, for it must bound a domain containing a maximum or minimum point not belonging to the level set, violating the maximum principle.

Our assumptions that the critical values of r are isolated and that the potential jump discontinuities of W only appear in the $\chi(\Sigma_t)$ term are unfounded, however. At best, we may guarantee that the set of critical values are contained in some interval $[t_{\min}, t_{\max}]$. We approximate r on $r^{-1}([t_{\min}, t_{\max}])$ to obtain a Morse function r_ϵ satisfying some "ϵ-close" criterion. As a result we must work with slightly perturbed functions W_ϵ, $\widetilde{W}_\epsilon$ and level sets Σ_t^ϵ. One now must show that

(a) the term $W_\epsilon(t) - \frac{t}{4}\chi(\Sigma_t^\epsilon)$ does not change across critical values of r_ϵ,

(b) as $\epsilon \to 0$, the integral $\int_{t_0}^{\infty} W_\epsilon'(t)\,\mathrm{d}t$ may be made arbitrarily small, and

(c) as $\epsilon \to 0$, the sum $\sum_{t_i \in \mathrm{Crit}(r)} t_i \Big[\lim_{t \to t_i^+} |\Gamma(\Sigma_t)| - \lim_{t \to t_i^-} |\Gamma(\Sigma_t)|\Big]$ may be made arbitrarily small.

For (c), we give an argument that relies on a "near"-maximum principle which allows us to pair positive and negative contributions from different critical points. The details of this argument are carried out in Section 5.

The paper is structured as follows. In Section 2, we discuss the p-Laplace boundary value problem and determine existence and regularity. In Section 3, we reintroduce the W function and determine that its limiting value is equal to the ADM mass; in Section 4, we establish (a) and (b) for approximations W_ϵ. In Section 5, we demonstrate the pairing argument and prove the positive mass theorem.

Acknowledgements.

The authors would like to thank faculty advisors Professor D. Jerison and Professor A. Moitra for having extended discussions and offering their insights. We would like to thank SPUR director Dr. S. Gerovitch as well as Dr. B. Peskin for making this (remote) research program possible. We thank J. Baldauf for introducing us to the positive mass theorem and its surrounding wealth of literature, and his mentoring, which we benefited from.

2. p-Harmonic Functions on the Exterior Region

Let (M^3, g) be an asymptotically flat manifold with a single end M_{end} contained in an exterior region M_{ext} with minimal boundary. Let $\Phi_0 \colon M_{\text{ext}} \to \mathbb{R}^3$ be an embedding of the exterior region satisfiying the decay conditions (1.1). Assume that we may extend the diffeomorphism Φ_0 to a coordinate system $\Phi \colon M_{\text{ext}} \to \mathbb{R}^3$.

For $1 \leq p < \infty$ and $\delta \in \mathbb{R}$ we define the weighted Lebesgue space $L^p_\delta(M_{\text{ext}}) \subset L^p_{\text{loc}}(M_{\text{ext}})$ to be the subspace of functions for which

$$\|u\|_{p,\delta} \overset{\text{def}}{=} \left(\int_{M_{\text{ext}}} |u|^p \sigma^{-\delta p - 3} \, \mathrm{d}V \right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

where $\sigma(x) = (1 + |\Phi(x)|^2)^{1/2}$ is the weight function. Proceeding, we can define weighted Sobolev spaces $W^{1,p}_\delta(M_{\text{ext}}) \subseteq L^p_\delta(M_{\text{ext}})$ to be the subspace of functions u with weak derivative ∇u for with

$$\|u\|_{1,\,p,\,\delta} \overset{\text{def}}{=} \|u\|_{p,\,\delta} + \|\nabla u\|_{p,\,\delta-1} < \infty, \tag{2.2}$$

where

$$\|\nabla u\|_{p,\,\delta-1} \overset{\text{def}}{=} \left(\int_{M_{\text{ext}}} |\nabla_i u|^p \sigma^{-(\delta-1)p-3} \, \mathrm{d}V \right)^{\frac{1}{p}}. \tag{2.3}$$

The definitions given are equivalent to those given in [Ba] for $n = 3$. We define the subspace of trace zero functions $W^{1,p}_{\delta,0}(M_{\text{ext}})$ to be the completion of $C^\infty_c(M^\circ_{\text{ext}})$ under the $\|\cdot\|_{1,\,p,\delta}$ norm. Without fullying defining a trace operator, we may meaningfully talk about solutions to a boundary value problem on $W^{1,p}_\delta(M_{\text{ext}})$. Suppose that $\partial M_{\text{ext}} \neq \varnothing$. Then for $u \in W^{1,p}_\delta(M_{\text{ext}})$, we say that $u \equiv 1$ on ∂M_{ext} if $u - \eta \in W^{1,p}_{\delta,0}(M_{\text{ext}})$ for some(any)[1] smooth function $\eta \in C^\infty_c(M_{\text{ext}})$ such that $\eta \equiv 1$ on ∂M_{ext}.

A function $u \in W^{1,p}_\delta(M_{\text{ext}})$ is said to be a weak solution to the p-Laplace equation if for all smooth φ compactly supported in the interior M°_{ext} of the exterior region.

$$\int_{M_{\text{ext}}} \langle \nabla\varphi, |\nabla u|^{p-2} \nabla u \rangle_g \, \mathrm{d}V = 0. \tag{2.4}$$

The central theorem in this section involves the existence and uniqueness of weak solutions. Shortly after, we will show our solution satisfies a regularity lemma, and give description of the asymptotic nature of solutions.

Theorem 2.1. *Let $1 < p < 3$ and $1 - 3/p < \delta < 0$. Then there exists a weak solution $u \in W^{1,p}_\delta(M_{\text{ext}}) \cap C^{1,\alpha}_{\text{loc}}(M_{\text{ext}})$ of the p-Laplace equation satisfying the boundary conditions $u \equiv 1$.*

The problem of existence and uniqueness of a solution to the p-Laplace equation may be rephrased in terms of finding a minimizer to the p-Dirichlet energy

$$\mathscr{E}_p[u] \overset{\text{def}}{=} \int_{M_{\text{ext}}} |\nabla u|^p \, \mathrm{d}V, \tag{2.5}$$

[1] Since compactly supported smooth functions which vanish on ∂M_{ext} are in $W^{1,p}_{\delta,0}(M_{\text{ext}})$, "some" may be replaced with "any."

after which we proceed with the direct method in the calculus of variations [?]. Note that $|\nabla u|$ is in general only *locally* integrable, so we consider the energy to be a functional $\mathscr{E}_p\colon W^{1,p}_{\delta}(M_{\mathrm{ext}}) \to [0,\infty]$.

Lemma 2.2. *Suppose that $u \in W^{1,p}_{\delta}(M_{\mathrm{ext}})$ minimizes $\mathscr{E}_p$ with $\mathscr{E}_p[u] < \infty$, subject to the boundary condition $u \equiv 1$ on $\partial M_{\mathrm{ext}}$. Then u is a weak solution to the p-Laplace equation.*

Proof. Let $\varphi \in C^{\infty}_{\mathrm{c}}(M_{\mathrm{ext}})$ and supp $\subset M_{\mathrm{ext}}{}^{\circ}$. Consider the variation $u_s = u + s\varphi$ for $s \in \mathbb{R}$. Let $\mathscr{E}_p(s) = \mathscr{E}_p[u_s]$. Since the variation φ has compact support, we may differentiate $\mathscr{E}_p$, which by minimality has

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{M_{\mathrm{ext}}} |\nabla u_s|^p \,\mathrm{d}V\right]_{s=0} &= \int_{M_{\mathrm{ext}}} \Big[p|\nabla u + s\nabla\varphi|^{p-2}\cdot\langle\nabla\varphi, \nabla u + \nabla s\varphi\rangle\Big]_{s=0} \mathrm{d}V. \\ &= \int_{M_{\mathrm{ext}}} \langle\nabla\varphi, |\nabla u|^{p-2}\nabla u\rangle \,\mathrm{d}V. \end{aligned} \tag{2.6}$$

Thus u is a weak solution to the p-Laplace equation. □

The direct method requires one to produce a minimizing sequence of the p-Dirichlet energy and extract a subsequence convergent in some topology of $W^{1,p}_{\delta}$ where $\mathscr{E}_p$ is lower semicontinuous, implying the limit is in fact a minimizer. The relevant topology will be the weak topology on $W^{1,p}_{\delta}$; therefore it is of interest to determine if such a subsequence can extracted via a weak compactness argument.

Lemma 2.3 (Hardy's Inequality)**.** *Let $1 < p < 3$. Then there exists a constant $C(p)$ such that for all $u \in C^{\infty}_{\mathrm{c}}(M_{\mathrm{ext}})$,*

$$\int_{M_{\mathrm{ext}}} |\sigma^{-1}u|^p \,\mathrm{d}V \leq C\cdot\int_{M_{\mathrm{ext}}} |\nabla u|^p \,\mathrm{d}V. \tag{2.7}$$

Proof. If we let $u \in C^{\infty}_{c}(M_{\mathrm{ext}})$, then since M_{ext} is diffeomorphic to $\mathbb{R}^n$ minus a union of finitely many balls, we can consider u as a function on $\mathbb{R}^n$, minus a finite number of balls. After a rigid motion, we may assume one of these balls contains the origin. If the number of balls is zero, then $M_{\mathrm{ext}} \cong \mathbb{R}^n$ and we assume $u \in C^{\infty}_{\mathrm{c}}(\mathbb{R}^n \setminus \{0\})$. Hardy's Inequality on Euclidean space then immediately gives that

$$\int_{M_{\mathrm{ext}}} \left|\frac{u}{r}\right|^p dV_{euc} \leq \frac{p}{n-p}\int_{M_{\mathrm{ext}}} |d\,|u||^2_{g_{euc}}\, dV_{euc}. \tag{2.8}$$

By asymptotic flatness, g and g_{euc} are uniformly equivalent on M_{ext}. Therefore, up to a constant, the above inequality must hold when $dV_{g_{euc}}$ is replaced with dV_g. In addition, g and g_{euc} are also uniformly equivalent on M_{ext}, so if we apply Kato's inequality we see that $|d\,|u||_{g_euc} \leq C\,|d\,|u||_g \leq C\,|\nabla u|_g$. We have thus shown that

$$\left\|r^{-1}u\right\|_{L^p(M_{\mathrm{ext}})} \leq C\,\|\nabla u\|_{L^p(M_{\mathrm{ext}})} \tag{2.9}$$

The lemma then immediately follows from the fact that r and σ are uniformly equivalent. □

Lemma 2.4. *Let $1 < p < 3$ and $\delta \geq 1 - 3/p$. Then there exists a constant $C(p)$ such that $\|u\|_{1,p,\delta} \leq C\cdot\mathscr{E}_p[u]$ for all $u \in W^{1,p}_{\delta}(M_{\mathrm{ext}})$.*

Proof. Since $C_c^\infty(M_{\text{ext}})$ is dense in $W_\delta^{1,p}(M_{\text{ext}})$ [Ba], it suffices to consider $u \in C_c^\infty(M_{\text{ext}})$. We may write for some $c(p)$ [Ba, Thm. 1.2(i)],

$$\|u\|_{1,\,p,\,\delta} = \|u\|_{p,\,\delta} + \|\nabla u\|_{p,\,\delta-1} \leq c \cdot \left[\|u\|_{p,\,1-\frac{3}{p}} + \cdot\|\nabla u\|_{p,-\frac{3}{p}}\right] \tag{2.10}$$

By (2.3), for some $C(p)$ we may write $\|\nabla u\|_{p,\,1-3/p} = \|\sigma^{-1}u\|_p \leq C\|\nabla u\|_p$, where $\|\cdot\|_p$ is the norm on $L^p(M_{\text{ext}})$. Furthermore, since $\|\cdot\|_{p,-3/p} = \|\cdot\|_p$, we have

$$\|u\|_{1,\,p,\,\delta} \leq C' \cdot \|\nabla u\|^p = C' \cdot \mathscr{E}_p[u], \tag{2.11}$$

which proves the claim. □

Lemma 2.5. *Let $1 < p < 3$. Then $W_\delta^{1,p}$ is reflexive; in particular, every bounded sequence has a weakly convergent subsequence.*

Proof. We follow the line of proof given in [Br, Prop. 8.1]. For $1 < p < \infty$, the Lebesgue space $L^p(M_{\text{ext}}, \mu_\delta) = L_\delta^p(M_{\text{ext}})$, where $\mu_\delta(U) = \int_U \sigma^{-\delta p-3}\,\mathrm{d}V$, is reflexive [Br, Thm. 4.10]. So consider the map

$$T\colon W_\delta^{1,p}(M_{\text{ext}}) \to L_\delta^p(M_{\text{ext}}) \times L_{\delta-1}^p(M_{\text{ext}};\mathbb{R}^3) \tag{2.12}$$

which takes $u \mapsto (u, \nabla u)$. From (2.1) it is clear that T is an isometric embedding. Since $W_\delta^{1,p}(M_{\text{ext}})$ is complete it follows that $\operatorname{im}(T)$ is complete and thus closed. Furthermore, since the product of reflexive spaces is reflexive, and closed subspaces of reflexive spaces are reflexive, it follows that $\operatorname{im}(T)$, and thus $W_\delta^{1,p}(M_{\text{ext}})$ is reflexive(see [Br, Section 3.5] for an account of the properties of reflexive spaces). In particular, every bounded sequence in $W_\delta^{1,p}$ has a weakly convergent subsequence [Br, Thm. 3.18]. □

Lemma 2.6. *Let $1 < p < 3$ and $\delta \in \mathbb{R}$. Then the p-Dirichlet energy functional $\mathscr{E}_p$ is weakly sequentially lower semicontinuous on $W_\delta^{1,p}(M_{\text{ext}})$.*

Proof. We follow the proof the Tonelli-Serrin theorem as done in [Ri, Thm. 2.6]. First, we prove the strong lower semicontinuity of $\mathscr{E}_p$. Consider the following well-known fact.

Fact 2.7. *Let a_n be a sequence of real numbers. Suppose there exists some real a such that for every subsequence $b_m = a_{n_m}$ of a_n, there exists a further subsequence b_{m_k} such that $a \leq \liminf_{k\to\infty} b_{m_k}$. Then $a \leq \liminf_{n\to\infty} a_n$.*

Now, let $u_n \to u$ be a norm-convergent sequence in $W_\delta^{1,p}(M_{\text{ext}})$, and $v_m = v_{n_m}$ be an arbitrary subsequence. Since $v_m \to u$, it follows from (2.1) that $\|\sigma^{-(\delta-1)p-3}|\nabla u - \nabla v_m|\|_p \to 0$; therefore there exists some subsequence v_{m_k} for which $\sigma^{-(\delta-1)p-3}|\nabla u - \nabla u_{m_k}| \to 0$ pointwise almost everywhere, so $\nabla u_{m_k} \to \nabla u$ pointwise a.e. as well. Thus by Fatou's lemma,

$$\mathscr{E}_p[u] = \int_{M_{\text{ext}}} |\nabla u|^p\,\mathrm{d}V \leq \liminf_{k\to\infty} \int_{M_{\text{ext}}} |\nabla v_{m_k}|^p\,\mathrm{d}V = \liminf_{k\to\infty} \mathscr{E}_p[v_{m_k}]. \tag{2.13}$$

Thus by (2.7), using the sequence $E_n = \mathscr{E}_p[u_n]$, we obtain $\mathscr{E}_p[u] \leq \liminf_{n\to\infty} \mathscr{E}_p[u_n]$. Since u_n was arbitrary, this shows the strong lower semicontinuity of $\mathscr{E}_p$.

To show weak sequential lower semicontinuity, we use the fact that a map $f\colon X \to \mathbb{R} \cup \{+\infty\}$ on a Banach space is (weakly) lower semicontinuous if and only if the epigraph, denoted by $\operatorname{epi}(f) = \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \geq \alpha\}$ is a (weakly) sequentially closed subset of $X \times \mathbb{R}$, and similarly, that f is convex if and only if $\operatorname{epi}(f)$ is convex. [Ri, Section 2.6]. Since $\mathscr{E}_p$ is strongly lower semicontinuous, $\operatorname{epi}\mathscr{E}_p$ is closed in the norm topology. Furthermore, given $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$ and $u_1, u_2 \in W^{1,p}_\delta(M_{\text{ext}})$, we have

$$\begin{aligned}\mathscr{E}_p[\lambda_1 u_1 + \lambda_2 u_2] = \int_{M_{\text{ext}}} |\lambda_1 \nabla u_1 + \lambda_2 \nabla u_2|^p \, \mathrm{d}V &\leq \lambda_1 \int\limits_{M_{\text{ext}}} |\nabla u_1|^p \, \mathrm{d}V + \lambda_2 \int\limits_{M_{\text{ext}}} |\nabla u_2|^p \, \mathrm{d}V \\ &= \lambda_1 \mathscr{E}_p[u_1] + \lambda_2 \mathscr{E}_p[u_2], \end{aligned} \tag{2.14}$$

which holds by the convexity of the integrand. Thus $\operatorname{epi}\mathscr{E}_p$ is convex. In particular, as it is closed in the norm topology and convex, it is closed in the weak topology [Br, Theorem 3.7]. Thus $\mathscr{E}_p$ is weakly lower semicontinuous. □

With these lemmas proven, all the sufficient conditions needed in order to proceed with the direct method have been established:

Proof of Theorem (2.1). By (2.2), it suffices to show that there exists a minimizer of the energy $\mathscr{E}_p$, subject to the given boundary condition. Let $\mathcal{A}_{p,\delta}(M_{\text{ext}}) = \{v \in W^{1,p}_{\delta,0}(M_{\text{ext}}) \mid v \equiv 1 \text{ on } \partial M_{\text{ext}}\}$ be the space of *admissible* functions, and note that for any smooth $\eta \in \mathcal{A}_{p,\delta}(M_{\text{ext}}) \cap C^\infty_{\mathrm{c}}(M_{\text{ext}})$ we may write $\mathcal{A}_{p,\delta}(M_{\text{ext}}) = \eta + W^{1,p}_{\delta,0}(M_{\text{ext}})$. In particular, $\inf_{\mathcal{A}_{p,\delta}(M_{\text{ext}})} \mathscr{E}_p \leq \mathscr{E}_p[\eta] < \infty$, since η has compact support. Now there exists a sequence of admissible functions $u_n \in \mathcal{A}_{p,\delta}(M_{\text{ext}})$ such that $\mathscr{E}_p[u_n] \leq E_0 + 1/n$, where $E_0 = \inf_{\mathcal{A}_{p,\delta}(M_{\text{ext}})} \mathscr{E}_p$. By (2.4), $\|u_n\|_{1,\,p,\,\delta} \leq C \cdot \mathscr{E}_p[u_n]$ for some constant $C(p)$, implying that u_n is a bounded sequence. Hence by (2.5), there exists a weakly convergent subsequence $u_{n_k} \in \mathcal{A}_{p,\delta}(M_{\text{ext}})$ with weak limit $u \in W^{1,p}_\delta(M_{\text{ext}})$. However, the plane $\mathcal{A}_{p,\delta}(M_{\text{ext}}) = \eta + W^{1,p}_{\delta,0}(M_{\text{ext}})$ is clearly convex and is closed in the norm topology, since $W^{1,p}_{\delta,0}(M_{\text{ext}})$ is closed. Thus it is closed in the weak topology [Br, Theorem 3.7]; in particular, $u \in \mathcal{A}_{p,\delta}(M_{\text{ext}})$. By (2.6), $\mathscr{E}_p$ is weakly sequentially lower semicontinuous, implying

$$E_0 \leq \mathscr{E}_p[u] \leq \liminf_{k\to\infty} \mathscr{E}_p[u_{n_k}] = E_0, \tag{2.15}$$

so $u \in \mathcal{A}_{p,\delta}(M_{\text{ext}})$ is a minimzer of $\mathscr{E}_p$ satisfying $u \equiv 1$ on ∂M_{ext}.

Now since the p-Laplace equation $\Delta_p u = 0$ takes on the *divergence form* $\operatorname{div}\vec{a}(x, u, \partial u) = 0$, where $a(x, u, \partial u) = |g|^{-1}(g^{ij}u_i u_j)^{\frac{p-2}{2}} g^{k\ell} u_\ell \, \partial_k$. We then calculate for $\xi \in \mathbb{R}^3$

$$a^k_{u_m} \xi^j \xi^k = \tfrac{p-2}{2} |g|^{-1/2} (g^{ij} u_i u_j)^{\frac{p-4}{2}} (g^{im} u_i)(g^{k\ell} u_\ell) \xi^j \xi^k, \tag{2.16}$$

$$|a^k_{u_m}| = \tfrac{p-2}{2} |g|^{-1/2} (g^{ij} u_i u_j)^{\frac{p-4}{2}} (g^{im} u_i)(g^{k\ell} u_\ell), \tag{2.17}$$

$$\begin{aligned}|a^k_{x_m}| &= \partial_m(|g|^{-1/2})(g^{ij} u_i u_j)^{\frac{p-2}{2}} (g^{k\ell} u_\ell) \\ &\quad + \tfrac{p-2}{2} |g|^{-1/2} (g^{ij} u_i u_j)^{\frac{p-4}{2}} [(\partial_m g^{ij}) u_i u_j + 2 g^{ij} u_{im} u_j](g^{k\ell} u_\ell),\end{aligned} \tag{2.18}$$

and $|a^k_u| = 0$. By asymptotic flatness, there exists continuous $\gamma_0, \gamma_1 \colon [0,\infty) \to [0,\infty)$ such that the hypotheses of [DB, Thm. 2] are satisfied. It follows that $u \in C^{1,\alpha}_{\text{loc}}(M_{\text{ext}})$ [DB]. □

Lemma 2.8. *Let $1 < p < 3$ and $u \in W^{1,p}_\delta(M_{\mathrm{ext}}) \cap C^{1,\alpha}_{\mathrm{loc}}(M_{\mathrm{ext}})$ be a weak solution to the p-Laplace equation satisfying the boundary conditions given in Lemma (2.1). Suppose $U \subset M_{\mathrm{ext}}$ is an open subset of M containing no critical points of u. Then $u|_U \in C^\infty(U)$.*

Proof. Let $K \subseteq U$ be compact. Then since $|\nabla u|$ is bounded below by a positive constant, the p-Laplace equation is uniformly elliptic and satisifies the conditions of [LU, Thm. 5.2, 6.3], which implies it is smooth. □

We will use an argument based on the comparison principle for supersolutions and subsolutions of the p-Laplace equation to handle the asymptotic behavior of a solution u. On $\mathbb{R}^n$, the *fundamental solution* of the p-Laplace equation is given by $|x|^{(p-n)/(p-1)}$.

Lemma 2.9. *Let $1 < p < 3$ and $u \in W^{1,p}_\delta(M_{\mathrm{ext}}) \cap C^{1,\alpha}_{\mathrm{loc}}(M_{\mathrm{ext}})$ be a weak solution to the p-Laplace equation satisfying the boundary conditions given in Lemma (2.1). Then the set of critical points $\mathrm{Crit}(u)$ is contained in a compact set K. In particular, u is smooth on $M_{\mathrm{ext}} \setminus K$.*

Lemma 2.10. *Let $1 < p < 3$ and $u \in W^{1,p}_\delta(M_{\mathrm{ext}}) \cap C^{1,\alpha}_{\mathrm{loc}}(M_{\mathrm{ext}})$ be a weak solution to the p-Laplace equation satisfying the boundary conditions given in Lemma (2.1). Then $\nabla u \neq 0$ on $\partial M_{\mathrm{ext}}$. In particular, this implies the existence of an open set $U \supset \partial M_{\mathrm{ext}}$ such that $\mathrm{Crit}(u) \cap U = \varnothing$.*

Lemma 2.11. *Let $1 < p < 3$ and $u \in W^{1,p}_\delta(M_{\mathrm{ext}}) \cap C^{1,\alpha}_{\mathrm{loc}}(M_{\mathrm{ext}})$ be a weak solution to the p-Laplace equation satisfying the boundary conditions given in Lemma (2.1) such that u is smooth on the complement $M_{\mathrm{ext}} \setminus K$ of a compact set K. Let $r = u^{(p-1)/(p-3)}$. Then on $M_{\mathrm{ext}} \setminus K$ we have*

$$\lim_{|x|\to\infty} |\nabla r| > 0, \tag{2.19}$$

$$\lim_{|x|\to\infty} \nabla_{\nabla r/|\nabla r|^2} |\nabla r| = 0. \tag{2.20}$$

3. Hawking and ADM Masses

In this section we recall known expressions for the mass and introduce a new quantity whose limiting value is bounded above by the ADM mass. Let M_{end} be the single end of an asymptotically flat manifold (M^3, g) with coordinates $\Phi\colon M_{\mathrm{end}} \to \mathbb{R}^3 \setminus B_1(0)$ satisfying (1.1) for $\tau \geq \frac{1}{2}$. Recall that the *ADM mass* is defined as

$$\mathfrak{m}(M, g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} ((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i}) \nu^i \,\mathrm{d}S_r.$$

Given a 2-surface $\Sigma \subset M_{\mathrm{end}}$ one has the *Hawking quasi-local mass* [Ha],

$$\mathfrak{m}_{\mathrm{H}}(\Sigma) = \sqrt{\frac{\mathrm{area}(\Sigma)}{16\pi}} \left(1 - \int_\Sigma H^2 \,\mathrm{d}\sigma\right), \tag{3.1}$$

which is asymptotic to the ADM mass for coordinate spheres $\Sigma_r = S_r(0)$ as $r \to \infty$. We have the following result, see [Ba, Prop. 4.1][2], which allows more generality in the choice of surface in (**??**).

[2]Note that the first condition in [Ba] is redundant and the connectedness assumption may be relaxed.

Proposition 3.1. *Let (M^3, g) be an asymptotically flat manifold with a single end M_{end} and asymptotically flat coordinates $\Phi\colon M_{\mathrm{end}} \to \mathbb{R}^3 \setminus B_1(0)$ satisfying (1.1) with $\tau \geq \frac{1}{2}$. Let $\{D_k\}_{k=1}^{\infty}$ be an exhaustion of M_{end} by compact sets such that D_k is a smooth 3-manifold with boundary $\Sigma_k = \partial D_k$, and $\mathrm{area}_g(\Sigma_k) = O(r_k^2)$ as $r_k \to \infty$, where $r_k = \inf_{x \in \Sigma_k}|x|$. Then*

$$\mathfrak{m}(M, g) = \lim_{k\to\infty} \frac{1}{16\pi} \int_{\Sigma_k} ((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i})\nu^i \, \mathrm{d}\sigma_k, \tag{3.2}$$

where $\mathrm{d}\sigma_k$ is the volume form induced on Σ_k by g.

Definition 3.2. Let $\{D_\rho\}_{\rho > \rho_0}$ be an exhaustion of M_{end} by compact 3-manifolds with smooth boundary Σ_ρ. We say that $\{\Sigma_\rho\}_{\rho>\rho_0}$ *approximates* $\mathbb{S}^2_\rho$ (compare with "nearly round" in [MT, Def. 2.1]) if

1. there exists $c_1, c_2 > 0$ such that $c_1\rho \leq |x| \leq c_2\rho$ for all $x \in \Sigma_\rho$,
2. $\mathrm{area}(\Sigma_\rho) = \mathrm{area}(\mathbb{S}^2_\rho)(1 + O(\rho^{-\tau}))$, where $\mathbb{S}^2_\rho = \{|x| = \rho\}$, and
3. $\sup_{\Sigma_\rho}|X - \rho\nu| = O(\rho^{1-\tau})$, where $X = x^i\partial_i$ and ν is the unit normal vector to Σ_ρ.

Note that the third condition is stronger than what the hypotheses taken by [MT, Def. 4.1] imply, as (3.2) further restricts the "center" of the approximate family to be the origin. We now determine an asymptotic relation between the Hawking and ADM mass sufficient for our purposes; to do this, we adapt the method of argument in [MT, Thm. 2.1]. Before doing so, we define a notion of asymptotic behavior for functions defined on the M_{end} using the index ρ.

Definition 3.3. Let $f\colon M_{\mathrm{end}} \to \mathbb{R}$ and $\xi\colon [\rho_0, \infty) \to \mathbb{R}$ be functions. We say that $f = O_\rho(\xi)$ as $\rho \to \infty$ if there exists a constant C and $\rho_1 \geq \rho_0$ such that $f(x) \leq C\xi(\rho)$ for all $x \in \Sigma_\rho$ for $\rho \geq \rho_1$. Likewise, we say that $f = o_\rho(\xi(\rho))$ as $\rho \to \infty$ if for every $c > 0$ there exists $\rho_1(c) \geq \rho_0$ such that $f(x) \leq c\xi(\rho)$ for all $x \in \Sigma_\rho$ for $\rho \geq \rho_1(c)$.

It follows immediately that $O(|x|) \subseteq O_\rho(\rho)$ and $o(|x|) \subseteq o_\rho(\rho)$. Furthermore, the usual rules of arithmetic apply.

Lemma 3.4. *Let (M^3, g) be manifold with a single asymptotically flat end M_{end} of order $\tau \geq \frac{1}{2}$. Let $\{D_\rho\}_{\rho>\rho_0}$ be an exhaustion of M_{end} by compact 3-manifolds with smooth boundary Σ_ρ such that $\{\Sigma_\rho\}_{\rho>\rho_0}$ approximates $\mathbb{S}^2_\rho$. Then*

$$\lim_{\rho\to\infty} \mathfrak{m}_{\mathrm{H}}(\Sigma_\rho) \leq \mathfrak{m}_{\mathrm{ADM}}(M, g). \tag{3.3}$$

Proof. Define the tensor $G = \frac{1}{2}R_g g - \mathrm{Ric}_g$. We wish to establish the inequality

$$\lim_{\rho\to\infty} \mathfrak{m}_{\mathrm{H}}(\Sigma_\rho) \leq \lim_{\rho\to\infty} \frac{1}{8\pi} \int_{\Sigma_\rho} G(\rho\nu, \nu)\, \mathrm{d}\sigma_\rho = \mathfrak{m}_{\mathrm{ADM}}(M, g), \tag{$\star$}$$

where ν is the unit normal to Σ_ρ. To study the curvatures we apply the Koszul formula $\Gamma^k_{ij} = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k})$ and use the fact that $g_{ij,k} = o_\rho(\rho^{-1-\tau})$ and $g_{ij,kl} = o(|x|^{-2-\tau}) = o_\rho(\rho^{-2-\tau})$ to obtain

$$
\begin{aligned}
R_{ij} = R^k{}_{ikj} &= \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{k\ell}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^\ell_{ki} && (3.4)\\
&= \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + o_\rho(\rho^{-2-2\tau}) && (3.5)\\
&= \tfrac{1}{2}(g_{ik,jk} + g_{kj,ik} - g_{ji,kk}) - \tfrac{1}{2}(g_{ik,kj} + g_{kk,ij} - g_{ki,kj}) + o_\rho(\rho^{-2-2\tau}) && (3.6)\\
&= \tfrac{1}{2}(g_{ik,jk} + g_{kj,ik} - g_{ji,kk} - g_{kk,ij}) + o_\rho(\rho^{-2-2\tau}). && (3.7)
\end{aligned}
$$

In particular, $R_{ij} = o_\rho(\rho^{-2-\tau})$. One also gets

$$
R = g^{ij}R_{ij} = R_{ii} + (g^{ij} - \delta^{ij})R_{ij} = R_{ii} + o_\rho(\rho^{-2-2\tau}) = g_{ij,ij} + g_{ii,jj} + o_\rho(\rho^{-2-2\tau}). \tag{3.8}
$$

In order to calculate G we "extend" the metric coefficients g_{ij} to smooth functions defined on all of $\mathbb{R}^3$. Let $\eta\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth cutoff function such that $\eta \equiv 0$ on $\overline{B_1}$ and $\eta_m \equiv 1$ on $\mathbb{R}^n \setminus B_2$. Define the functions

$$
\overline{g}_{ij}(x) = \begin{cases} \eta(x) g_{ij}(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{B_1} \\ 0 & \text{if } x \in \overline{B_1}. \end{cases} \tag{3.9}
$$

Now for $\rho \geq \rho_1$, where $B_2(0) \subseteq D_{\rho_1}$, we may calculate

$$
\begin{aligned}
&\int_{\Sigma_\rho} \tfrac{1}{2}(g_{ik,jk} + g_{kj,ik} - g_{ji,kk} - g_{kk,ij}) x^i \nu^j \,\mathrm{d}\sigma_\rho && (3.10)\\
={}& \int_{D_\rho} \tfrac{1}{2}\partial_j \Big[(\overline{g}_{ik,jk} + \overline{g}_{kj,ik} - \overline{g}_{ji,kk} - \overline{g}_{kk,ij}) x^i \Big] \mathrm{d}V && (3.11)\\
={}& \int_{D_\rho} \tfrac{1}{2}(\overline{g}_{ik,jkj} + \overline{g}_{kj,ikj} - \overline{g}_{ji,kkj} - \overline{g}_{kk,ijj}) x^i \,\mathrm{d}V + \int_{D_\rho} (\overline{g}_{kj,kj} - \overline{g}_{kk,jj})\,\mathrm{d}V && (3.12)\\
={}& \int_{D_\rho} \tfrac{1}{2}(\overline{g}_{kj,ikj} - \overline{g}_{kk,ijj}) x^i \mathrm{d}V + \int_{D_\rho} (\overline{g}_{kj,kj} - \overline{g}_{kk,jj})\,\mathrm{d}V && (3.13)\\
={}& \int_{D_\rho} \tfrac{1}{2}\partial_i [(\overline{g}_{kj,kj} - \overline{g}_{kk,jj}) x^i] \,\mathrm{d}V + \int_{D_\rho} \tfrac{1}{2}(\overline{g}_{kk,jj} - \overline{g}_{kj,kj})\,\mathrm{d}V && (3.14)\\
={}& \int_{\Sigma_\rho} \tfrac{1}{2}(g_{kj,kj} - g_{kk,jj}) x^i \nu^i \,\mathrm{d}\sigma_\rho + \int_{\Sigma_\rho} \tfrac{1}{2}(g_{kk,j} - g_{kj,k}) \nu^j \,\mathrm{d}\sigma_\rho, && (3.15)
\end{aligned}
$$

Note that $\int_{\Sigma_\rho} o_\rho(\rho^{-2-2\tau})\,\mathrm{d}\sigma_\rho = o(\rho^{-2-2\tau})\,\mathrm{area}(\Sigma_\rho) = o(\rho^{-2\tau}) = o(1)$, since $\tau \geq \frac{1}{2}$. Furthermore, since $|x^i - \rho\nu^i| = O_\rho(\rho^{-\tau})$, and $g_{ij,kl} = o_\rho(\rho^{-2-\tau})$, we find from (3.10), (3.15), and (3.1),

$$
\lim_{\rho\to\infty} \frac{1}{8\pi} \int_{\Sigma_\rho} G(\rho\nu, \nu)\mathrm{d}\sigma_\rho = \lim_{\rho\to\infty} \frac{1}{16\pi} \int_{\Sigma_\rho} (g_{kk,j} - g_{kj,k})\nu^j \,\mathrm{d}\sigma_\rho = \mathfrak{m}_{\mathrm{ADM}}(M, g). \tag{3.16}
$$

Now we use the Gauss-Codazzi equations to write

$$
\int_{\Sigma_\rho} G(\rho\nu, \nu)\mathrm{d}\sigma_\rho = \rho \int_{\Sigma_\rho} \left[\frac{1}{2}R - \mathrm{Ric}(\nu, \nu) \right] \mathrm{d}\sigma_\rho = \frac{\rho}{2} \int_{\Sigma_\rho} \left[R_{\Sigma_\rho} - \frac{1}{2}H_\rho^2 + \underbrace{\left(\|A_\rho\|^2 - \frac{1}{2}H_\rho^2 \right)}_{\geq 0} \right] \mathrm{d}\sigma_\rho, \tag{3.17}
$$

where R_{Σ_ρ} and H_ρ are the scalar and mean curvatures on Σ_ρ respectively, and A_ρ is the second fundamental form on Σ_ρ. Finally, the area estimate in (3.2) yields

$$\lim_{\rho\to\infty}\frac{1}{8\pi}\int_{\Sigma_\rho} G(\rho\nu,\nu)\mathrm{d}\sigma_\rho \geq \lim_{\rho\to\infty}\sqrt{\frac{\mathrm{area}(\Sigma_\rho)}{16\pi}}\int_{\Sigma_\rho}\left[R_{\Sigma_\rho}-\frac{1}{2}H_\rho^2\right]\mathrm{d}\sigma_\rho = \lim_{\rho\to\infty}\mathfrak{m}_{\mathrm{H}}(\Sigma_\rho), \tag{3.18}$$

so the conclusion follows. $\square$

Note that the second condition in (3.2) will not be necessary for our purposes; it will suffice to only show the first and the third conditions hold, along with the weaker condition $\mathrm{area}(\Sigma_\rho) = O(\rho^2)$, which gives

$$\lim_{\rho\to\infty}\frac{1}{16\pi}\int_{\Sigma_\rho}\rho\left[R_{\Sigma_\rho}-\frac{1}{2}H_\rho^2\right]\mathrm{d}\sigma_\rho \leq \mathfrak{m}_{\mathrm{ADM}}(M,g). \tag{3.19}$$

In our setting however, the second condition can be shown to follow from the third, so (3.4) still holds.

3.1. Level Sets of a p-Harmonic Function

Consider the function $r(x) = \tau_0\cdot u(x)^{\frac{p-1}{p-3}}$, where $u\in W^{1,p}_\delta(M_{\mathrm{ext}})\cap C^{1,\alpha}_{\mathrm{loc}}(M_{\mathrm{ext}})$ is a solution to the p-Laplace equation satisfying the boundary conditions in (2.1), and $\tau_0>0$ is some undetermined constant. Then by (2.9), r is smooth outside some compact set $K\supseteq M_{\mathrm{ext}}\setminus M_{\mathrm{end}}$. It follows that t is a regular value of r for $t>t_{\max}=\sup_K r$. We wish to show that for a particular value of τ_0, the family of surfaces $\{\Sigma_t\}_{t>t_{\max}}$ approximates $\mathbb{S}^2_t$, where $\Sigma_t = r^{-1}(\{t\})$.

In general, let $\Sigma_{t\geq t_0} = r^{-1}(\{t\geq t_0\})$, and $\Sigma_{t\leq t_0} = r^{-1}(\{t\leq t_0\})$, and define $\Sigma_{t>t_0}$ and $\Sigma_{t<t_0}$ similarly.

Lemma 3.5. *There exists $c_1, c_2>0$ such that $c_1 r(x)\leq |x|\leq c_2 r(x)$ for all $x\in\Sigma_{t>t_{\max}}$.*

Lemma 3.6. *Suppose* $\lim_{|x|\to\infty}|\nabla r| = 1$. *Then* $\mathrm{area}(\Sigma_t) = \mathrm{area}(\mathbb{S}^2_t)(1+O(t^{-\tau}))$.

Lemma 3.7. *Suppose* $\lim_{|x|\to\infty}|\nabla r| = 1$. *Then $\sup_{\Sigma_t}|X-t\nu| = O(t^{1-\tau})$, where $X = x^i\partial_i$ and ν is the unit normal vector to Σ_t.*

In view of these lemmas, we choose $\tau_0 = (\lim_{|x|\to\infty}|\nabla r|)^{-1}$, which is possible by (2.11). Now, define for regular values $t>t_{\min}$

$$W(t) = \frac{1}{16\pi}\int_{\Sigma_t} t\left[\left(R_{\Sigma_t}-\frac{1}{2}H^2\right)+\frac{1}{2}\left(H-\frac{2|\nabla r|}{t}\right)^2\right]\mathrm{d}\sigma_t. \tag{3.20}$$

As noted in the Introduction, we seek to relate this quantity to the Hawking quasi-local mass or ADM mass and afterwards produce a monotonicity formula.

Lemma 3.8. *Let $f\colon M\to\mathbb{R}$ be a smooth function on a Riemannian manifold M. Define $s=\lambda f^{\frac{p-1}{p-3}}$ for some constant $\lambda>0$, and set $\nu=\nabla s/|\nabla s|$, $\varphi = 1/|\nabla s|$. Let Σ be a regular level set of s. Then on Σ,*

$$\nabla_{\varphi\nu}|\nabla s| = \frac{1}{p-1}\left[\frac{2|\nabla s|}{s}-H\right]+\frac{1}{p-3}\cdot\frac{s^2}{|\nabla s|^{p-1}}\cdot\Delta_p f, \tag{3.21}$$

where H is the mean curvature of Σ.

Proof. Assume at first that $\lambda = 1$. We may calculate $\nabla f = \nabla s^{\frac{p-3}{p-1}} = \frac{p-3}{p-1} \cdot s^{\frac{-2}{p-1}} \nabla s$. Now by the product rule

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f) = \tfrac{p-3}{p-1}\Big[\Big(|\nabla s|^{p-2}s^{-2}\Big)\operatorname{div}(\nabla s) + \Big\langle \operatorname{grad}(|\nabla s|^{p-2}s^{-2}), \nabla s\Big\rangle\Big]. \tag{3.22}$$

Now $\operatorname{div}(\nabla s) = \Delta_2 s$ is the usual 2-Laplacian and

$$\operatorname{grad}(|\nabla s|^{p-2}s^{-2}) = -2s^{-3}|\nabla s|^{p-2}\nabla s + (p-2)s^{-2}|\nabla s|^{p-3}\operatorname{grad}|\nabla s|. \tag{3.23}$$

This yields

$$\Delta_p f = \frac{|\nabla s|^{p-1}}{s^2}\left(\frac{p-3}{p-1}\right)\left[(p-2)\nabla_{\phi\nu}|\nabla s| + \frac{\Delta_2 s}{|\nabla s|} - \frac{2|\nabla s|}{s}\right]. \tag{3.24}$$

Now the mean curvature of Σ is given by

$$H = \operatorname{tr} A = \frac{\operatorname{tr}\operatorname{Hess}^M_s - \operatorname{Hess}^M_s(\nu,\nu)}{|\nabla s|} = \frac{\Delta_2 s}{|\nabla s|} - \nabla_{\varphi\nu}|\nabla s|. \tag{3.25}$$

Substituting into (3.24) and rearrangment yields the conclusion for $\lambda = 1$. Now one verifies that (3.21) is invariant under the scaling $s \mapsto \lambda s$. $\square$

Our main result in this subsection is:

Lemma 3.9. $\lim_{t\to\infty} W(t) \le \mathfrak{m}_{\mathrm{ADM}}(M,g)$.

4. Morse Approximations to a p-Harmonic Function

In this section we study approximations by Morse functions to the function r defined in the previous section. Following the program detailed in the Introduction, we introduce modifications to W and show an approximate monotonicty formula.

Lemma 4.1. *Let $f\colon M \to \mathbb{R}$ be a smooth function on a Riemannian n-manifold M where $n > 2$. Suppose $p_0 \in M$ is a nondegenerate critical point of f with value t_0, and that there exists $\varepsilon > 0$ such that $f^{-1}([t_0-\varepsilon, t_0+\varepsilon])$ is compact and contains no critical point of f other than p. Let $\Omega_t = f^{-1}(\{t\})$ and ω_t be the volume form on Ω_t for $t \in [t_0-\varepsilon, t_0+\varepsilon] - \{t_0\}$. For a smooth function $\eta\colon f^{-1}([t_0-\varepsilon, t_0+\varepsilon]) - \{p_0\} \to \mathbb{R}$, define*

$$I_\eta(t) = \int_{\Omega_t} \eta \cdot \omega_t, \tag{4.1}$$

for $t \in (t_0-\varepsilon, t_0+\varepsilon) - \{t_0\}$. Let (U,ψ) be a Morse chart around p_0. If $\eta = O(|\psi|^{2-n})$ as $\psi(x) \to 0$, then I_η is continuous at t.

Proof. Without loss of generality, let $t_0 = 0$, assume that $\overline{B_1} \subset \psi(U)$, and give f the form

$$f = t_0 - |\mathbf{x}|^2 + |\mathbf{y}|^2; \quad \mathbf{x} = (\psi_1,\ldots,\psi_k),\ \mathbf{y} = (\psi_{k+1},\ldots,\psi_n). \tag{4.2}$$

Let $\mu\colon [0,\infty) \to \mathbb{R}$ be a smooth function satisfying the properties:

1. $\mu(0) > 0$ and $\mu^{(j)}(0) = 0$ for all $j \geq 1$.

2. $\mu(x) = 0$ for all $x \geq 1$.

3. $-1 < \mu'(x) \leq 0$ for all $x \geq 0$.

Using this we define the family of functions $\mu_\delta(x) = \delta\mu(x/\delta)$ for $\delta \in (0,1]$. Define $F_\delta \colon M \to \mathbb{R}$ by

$$F_\delta(p) = \begin{cases} f(\mathbf{x},\mathbf{y}) + \mu_\delta(|\mathbf{x}|^2 + |\mathbf{y}|^2) & \text{on } U \\ f(p) & \text{on } M - U. \end{cases} \tag{4.3}$$

One checks easily that F_δ is smooth. We calculate that on U,

$$\frac{\partial F_\delta}{\partial \psi_i} = 2\psi_i \cdot \left[\mu_\delta'(|\mathbf{x}|^2 + |\mathbf{y}|^2) \pm 1\right],$$

which only vanishes when $\psi_i = 0$. It follows that the only critical point of F_δ in U is p_0. Since $F_\delta \equiv f$ on $M - U$, the critical points of F_δ and f coincide. Since $F_1(p_0) > f(p_0) = 0$, it follows that 0 is not a critical value of F_1, and thus lies in some interval $[-\varepsilon_1, \varepsilon_1]$ of regular values. Since on $B_{\sqrt{\delta}}$,

$$F_\delta(\mathbf{x},\mathbf{y}) = \delta \cdot F_1\left(\frac{\mathbf{x}}{\sqrt{\delta}}, \frac{\mathbf{y}}{\sqrt{\delta}}\right), \tag{4.4}$$

and $F_\delta \equiv f$ outside of $B_{\sqrt{\delta}}$, it follows that $[-\delta\varepsilon_1, \delta\varepsilon_1]$ is an interval of regular values of F_δ. Using these intervals we define $I_\eta^\delta \colon [-\varepsilon_1, \varepsilon_1] \to \mathbb{R}$ for $\delta \in [0,1]$ by

$$I_\eta^\delta(\tau) = \begin{cases} \displaystyle\int_{\Omega_\tau^\delta} \eta \cdot \omega_\tau^\delta & \text{if } \delta > 0 \\ \lim_{t\to 0^-} I_\eta(t) & \text{if } \delta = 0 \text{ and } \tau \leq 0 \\ \lim_{t\to 0^+} I_\eta(t) & \text{if } \delta = 0 \text{ and } \tau > 0 \end{cases} \tag{4.5}$$

where $\Omega_\tau^\delta = F_\delta^{-1}(\{\tau\delta\})$ and ω_τ^δ is the volume form on Ω_τ^δ. Now one has $\Omega_\tau^\delta - B_{\sqrt{\delta}} = \Omega_{\tau\delta} - B_{\sqrt{\delta}}$ for $\tau \in [-\varepsilon_1, \varepsilon_1]$ and $\delta \in (0,1]$, which implies for $\tau \neq 0$,

$$\left|I_\eta^\delta(\tau) - I_\eta(\tau\delta)\right| \leq \int\limits_{\Omega_\tau^\delta \cap B_{\sqrt{\delta}}} |\eta| \cdot \omega_\tau^\delta + \int\limits_{\Omega_{\tau\delta} \cap B_{\sqrt{\delta}}} |\eta| \cdot \omega_{\tau\delta}. \tag{4.6}$$

Let $S_\tau^\delta = \Omega_\tau^\delta \cap B_{\sqrt{\delta}}$ and $T_\tau^\delta = \Omega_{\tau\delta} \cap B_{\sqrt{\delta}}$. Then by the scaling properties of F_δ and f,

$$F_\delta(\mathbf{x},\mathbf{y}) = \delta \cdot F_1\left(\frac{\mathbf{x}}{\sqrt{\delta}}, \frac{\mathbf{y}}{\sqrt{\delta}}\right) \text{ for } (\mathbf{x},\mathbf{y}) \in B_{\sqrt{\delta}} \implies S_\tau^\delta = \sqrt{\delta} \cdot S_\tau^1, \tag{4.7}$$

$$f(\mathbf{x},\mathbf{y}) = \delta \cdot f\left(\frac{\mathbf{x}}{\sqrt{\delta}}, \frac{\mathbf{y}}{\sqrt{\delta}}\right) \text{ for } (\mathbf{x},\mathbf{y}) \in B_{\sqrt{\delta}} \implies T_\tau^\delta = \sqrt{\delta} \cdot T_\tau^1. \tag{4.8}$$

Thus we may write for some constant $c(\tau) > 0$,

$$\left|I_\eta^\delta(\tau) - I_\eta(\tau\delta)\right| \leq \int_{S_\tau^\delta} |\eta| \cdot \omega_\tau^\delta + \int_{T_\tau^\delta} |\eta| \cdot \omega_{\tau\delta} \tag{4.9}$$

$$\leq \sup_{S_\tau^\delta} |\eta| \cdot \mathrm{vol}(\sqrt{\delta} \cdot S_\tau^1) + \sup_{T_\tau^\delta} |\eta| \cdot \mathrm{vol}(\sqrt{\delta} \cdot T_\tau^1) \tag{4.10}$$

$$\leq c(\tau) \cdot \delta^{\frac{n-1}{2}} \left[\sup_{S_\tau^\delta} \rho^{2-n} + \sup_{T_\tau^\delta} \rho^{2-n} \right]. \tag{4.11}$$

For $\tau \neq 0$ we have $\sup_{T_\tau^\delta} \rho^{2-n} = |\tau\delta|^{2-n}$, and by scaling,

$$\sup_{S_\tau^\delta} \rho^{2-n} = \delta^{2-n} \cdot \sup_{\bigcup_{\tau \in [-\varepsilon, \varepsilon]} S_\tau^1} \rho^{2-n}. \tag{4.12}$$

Together this shows that for $\tau \neq 0$,

$$\left|I_\eta^\delta(\tau) - I_\eta(\tau\delta)\right| \leq C(\tau) \left(1 + |\tau|^{2-n}\right) \sqrt{\delta}. \tag{4.13}$$

By taking a limit as $\delta \to 0^\pm$, this implies the pointwise convergence of $I_\eta^\delta(\tau)$ to $I_\eta^0(\tau)$ for $\tau \neq 0$. A scaling lemma will ultimately produce a uniform $c(\tau)$, which should rely on the fact that S_τ^δ is a "continuous" family of surfaces in some sense. Furthermore, a more clever way to handle the term $\int_{T_\tau^\delta} |\eta| \cdot \omega_{\tau\delta}$, which should not blow up as $\tau \to 0$, will ultimately yield uniform convergence, implying the conclusion. □

Lemma 4.2. *Let $f\colon M \to \mathbb{R}^3$ be a Morse function on a Riemannian manifold M and let $s = f^{\frac{p-1}{p-3}}$. Let x_0 be a critical point of f and (U, ψ) be a Morse chart around x_0. Then $|\nabla s| = O(|\psi|)$ and $|H| = O(|\psi|^{-1})$ as $\psi(x) \to 0$, where $H(x)$ is the mean curvature of the level set of s at a point x.*

Lemma 4.3. *Let $K \subset M_{\mathrm{ext}}$ be compact, and $u \in W_\delta^{1,p}(M_{\mathrm{ext}}) \cap C_{\mathrm{loc}}^{1,\alpha}(M_{\mathrm{ext}})$ be a weak solution to the p-Laplace equation satisfying (2.1). Then for any $\epsilon > 0$, there exists a Morse function $u_\epsilon \in W_\delta^{1,p}(M_{\mathrm{ext}}) \cap C^\infty(M_{\mathrm{ext}})$ such that $u \equiv u_\epsilon$ on $M_{\mathrm{ext}} \setminus K$, $\inf_K u_\epsilon > 1$, $\sup_K \|u - u_\epsilon\| < \epsilon$, and*

$$\left| \int_K \frac{r_\epsilon^2}{|\nabla r_\epsilon|^{p-2}} \left(\frac{2}{r_\epsilon} - \frac{H_\epsilon}{|\nabla r_\epsilon|} \right) \Delta_p u_\epsilon \, \mathrm{d}V \right| < \epsilon \tag{4.14}$$

where $r_\epsilon = \tau_0 \cdot u_\epsilon^{\frac{p-1}{p-3}}$ and H_ϵ is the mean curvature of a level set of r_ϵ at a point.

Proof. □

For $\epsilon > 0$, take such an approximation u_ϵ with respect to a compact set K such that $\mathrm{Crit}(f) \subseteq K$ and $\partial M_{\mathrm{ext}} = \varnothing$, which exists by (2.9) and (2.10). Then we may define for all regular values $t \geq \tau_0$,

$$W_\epsilon(t) = \frac{1}{16\pi} \int_{\Sigma_t^\epsilon} t \left[\left(R_{\Sigma_t^\epsilon} - \frac{1}{2} H^2 \right) + \frac{1}{2} \left(H - \frac{2|\nabla r_\epsilon|}{t} \right)^2 \right] \mathrm{d}\sigma_t^\epsilon, \tag{4.15}$$

where $\Sigma_t^\epsilon = r_\epsilon^{-1}(\{t\})$. By the the maximum principle, $u > 1$ on M_{ext}°. Then since $u_\epsilon > 1$ on K, and $u \equiv u_\epsilon$ on $M_{\text{ext}} \setminus K$, we have $u_\epsilon > 1$ on M_{ext}° as well. Thus $\Sigma_{\tau_0} = \Sigma_{\tau_0}^\epsilon = \partial M_{\text{ext}}$. In particular, we may calculate

$$W_\epsilon(\tau_0) = \frac{1}{16\pi}\int_{\partial M_{\text{ext}}} \tau_0 \left[R_{\partial M_{\text{ext}}} + \frac{2|\nabla r|^2}{\tau_0^2}\right] \mathrm{d}\sigma_{\tau_0} = \frac{\tau_0}{2} + \frac{1}{8\pi\tau_0}\int_{\partial M_{\text{ext}}} |\nabla r|^2 \, d\sigma_{\tau_0} > 0. \tag{4.16}$$

Definitionally, it is clear that $W_\epsilon(t) = W(t)$ for $t > \sup_K r$, so $\lim_{t\to\infty} W_\epsilon(t) = \lim_{t\to\infty} W(t) \leq \mathfrak{m}(M, g)$.

Lemma 4.4. *Let $[t_0, t_1]$ be an closed interval of regular values of r_ϵ. Then for $t \in (t_0, t_1)$,*

$$W_\epsilon'(t) \geq \frac{1}{8\pi}\int_{\Sigma_t^\epsilon} \left[\frac{1}{p-3}\cdot\frac{t^2}{|\nabla r_\epsilon|^{p-2}}\left(\frac{2}{t} - \frac{H}{|\nabla r_\varepsilon|}\right)\Delta_p u_\epsilon\right] \mathrm{d}\sigma_t^\epsilon. \tag{4.17}$$

Proof. Note that in this interval, we may write

$$W_\epsilon(t) = \frac{t}{4}\chi(\Sigma_t^\epsilon) + \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left(\frac{|\nabla r_\epsilon|^2}{t} - H|\nabla r_\epsilon|\right)\mathrm{d}\sigma_t^\epsilon. \tag{4.18}$$

Let $\nu = \nabla r_\epsilon / |\nabla r_\epsilon|$ and $\varphi = 1/|\nabla r_\epsilon|$. Since r_ϵ is proper and has no critical values in $[t_0, t_1]$, there exists a flow along the vector field $\phi\nu$ pushing $\Sigma_{t_0}^\epsilon$ diffeomorphically onto Σ_t^ϵ for $t \in [t_0, t_1]$. Thus the topology of Σ_t^ϵ does not change in $[t_0, t_1]$. Furthermore, we may differentiate under the integral to obtain

$$W'(t) = \frac{1}{4}\chi(\Sigma_t^\epsilon) + \frac{1}{8\pi}\left[\int_{\Sigma_t}\left(\frac{|\nabla r|}{t} - H\right)\mathcal{L}_\nu(\mathrm{d}\sigma_t) + \int_{\Sigma_t}\mathcal{L}_{\varphi\nu}\left(\frac{|\nabla r|^2}{r} - H|\nabla r|\right)\mathrm{d}\sigma_t\right]. \tag{4.19}$$

The variation of area is $\mathcal{L}_\nu(\mathrm{d}\sigma_t) = H\mathrm{d}\sigma_t$. Now we separately calculate using (3.8),

$$\begin{aligned}\mathcal{L}_{\varphi\nu}\left(\frac{|\nabla r_\epsilon|^2}{r_\epsilon}\right) &= \frac{2|\nabla r_\epsilon|\mathcal{L}_{\varphi\nu}|\nabla r_\epsilon|}{r_\epsilon} - \frac{|\nabla r_\epsilon|^2}{r_\epsilon^2} &(4.20)\\ &= \left(\frac{1}{p-1} - \frac{1}{4}\right)\cdot\frac{4|\nabla r_\epsilon|^2}{r_\epsilon^2} - \frac{1}{p-1}\cdot\frac{2H|\nabla r_\epsilon|}{r_\epsilon} + \frac{2}{p-3}\cdot\frac{r}{|\nabla r_\epsilon|^{p-2}}\cdot\Delta_p u_\epsilon &(4.21)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{\varphi\nu}(H|\nabla r_\epsilon|) &= \mathcal{L}_{\varphi\nu}|\nabla r_\epsilon|\cdot H + |\nabla r_\epsilon|\cdot\mathcal{L}_{\varphi\nu}H &(4.22)\\ &= \frac{1}{p-1}\cdot\frac{2H|\nabla r_\epsilon|}{r_\epsilon} - \frac{1}{p-1}\cdot H^2 + \frac{1}{p-3}\cdot\frac{r_\epsilon^2}{|\nabla r_\epsilon|^{p-1}}\cdot H\Delta_p u_\epsilon + |\nabla r_\epsilon|\cdot\mathcal{L}_{\varphi\nu}H. &(4.23)\end{aligned}$$

Together, this gives

$$\begin{aligned}W_\epsilon'(t) &= \frac{1}{4}\chi(\Sigma_t^\epsilon)\\ &+ \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left[\left(\frac{1}{p-1}-\frac{1}{4}\right)\cdot H^2 - 2\cdot\left(\frac{1}{p-1}-\frac{1}{4}\right)\cdot\frac{2H|\nabla r_\epsilon|}{t} + \left(\frac{1}{p-1}-\frac{1}{4}\right)\cdot\frac{4|\nabla r_\epsilon|^2}{t^2}\right]\mathrm{d}\sigma_t^\epsilon\\ &- \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left[\frac{3}{4}H^2 + |\nabla r_\epsilon|\cdot\mathcal{L}_{\varphi\nu}H\right]\mathrm{d}\sigma_t^\epsilon\\ &+ \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left[\frac{1}{p-3}\cdot\frac{t^2}{|\nabla r_\epsilon|^{p-2}}\left(\frac{2}{t}-\frac{H}{|\nabla r_\varepsilon|}\right)\Delta_p u_\epsilon\right]\mathrm{d}\sigma_t^\epsilon. \end{aligned}\tag{4.24}$$

Simplifying and applying the traced Riccati equation yields

$$\begin{aligned}W'(t) &= \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left[\left(\frac{1}{p-1}-\frac{1}{4}\right)\left(H^2-\frac{2|\nabla r_\epsilon|}{t}\right)^2 + \frac{1}{2}\left(\|h\|^2-\frac{1}{2}H^2\right) + \frac{1}{2}R_M + \frac{\Delta_{\Sigma_t^\epsilon}\varphi}{\varphi}\right]\mathrm{d}\sigma_t^\epsilon\\ &+ \frac{1}{8\pi}\int_{\Sigma_t^\epsilon}\left[\frac{1}{p-3}\cdot\frac{t^2}{|\nabla r_\epsilon|^{p-2}}\left(\frac{2}{t}-\frac{H}{|\nabla r_\varepsilon|}\right)\Delta_p u_\epsilon\right]\mathrm{d}\sigma_t^\epsilon. \end{aligned}\tag{4.25}$$

Since each of the terms in the first integrand are nonnegative, the inequality follows. □

Define $\widetilde{W}_\epsilon(t) = W_\epsilon(t) - t\left[\frac{1}{4}\chi(\Sigma_t^\epsilon) - \frac{1}{2}\Gamma(\Sigma_t^\epsilon)\right]$. Note that the additional term may be interpreted as the number of *handles* of the level set(see [Ja]).

Lemma 4.5. *$\widetilde{W}_\epsilon(\tau_0) = W(\tau_0)$ and $\lim_{t\to\infty}\widetilde{W}_\epsilon(t) = \lim_{t\to\infty} W(t)$.*

Proof. Since $\partial M_{\text{ext}} \cong \mathbb{S}^2$, it immediately follows that $\widetilde{W}_\epsilon(\tau_0) = W(\tau_0) > 0$. Now pick $t_0 > t_{\max}$. Then there exists a flow along the vector field $\phi\nu$ yielding diffeomorphisms from $\Sigma_{t_0}^\epsilon$ to Σ_t^ϵ for $t \geq t_0$. Thus one has a deformation retract $\Sigma_{t\geq t_0}^\epsilon \to \Sigma_{t_0}^\epsilon$ which may be extended to a retract $\mathbb{R}^3 \to \Sigma_{t\leq t_0}$. It follows that $\chi(\mathbb{R}^3) = \chi(\Sigma_{t\leq t_0}^\epsilon \cup \mathbb{B}^3) = 1$. For compact 3-manifolds N, one has $\chi(\partial N) = 2\chi(N)$; in particular, $\chi(\Sigma_{t_0}^\epsilon) = \chi(\partial(\Sigma_{t\leq t_0}^\epsilon \cup \mathbb{B}^3)) = 2\chi(\Sigma_{t\leq t_0}^\epsilon \cup \mathbb{B}^3) = \chi(\mathbb{S}^2)$.

Now let Q be a connected component of $\Sigma_{t\geq t_0}^\epsilon$. Assume, for the sake of contradiction, that Q is compact. Then there exists a point $x \in Q$ such that $r_\epsilon(x) = \sup_Q r_\epsilon$. But then the flow starting at x yields a curve $\gamma\colon [0,1] \to \Sigma_{t\geq t_0}^\epsilon$ such that $\gamma(0) = x$ and $\gamma(1) = y$ with $r_\epsilon(y) > t_0$. By path-connectedness, $y \in Q$ and $r_\epsilon(y) > r_\epsilon(x) = \sup_Q r_\epsilon$, which is a contradiction. Thus Q is noncompact, and in particular, unbounded as Q is closed. Now since $\Sigma_{t<t_0}^\epsilon$ is bounded, there exists a ball $B_R(0)$ such that $\Sigma_{t\leq t_0}^\epsilon \subset B_R(0)$, or equivalently, $\mathbb{R}^3 \setminus B_R(0) \subset \Sigma_{t\geq t_0}^\epsilon$. Finally, assume for the sake of contradiction that $Q_1 \neq Q_2$ are distinct connected components of $\Sigma_{t\geq t_0}^\epsilon$. Then without loss of generality, since $\mathbb{R}^3 \setminus B_R(0)$ is connected, we may assume $\mathbb{R}^3 \setminus B_R(0) \subseteq Q_1$. Then as $Q_1 \cap Q_2 = \varnothing$, we must have $Q_2 \subseteq B_R(0)$, implying Q_2 is bounded and hence compact, which is a contradiction. Thus $\Sigma_{t\geq t_0}^\epsilon$ is connected. By the deformation retract $\Sigma_{t\geq t_0}^\epsilon \to \Sigma_{t_0}^\epsilon$, we obtain $\Gamma(\Sigma_{t_0}^\epsilon) = \Gamma(\Sigma_{t\geq t_0}^\epsilon) = \Gamma(\mathbb{S}^2)$. The conclusion follows. □

We think of $\left[\frac{1}{4}\chi(\Sigma_t^\epsilon) - \frac{1}{2}\Gamma(\Sigma_t^\epsilon)\right]$ as a "topological" coefficient. As described in the Introduction, we would like to better understand W_ϵ' by first moving the discontinuous nature of W_ϵ into a single term.

Lemma 4.6. *Define $V_\epsilon(t) = W_\epsilon(t) - \frac{t}{4}\chi(\Sigma_t^\epsilon)$. Then V_ϵ admits a continuous extension to all of $[\tau_0, \infty)$.*

Proof. Let $\eta_\epsilon = |\nabla r_\epsilon|^2/r_\epsilon - H|\nabla r_\epsilon|$. Then for regular values t, write

$$V_\epsilon(t) = \frac{1}{8\pi}\int_{\Sigma_t^\epsilon} \eta_\epsilon \, \mathrm{d}\sigma_t^\epsilon. \tag{4.26}$$

When t is contained in an interval of regular values, it is clear that V_ϵ is continuous at t. As u_ϵ is a Morse function, the critical values of u_ϵ are isolated and are strictly greater that τ_0. Hence for a crtical value s_0 of u_ϵ, there exists $\delta > 0$ such that $u_\varepsilon^{-1}([s_0-\delta, s_0+\delta])$ is compact and contains no critical point of f other than $u_\epsilon^{-1}(\{s_0\})$. By (4.2) and (4.1), we observe that $V_\varepsilon(\tau_0 \cdot s^{(p-1)/(p-3)})$ may be extended continuously at $s = s_0$. It follows that V_ϵ may be extended continuously to all critical values. □

5. A Proof of the Positive Mass Theorem

We recall the statement of (5.1) and give its proof.

Theorem 5.1. *Let (M^3, g) be a complete, asymptotically flat, manifold with nonnegative scalar curvature and a single end contained in an exterior region with boundary homeomorphic to a 2-sphere. Then $\mathfrak{m}(M, g) \geq 0$.*

Proof. We have from our previous discussions that

$$\mathfrak{m}(M,g) \geq \lim_{t\to\infty} \widetilde{W}_\epsilon(t) = \lim_{t\to\infty}[V_\epsilon(t) + \tfrac{t}{2}\Gamma(\Sigma_t^\epsilon)] = \lim_{t\to\infty}[V_\epsilon(t) + \tfrac{t}{2}(\Gamma(\Sigma_{\leq t}^\epsilon) + \Gamma(\Sigma_{\geq t}^\epsilon) - 1)]. \tag{5.1}$$

Now by (4.6) we may write

$$\begin{aligned}\mathfrak{m}(M,g) \geq \widetilde{W}_\epsilon(\tau_0) + \int_{[\tau_0,\infty)} \left[\frac{\mathrm{d}V_\epsilon}{\mathrm{d}t} + \frac{1}{2}\Gamma(\Sigma_t^\epsilon)\right] \mathrm{d}t + \sum_i \frac{t_i}{2}\left[\lim_{t\to t_i^+}\Gamma(\Sigma_{\leq t}^\epsilon) - \lim_{t\to t_i^-}\Gamma(\Sigma_{\leq t}^\epsilon)\right] \\ + \sum_i \frac{t_i}{2}\left[\lim_{t\to t_i^+}\Gamma(\Sigma_{\geq t}^\epsilon) - \lim_{t\to t_i^-}\Gamma(\Sigma_{\geq t}^\epsilon)\right],\end{aligned} \tag{5.2}$$

where the sums are over critical values t_i. Now $\widetilde{W}_\epsilon(\tau_0) = W(\tau_0)$, and wherever defined, $V_\epsilon'(t) = W_\epsilon'(t) - \frac{1}{4}\chi(\Sigma_t^\epsilon)$. For ease, we make the redefinitions $\Gamma_\leq(t) = \Gamma(\Sigma_{\leq t}^\epsilon)$ and $\Gamma_\geq(t) = \Gamma(\Sigma_{\geq t}^\epsilon)$. Then we may rewrite the inequality as

$$\begin{aligned}\mathfrak{m}(M,g) \geq W(\tau_0) + \int_\tau^\infty W_\epsilon'(t)\,\mathrm{d}t + \frac{1}{2}\int_\tau^\infty \left[\Gamma(\Sigma_t^\epsilon) - \tfrac{1}{2}\chi(\Sigma_t^\epsilon)\right]\mathrm{d}t + \sum_i \frac{t_i}{2}\Big[\Gamma_\leq(t_i+) - \Gamma_\leq(t_i-)\Big] \\ + \sum_i \frac{t_i}{2}\Big[\Gamma_\geq(t_i+) - \Gamma_\geq(t_i-)\Big].\end{aligned} \tag{5.3}$$

The first term is positive by (4.16), and the the third term is nonnegative since $\Gamma(X) - \frac{1}{2}\chi(X) \geq 0$ for 2-surfaces. By (4.4),

$$\begin{aligned}\int_\tau^\infty W_\epsilon'(t)\,\mathrm{d}t &= \frac{1}{8\pi}\int_\tau^\infty \int_{\Sigma_t^\epsilon} \left[\frac{1}{p-3}\cdot\frac{t^2}{|\nabla r_\epsilon|^{p-2}}\left(\frac{2}{t} - \frac{H}{|\nabla r_\varepsilon|}\right)\Delta_p u_\epsilon\right] \mathrm{d}\sigma_t^\epsilon\,\mathrm{d}t \\ &= \frac{1}{8\pi}\int_{\Sigma^\epsilon_{t\leq t_{\max}}} \left[\frac{1}{p-3}\cdot\frac{t^2}{|\nabla r_\epsilon|^{p-3}}\left(\frac{2}{t} - \frac{H}{|\nabla r_\varepsilon|}\right)\Delta_p u_\epsilon\right] \mathrm{d}V \\ &\geq -\frac{\epsilon}{8\pi}. \end{aligned} \tag{5.4}$$

It remains to handle the discontinuities arising from changes in the number of connected components of the level sets. In spirit, the core of this argument is the same "near"-maximum principle used by Jang [Ja]. However, our formulation in terms of sublevel and superlevel sets simplifies matters. □

References

[Ba] Bartnik, R. (1986). The mass of an asymptotically flat manifold. *Communications on pure and applied mathematics, 39*(5), 661-693.

[Br] Brezis, H. (2010). *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media.

[CP] Corvino, J., & Pollack, D. (2011). Scalar curvature and the Einstein constraint equations. *arXiv preprint arXiv:1102.5050*

[DB] DiBenedetto, E. (1983). $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Analysis: Theory, Methods & Applications, 7*(8), 827-850.

[LU] Ladyzhenskaya, O. A., & Uraltseva, N. (1968). *Linear and quasilinear elliptic equations.* Academic Press, New York.

[Ha] S. Hawking. (1968). *Gravitational radiation in an expanding universe.* J. Math. Phys. 9 598–604

[HI] Huisken, G., & Ilmanen, T. (2001). The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry, 59*(3), 353-437.

[Ja] Jang, P. S. (1979). On positivity of mass for black hole space-times. *Communications in Mathematical Physics, 69*(3), 257-266.

[MT] Miao, P. & Tam, L. (2015) Evaluation of the ADM mass and center of mass via the Ricci tensor. *arXiv preprint arXiv:1408.3893*

[Ri] Rindler, F. (2018). *Calculus of variations.* Universitext, Springer, Cham, 2018. MR3821514