A new approach to the upper estimate of lattice points on a curve via ℓ^2 decoupling

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1. Introduction

In this paper we examine the number of $\frac{1}{N}$-integral points on a fixed curve Γ. That is, we start with a fixed curve Γ and consider the set $\Lambda = \Gamma \cap (\frac{1}{N}\mathbb{Z})^2$. It turns out that a higher regularity of the curve, under an additional analytic condition, implies a significantly better upper bound for $|\Lambda|$. Throughout the paper, we will write $A \lesssim_\epsilon B$ to denote $A \le CB$ for an implicit constant that depends on the parameter ϵ.

Let s be the smallest positive integer such that $n \le \frac{1}{2}(s+1)(s+2) - 1$, and define $\Delta n = n - (\frac{1}{2}s(s+1) - 1)$. Let us denote by $\mathcal{M}$ the set of all monomials with a positive degree about two variables x and y. Then for each finite subset $M \subset \mathcal{M}$, we can define its total degree $\deg(M)$ as the sum $\sum_{m\in M} \deg(m)$. We denote by $m(n)$ the minimal value possible for a total degree of n distinct monomials. For example, if $n = \frac{1}{2}(s+1)(s+2) - 1$, then $m(n) = \frac{s(s+1)(s+2)}{3}$. We call $M_n \subset \mathcal{M}$ a minimal collection of n monomials if it satisfies $|M_n| = n$ and $\deg(M_n) = m(n)$.

Given a planar curve Γ parameterized by $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $t \in [0,1]$ and a minimal collection of n monomials, $M_n = \{m_1, \cdots, m_n\}$, we can consider the $n \times n$ Wronskian determinant, which we denote by $W^{M_n}(\Gamma)(t)$,

$$W(m_1(\gamma_1,\gamma_2)', \cdots, m_n(\gamma_1,\gamma_2)')(t) = \begin{vmatrix} m_1(\gamma_1,\gamma_2)'(t) & \cdots & m_n(\gamma_1,\gamma_2)'(t) \\ \vdots & \ddots & \vdots \\ m_1(\gamma_1,\gamma_2)^{(n)}(t) & \cdots & m_n(\gamma_1,\gamma_2)^{(n)}(t) \end{vmatrix}$$

where we take derivatives about t. Now we can state our main result.

Theorem 1. *Suppose that Γ is a compact $C^{n,\alpha}$ curve such that $W^{M_n}(\Gamma)$ is nonvanishing for some minimal collection M_n of n monomials. Then we have*

$$|\Lambda| \lesssim N^{e_1(n)+\epsilon}$$

for each $\epsilon > 0$, where the exponent $e_1(n)$ is given by $e_1(n) = \frac{2}{n(n+1)}(\frac{(s-1)s(s+1)}{3} + s \cdot \Delta n)$.

The statement with $n = 2$ can be thought of as a weak version of Jarník's result [10]. The original theorem by Jarník states that a strictly convex curve Γ of length ℓ contains at most $3(4\pi)^{-1/3}\ell^{2/3} + O(\ell^{1/3})$ integral points. In Section 2 we prove Theorem 1 for $n = 2$ in detail using a decoupling inequality for strictly convex curves.

Another interesting case is $n = \frac{1}{2}(s+1)(s+2) - 1$, when there is a unique minimal collection of n monomials, that is $M_{\le s} = \{x^i y^j : 1 \le i + j \le s\}$. In such cases we have a much simpler expression for the exponent.

Corollary 1. *Let n be an integer of the form $\frac{1}{2}(s+1)(s+2) - 1$. Suppose that Γ is a compact $C^{n,\alpha}$ planar curve such that $W^{M_{\le s}}(\Gamma)$ is nonvanishing. Then we have*

$$|\Lambda| \lesssim N^{\frac{8}{3(s+3)}+\epsilon}.$$

It is worth noting that Bombieri and Pila [2] obtained the same upper bound, with exclusion of ϵ, for these values of n. They obtained the result by evaluating the number of lattice points that an algebraic curve of degree s can contain. Pila furthered the study in this direction [12] and found a similar Wronskian condition to the one in Theorem 1. On the other hand, our approach does not restrict ourselves to special values of n.

Schmidt [14] conjectured that $|\Lambda| \lesssim N^{\frac{1}{2}+\epsilon}$ is true for any C^2 curve $\Gamma \subset [0,N]^2$ given as $y = f(x)$, provided that f'' is weakly monotonic and vanishs at most one value of x. An interesting case in the light of this conjecture is $n = 7$, when Theorem 1 implies upper bound $|\Lambda| \lesssim N^{1/2+\epsilon}$ under a certain analytic condition. For instance, if we choose $M_7 = \{x, y, x^2, xy, y^2, x^3, x^2y\}$, then we impose the analytic condition that the determinant

$$W(x) = \begin{vmatrix} f^{(4)} & 4f^{(3)} & 12f^{(2)} & (f^2)^{(4)} \\ f^{(5)} & 5f^{(4)} & 20f^{(3)} & (f^2)^{(5)} \\ f^{(6)} & 6f^{(5)} & 30f^{(4)} & (f^2)^{(6)} \\ f^{(7)} & 7f^{(6)} & 42f^{(5)} & (f^2)^{(7)} \end{vmatrix}$$

is nonvanishing. This provides a simpler, alternative condition on the curve, while Pila [12] gave a condition $\Gamma \in C^{104}$ and a nonvanishing condition of a determinant.

There are, however, some earlier upper estimates that Theorem 1 does not imply. For example, Swinnerton-Dyer [15] proved the upper bound $|\Lambda| \lesssim N^{3/5+\epsilon}$ for any C^3 strictly convex curve.

We introduce decoupling inequalities for curves in a higher dimensional space in Section 3 and prove the main theorem in Section 4.

In Section 5, we explore the upper estimate of lattice points on a hypersurface $S \subset \mathbb{R}^{d+1}$ with use of the $\ell^p L^p$ decoupling theorem due to Guo and Zhang [9]. Setting $\Lambda = S \cap (\frac{1}{N}\mathbb{Z})^{d+1}$, we prove the upper bound for any C^{k+1} hypersurface S_d which satisfies a certain analytic condition

$$|\Lambda| \lesssim N^{e_d(k)+\epsilon}$$

where $e_d(k) = \frac{d}{2} + O(k^{-\frac{1}{d+1}})$.

It is known that one can construct a uniform Jarník curve which is strictly convex that satisfies $|\Lambda| \leq N^{\frac{2}{3}-\epsilon}$ for infinitely many N for any given $\epsilon > 0$. (See [13] for the proof) But the curve is not constructed to be C^1. In Appendix we construct a C^1, strictly convex curve C such that $|C \cap (\frac{1}{N})^2| \geq \frac{1}{2} N^{\log_3 2}$ for infinitely many N.

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2. Weak Jarnik's theorem via decoupling for parabolas

In this section we prove Theorem 1 for $n = 2$. More precisely, we can prove a slightly stronger statement in this case. Recalling the notation $\Lambda = \Gamma \cap (\frac{1}{N}\mathbb{Z})^2$, we aim at the following upper bound for $|\Lambda|$.

Proposition 1. *For a C^2, strictly convex curve Γ, we have $|\Lambda| \lesssim N^{2/3+\epsilon}$ for each $\epsilon > 0$.*

We first state the decoupling inequality for curves and see how it implies weak Jarník's theorem.

2.1. Decoupling theory for parabolas.

Throughout this paper, we use the notation $e(\alpha) = \exp(2\pi i\alpha)$.

We recall the following corollary of decoupling theory for the parabola [3], [4].

Theorem 2. *Let Γ be a C^2, strictly convex curve, and suppose we are given a δ-separated set Λ on Γ. Then we have*

$$(\frac{1}{|B_R|}\int_{B_R}|\sum_{\xi\in\Lambda}a_\xi e(\xi\cdot x)|^6dx)^{\frac{1}{6}}\lesssim\delta^{-\epsilon}\|a_\xi\|_{\ell^2}$$

for each $\epsilon>0$, each $a_\xi\in\mathbb{C}$ and each ball B_R of radius $R\gtrsim\delta^{-2}$.

Remark 1. *As we take R very large, the left hand side in Theorem 2 tends to a combinatorial quantity called $\frac{p}{2}$-energy. We define the k-energy of a discrete set Λ by $\mathbb{E}_k(\Lambda)=|\{(\lambda_1,\cdots,\lambda_k,\lambda'_1,\cdots,\lambda'_k)\in\Lambda^{2k}:\lambda_1+\cdots+\lambda_k=\lambda'_1+\cdots+\lambda'_k\}|$.*

2.2. Proof of weak Jarník's theorem. We provide the proof of Proposition 1 using the decoupling inequality.

Proof of Proposition 1. We denote by $F(x)$ the sum of exponential functions $\sum_{\xi\in\Lambda}e(\xi\cdot x)$. Since any two distinct $\frac{1}{N}$-integral points are separated by at least $\frac{1}{N}$, the set Λ satisfies the separation condition with $\delta=\frac{1}{N}$. Therefore, we can apply Theorem 2 with $a_\xi=1$ and $\delta=\frac{1}{N}$ then we obtain

$$(\frac{1}{|B_R|}\int_{B_R}|F(x)|^6)^{\frac{1}{6}}\lesssim N^\epsilon|\Lambda|^{\frac{1}{2}}$$

for each ball B_R of radius $R\gtrsim N^2$.

Since the curve is compact, we can assume Γ is inside the unit square. The value $e(\alpha)$ has its real part at least $\frac{1}{2}$ for each $\alpha\in B_{\frac{1}{6}}$, so each exponential $e(\xi\cdot x)$ where $\xi\in\Lambda$ has its real part at least $\frac{1}{2}$ for every point x in the ball of radius $\frac{1}{6\sqrt{2}}$. Therefore, in this range of x we have $|F(x)|\geq\frac{1}{2}|\Lambda|$.

We notice the periodic relation $e(\xi\cdot x)=e(\xi\cdot x')$ for each pair such that $x-x'\in(N\mathbb{Z})^2$ because $\Lambda\subset(\frac{1}{N}\mathbb{Z})^2$. Therefore, the local estimate above applies around each point $x_0\in(N\mathbb{Z})^2$. This leads to the following lower bound for the weighted L^p norm of $F(x)$ up to an absolute constant:

$$c_0\,(N^{-2}|\Lambda|^6)^{\frac{1}{6}}\leq(\frac{1}{|B_R|}\int_{B_R}|F(x)|^6dx)^{\frac{1}{6}}.$$

Combining the estimate of the weighted L^p norm of $F(x)$ from two sides, we obtain

$$(N^{-2}|\Lambda|^6)^{\frac{1}{6}}\lesssim N^\epsilon|\Lambda|^{\frac{1}{2}}.$$

This completes the proof. □

Remark 2. *By taking the ball B_R of radius $R=CN^2$ with a constant C independent of N, which is allowed by Theorem 2, we can see that the same upper bound holds when we replace Λ by a set of $\frac{1}{N}$-integral points which are of distance at most $c_0\frac{1}{N^2}$ to the curve Γ.*

3. More results of ℓ^2 decoupling theory for curves

3.1. The case with moment curves. We fix the dimension $n>1$. The moment curve $M\subset\mathbb{R}^n$ is defined as the curve parameterized by

$$\Phi(t)=(t,t^2,\cdots,t^n)$$

for $t \in [0,1]$. Given $g : [0,1] \to \mathbb{C}$ and an interval $J \subset [0,1]$ we define the extension operator in $\mathbb{R}^n$ as

$$E_J g(x) = \int_J g(t) e(x \cdot \Phi(t)) dt.$$

The decoupling constant $V_{p,2}(\delta)$ is the smallest constant such that

$$\|E_{[0,1]} g\|_{L^p(\omega_B)} \leq V_{p,2}(\delta) \Big(\sum_{\substack{J:\ \text{interval in } [0,1] \\ \iota(J)=\delta}} \|E_J g\|^2_{L^p(\omega_B)} \Big)^{\frac{1}{2}}$$

for each ball $B \subset \mathbb{R}^n$ of radius δ^{-n}.

Bourgain, Demeter and Guth gave the following estimate of $V_{(2,p)}(\delta)$ for the critical value $p = n(n+1)$. See [6] and [8] for the proof.

Theorem 3. *Let $p = n(n+1)$. Then for every $\epsilon > 0$ there exists a constant C_ϵ such that*

$$V_{(p,2)}(\delta) \leq C_\epsilon \delta^{-\epsilon}$$

for every $\delta \in (0,1)$.

3.2. Equivalent formulation of decoupling. We can state the ℓ^2 decoupling theorem in terms of Fourier restrictions instead of extension operators. For the detail on the equivalence of these formulations, we refer to [5]. For each $f : \mathbb{R}^n \to \mathbb{C}$ and $R \subset \mathbb{R}^n$ we denote by f_R the Fourier restriction of f to R

$$f_R(x) = \int_R \hat{f}(\xi) e(x \cdot \xi) d\xi.$$

Let $N_\delta(M)$ be the union of the δ-neighborhoods of all $\delta^{\frac{1}{n}}$-arcs of M. (See [8] for a precise formulation) For $\delta \in (0,1)$, we denote by $D_{(p,2)}(\delta)$ the smallest constant such that

$$\|f\|^2_{L^p} \leq D_{(p,2)}(\delta) \cdot \sum \|f_\theta\|^2_{L^p}.$$

In this definition, we have the same upper estimate $D_{(p,2)}(\delta) \lesssim \delta^{-\epsilon}$ for $p = n(n+1)$.

It has been observed in [3] that the above upper estimate implies the following result by a limiting procedure that we replace f with a sum of Dirac deltas.

Theorem 4. *Let $p = n(n+1)$. Then for each δ-separated set Λ of points on M, we have*

$$\Big(\frac{1}{|B_R|} \int_{B_R} \Big|\sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x)\Big|^p dx\Big)^{\frac{1}{p}} \lesssim \delta^{-\epsilon} \|a_\xi\|_{\ell^2}$$

for each $\epsilon > 0$, each $a_\xi \in \mathbb{C}$ and each ball B_R of radius $R \gtrsim \delta^{-n}$.

3.3. The case with more general curves. Consider a compact curve $C \subset \mathbb{R}^n$ parameterized by

$$\Gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$$

for $t \in [0,1]$ with $\gamma_i \in C^{n,\alpha}([0,1])$ and such that the Wronskian $W(\gamma_1', \cdots, \gamma_n')(t)$ is nonvanishing.

Theorem 5. *Let C be a curve as above. For $p = n(n+1)$, we have*

$$V_{C(p,2)}(\delta) \lesssim \delta^{-\epsilon}.$$

Here we provide a proof for Theorem 5. A similar argument for C^{n+1} curves can be found in [8]. The key assumption we have impose on the curve is that we have a good control over the remainder term when we conduct the Taylor expansion to coordinate functions γ_i.

Proof. In the proof, we adopt the formulation of decoupling mentioned in Section 3.2. The condition $f \in C^{n,\alpha}$ implies that the remainder term $R_n(t_0+\Delta t)$ in the n-th Taylor series at t_o satisfies $\|R_n(t_0+\Delta t)\| \le M\|\Delta t\|^{n+\alpha}$ with a constant M independent of the choice of t_o. Denote by T a curve with parameterization $T(t) = (m_1(t), \cdots, m_n(t))$ where $m_i(t)$ is the n-th Taylor series of $\gamma_i(t)$ at the point t_0 for each $1 \le i \le n$. Then we have $|T(t_0+\Delta t) - \Gamma(t_o+\Delta t)| < \delta$ for $\|\Delta t\| \le \delta^{\frac{1}{n+\alpha}}$. By the Wronskian condition on C, the curve T defined as above for t_o in an $\delta^{\frac{1}{n+\alpha}}$-arc maps to the moment curve M under a linear change of variables. Denoting by τ a δ-neighborhood of a $\delta^{\frac{1}{n+\alpha}}$-arc of C, we obtain

$$\begin{aligned}\|f\|_{L^p}^2 &\le V_{C(p,2)}(\delta^{\frac{n}{n+\alpha}}) \cdot \sum_{\tau} \|f_\tau\|_{L^p}^2 \\ &\le V_{C(p,2)}(\delta^{\frac{n}{n+\alpha}}) \cdot V_{(p,2)}(\delta) \cdot \sum_{\tau} \sum_{\theta\subset\tau} \|f_\theta\|_{L^p}^2\end{aligned}$$

By Theorem 3, we obtain $V_{C(p,2)}(\delta) \lesssim \delta^{-\epsilon} \cdot V_{C(p,2)}(\delta^{\frac{n}{n+\alpha}})$. By iteration, we conclude $V_{C(p,2)}(\delta) \lesssim \delta^{-\epsilon}$ □

By the same procedure with moment curves, Theorem 5 leads to the following result.

Theorem 6. *Fix C as above, and let $p = n(n+1)$. Then for each δ-separated set Λ of points on C. we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} |\sum_{\xi\in\Lambda} a_\xi e(\xi\cdot x)|^p dx\right)^{\frac{1}{p}} \lesssim \delta^{-\epsilon}\|a_\xi\|_{\ell^2}$$

for each $\epsilon > 0$, each $a_\xi \in \mathbb{C}$ and each ball $B_R \subset \mathbb{R}^n$ of radius $R \gtrsim \delta^{-n}$.

4. Proof of the main theorem

4.1. Skewed lattice points on a curve. Suppose that we have a curve Γ inside $\mathbb{R}^n$. Given an n-tuple $\mathbf{s} = (s_1, \cdots, s_n)$ of positive integers, we can define the set of skewed $\frac{1}{N}$-integral points on the curve as $\Lambda_{\mathbf{s}}(N) = \Gamma \cap (\frac{1}{N^{s_1}}\mathbb{Z} \times \cdots \times \frac{1}{N^{s_n}}\mathbb{Z})$. With notation $|\mathbf{s}| = s_1 + \cdots + s_n$, we have the following result:

Theorem 7. *For any $C^{n,\alpha}$ curve Γ with a nonvanishing Wronskian, we have*

$$|\Lambda_{\mathbf{s}}(N)| \lesssim N^{\frac{2|\mathbf{s}|}{n(n+1)}+\epsilon}.$$

Proof. Define again $F(x) = \sum_{\xi\in\Lambda} e(x\cdot\xi)$. By the assumption on Γ, we can apply Theorem 6 with $\delta = \frac{1}{N}$ and obtain

$$\left(\frac{1}{|B_R|}\int_{B_R} |F(x)|^p\right)^{\frac{1}{p}} \lesssim N^{\epsilon}|\Lambda|^{\frac{1}{2}}$$

for $p = n(n+1)$ and each ball $B_R \subset \mathbb{R}^n$ of radius $R \gtrsim N^n$. We observe that $F(x)$ is a periodic function of period $(N^{s_1}, \cdots, N^{s_n})$ since each point $\xi \in \Lambda$ lies in $\frac{1}{N^{s_1}}\mathbb{Z} \times \cdots \times \frac{1}{N^{s_n}}\mathbb{Z}$. Therefore we can apply the local estimate around every point in $N^{s_1}\mathbb{Z} \times \cdots \times N^{s_n}\mathbb{Z}$ and obtain the lower bound for the weighted L^p norm

$$\left(N^{-(s_1+\cdots+s_n)}|\Lambda|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{|B_R|}\int_{B_R} |F(x)|^p\right)^{\frac{1}{p}}.$$

Combining these inequalities finishes the proof. □

4.2. Proof of Theorem 1. Suppose that we are given a planar curve Γ parameterized by $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$.

Definition 1. *A lift-up of* Γ *into* $\mathbb{R}^n$ *is a curve* $\widetilde{\Gamma} \subset \mathbb{R}^n$ *parameterized by*

$$\widetilde{\Gamma}(t) = (m_1(\gamma_1, \gamma_2)(t), \cdots, m_n(\gamma_1, \gamma_2)(t))$$

for some minimal collection of n *monomials* $M_n = \{m_1, \cdots, m_n\}$.

For instance, a lift-up of Γ into $\mathbb{R}^5$ is unique up to an order of coordinates, and it is given by $\widetilde{\Gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_1(t)^2, \gamma_1(t)\gamma_2(t), \gamma_2(t)^2)$. Now we are ready to prove the main theorem.

(Proof of Theorem 1). Let $\widetilde{\Gamma}$ be any lift-up of the planar curve Γ with the given minimal collection of n monomials M_n. For simplicity, we list $m_1, \cdots, m_n$ in the order such that the sequence of degrees $s_i = \deg(m_i)$ is weakly decreasing. In particular, we can take $m_1 = x$ and $m_2 = y$. Then we observe that the point $(\gamma_1(t), \gamma_2(t))$ on Γ is an $\frac{1}{N}$-integral point if and only if the corresponding point $\widetilde{\Gamma}(t) = (\gamma_1(t), \gamma_2(t), \cdots, m_n(\gamma_1, \gamma_2)(t))$ on the lift-up $\widetilde{\Gamma}$ is a skewed $\frac{1}{N}$-integral point with degree $\mathbf{s} = (s_1, s_2, \cdots, s_n)$. The assumption on Γ implies that the Wronskian $W(\widetilde{\Gamma})$ is nonvanishing, so Theorem 7 applies to the lift-up $\widetilde{\Gamma}$ and yields $|\widetilde{\Gamma}| \lesssim N^{\frac{2|\mathbf{s}|}{n(n+1)}+\epsilon}$ where $|\mathbf{s}|$ is the total degree of M_n.

By definition, a minimal collection of n monomials attains the minimal total degree $m(n)$ among the choices of n distinct monomials, so it must consist of all monomials with degree up to some positive integer d and monomials with degree $d+1$ if allowed. The number of monomials with degree at most d is given by $\frac{1}{2}(d+1)(d+2)-1$ because there are $i+1$ distinct monomials with degree i for each $i \geq 1$. By our choice of s, it is clear that $d = s-1$ and hence M_n consists of all monomials with degree up to $s-1$ and $\Delta n = n - (\frac{1}{2}s(s+1)-1)$ monomials with degree s. We finally obtain the formula for $m(n)$ by $m(n) = (\sum_{i=1}^{s-1} i(i+1)) + s \cdot \Delta n = \frac{1}{3}(s-1)s(s+1) + s \cdot \Delta n$, which completes the proof. $\square$

5. Extension of the results to surfaces

5.1. Preliminary estimates of lattice points on a hypersurface. The decoupling approach to lattice points extends to the case when we are given a fixed hypersurface. Suppose that we are given a hypersurface S in $\mathbb{R}^n$. Abusing notation, we use $\Lambda = (\frac{1}{N}\mathbb{Z})^2 \cap S$. The decoupling inequalities for hypersurfaces have been settled in [3] and [4]. using decoupling inequalities for paraboloids.

Proposition 2. *Let* S *be a compact* C^2 *hypersurface in* $\mathbb{R}^n$ *with positive definite second fundamental form, and let* $\Lambda \subset S$ *be a* δ*-separated set. For* $p = \frac{2(n+1)}{n-1}$ *we have*

$$(\frac{1}{|B_R|}\int_{B_R} |\sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x)|^p)^{\frac{1}{p}} \lesssim \delta^{-\epsilon} \|a_\xi\|_{\ell^2}$$

for each $\epsilon > 0$*, each* $a_\xi \in \mathbb{C}$ *and each ball* B_R *of radius* $R \gtrsim \delta^{-2}$.

The lower estimate for the weighted L^p norm using local estimates and the periodicity immediately leads to the following result.

Proposition 3. *Let C as above. Then we have*

$$|\Lambda| \lesssim N^{\frac{n(n-1)}{n+1}+\epsilon}.$$

For example if we set $n=3$ then we obtain $|\Lambda| \lesssim N^{3/2+\epsilon}$. One can also derive Proposition 2 using the main theorem in [1].

5.2. $\ell^p L^p$ decoupling for $S_{d,k}$. Before we further the upper estimate of lattice points on a hypersurface, we prepare $\ell^p L^p$ decoupling inequalities for d-dimensional manifolds.

For each $d \leq 1$ and $k \leq 2$, we define a compact d-manifold $S_{d,k}$ by

$$S_{d,k} = \{\Phi_{d,k}(t_1,\cdots,t_d) = (t_1,\cdots,t_d,\cdots,t_1^d,\cdots,t_k^d) : (t_1,\cdots,t_d) \in [0,1]^d\}$$

where the entries consist of all monomias $t_1^{s_1}\cdots t_k^{s_k}$ with $1 \leq s_1+\cdots+s_k \leq k$. The dimension of space $\mathbb{R}^n$ in which $S_{d,k}$ lies is given in the formula $n = \binom{k+d}{d} - 1$. Following the notation in [11], we denote by $\mathcal{K}_{d,k}$ the number $\frac{d\cdot k}{d+1}\binom{d+k}{d}$. Then we can see that $\mathcal{K}_{d,k}$ gives the total degree of the monomials used as the coordinate functions for $S_{d,k}$.

As with the case of moment curves, we can define the decoupling constant for $S_{d,k}$. For $R \subset [0,1]^d$, we define the extension operator associated to the set R

$$E_R^{(d,k)} g(x) = \int_R g(t) e(x\cdot\Phi_{d,k}(t))dt.$$

Also for a ball $B \subset \mathbb{R}^n$ of radius r_B centered at c_B we will use the weight $\omega_B(x) = (1+\frac{|x-c_B|}{r_B})^{-C}$ with an unspecified large constant C. Let $V_{(p,p)}^{(d,k)}(\delta)$ be the smallest constant such that

$$\|E_{[0,1]^d}^{(d,k)} g\|_{L^p(\omega_B)} \lesssim V_{(p,p)}^{(d,k)}(\delta)\Big(\sum_{\substack{\Delta:\ \text{cube inside } [0,1]^d \\ \iota(\Delta)=\delta}} \|E_\Delta^{(d,k)} g\|_{L^p(\omega_B)}^p\Big)^{\frac{1}{p}}$$

for each ball $B \subset \mathbb{R}^n$ of radius δ^{-k}. For each $p \geq 2$ define $\Gamma_{d,k}(p)$

$$\Gamma_{d,k}(p) = \max\{d(\frac{1}{2}-\frac{1}{p}),\ \max_{1\leq i\leq d}\{(1-\frac{1}{p})i - \frac{\mathcal{K}_{i,k}}{p}\}\}.$$

Now we can state the $\ell^p L^p$ decoupling inequality for $S_{d,k}$ [9]:

Theorem 8. *We have*

$$V_{(p,p)}^{(d,k)}(\delta) \lesssim \delta^{-\Gamma_{d,k}(p)-\epsilon}.$$

5.3. $\ell^p L^p$ decoupling for more general d-dimensional manifolds. We start with the definition of the decoupling constant for d-manifolds which lie in the same Euclidean space as $S_{d,k}$. Let S be compact d-manifold inside $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$. We define the decoupling constant $V_{S_{(p,p)}^{(d,k)}}$ for S inside $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$ by the same inequality as with $S_{d,k}$, but now the constant $V_{(p,p)}^{(d,k)}(\delta)$ must work for all of the local coordinates if there are multiple ones defining S.

Let S be a compact, C^k d-manifold inside $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$. For each local coordinate system $\Gamma : U \subset \mathbb{R}^d \to \mathbb{R}^n$

$$\Gamma(x) = (\gamma_1(x),\cdots,\gamma_n(x))$$

we define the $n \times n$ determinant $W_d(\Gamma)(x)$

$$W_d(\Gamma)(x) = \begin{vmatrix} \frac{\partial}{\partial x_1}(\gamma_1)(x) & \frac{\partial}{\partial x_1}(\gamma_2)(x) & \cdots & \frac{\partial}{\partial x_1}(\gamma_n)(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_d}(\gamma_1)(x) & \frac{\partial}{\partial x_d}(\gamma_2)(x) & \cdots & \frac{\partial}{\partial x_d}(m_n)(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial {x_1}^k}(\gamma_1)(x) & \frac{\partial^k}{\partial {x_1}^k}(\gamma_1)(x) & \cdots & \frac{\partial^k}{\partial {x_1}^k}(\gamma_n)(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial {x_d}^k}(\gamma_1)(x) & \frac{\partial^k}{\partial {x_d}^k}(\gamma_2)(x) & \cdots & \frac{\partial^k}{\partial {x_d}^k}(\gamma_n)(x) \end{vmatrix}$$

where we take all partial derivatives $\frac{\partial^{\mathbf{i}}}{\partial {x_1}^{i_1} \cdots \partial {x_d}^{i_d}}$ for $1 \leq i_1 + \cdots + i_d \leq k$.

Let C be a compact, C^{k+1} d-manifold in $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$ such that for each local coordinate system $\Gamma : U \in \mathbb{R}^n$ the function $W_d(\Gamma)$ is nonvanishing on U.

By the assumption each coordinate function γ_i is C^{k+1}, and so we have an upper estimate $|R_n(\mathbf{x})| \lesssim \|x\|^{k+1}$ for the remainder $R_n(x_1, \cdots, x_d)$ in the k-th Taylor series of γ_i. The same argument as in Section 3 works by replacing the moment curves by the d-manifolds $S_{d,k}$, and we obtain the inequality $V_{C^{(d,k)}_{(p,p)}}(\delta) \lesssim V^{(d,k)}_{(p,p)}(\delta) V_{C^{(d,k)}_{(p,p)}}(\delta^{\frac{k}{k+1}})$. By iteration, this leads to $V_{C^{(d,k)}_{(p,p)}}(\delta) \lesssim \delta^{-\Gamma_{d,k}(p)}$. Thus we obtain the following result.

Corollary 2. *Let C be a compact, C^{k+1} d-manifold inside $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$ such that for each local coordinate system $\Gamma : U \subset \mathbb{R}^d \to \mathbb{R}^n$ the function $W_d(\Gamma)$ is nonvanishing on U. Then we have*

$$V_{C^{(d,k)}_{(p,p)}}(\delta) \lesssim \delta^{-\Gamma_{d,k}(p)}.$$

Remark 3. *Here we impose the condition that S is C^{k+1} in order to have a control over the remainder of coordinate functions when we take k-th Taylor series. It is possible that a weaker condition than C^{k+1} is sufficient,*

Now apply the decoupling inequality with the critical value $p = \frac{2\mathcal{K}_{d,k}}{d}$ (See [GZ] for a detail) and we obtain the following result.

Proposition 4. *Let $p = \frac{2\mathcal{K}_{d,k}}{d}$. For each δ-separated set Λ of points on S, we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} |\sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x)|^p dx\right)^{\frac{1}{p}} \lesssim \delta^{-d(\frac{1}{2}-\frac{1}{p})-\epsilon} \|a_\xi\|_{\ell^p}$$

for each $\epsilon > 0$, each $a_\xi \in \mathbb{C}$ and each ball $B_R \subset \mathbb{R}^{\mathcal{K}_{d,k}}$ of radius $R \gtrsim \delta^{-k}$.

5.4. Skewed lattice points on a d-dimensional manifold. Let S be a compact, C^{k+1} d-manifold in $\mathbb{R}^n$ where $n = \binom{k+d}{d} - 1$, For $\mathbf{s} = (s_1, \cdots, s_n)$ be a list of degrees, denote by $\Lambda_{\mathbf{s}}$ the set of skewed $\frac{1}{N}$-integral points of degree $\mathbf{s}$ on S. Then we have the following upper estimate.

Theorem 9. *Let $\Lambda_{\mathbf{s}}$ as above for a list of degrees $\mathbf{s}$ which contains 1. Then we have*

$$|\Lambda_{\mathbf{s}}| \lesssim N^{f(\mathbf{s})+\epsilon}$$

where $f(s) = \frac{2\mathcal{K}_{d,k}}{2\mathcal{K}_{d,k}-d}(\frac{d}{2} + \frac{d(|\mathbf{s}|-d)}{2\mathcal{K}_{d,k}})$.

Remark 4. *We can see that the above upper bound is sharp for skewed $\frac{1}{N}$-integral points with order $(1, 1, \cdots, k, \cdots, k)$ on $S_{d,k}$.*

Proof. The assumption $\mathbf{s}$ contains 1 implies that $\Lambda_{\mathbf{s}}$ is δ-separated with $\delta = \frac{1}{N}$. By Proposition 4, we obtain

$$(\frac{1}{|B_R|}\int_{B_R}|\sum_{\xi\in\Lambda} a_\xi e(\xi\cdot x)|^p dx)^{\frac{1}{p}} \lesssim N^{d(\frac{1}{2}-\frac{1}{p})+\epsilon}\|a_\xi\|_{\ell^p}$$

for $p = \frac{2\mathcal{K}_{d,k}}{d}$. On the other hand we have the lower bound

$$(N^{-|\mathbf{s}|}|\Lambda|^p)^{\frac{1}{p}} \lesssim (\frac{1}{|B_R|}\int_{B_R}|\sum_{\xi\in\Lambda} e(\xi\cdot x)|^p dx)^{\frac{1}{p}}.$$

Combining these inequalities we obtain the desired result. □

5.5. Lattice points on a surface. Suppose that we are given a fixed hypersurface $S \subset \mathbb{R}^{d+1}$.

Let $\Gamma(x) = (\gamma_1(x), \gamma_2(x), \cdots, \gamma_{d+1}(x))$ for $x \in U \subset \mathbb{R}^d$ be a local chart for S.

Let M_n be a minimal collection of n monomials about $d+1$ variables $x_1, x_2, \cdots, x_{d+1}$, and define $m^{(d+1)}(n)$ to be the minimal total degree for a collection of distinct n monomials about $d+1$ variables.

By definition, $m^{(2)}(n)$ is the function we denote by $m(n)$. Similar to this case, $m^{(d+1)}(n)$ has an explicit formula for each fixed $d \leq 2$. Let k' be the minimal positive integer such that $n \leq \binom{k'+d+1}{d+1} - 1$, and denote $\Delta n = n - \binom{k'+d}{d+1} + 1$. Since $\binom{k'+d}{d+1} - 1$ counts the number of monomals with degree at most $k'-1$ used in a minimal collection M_n of n monomials, it is clear that Δn counts the number of monomials with degree k' in M_n. Then we have

$$m^{(d+1)}(n) = \mathcal{K}_{d+1,k'-1} + k\cdot\Delta n.$$

In particular we observe that $m^{(d+1)}(n)$ is asymptotically $n^{\frac{d+2}{d+1}}$.

For the value $n = \binom{k+d}{d} - 1$ and each minimal collection M_n of monomials about n variables, we can define the generalized Wronskian $W_d^{M_n}(S)$ as the $n \times n$ determinant

$$W_d^{M_n}(S) = \begin{vmatrix} \frac{\partial}{\partial x_1}(m_1) & \frac{\partial}{\partial x_1}(m_2) & \cdots & \frac{\partial}{\partial x_1}(m_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_d}(m_1) & \frac{\partial}{\partial x_d}(m_2) & \cdots & \frac{\partial}{\partial x_d}(m_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial {x_1}^k}(m_1) & \frac{\partial^k}{\partial {x_1}^k}(m_1) & \cdots & \frac{\partial^k}{\partial {x_1}^k}(m_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial {x_d}^k}(m_1) & \frac{\partial^k}{\partial {x_d}^k}(m_2) & \cdots & \frac{\partial^k}{\partial {x_d}^k}(m_n) \end{vmatrix}$$

where we take all partial derivatives $\frac{\partial^{\mathbf{i}}}{\partial {x_1}^{i_1}\cdots\partial {x_d}^{i_d}}$ for $1 \leq i_1 + \cdots + i_d \leq k$ and m_i denotes the function $m_i(\gamma_1, \cdots, \gamma_{d+1})$.

Theorem 10. *Let $S \subset \mathbb{R}^{d+1}$ be a C^{k+1} hypersurface such that $W_d^{M_n}(S)$ is nonvanishing for some minimal collection M_n of $n = \binom{k+d}{d} - 1$ monomials about $d+1$ variables. Then we have*

$$|\Lambda| \lesssim N^{e_d(k)+\epsilon}$$

where $e_d(k) = \frac{2\mathcal{K}_{d,k}}{2\mathcal{K}_{d,k}-d}(\frac{d}{2}+\frac{d(s-d)}{2\mathcal{K}_{d,k}})$ *for* $s = m^{(d+1)}(\binom{k+d}{d}-1)$*. Moreover, we have the asymptotic expression*

$$e_d(k) = \frac{d}{2} + O(k^{-\frac{1}{d+1}}).$$

Proof. We will write $n = \binom{k+d}{d} - 1$. We define a lift-up $\widetilde{S}$ of the hypersurface S into $\mathbb{R}^n$ associated with the given minimal collection of monomials M_n as

$$\widetilde{\Gamma}(x) = (m_1(\gamma_1, \cdots, \gamma_{d+1})(x), m_2(\gamma_1, \cdots, \gamma_{d+1})(x), \cdots, m_n(\gamma_1, \cdots, \gamma_{d+1})(x))$$

for each local coordinate system $\Gamma(x) = (\gamma_1(x), \cdots, \gamma_{d+1}(x))$ of S. Then $\frac{1}{N}$-integral points on S correspond to the skewed $\frac{1}{N}$-integral points with order $(\deg m_1, \deg m_2, \cdots, \deg m_n)$ on the lift-up $\widetilde{S}$. By the assumption on S, we can apply Theorem 9 to the lift-up $\widetilde{S}$. Since the sum of degrees $\deg m_1 + \deg m_2 + \cdots + \deg m_n$ is just the total degree of M_n denoted by $m^{(d+1)}(n)$, we obtain the desired result.

Since the function $n = \binom{k+d}{d} - 1$ is asymptotically k^d and the function $\mathcal{K}_{d,k}$ is asymptotically k^{d+1},

$$\begin{aligned} e_d(k) &= \frac{d}{2} + O(\frac{k^{\frac{d(d+2)}{d+1}}}{k^{d+1}}) \\ &= \frac{d}{2} + O(k^{-\frac{1}{d+1}}). \end{aligned}$$

This completes the proof. □

Appendix: construction of a C^1 curve with many lattice points

In this appendix we construct a C^1 curve such that Λ contains $N^{\log_3(2)}$ integral points for infintiely many N. A similar but less concrete construction of such curve attaining the exponent $\log_3(2)$ can be found in [7]. The construction here is purely number theoretic and exploits the idea of sorting rational numbers. We start with the following notation:

For each nonnegative integer n we construct a collection of 2^n+1 points $P_0^n, \cdots, P_{2^n}^n$. Then we have $P_m^n = P_{2m}^{n+1}$ for each n and m.

$$P_m^n = \frac{1}{3^n}\sum_{i=1}^{m} v_i^{(n)}.$$

For instance, $A^{(1)} = \{(1,1)\}$ and $A^{(2)} = \{(2,1),(1,2)\}$. Let F_i be a collection of 2^i+1 vectors defined recursively by $F_0 = \{(1,0),(0,1)\}$ and $F_n = \{f_0^{(n)}, \cdots, f_{2^n}^{(i)}\}$:

$$\begin{aligned} f_{2i}^{(n+1)} &= f_i^{(n)} \\ f_{2i+1}^{(n+1)} &= f_i^{(n)} + f_{i+1}^{(n)} \end{aligned}$$

For instance, we see that $F_1 = \{(1,0),(1,1),(0,1)\}$ and $F_2 = \{(1,0),(2,1),(1,1),(1,2),(0,1)\}$. Then we define the set of vectors in generation n as $A_n = F_n \setminus F_{n-1}$ for each $n \geq 1$. We sort by their slope $A_n = \{v_1^{(n)}, \cdots, v_{2^{n-1}}^{(n)}\}$. Now we can define the points

$$P_m^{(n)} = \frac{1}{3^{n-1}}\sum_{i=1}^{m} v_i^{(n)}$$

for each $1 \leq m \leq 2^{n-1}$.

Lemma 1. *The set of vertices $P^{(n)}$ is a subset of $P^{(n+1)}$ for each n.*

Proof. This is straightforward from the fact $v_{2i-1}^{(n+1)} + v_{2i}^{(n+1)} = 3v_i^n$ for each $1 \leq i \leq 2^{n-1}$. □

Denote by P the union of $P^{(n)}$.

The above lemma implies that there is a unique curve C_0 which contains all points in P. Consider the curve C defined as $C_0 \cap [0, \frac{2}{3}] \times [0, \frac{1}{3}]$, then it turns out that C is a C^1 strictly convex curve with many $\frac{1}{N}$-integral points for infinitely many N.

Proposition 5. *C is a C^1, strictly convex curve, and it satisfies*

$$|\Lambda| \geq \frac{1}{2} N^{\log_3 2}$$

for infinitely many N.

The strictly convexity and C^1 follow from the observation that given any point $x_0 \in (0, \frac{2}{3})$ any $\epsilon > 0$ we can find vertices P_1 and P_2 in P on each side such that $|x(P_1) - x_0|, |x(P_2) - x_0| < \epsilon$. The last assertion is clear from the construction for each $N = 3^n$.

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