SPLIT-MULTIPLICITY-FREE FLAGGED SCHUR POLYNOMIALS
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Abstract. Quiver coefficients come from the study of a general kind of degeneracy locus associated to an oriented quiver of type A. They can be obtained by expanding the Schubert polynomials into the split-Schur polynomials, and possess very rich combinatorial structures. In this paper, we investigate the problem of determining which Schubert polynomials are split-multiplicity-free by looking at two meaningful special cases: the flagged Schur polynomials, which are Schubert polynomials of vexillary permutations, and the Stanley symmetric polynomials, which are stable limits of Schubert polynomials. Specifically, we present a necessary and sufficient condition on a shape λ for the flagged Schur polynomials s_λ^b to be split-multiplicity-free, given a generic flag. We also discuss progress on the Stanley symmetric polynomials via Rothe diagram.

1. Introduction

The study of multiplicity-freeness of various polynomials in algebraic combinatorics has received a lot of attention in recent years, with meaningful implications to algebraic geometry and representation theory. We say that a polynomial f is *multiplicity-free* with respect to a basis $\mathcal{B}$, if the coefficients of the expansion of f into $\mathcal{B}$ all belong to $\{0, 1\}$. Knowing that a polynomial is multiplicity-free, together with the information of its support, which is typically a saturated lattice polytope, one can uniquely recover this polynomial. Therefore, multiplicity-freeness provides us with a great handle on various families of polynomials arising from combinatorics and geometry.

Fink, Mészáros and St. Dizier [5] characterized the permutations w, for which the corresponding Schubert polynomial $\mathfrak{S}_w$ is multiplicity-free with respect to the monomial basis, via pattern avoidance. As a consequence, in this case, $\mathfrak{S}_w$ is the integer point transform of a generalized permutahedron. Hodges and Yong [8, 9] characterized which key polynomials are multiplicity-free with respect to the monomial basis in the context of spherical geometry. With further consideration into the split-Schur basis, Gao, Hodges and Yong [7] characterized which Schubert varieties are Levi-spherical. Another classical problem in the study of multiplicity-freeness involves Littlewood-Richardson coefficients. A result dates back to Stembridge [14] provides necessary and sufficient condition for the product of two Schur functions $s_\lambda s_\mu$ to be multiplicity-free with respect to the Schur basis. This result was later refined by Thomas and Yong [15] with consideration to Grassmannian, and further refined by Gao, Hodges and Orelowitz [6] to Schur polynomials.

The main question considered by this paper includes many of the aforementioned work as important special cases. We explain the necessary background and notations in Section 2. Let $w \in S_n$ be a permutation with descents $\mathrm{Des}(w) = \{b_1, \ldots, b_{k-1}\}$ and let $b_k = n$. Then its corresponding Schubert polynomial $\mathfrak{S}_w$ lives in the split-symmetric ring Λ^b, which has the split-Schur polynomials as a natural basis. The coefficients $c_{\underline{\lambda}}(w)$ in the expansion

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\underline{\lambda}=(\lambda^1,\ldots,\lambda^k)} c_{\underline{\lambda}}(w) \cdot s_{\lambda^1}(x_1, \ldots, x_{b_1}) \otimes \cdots \otimes s_{\lambda^k}(x_{b_{k-1}+1}, \ldots, x_{b_k})$$

are called *quiver coefficients*, which arise from the study of degeneracy locus of the quiver varieties of the oriented quiver of type A [3]. There are combinatorial formula for quiver coefficients [2]. Here is the main question of interest.

Question 1.1. For which permutation w is the Schubert polynomial $\mathfrak{S}_w$ multiplicity-free with respect to the split-Schur basis?

While Question 1.1 seems intractable at the moment, its special cases are already of interest and are rich in its combinatorial nature.

If w is vexillary, i.e. w avoids the pattern 2143, then its corresponding Schubert polynomial $\mathfrak{S}_w$ is a *flagged Schur polynomial* (see for example [11]). Our main theorem (Theorem 3.2) provides a necessary and sufficient condition for a flagged Schur polynomial to be multiplicity-free with respect to the split-Schur basis, when the flag is generic. We also discuss necessary conditions when the flag b is not generic. The content related to flagged Schur polynomials will be in Section 3.

Another special case of Question 1.1 involves the *Stanley symmetric polynomials* F_w, labeled by permutations w, as they are the stable limits of Schubert polynomials. Specifically,

$$F_w(x) = \lim_{n\to\infty} \mathfrak{S}_{\mathrm{id}_n \oplus w}(x).$$

Here, Question 1.1 is asking when is F_w multiplicity-free with respect to the Schur basis. Expanding Stanley symmetric polynomials into Schur polynomials gives rise to the Edelman-Greene coefficients [4]. By Theorem 4.4 of [1], this question is known to be governed by pattern avoidance. However, the list of patterns is too large to be meaningful. Thus, we propose another viewpoint to this question by investigating the Rothe diagram of a permutation, in Section 4.

2. Preliminaries

Let S_n be the symmetric group on n elements. For a permutation $w \in S_n$, its (right) *descent set* is defined as $\mathrm{Des}(w) := \{i \,|\, w(i) > w(i+1)\}$. Let $\ell(w)$ be the standard Coxeter length of w, which equals the number of inversions of w.

2.1. Schubert polynomials and the ring of split-symmetric polynomials.

To define Schubert polynomials, we first define the *divided difference operator* ∂_i, for $i = 1, \ldots, n-1$, acting on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, as follows:

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$$

where s_i acts by swapping the variable x_i with x_{i+1}. It is easy to check that $\partial_i^2 = 0$, $\partial_i\partial_j = \partial_j\partial_i$ if $|i-j| \geq 2$ and $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.

Definition 2.1. The Schubert polynomials $\{\mathfrak{S}_w | w \in S_n\}$ are defined recursively as follows:

$$\mathfrak{S}_w := \begin{cases} x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1} & \text{if } w = n\ n{-}1\ \cdots\ 1, \\ \partial_i \mathfrak{S}_{ws_i} & \text{if } \ell(w) = \ell(ws_i) - 1. \end{cases}$$

The theory of symmetric functions has been developed thoroughly in the past, with direct connections to the representation theory of S_n and GL_n and the geometry of Grassmannians. Readers are referred to [10, 13] for detailed expositions.

The *ring of symmetric polynomials* $\Lambda_{\mathbb{Z}}[x_1, \ldots, x_n]$ is a subring of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ that consists of all the polynomials that are invariant under the natural action of S_n. We omit writing the base ring $\mathbb{Z}$ when it is clear in its context. As a graded vector space, $\Lambda[x_1, \ldots, x_n]$ has a few notable basis. We focus on the Schur polynomials s_λ's, indexed by a partition λ.

Definition 2.2. A *partition* of an integer m is a sequence of weakly decreasing positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = m$. We write $|\lambda| = m$ and $\ell(\lambda) = k$. And a *Young diagram* of such shape λ, is a collection of m left-justified boxes in k rows such that there are λ_i boxes in row i.

Definition 2.3. Let μ and λ be partitions, and suppose the Young diagram of μ is entirely contained in λ. A *skew shape* $\lambda \setminus \mu$ is the set of boxes contained in λ and not in μ.

Definition 2.4. A *(skew) semi-standard Young tableau* (or SSYT) T of shape $\lambda \setminus \mu$ and order n, is a filling of the Young diagram of skew shape $\lambda \setminus \mu$ with $\{1, 2, \ldots, n\}$ such that the entries are weakly increasing along the rows and strictly increasing down the columns. We say that T is a *skew semi-standard Young tableau* when $\mu \neq \varnothing$, and a *non-skew semi-standard Young tableau* or simply a *semi-standard Young tableau* when $\mu = \varnothing$. When we refer to an SSYT that may or may not be skew, we say a *(skew) semi-standard Young tableau.*

$$\begin{ytableau} 3 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 6 \\ 5 & 6 \\ 7 \end{ytableau} \qquad \begin{ytableau} \none & \none & 2 & 4 & 5 \\ \none & \none & 3 & 5 \\ 1 & 1 & 4 \\ 2 & 3 \end{ytableau}$$

FIGURE 1. A non-skew SSYT (to the left) and a skew SSYT (to the right).

Given a (skew) SSYT T, let $T(i,j)$ be the entry in row i and column j. Then the *weight* of T is defined as

$$\operatorname{wt}(T) := \prod_{(i,j) \in \lambda \setminus \mu} x_{T(i,j)}$$

where the product is taken over all the boxes (i,j) in the skew shape $\lambda \setminus \mu$.

Definition 2.5. The *(skew) Schur polynomial* $s_{\lambda\setminus\mu}(x_1, \ldots, x_n)$ of shape $\lambda \setminus \mu$ and order n is defined as $s_{\lambda\setminus\mu}(x_1, \ldots, x_n) = \sum_T \operatorname{wt}(T)$, where T ranges over all (skew) SSYT of shape $\lambda \setminus \mu$ and order n. We say that $s_{\lambda\setminus\mu}(x_1, \ldots, x_n)$ is a *skew Schur polynomial* when $\mu \neq \varnothing$, and a *non-skew Schur polynomial* or simply a *Schur polynomial* if $\mu = \varnothing$. When $s_{\lambda\setminus\mu}(x_1, \ldots, x_n)$ is not skew, we write $s_\lambda(x_1, \ldots, x_n)$. When we refer to a Schur polynomial which may or may not be skew, we say a *(skew) Schur polynomial.*

It is well-known that the Schur polynomials $\{s_\lambda(x_1, \ldots, x_n) \mid \ell(\lambda) \leq n\}$ form a basis of $\Lambda[x_1, \ldots, x_n]$.

Let $b = (b_1, \ldots, b_k)$ be a weakly increasing sequence of positive integers. Define

$$\Lambda^b := \Lambda[x_1, \ldots, x_{b_1}] \otimes \Lambda[x_{b_1+1}, \ldots, x_{b_2}] \otimes \cdots \otimes \Lambda[x_{b_{k-1}+1}, \ldots, x_{b_k}]$$

to be the *ring of split-symmetric polynomials.* It is a subring of $\mathbb{Z}[x_1, \ldots, x_{b_k}]$, consisting of polynomials that are symmetric in each of the blocks $x_{b_{i-1}+1}, \ldots, x_{b_i}$, i.e. the polynomials invariant under the Young's subgroup $S_{b_1} \times S_{b_2-b_1} \times \cdots \times S_{b_k-b_{k-1}}$. The ring Λ^b has a basis

$$\{s_{\underline{\lambda}=(\lambda^1,\ldots,\lambda^k)} := s_{\lambda^1}(x_1, \ldots, x_{b_1}) \otimes \cdots \otimes s_{\lambda^k}(x_{b_{k-1}+1}, \ldots, x_{b_k}) \mid \ell(\lambda^i) \leq b_i - b_{i-1}\}$$

called the *split-Schur* polynomials, where $\underline{\lambda}$ is a tuple of partitions.

Definition 2.6. A polynomial $f \in \Lambda^b$ is *split-multiplicity-free* if f is multiplicity-free with respect to the basis of split-Schur polynomials.

We note that the Schubert polynomials live in the ring of split-symmetric polynomials in the sense of Lemma 2.7, which is classical and simple. We include the proof for completeness.

Lemma 2.7. *Let $w \in S_n$ and let $\mathrm{Des}(w) = (b_1, \ldots, b_{k-1})$. Let $b_k = n$ and write $b = (b_1, \ldots, b_k)$. Then $\mathfrak{S}_w \in \Lambda^b$.*

Proof. It suffices to show that $\mathfrak{S}_w$ is invariant under s_i, if $i \notin \mathrm{Des}(w)$. Since $i \notin \mathrm{Des}(w)$, we have $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$ by definition. Then $\partial_i \mathfrak{S}_w = \partial_i^2 \mathfrak{S}_{ws_i} = 0$, meaning that $s_i \mathfrak{S}_w = \mathfrak{S}_w$. $\square$

2.2. **Flagged Schur polynomials.**

Definition 2.8. Let $b = (b_1 \leq b_2 \leq \ldots \leq b_k)$ be a sequence of weakly increasing positive integers. We call b a *flag*, and we call b a *generic flag* if $b_i - b_{i-1} \geq 3 \,\forall i \in \{1, 2, ..., k-1\}$. We take $b_0 = 0$ by convention. Then, we say a (skew) SSYT T of shape $\lambda \setminus \mu$ *respects the flag* b if $T(i,j) < b_i \,\forall (i,j) \in \lambda \setminus \mu$.

Definition 2.9. A *flagged (skew) Schur polynomial* of shape $\lambda \setminus \mu$ and flag $b = (b_1 \leq b_2 \leq \ldots \leq b_{\ell(\lambda)})$ is defined as $s^b_{\lambda \setminus \mu} = \sum_T \mathrm{wt}(T)$, where T ranges over all (skew) SSYT of shape $\lambda \setminus \mu$ that respects the flag b.

Definition 2.10. The *Rothe diagram* of a permutation $w \in S_n$ is defined as the set of tuples $D(w) = \{(i,j) \mid 1 \leq i,j \leq n, w(i) > j, w^{-1}(j) > i\}$. Graphically, $D(w)$ can be seen as the complement of the hooks $(i, w(i))$, $\forall i \in \{1, 2, \ldots, n\}$. Let c_i be $\#\{(x,y) \in D(w) \mid x = i\}$. Then, $c = (c_1, c_2, \ldots, c_n)$ is called the *Lehmer code* of w. If we order the c_i to be weakly decreasing, the partition we get is called the *shape* of w, which we denote $\lambda(w)$. For all $c_i > 0$, let e_i be the greatest integer $j \geq i$ such that $c_j(w) \geq c_i(w)$. The sequence of e_i ordered to be weakly increasing is called the *flag* of w, which we denote $b(w)$.

Definition 2.11. The *essential set* of a Rothe diagram $D(w)$ is the collection of elements (i,j) such that $(i,j) \in D(w)$, and $(i+1,j), (i,j+1), (i+1,j+1) \notin D(w)$.

In Figure 2, the essential set is given by $\{(2,2), (4,2)\}$.

Definition 2.12. A permutation w is called *2143-avoiding*, or *vexillary*, if there does not exist a sequence of positive integers $i < j < k < l$ such that $w(j) < w(i) < w(l) < w(k)$.

Lemma 2.13. *(Manivel [11]) Let w be a vexillary permutation. Then, $\mathfrak{S}_w = s^{b(w)}_{\lambda(w)}$.*

Definition 2.14. A sequence of positive integers $a_1, a_2, ..., a_n$ is a *lattice permutation* if for any left factor, that is, for any sequence $a_1, a_2, ..., a_m$, $m \in \{1, ..., n\}$, the number of is is greater than or equal to the number of $i+1$s.

Definition 2.15. The *reverse reading* of a (skew) SSYT T is the sequence of entries of T, read from right to left and then from top to bottom.

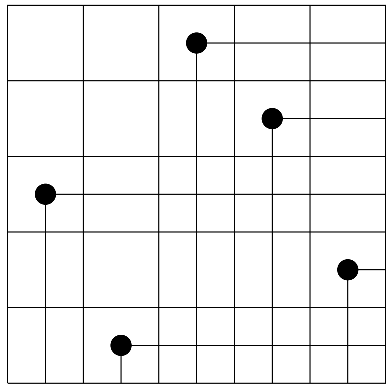

FIGURE 2. The Rothe diagram of the permutation 34152. We can observe that $c = (2, 2, 0, 1, 0, ...)$, $\lambda(w) = (2, 2, 1)$, and $b(w) = (2, 2, 4)$.

		1	1	3	5
	2	2	3	4	
1	3				

FIGURE 3. A skew SSYT whose reverse reading is 5311432231.

Definition 2.16. The *type* of a (skew) SSYT T of order n and weight $wt(T) = \prod_{i=1}^{n} x_i^{\alpha_i}$ is the sequence of positive integers $\alpha_1, \alpha_2, ..., \alpha_n$.

Lemma 2.17. *(Littlewood-Richardson) Let λ, μ be partitions, with λ containing μ. Then, $s_{\lambda\setminus\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu$, where the coefficient $c^\lambda_{\mu,\nu}$ is the number of (skew) SSYTs of shape $\lambda \setminus \mu$ and type ν such that its reverse reading is a lattice permutation.*

An immediate consequence of the Littlewood-Richardson rule motivates our approach in expressing flagged Schur polynomials in the split-Schur basis. Firstly, since each $c^\lambda_{\mu,\nu}$ is a non-negative integer, we can say that the skew Schur polynomials are *Schur-positive*, that is, when expressed in the Schur basis as $s_{\lambda\setminus\mu} = \sum_\nu k_\nu s_\nu$, each $k_\nu \geq 0$. This allows us to easily spot multiplicity by first expressing a flagged Schur polynomial s_{λ^b} as a sum of tensor products of skew Schur polynomials. Because the skew Schur polynomials are Schur-positive, multiplicity in s^λ_b can only occur in two ways: as multiplicity in the skew-Schur basis of one of the summands, or if two summands share some basis element in the skew-Schur basis.

Lemma 2.18. *If T is a (skew) SSYT of shape $\lambda \setminus \mu$ whose reverse reading is a lattice permutation, $T_{ij} \leq i$.*

Proof. We prove this assertion by induction. The entry furthest to the right in the first row must be 1, since the reverse reading of T is a lattice permutation. Then, since the entries of a row of T weakly increase, all entries in the first row must be 1, so the statement holds. Now, suppose the entries in row k are less than or equal to k. Then, in order for the reverse reading of T to be a lattice permutation, the entry furthest to the right in the $k+1$th row must be $k+1$. Again, as the entries of the rows of T weakly increase, this means all entries of row $k+1$ must be less than or equal to $k+1$. This proves the assertion. $\square$

2.3. **Stanley symmetric polynomials.** The Stanley symmetric polynomials F_w, indexed by permutations w, are introduced by Stanley [12], to study the number of reduced words of permutations. It is a symmetric polynomial with the crucial property that the coefficient of $x_1x_2\cdots x_\ell$ in F_w equals the number of reduced words of w, where $\ell = \ell(w)$ is the Coxeter length of w. The Stanley symmetric polynomials are also the stable limits of Schubert polynomials. Moreover, they expand positively into the Schur basis, $F_w = \sum_\nu j^w_\nu s_\nu$, with coefficients j^w_ν called the *Edelman-Greene coefficients* [4]. For the sake of this report, we will not go into details of how F_w's are defined and how the Edelman-Greene coefficients are computed.

The Stanley symmetric polynomials are symmetric functions which are the stable limits of Schubert polynomials. Hence, we can examine how these polynomials expand into the Schur basis. We see that the Stanley symmetric polynomials are indeed Schur-positive, and we can characterize the coefficients j^w_ν in $F_w = \sum_\nu j^w_\nu s_\nu$ as the *Edelman-Greene coefficients.* Moreover, a result of Billey and Pawlowski implies that if w is not multiplicity-free, and a permutation v contains w as a pattern, then v too is not multiplicity-free.

Lemma 2.19. *(Edelman-Greene [4]) Let $F_w(x) = \lim_{n\to\infty} \mathfrak{S}_w(x)$. Then, we know that $F_w(x) = \sum_\nu j^w_\nu s_\nu$, where each j^w_ν is a non-negative integer, i.e., the Stanley symmetric polynomials are Schur-positive. Moreover, we call j^w_ν the Edelman-Greene coefficient of w at ν.*

Definition 2.20. Suppose we have two permutations $v \in S_m$, $w \in S_n$, with $m \geq n$. Then, we say that v *contains w as a pattern* if there is a subsequence of v that is *order-isomorphic* to w. In other words, when we consider v, w as sequences $v_1, v_2, ..., v_m$ and $w_1, w_2, ..., w_n$, there exists a subsequence of v, $v_{s_1}, v_{s_2}, ...v_{s_m}$ such that $w_i < w_k \iff v_{s_i} < v_{s_k}$.

Lemma 2.21. *(Billey-Pawlowski [1]) Suppose $v \in S_m$ and $w \in S_n$ with $m \geq n$. If w is not multiplicity-free and v contains w as a pattern, then v is also not multiplicity-free.*

In order to use the Rothe diagram to examine a permutation, we must introduce some graph theoretic terms. We construct the NW-SE graph of a permutation in order to pass from permutations to an environment where we can make use of our graph theoretic tools.

Definition 2.22. Let G be a graph. Let v be a vertex of G. Then, the *degree* of v is given by the number of edges of G that contain v. We write the degree of v as $d(v)$.

Definition 2.23. The *complete graph* on n vertices, K_n, is the graph with n vertices such that any two vertices are connected by an edge. Moreover, the *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $V = A \cup B$, where $\#A = m$, $\#B = n$, and $A \cap B = \varnothing$, and edge set $\{(v, w) \mid v \in A, w \in B.\}$.

Motivated by the idea of pattern avoidance, we make use of the NW-SE graph of the Rothe diagram of a permutation to examine the multiplicity-freeness of Stanley symmetric polynomials. Properties of the NW-SE graph like containment of subgraphs may characterize collections of permutations which contain patterns that generate these properties of the NW-SE graph.

Definition 2.24. The *NW-SE graph* of a permutation w is the graph whose vertices are the essential set of $D(w)$, and two vertices $(i, j), (k, l)$ are adjacent if $i < k$ and $j < l$, or $i > k$ and $j > l$.

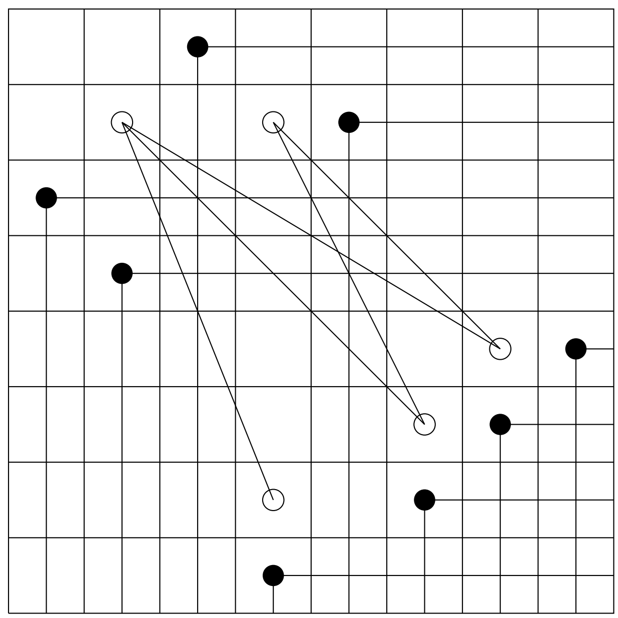

FIGURE 4. The Rothe diagram of the permutation 35128764, with elements of the essential set labelled with empty circles. The graph connecting elements of the essential set is the NW-SE graph of the permutation.

3. Split-multiplicity-free flagged Schur polynomials

Lemma 3.1. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ be a partition, and $b = (b_1, b_2, ..., b_k)$ be a flag. Then,*

$$s_\lambda^b = \sum_{\mu^{(1)} \subseteq \mu^{(2)} \subseteq \cdots \subseteq \mu^{(k)} = \lambda} s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes \cdots \otimes s_{\mu^{(k)} \setminus \mu^{(k-1)}} \tag{1}$$

where each $\mu^{(i)}$ contains the first i rows of λ and has at most b_i rows.

Proof. Recall that $s^b_\lambda = \sum_T x^T$, where T ranges over the SSYT of shape λ whose entries in the ith row are less than or equal to b_i. Then,

$$s^b_\lambda = \sum_T x^{T'} \otimes x^{T \setminus T'},$$

where T' is the SSYT contained in T whose entries are all less than or equal to b_i. We say that T' has shape $\mu^{(i)}$. Any SSYT T can be broken into a tensor product of the SSYT T' as defined above and a skew SSYT $T \setminus T'$. Moreover, any T' that contains the first i rows of λ, has at most b_i rows, and whose entries in row i are less than or equal to b_i, along with any skew SSYT of shape $T \setminus T'$ whose entries are all in $\{b_i + 1, ..., b_k\}$ can be multiplied to get an SSYT of shape λ that respects the flag. This establishes a bijection between SSYTs of shape λ that respect the flag and pairs $(T', T \setminus T')$, where T' satisfies the conditions above and $T \setminus T'$ is a skew SSYT whose entries are in $\{b_i + 1, ..., b_k\}$ Hence,

$$s^b_\lambda = \sum_{\mu^{(i)} \subseteq \mu^{(k)} = \lambda} s^b_{\mu^{(i)}} \otimes s^b_{\mu^{(k)} \setminus \mu^{(i)}} \in \Lambda[x_1, ..., x_{b_i}] \otimes \Lambda[x_{b_i+1}, ..., x_{b_k}] \tag{2}$$

where $\mu^{(i)}$ contains the first i rows of λ, respects the flag b, and has b_k or fewer rows. We can set $i = k - 1$ to get

$$s^b_\lambda = \sum_{\mu^{(k-1)} \subseteq \mu^{(k)} = \lambda} s^b_{\mu^{(k-1)}} \otimes s_{\mu^{(k)} \setminus \mu^{(k-1)}}. \tag{3}$$

Because equation 2 implies that $s_{\mu^{(k)} \setminus \mu^{(k-1)}} \in \Lambda[x_{b_{k-1}+1}, ..., x_{b_k}]$, $s_{\mu^{(k)} \setminus \mu^{(k-1)}} = s^b_{\mu^{(k)} \setminus \mu^{(k-1)}}$, i.e., the Schur polynomial already respects the flag b. Since each $\mu^{(i)}$ contains the first i rows of λ, we can iterate the process by repeatedly expanding the flagged schur in equation 3 as follows.

$$s^b_\lambda = \sum_{\mu^{(1)} \subseteq \cdots \subseteq \mu^{(k)} = \lambda} s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes \cdots \otimes s_{\mu^{(k)} \setminus \mu^{(k-1)}} \tag{4}$$

where $\mu^{(i)}$ contains the first i rows of λ, and has at most b_i rows. Equation 2 implies that $s_{\mu^{(1)}} \in \Lambda[x_1, ..., x_{b_1}]$, so $s^b_{\mu^{(1)}} = s_{\mu^{(1)}}$, which proves the assertion. □

Theorem 3.2. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ be a partition and let b be a generic flag. Then, s^b_λ is split-multiplicity-free if and only if λ contains $(2, 2, 2, 1)$.*

Proof. We begin by showing that if λ contains $(2, 2, 2, 1)$, s^b_λ has multiplicity. We know by the Littlewood-Richardson rule that the skew Schur polynomials are Schur-positive. Hence, if we can find $\mu = (\mu^{(1)} \subseteq \mu^{(2)} \subseteq \cdots \subseteq \mu^{(k)} = \lambda)$ and $\mu' = (\mu'^{(1)} \subseteq \mu'^{(2)} \subseteq \cdots \subseteq \mu'^{(k)} = \lambda)$ which share a term $s_{\xi_1} \otimes s_{\xi_2} \otimes \cdots \otimes s_{\xi_k}$ when expanded into the split-Schur basis, we can guarantee that s^b_λ has multiplicity at that term. I construct such a term as follows. Let

$$\begin{aligned}
\mu^{(1)} &= \mu'^{(1)} = (\lambda_1, \lambda_2, 1)\\
\mu^{(2)} &= (\lambda_1, \lambda_2, 2)\\
\mu'^{(2)} &= (\lambda_1, \lambda_2, 1, 1)\\
\mu^{(3)} &= \mu'^{(3)} = (\lambda_1, \lambda_2, \lambda_3, 1)\\
\mu^{(n)} &= \mu'^{(n)} = (\lambda_1, ..., \lambda_n) \quad \forall n \geq 4.
\end{aligned}$$

Then, we have two elements of the skew Schur expansion:

$$\begin{aligned}
(5) \quad & s_{\mu^{(1)}} \otimes s_{\mu^{(2)}\setminus\mu^{(1)}} \otimes s_{\mu^{(3)}\setminus\mu^{(2)}} \otimes s_{\mu^{(4)}\setminus\mu^{(3)}} \otimes s_{\mu^{(5)}\setminus\mu^{(4)}} \otimes \cdots \otimes s_{\mu^{(k)}\setminus\mu^{(k-1)}} = \\
& \qquad s_{(\lambda_1,\lambda_2,1)} \otimes s_1 \otimes s_{(\lambda_3-2,1)} \otimes s_{\lambda_4-1} \otimes s_{\lambda_5} \otimes \cdots \otimes s_{\lambda_k}
\end{aligned}$$

$$\begin{aligned}
(6) \quad & s_{\mu'^{(1)}} \otimes s_{\mu'^{(2)}\setminus\mu'^{(1)}} \otimes s_{\mu'^{(3)}\setminus\mu'^{(2)}} \otimes s_{\mu'^{(4)}\setminus\mu'^{(3)}} \otimes s_{\mu'^{(5)}\setminus\mu'^{(4)}} \otimes \cdots \otimes s_{\mu'^{(k)}\setminus\mu'^{(k-1)}} = \\
& \qquad s_{(\lambda_1,\lambda_2)} \otimes s_1 \otimes s_{\lambda_3-1} \otimes s_{\lambda_4-1} \otimes s_{\lambda_5} \otimes \cdots \otimes s_{\lambda_k}
\end{aligned}$$

In equation 6, we see that each factor is indeed a non-skew Schur polynomial, so its expansion into the split-Schur basis is simply itself. In equation 5, we see that each factor except for $s_{(\lambda_3-2,1)}$ is a non-skew Schur polynomial. Hence, we can expand this into the Schur basis as

$$\begin{aligned}
(7) \quad & s_{(\lambda_1,\lambda_2,1)} \otimes s_1 \otimes s_{(\lambda_3-2,1)} \otimes s_{\lambda_4-1} \otimes s_{\lambda_5} \otimes \cdots \otimes s_{\lambda_k} = \\
& \qquad \sum_\nu c^{\mu^{(3)}}_{\mu^{(2)},\nu} s_{(\lambda_1,\lambda_2,1)} \otimes s_1 \otimes s_\nu \otimes s_{\lambda_4-1} \otimes s_{\lambda_5} \otimes \cdots \otimes s_{\lambda_k}
\end{aligned}$$

It is now sufficient to show that $c^{\mu^{(3)}}_{\mu^{(2)},\lambda_3-1} > 0$. By the Littlewood-Richardson rule, it is sufficient to show that a tableau of shape $\mu^{(3)} \setminus \mu^{(2)}$ is an SSYT whose reverse reading is a lattice permutation. Since such a reverse reading contains only ones, it must be a lattice permutation. Moreover, since no column contains more than one entry, such a tableau is an SSYT of type $(\lambda_3 - 1, 0, ...)$. Since any other SSYT will have fewer ones, this is the only SSYT of shape $\mu^{(3)} \setminus \mu^{(2)}$ whose reverse reading is a lattice permutation. Hence, $c^{\mu^{(3)}}_{\mu^{(2)},\lambda_3-1} = 1$. Hence, s^λ_b has multiplicity, as the coefficient on the term in equation 6 will be at least 2.

Conversely, we show that if λ does not contain $(2, 2, 2, 1)$, s^b_λ does not have multiplicity. This will only occur in four cases:

(1) λ has one row.
(2) λ has two rows.
(3) λ has three rows.
(4) λ is of the form $(\lambda_1, \lambda_2, 1, ..., 1)$.

In case 1, since λ has only one row, $s^b_\lambda = s_\lambda(x_1, ..., x_{b_1})$. Hence, it does not have multiplicity. In case 2, we make use of corollary 3.1 to see that

$$s^b_\lambda = \sum_{i=0}^{\lambda_2} s_{(\lambda_1,i)} \otimes s_{\lambda_2 - i}.$$

Since both $s_{(\lambda_1,i)}$ and s_{λ_2-i} are both non-skew Schur polynomials, this is precisely the split-Schur expansion and we see that s^b_λ does not contain multiplicity.

In case 3, we use corollary 3.1 again to see that

$$s^b_\lambda = \sum_{\mu^{(1)} \subseteq \mu^{(2)} \subseteq \mu^{(3)} = \lambda} s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes s_{\mu^{(3)} \setminus \mu^{(2)}} \tag{8}$$

where $\mu^{(i)}$ contains the first i rows of λ. Both $s_{\mu^{(1)}}$ and $s_{\mu^{(3)} \setminus \mu^{(2)}}$ are non-skew Schur polynomials. I claim that for $\mu = (\mu^{(1)} \subseteq \mu^{(2)} \subseteq \mu^{(3)})$ and $\mu' = (\mu'^{(1)} \subseteq \mu'^{(2)} \subseteq \mu'^{(3)})$, $s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes s_{\mu^{(3)} \setminus \mu^{(2)}}$ and $s_{\mu'^{(1)}} \otimes s_{\mu'^{(2)} \setminus \mu'^{(1)}} \otimes s_{\mu'^{(3)} \setminus \mu'^{(2)}}$ cannot share a term. We already know that $\mu^{(3)} = \mu'^{(3)} = \lambda$. Moreover, if they share a term, since $\mu^{(1)}$ is a non-skew Schur polynomial, $\mu^{(1)} = \mu'^{(1)}$. Similarly, since $s_{\mu^{(3)} \setminus \mu^{(2)}}$ is also a non-skew Schur polynomial, $\mu^{(3)} \setminus \mu^{(2)} = \mu'^{(3)} \setminus \mu'^{(2)}$. Hence, $\mu^{(2)} = \mu'^{(2)}$ as well and we have a contradiction. Now, it is enough to show that each summand of 8, when expanded in the split-Schur basis, does not have multiplicity. Since all terms are non-skew Schur polynomials except $s_{\mu^{(2)} \setminus \mu^{(1)}}$,

$$s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes s_{\mu^{(3)} \setminus \mu^{(2)}} = \sum_\nu c^{\mu^{(2)}}_{\mu^{(1)},\nu} s_{\mu^{(1)}} \otimes s_\nu \otimes s_{\mu^{(3)} \setminus \mu^{(2)}}. \tag{9}$$

It is now sufficient to show that each $c^{\mu^{(2)}}_{\mu^{(1)},\nu} \leq 1$. By the Littlewood-Richardson rule, it is sufficient to show that any SSYT of shape $\mu^{(2)} \setminus \mu^{(1)}$ whose reverse reading is a lattice permutation has a unique type. $\mu^{(2)} \setminus \mu^{(1)}$ contains at most two rows. Moreover, since the reverse readings must be lattice permutations, all entries in the first row must be 1, and the last entry in the second row must be either a two or a one. If it is a one, then the only SSYT that is possible is filled with all ones. Such a case only occurs when no column of $\mu^{(2)} \setminus \mu^{(1)}$ has more than one entry. This is the only SSYT of shape $\mu^{(2)} \setminus \mu^{(1)}$ with its type, as no other SSYT will contain only ones. Now, suppose the second row ends in a two. By the weakly increasing condition on rows of SSYTs, any two distinct SSYTs satisfying this must have distinct types. This is because the only way to fill an SSYT of this shape with ones and twos such that the first row is ones and the second row ends with a two is by placing some number of twos in the second row, such that the second row is of the form $(1, ..., 1, 2, ..., 2)$. Hence, each $c^{\mu^{(2)}}_{\mu^{(1)},\nu} \leq 1$ and so s^b_λ is multiplicity-free.

In case 4, we use corollary 3.1 again to expand s^λ_b into the split-Schur basis.

$$s^b_\lambda = \sum_{\mu^{(1)} \subseteq \cdots \subseteq \mu^{(2)}} s_{\mu^{(1)}} \otimes s_{\mu^{(2)} \setminus \mu^{(1)}} \otimes s_{\mu^{(3)} \setminus \mu^{(2)}} \otimes \cdots s_{\mu^{(k)} \setminus \mu^{(k-1)}} \tag{10}$$

$\mu^{(1)}$ is certainly a non-skew Schur polynomial, as are $s_{\mu^{(n)}\setminus\mu^{(n-1)}}$ for $n \geq 3$, as $\mu^{(n)} \setminus \mu^{(n-1)}$ has only one column. Because of this, two summands $\mu = (\mu^{(1)} \subseteq \mu^{(2)} \subseteq \cdots \mu^{(k)} = \lambda)$ and $\mu' = (\mu'^{(1)} \subseteq \mu'^{(2)} \subseteq \cdots \mu'^{(k)} = \lambda)$ can share a term only if $\mu^{(1)} = \mu'^{(1)}$, and $\mu^{(n)} \setminus \mu^{(n-1)} = \mu'^{(n)} \setminus \mu'^{(n-1)}$ for all $n \in \{3, ..., k\}$. But since $\mu^{(k)} = \mu'^{(k)}$, this means that $\mu^{(n)} = \mu'^{(n)}\ \forall n \in \{1, 2, ...k\}$. Therefore, no two summands of 10 share a term. Moreover, since

$$s_{\mu^{(1)}} \otimes s_{\mu^{(2)}\setminus\mu^{(1)}} \otimes s_{\mu^{(3)}\setminus\mu^{(2)}} \otimes \cdots s_{\mu^{(k)}\setminus\mu^{(k-1)}} = \sum_{\nu} c^{\mu^{(2)}}_{\mu^{(1)},\nu} s_{\mu^{(1)}} \otimes s_{\nu} \otimes s_{\mu^{(3)}\setminus\mu^{(2)}} \otimes \cdots s_{\mu^{(k)}\setminus\mu^{(k-1)}}, \tag{11}$$

it is sufficient to show that each $c^{\mu^{(2)}}_{\mu^{(1)},\nu} \leq 1$. Suppose $\mu^{(1)} = \lambda_1$. Then this is certainly true because $s_{\mu^{(2)}\setminus\mu^{(1)}}$ would be a non-skew Schur polynomial. Otherwise, $\mu^{(2)} \setminus \mu^{(1)}$ is made up of two disconnected pieces, one vertical piece in the first column and below the second row, and one horizonal piece in the second row and to the right of the first column. The only way to fill a tableau of such a shape such that it is an SSYT and its reverse reading is a lattice permutation is to fill the horizontal piece with ones, and then fill the vertical piece with either $(1, 2, ..., m)$ or $(2, 3, ..., m + 1)$, where the vertical piece has m entries. Since these two SSYTs do not share the same number of ones, they do not share a type, so each $c^{\mu^{(2)}}_{\mu^{(1)},\nu} \leq 1$ and hence s^b_λ has no multiplicity. Therefore, our assertion that s^b_λ where b is a generic flag has multiplicity in the split-Schur basis if and only if λ contains $(2, 2, 2, 1)$. $\square$

The general approach of Theorem 3.2 in the necessary direction is to show that if λ contains $(2, 2, 2, 1)$, we can find a specific example of multiplicity using lemma 3.1. We will expand this approach to find a more generic example of multiplicity in some flagged Schur polynomial, and then characterize the flags which still give rise to this multiplicity.

Theorem 3.3. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a shape that contains $(2, 2, 2, 1)$. Then, s^b_λ has multiplicity if $b_\alpha \geq \alpha\ \forall \alpha \in \{1, 2, ..., k\}$ and there exists some $n \in \{1, 2, ..., k - 3\}$ such that $\exists i < j \leq n + 1$ such that $b_i \geq n + 1$, $b_j \geq n + 3$.*

Proof. By lemma 3.1, we can choose $\mu = (\mu^{(1)} \subseteq \mu^{(2)} \subseteq \cdots \subseteq \mu^{(k)} = \lambda)$ and $\mu' = (\mu'^{(1)} \subseteq \mu'^{(2)} \subseteq \cdots \subseteq \mu'^{(k)} = \lambda)$ as follows.

$$\begin{aligned}
\mu^{(i)} = \cdots = \mu^{(j-1)} = \mu'^{(i)} = \cdots = \mu^{(j-1)} &= (\lambda_1, ..., \lambda_n, 1)\\
\mu^{(j)} = \cdots = \mu^{(n+1)} &= (\lambda_1, ..., \lambda_{n+1}, 2)\\
\mu'^{(j)} = \cdots = \mu'^{(n+1)} &= (\lambda_1, ..., \lambda_{n+1}, 1, 1)\\
\mu^{(n+2)} = \mu'^{(n+2)} &= (\lambda_1, ..., \lambda_{n+2}, 1)\\
\mu^{(x)} = \mu'^{(x)}\ \forall x \notin \{j, j+1, ..., n+1\}&.
\end{aligned}$$

Then, the expansion of s_λ^b in the skew Schur basis contains two elements of the form:

$$s_{\mu^{(1)}} \otimes s_{\mu^{(2)}\setminus\mu^{(1)}} \otimes \cdots \otimes s_{\mu^{(i)}\setminus\mu^{(i-1)}} \otimes s_\varnothing \otimes \cdots \otimes \\ s_\varnothing \otimes s_{\mu^{(j)}\setminus\mu^{(j-1)}} \otimes s_\varnothing \otimes \cdots \otimes s_\varnothing \otimes s_{\mu^{(n+2)}\setminus\mu^{(n+1)}} \otimes s_{\mu^{(n+3)}\setminus\mu^{(n+2)}} \otimes \cdots$$

$$s_{\mu'^{(1)}} \otimes s_{\mu'^{(2)}\setminus\mu'^{(1)}} \otimes \cdots \otimes s_{\mu'^{(i)}\setminus\mu'^{(i-1)}} \otimes s_\varnothing \otimes \cdots \otimes \\ s_\varnothing \otimes s_{\mu'^{(j)}\setminus\mu'^{(j-1)}} \otimes s_\varnothing \otimes \cdots \otimes s_\varnothing \otimes s_{\mu'^{(n+2)}\setminus\mu'^{(n+1)}} \otimes s_{\mu'^{(n+3)}\setminus\mu'^{(n+2)}} \otimes \cdots$$

Which agree everywhere except for at $s_{\mu^{(j)}\setminus\mu^{(j-1)}} \neq s_{\mu'^{(j)}\setminus\mu'^{(j-1)}}$ and $s_{\mu^{(n+2)}\setminus\mu^{(n+1)}} \neq s_{\mu'^{(n+2)}\setminus\mu'^{(n+1)}}$. It is sufficient to show that each of these pairs share a term. The shapes $\mu^{(j)} \setminus \mu^{(j-1)}$ and $\mu'^{(j)} \setminus \mu'^{(j-1)}$ are of the form $(\lambda_n, \lambda_{n+1}, 2) \setminus (\lambda_n, 1)$ and $(\lambda_n, \lambda_{n+1}, 1, 1) \setminus (\lambda_n, 1)$. Both of these shapes share a term in their Schur basis expansion, as both shapes admit a skew SSYT of the same type whose reverse reading is a lattice permutation. Specifically, both skew SSYTs have type $(\lambda_{n+1}, 1)$. An example of this pair of SSYTs is given in Figure 5.

$$\begin{array}{ccccc} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{2} & & & \end{array} \qquad \begin{array}{ccccc} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & & & & \\ \boxed{2} & & & & \end{array}$$

FIGURE 5. Skew SSYTs of shape $\mu^{(j)} \setminus \mu^{(j-1)}$ (left) and $\mu'^{(j)} \setminus \mu'^{(j-1)}$ (right) whose reverse readings are lattice permutations and share a type.

The shapes of $\mu^{(n+2)} \setminus \mu^{(n+1)}$ and $\mu'^{(n+2)} \setminus \mu'^{(n+1)}$ are of the form $(\lambda_{n+1}, \lambda_{n+2}, 1) \setminus (\lambda_{n+1}, 2)$ and $(\lambda_{n+1}, \lambda_{n+2}, 1) \setminus (\lambda_{n+1}, 1, 1)$ respectively. Both of these shapes can be filled with all ones as in Figure 6, and so they both admit (skew) SSYTs of the same type whose reverse reading is a lattice permutation. Hence, the two elements of the skew Schur basis expansion we considered share a term in the split-Schur basis, and so s_λ^b has multiplicity.

$$\begin{array}{ccccc} & & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & & & & \end{array} \qquad \begin{array}{cccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{array}$$

FIGURE 6. Skew SSYTs of shape $\mu^{(n+2)} \setminus \mu^{(n+1)}$ (left) and $\mu'^{(n+2)} \setminus \mu'^{(n+1)}$ (right) whose reverse readings are lattice permutations and share a type.

$\square$

4. MULTIPLICITY-FREE STANLEY SYMMETRIC POLYNOMIALS

Via the method described in [11], section 2.7.4, we developed a program in SAGE that expresses a Stanley symmetric polynomial F_w in the Schur basis. We ran this program to generate a list of patterns that cause multiplicity to occur, and provide the list of these patterns along with some observations thereof, and a conjecture regarding the NW-SE diagram of a permutation w where F_w is not multiplicity-free.

Fact 4.1. The complete list of permutations $v \in S_n$, $n \leq 9$ such that w including v as a pattern causes F_w to have multiplicity is given by:

214365	321654	2413765
2416375	2417365	3142765
3152746	3152764	3215746
3216475	3251746	4216375
24136857	24137586	31426857
31427586	34128765	34172865
34182765	35128746	35128764
35172846	35172864	35182746
35182764	35217846	35271846
35281746	43172856	43217856
43271856	53172846	53172864
53217846	53271846	246138957
254138967	264138957	341269857
341279586	341279658	341279685
341285976	341286957	341286975
341287596	341296857	341297586
341728596	341826957	341827596
341926857	341927586	351279468
351279648	351279684	351286947
351286974	351428967	351482967
352148967	352418967	352481967
361428957	415283967	415328967
415382967	425138967	425318967
425381967	431528967	524138967
531428967	624138957	631428957

Upon generating this list of patterns, we generated their Rothe diagrams and NW-SE graphs. The following fact can be verified by reproducing these diagrams.

Fact 4.2. Suppose v is a pattern in the list from Fact 4.1. Then, its NW-SE graph contains either $K_3, K_{2,2}$, or there exists some vertex (i, j) such that $(i-1, j-1) \in D(w)$ and $d((i, j)) \leq 3$.

One can easily generate examples of multiplicity-free Stanley symmetric polynomials whose permutations contain $K_{2,2}$ or satisfy the degree condition.

Proposition 4.3. *There exist permutations v, w such that the NW-SE graph of v contains $K_{2,2}$ and the NW-SE graph of w contains a vertex (i, j) such that $(i-1, j-1) \in D(w)$ and $d((i, j)) \geq 3$.*

Proof. Let $v = 2417653$ and $w = 5327164$. The Rothe diagrams (with NW-SE graphs indicated) of v and w are given as follows.

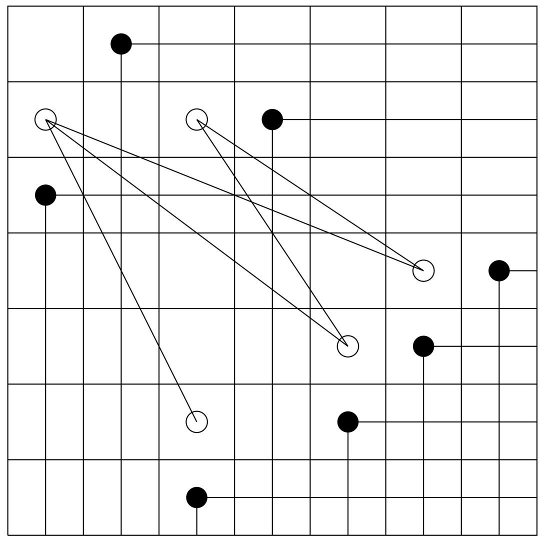

FIGURE 7. The Rothe diagram and NW-SE graph of v.

We observe that the NW-SE graph of v contains a copy of $K_{2,2}$ with vertices $\{(1,2),(3,2),(5,5),(6,4)\}$. Moreover, the NW-SE graph contains the vertex $(2,2)$. We observe that $(1,1) \in D(w)$, and $d((2,2)) \geq 3$. However, we see that both F_v and F_w are multiplicity free when we expand them.

$$F_v = s_{(4,2,1,1)} + s_{(3,3,1,1)} + s_{(3,2,2,1)} + s_{(3,2,1,1,1)}$$

$$F_w = s_{(4,3,2,2)} + s_{(4,4,1,1,1)} + s_{(5,3,2,1)} + s_{(5,2,2,1,1)} + s_{(5,3,1,1,1)} + \\ s_{(4,3,3,1)} + s_{(4,3,2,1,1)} + s_{(4,4,2,1)} + s_{(5,2,2,2)}$$

Our choices of v, w satisfy the proposition and our assertion is proven. □

Despite Proposition 4.3, we were unable to generate an example of a permutation w such that F_w is multiplicity-free and the NW-SE diagram contains a copy of K_3. This leads us to our conjecture about the multiplicity of Stanley symmetric polynomials.

Conjecture 4.4. If w is multiplicity-free, the NW-SE graph of w does not contain a copy of K_3.

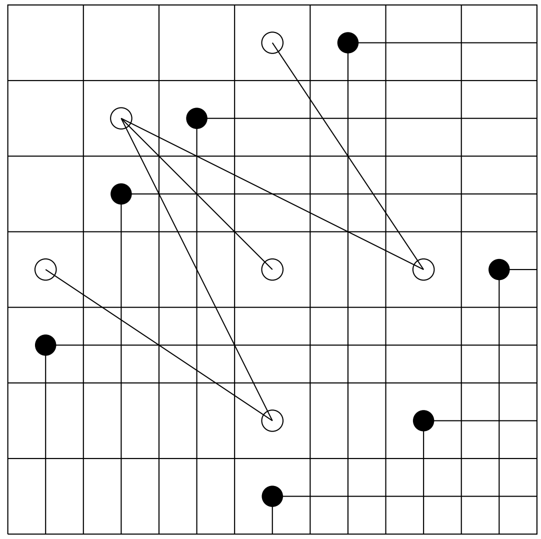

FIGURE 8. The Rothe diagram and NW-SE graph of w.

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