Relating counts in different characteristics for Steiner's conic problem

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Abstract

A well-known problem in enumerative geometry is to calculate the number of smooth conics tangent to a general set of five conics in a projective plane. This count is known in all characteristics, and differs from characteristic 2 (where it is 51) to characteristic 0 (where it is 3264) by a factor of 2^6. Following in the lines of Pacini and Testa, we give a new proof of the characteristic 2 count, dependent on the count in characteristic 0, that explains this factor. In particular, we show that half of the 3264 conics, when taken modulo 2, merge into 51 groups of 2^5, while the other half degenerate. By considering the flat limit of the scheme of complete conics into characteristic 2, we interpret these degenerate conics to be inside a dual space, which explains this split.

1 Problem and History

The problem of counting conics tangent to a general set of 5 conics was first formulated by Steiner in 1848. Since tangency to an arbitrary conic can be written directly as a degree 6 equation, Steiner [Ste48] first thought that this number should be $6^5 = 7776$ via Bézout's Theorem. However, the five degree 6 equations defined by tangency to 5 general conics are not general enough (in addition to some smooth conics, they are also satisfied by an infinite family of degenerate conics), and the true count was first shown to be 3264 by Chasles in 1864 [EH16, p. 290].

This count can also be computed over fields with finite characteristic. Using methods similar to those used for the characteristic 0 case, Vainsencher showed that the answer is 3264 in all characteristics besides 2, and 51 in characteristic 2 [Vai78]. It is not immediately clear from these computations why these two answers should differ a factor of a power of 2. Our results give an alternative computation in characteristic 2 that sheds more light on why one might expect the answers in characteristics 2 and 0 to be so related, based on the ideas in a recent paper of Pacini and Testa [PT20].

2 Methods and Intuition

In this section, we give an informal description of our strategy and some intuition for why one should expect there to be 1/64 as many conics tangent to five arbitrary conics in characteristic 2 as in characteristic 0. We also give some useful results about conic–line tangency in characteristic 2 that will be important intuition for the details of the latter stages of this paper. In most of this section, the word "multiplicity" will be used rather loosely to denote a concept that will be made rigorous in later sections. We will first describe the technique of Pacini and Testa, off of which we have based our arguments, then discuss a related example from the field of numerical computation, and finally give an outline of our proof.

2.1 The method of Pacini and Testa

It is a classical result that the number of inflection points of a general degree d plane curve is $3d(d-2)$ over all characteristics besides 3, and $d(d-2)$ over characteristic 3. In their recent paper, Pacini and Testa give an alternative proof of the result for characteristic 3 assuming the result for characteristic 0. To do this, they show that, in some precise sense, each inflection point in the characteristic 3 case has a multiplicity of 3. Specifically, one may consider each inflection point as an element of a fiber of a morphism between projective schemes over $\mathrm{Spec}(\mathbb{Z})$, so that restricting to the fibers of the target scheme over $\mathrm{Spec}(\mathbb{Z})$ is the same as choosing to work in a particular characteristic.

2.2 Conics over reals

The motivation for some of our steps comes from Sottile's exposition in [Sot08] of a construction for Steiner's conic problem in which all 3264 conics are real [BST19]. In this section, we describe the background behind this approach and apply it to our setting.

When defining schemes with tangency conditions, the conditions for two schemes X and Y to be tangent at a point p is that $X \cap Y$ has multiplicity at least 2 at p. As a result, a conic C is tangent to a singular conic $\ell_1 \sqcup \ell_2$ if and only if it is tangent to ℓ_1, is tangent to ℓ_2, or passes through $\ell_1 \cap \ell_2$. So, if ℓ_1 and $p = \ell_1 \cap \ell_2$ are kept constant, and ℓ_2 approaches ℓ_1, the result is a pair (p, ℓ_1) with p a point on ℓ_1, so that a conic C is tangent if and only if it passes through p or is tangent to ℓ_1. For a suitable definition of multiplicity, each of these occurs with multiplicity 2. This can also be viewed as the limiting case of hyperbolas with a fixed center and asymptotes approaching a fixed line; this is important as it allows five real smooth conics to be chosen.

To this end, Breiding, Sturmfels, and Timme choose five lines $\ell_1, \dots, \ell_5$ forming a convex pentagon and five points $p_1, \dots, p_5$ so that p_i lies on ℓ_i for each i, and determine how many conics are tangent to each of these point–line flags. We reproduce these calculations, as they will form the baseline for ours. The following preliminary lemma is required, for which we will should provide some general intuition.

Lemma 2.1. Given a generic set P of i points in $\mathbb{P}^2_{\mathbb{C}}$ and a set L of $5-i$ lines over $\mathbb{P}^2_{\mathbb{C}}$, there exist $2^{\min(i,5-i)}$ smooth conics passing through all points in P and tangent to all lines in L.

Proof Sketch. Let $\mathbb{P}^5$ denote the moduli space of conics over $\mathbb{P}^2$ where the point $(a_{00}, a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$ corresponds to the conic $\sum_{i,j} a_{ij}x_ix_j = 0$. For a point $p \in \mathbb{P}^2$, the condition $p \in C$ is cut out by an equation of degree 1 in $\mathbb{P}^5$, while for a line ℓ over $\mathbb{P}^2$, the condition that ℓ is tangent to C is cut out by an equation of degree 2. Thus, Bézout's Theorem suggests that we should expect there to be 2^{5-i} conics passing through every point in P and tangent to every line in L.

However, for $i \in \{0, 1, 2\}$, taking the dual gives a different picture. Since the dual of a smooth conic over $\mathbb{C}$ is another smooth conic over $\mathbb{C}$, the dual of a point is a line, and the dual of a line is a point, we expect there to be exactly as many smooth conics passing through all points in

$$L^\vee = \left\{\ell^\vee \ : \ \ell \in L\right\}$$

and tangent to all lines in

$$P^\vee = \left\{p^\vee \ : \ p \in P\right\}$$

as there are conics passing through all points in P and tangent to all lines in L, which tells us that it should be at most 2^i, and thus at most $2^{\min(i,5-i)}$ in general. It is not too hard to see that this is achievable in some cases for $i \in \{3, 4, 5\}$ and thus for $i \in \{0, 1, 2\}$ as well by taking the dual. □

Using this, we have that, if we pick a subset $T \subset \{1, 2, 3, 4, 5\}$, we have $2^{\min(|T|,5-|T|)}$ conics through p_i for $i \in T$ and tangent to l_i for $i \notin T$. Thus, in total, we have

$$\sum_{T\subset\{1,2,3,4,5\}} 2^{\min(|T|,5-|T|)} = \sum_{i=0}^{5} \binom{5}{i} 2^{\min(i,5-i)} = 1 + 10 + 40 + 40 + 10 + 1 = 102$$

conics. Now, each point–line flag (p, ℓ) may be pulled apart into a hyperbola centered at p and whose asymptotes are very close to ℓ. As these asymptotes approach ℓ, the conics tangent to such a hyperbola approach conics that are either tangent to ℓ or pass through p, and each such case becomes two separate conics tangent to the 5 hyperbolas. As a result, the total number of conics is multiplied by $2^5 = 32$ for a final result of 3264.

2.3 Transferring to characteristic 2

Now, we investigate what happens when this informal argument is transferred into characteristic 2. The main issue is that of the dual, which looks very different in characteristic 2 than in characteristic 0. This is shown by the following proposition.

Definition-Proposition 2.2. Let F be a field of characteristic 2 and C be a smooth conic in $\mathbb{P}^5_F$ represented by the point $(c_{00}, c_{11}, c_{22}, c_{01}, c_{02}, c_{12})$. The *strange point* $\mathrm{st}(C) \in \mathbb{P}^2_F$ is defined to be (c_{12}, c_{02}, c_{01}). A line ℓ is tangent to C if and only if $\mathrm{st}(C) \in \ell$.

Proof. This can be found in Vainsencher [Vai78]. □

Algebraically, this means that the equation for a conic to be tangent to a line, when written in characteristic 2, is the square of a degree 1 equation, and thus gives only one solution with multiplicity 2. This concept gives us the following analogue of Lemma 2.1, for which we offer a rigorous proof.

Proposition 2.3. Let F be a field with characteristic 2. Given a generic set P of i points in $\mathbb{P}^2_F$ and a set L of $5-i$ lines over $\mathbb{P}^2_F$, the number of smooth conics passing through all points in P and tangent to all lines in L is 1 (with multiplicity 2^i) if $i \leq 2$ and 0 otherwise.

Proof. First, we note that there is no smooth conic C tangent to 3 generic lines, since this would require the three lines to concur at $\mathrm{st}(C)$. This resolves the $i \geq 3$ case. Now, in characteristic 2, being tangent to a line is equivalent to the strange point lying on the line. So the equation for tangency is $c_{12}\ell_0 + c_{02}\ell_1 + c_{01}\ell_2 = 0$, which is a linear condition in the coefficients of the conic. Therefore, in characteristic 2, we always have 5 linear conditions on the coefficients of the conic, and thus 1 conic satisfying all 5 tangency and point conditions. With each tangency condition, the point of tangency gives a multiplicity of 2. □

When these point–line flags are pulled apart to hyperbolas, the above multiplicities of 2^i split apart, and give us 51 conics tangent to 5 fixed hyperbolas. Since conic–line tangency is a "multiplicity 2 concept" (in that the degree 2 equation for a conic to be tangent to a line in characteristic 0 factors as a square in characteristic 2), we get the factor of 32 that Pacini and Testa predicted in [PT20, 4.1].

Furthermore, we see from this example that exactly half of the conics in the above construction of 3264 real conics disappear. However, taking the dual, which swaps lines and points, would make these conics reappear but hide away our original conics. This leads us to suspect that half of the conics are "hiding in the dual space;" much of this paper is dedicated to making this notion precise.

2.4 Proof Outline

We prove that there are 51 smooth conics tangent to 5 general smooth conics in characteristic 2 using the method of Pacini and Testa [PT20]. First, we define a moduli space Γ which naturally extends the scheme of complete conics over $\mathbb{Q}$ from [EH16] into characteristic 2. This natural extension is seen by flatness over $\mathrm{Spec}(\mathbb{Z})$, which will guarantee that our counts should be the same across different characteristics. The fiber of Γ in characteristic 2 has 2 isomorphic irreducible components, and the scheme of the usual smooth conics is a dense open subset of one component. We will see from an example that the 3264 tangent conics split evenly along the two irreducible components, although the scheme structure in both components are very different (in the dual component, there are generally 1632 tangent "conics," each with multiplicity 1). We then show, using a method similar to [PT20, 3.2] that in the component containing the smooth conics, the multiplicity of each tangent conic is at least 32. We then use an example and upper-semicontinuity of length to conclude that there are, in general, $\frac{1632}{32} = 51$ smooth conics tangent to 5 given conics.

3 The proof

All schemes and rings we consider in this paper are Noetherian.

Definition 3.1. If X is a scheme over $\mathrm{Spec}(\mathbb{Z})$, and R be an arbitrary ring, we'll use X_R to denote the base change $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R)$. In particular, $X_{\mathbb{F}_p}$(or $X_{\mathbb{Q}}$) are the fibers of X over $p \in \mathrm{Spec}(\mathbb{Z})$ (or 0).

Definition 3.2. If X_R is a scheme over $\mathrm{Spec}(R)$, we'll use X_R^n to denote the fibered product of X_R with itself n times over $\mathrm{Spec}(R)$.

We now define an important concept that will be central to our discussion: the length of a point in the domain of a morphism.

Definition 3.3. Given a morphism $\pi : X \to Y$ of finite type k-schemes, let $x \in X_{\bar{k}}(\bar{k})$ be a geometric point. Suppose the scheme-theoretic fiber $X_{\bar{k},\pi_{\bar{k}}(x)}$ has dimension 0. Then the π*-length* $\lambda_\pi(x)$ of x is defined as

$$\lambda_\pi(x) = \dim_{\bar{k}} \left(\mathcal{O}_{X_{\bar{k},\pi_{\bar{k}}(x)},x} \right).$$

We see that π-length is multiplicative under fibered products and preserved under base change. In absolute terms, the length of a point x in a zero-dimensional scheme is defined as $\lambda(x) = \dim_{k(x)}(\mathcal{O}_{X,x})$, then the relative length of x is simply the length of the geometric fiber containing x.

Proposition 3.4. Let $\sigma_1 : X \to S$, $\sigma_2 : Y \to S$ be two morphisms. Let $Z = X \times_S Y$ with projections π_1 and π_2 X and Y. Then, for all $z \in Z$, letting $x = \pi_1(z)$ and $y = \pi_2(z)$, we have that

$$\lambda_{\sigma_1 \circ \pi_1}(z) = \lambda_{\sigma_1}(x)\lambda_{\sigma_2}(y)$$

and

$$\lambda_{\pi_1}(z) = \lambda_{\sigma_2}(y),$$

as desired.

Proof. By base change, we may assume $S = \mathrm{Spec}(\overline{k})$. We may further assume without loss of generality that X and Y are the connected components at x and y and contain only x and y, respectively. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Then, considering vector space dimensions over $\overline{k}$,

$$\dim_{\overline{k}}(\mathcal{O}_Z(Z)) = \dim_{\overline{k}}(A \otimes_{\overline{k}} B) = \dim_{\overline{k}}(A) \dim_{\overline{k}}(B)$$

and

$$\dim_{\overline{k}}(\mathcal{O}_{Z_x} Z_x) = \dim_{\overline{k}}(B \otimes_{\overline{k}} A \otimes_A k(x)) = \dim_{\overline{k}}(B).$$

□

Unfortunately, π-length is not multiplicative under composition, but we can prove some basic facts about π-lengths of compositions.

Proposition 3.5. Let $\pi : X \to Y$ and $\sigma : Y \to Z$ be two morphisms, and let $x \in X_{\bar{k}}(\bar{k})$ be a geometric point of X. We have that $\lambda_{\sigma\circ\pi}(x) \geq \lambda_\pi(x)$. Furthermore, if $\lambda_\sigma(x) = 1$, equality holds.

Proof. Assume without loss of generality that $Z = \mathrm{Spec}(\overline{k})$. Furthermore, we may assume that X and Y each only have 1 point by restricting to the connected components at x and y, respectively. Let $Y = \mathrm{Spec}(B)$, and let m_y be the maximal ideal of B at y. Let $k(y) = B/m_y$ and $X = \mathrm{Spec}(A)$. Then $X_y = A/m_y A$, so we have $\dim(A) \geq \dim(A/m_y A)$ and equality if $B = k(y)$. □

Remark 3.6. When π is a finite morphism, π-length is uppersemicontinuous due to a result in [LJT74]. Similarly, the total fiber length $\mu_\pi(y)$, defined as $\mu_\pi(y) = \sum_{f(x)=y} \lambda_\pi(x) = \dim(\pi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} k(y))$ is uppersemicontinuous on Y [Har77, 3.12.7.2].

Furthermore, we have the following result on the general π-length of some morphisms.

Lemma 3.7. Given integral schemes X and Y over $\overline{\mathbb{F}_p}$ with X normal and a finite, dominant morphism $\pi : X \to Y$, there exist $d \in \mathbb{N}$ and a dense open set $U \subset X$ such that $\lambda_\pi(x) = p^d$ for all $x \in U$.

Proof. Let K and L be the function fields on X and Y, respectively. Let K^s be the separable closure of K, and take $J = L \cap K^s$. Then J/L is separable extension, and K/L is purely inseparable of degree p^d. Now, we may assume that $Y = \mathrm{Spec}(B)$ is affine by taking an open subset; then, since finite morphisms are affine, $X = \mathrm{Spec}(A)$ is also affine.

Now, take $\sigma : Y^\# = \mathrm{Spec}(C) \to Y$ to be the normalization of Y in J; since X is normal, we have a morphism $\tau : X \to Y^\#$. Since τ is dominant and closed, it is surjective. Since C is integrally closed and A is integral over C, the minimal polynomial of elements in A has coefficients in C, so $A^{p^d} \subset C$. Let $\mathfrak{p}$ be a prime ideal in A, and $\mathfrak{p} \cap C = \mathfrak{q}$. Since $\mathfrak{p}^{p^d} \in \mathfrak{q}$,

$$\mathfrak{p}^{p^d} = \sqrt{\mathfrak{q}A},$$

and so the fibers of $X \to Y^\#$ always consist of 1 point. Now, the fiber of the generic point is

$$A \otimes_C J = A_{C\setminus 0} = K,$$

of degree p^d over L. By [Liu02, 5.1.25a], the length over each point is at least p^d, with equality on an open set containing the generic point of X. The morphism $Y^\# \to Y$ is étale at the generic point, since the field extension is separable. By [Liu02, 4.4.12], we thus know that it is étale on a dense open set. This implies that all fibers are reduced, and so the lengths are all 1. It follows from Proposition 3.5 that the length of $X \to Y$ is p^d on some dense open set, and no less elsewhere. □

Let $\mathbb{P}^5_{\mathbb{Z}}$ denote the moduli space of conics over $\mathbb{P}^2_{\mathbb{Z}}$ where the point $(a_{00}, a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$ corresponds to the conic $\sum_{i,j} a_{ij} x_i x_j = 0$. Similarly, let $\mathbb{P}^{5\vee}_{\mathbb{Z}}$ denote the moduli space of conics over $\mathbb{P}^{2\vee}_{\mathbb{Z}}$, the dual space of $\mathbb{P}^2_{\mathbb{Z}}$. The open subset $\mathcal{S} \subset \mathbb{P}^5_{\mathbb{Z}}$ given by the equation

$$a_{00}a_{12}^2 + a_{11}a_{02}^2 + a_{22}a_{01}^2 - a_{01}a_{02}a_{12} - 4a_{00}a_{11}a_{22} \neq 0$$

parametrizes the smooth conics. $\mathcal{S}$ is flat over $\mathbb{P}^5_{\mathbb{Z}}$, and thus over $\mathrm{Spec}\,\mathbb{Z}$ as well.

We now define a particular subset of conics which will be useful to our discussions.

Definition 3.8. The *double lines* is the closed subscheme $\mathcal{L}_F \in \mathbb{P}^5_F$ given by equations

$$a_{01} = a_{02} = a_{12} = 0.$$

We define similarly $\mathcal{L}_F^\vee \in {\mathbb{P}^5_F}^\vee$.

When trying to calculate the number of conics tangent to 5 given conics, one runs into the problem that the double lines will be tangent to every conic, since its intersections all have multiplicity at least 2. We may not remove the double lines solely by considering $\mathcal{S}$, since it is not proper. Instead, we eliminate these extra tangencies by blowing up $\mathbb{P}^5$ along the subscheme of double lines. This works well in fixed characteristic, but the closed subscheme of $\mathbb{P}^5_\mathbb{Z}$ corresponding to the double lines has a nonreduced fiber over $2 \in \operatorname{Spec}\mathbb{Z}$. Furthermore, this blowup fails to be flat over $\operatorname{Spec}\mathbb{Z}$. Instead, we consider the following scheme, which will allow us to properly deal with these double lines, and is consistent across characteristics.

Definition 3.9. The *scheme of complete conics* is the closed subscheme $\Gamma \subset \mathbb{P}^5_\mathbb{Z} \times {\mathbb{P}^5_\mathbb{Z}}^\vee$, where the first $\mathbb{P}^5_\mathbb{Z}$ is parametrized by (a_{ij}) and the second ${\mathbb{P}^5_\mathbb{Z}}^\vee$ is parametrized by (b_{ij}) given by the equations

$$2b_{ii}a_{ij} + 2a_{jj}b_{ij} + a_{jk}b_{ik} = 0 \tag{3.9.1}$$

$$4a_{ii}b_{ii} + a_{ik}b_{ik} - 4a_{jj}b_{jj} - a_{jk}b_{jk} = 0 \tag{3.9.2}$$

$$b_{jj}\left(a_{jk}^2 - 4a_{jj}a_{kk}\right) - b_{ii}\left(a_{ik}^2 - 4a_{ii}a_{kk}\right) = 0 \tag{3.9.3}$$

$$a_{jj}\left(b_{jk}^2 - 4b_{jj}b_{kk}\right) - a_{ii}\left(b_{ik}^2 - 4b_{ii}b_{kk}\right) = 0 \tag{3.9.4}$$

where i, j, k is any permutation of the indices $0, 1, 2$. A point in Γ is written as $(C, D^{(\vee)})$; in general, we use the notation $D^{(\vee)}$ to denote a conic in a dual space which is not necessarily the dual of any conic.

There is a canonical open immersion $\mathcal{S} \to \Gamma$ which sends a smooth conic $C = (c_{00}, c_{11}, c_{22}, c_{01}, c_{02}, c_{12})$ to $(C, C^\vee)$, where $C^\vee$ is the dual conic of C, given by

$$\begin{aligned}\big(&c_{12}^2 - 4c_{11}c_{22},\ c_{02}^2 - 4c_{00}c_{22},\ c_{01}^2 - 4c_{00}c_{11},\\ &4c_{01}c_{22} - 2c_{02}c_{12},\ 4c_{02}c_{11} - 2c_{01}c_{12},\ 4c_{00}c_{12} - 2c_{01}c_{02}\big).\end{aligned} \tag{3.9.5}$$

The image of this open immersion, as an open subset of Γ, will be denoted $\mathcal{S}'$.

Example 3.10. The *scheme of complete conics over* $\mathbb{Q}$ is the fiber $\Gamma_\mathbb{Q}$ of Γ over $0 \in \operatorname{Spec}\mathbb{Z}$ (see Definition 3.1). It is the closure of

$$\{(C, C^\vee) : C \text{ smooth}\} \subset \mathbb{P}^5_\mathbb{Q} \times {\mathbb{P}^5_\mathbb{Q}}^\vee$$

This scheme is isomorphic to the blowup of $\mathbb{P}^5_\mathbb{Q}$ along the (reduced subvariety of) double lines. In particular, $\Gamma_\mathbb{Q}$ is independent of coordinates of $\mathbb{P}^2_\mathbb{Q}$ [EH16, p. 301].

Example 3.10 is true for all fields with characteristic distinct from 2. However, it fails in characteristic 2, since the equations determining the double lines are different. Over a field F of characteristic 2 (for example $F = \mathbb{F}_2$), we will work with the corresponding scheme Γ_F.

Remark 3.11. Due to Remark 3.6, the most natural choice of Γ is to make it flat over $\operatorname{Spec}(\mathbb{Z})$. In fact, the construction in Example 3.10 is natural in any characteristic different from 2, and fails in characteristic 2 since the dual of a smooth conic is no longer smooth. Hence there exists a unique scheme $\Gamma^f = \overline{\Gamma_\mathbb{Q}} \in \mathbb{P}^5$ that is flat over $\operatorname{Spec}(\mathbb{Z})$ and is the scheme of complete conics in other characteristics.

Unfortunately, we were unable to determine Γ^f. On the other hand, we clearly have $\Gamma^f \subset \Gamma_{\mathbb{F}_2}$ by universal property of the closure. We will see that $\Gamma^f \supset (\Gamma_{\mathbb{F}_2})_{\text{red}}$ in Lemma 3.13. In other words, Γ^f is very close to what we have right now, and may only differ on the component of double lines $\mathcal{L}_{\mathbb{F}_2}$.

By considering the points in $\mathbb{P}^2_F$ as lines over ${\mathbb{P}^2_F}^\vee$, we obtain a canonical isomorphism $\mathbb{P}^2_F \cong \mathcal{L}_F$. Thus it makes sense to speak of the strange point of a conic as a degenerate conic in the moduli space ${\mathbb{P}^5_F}^\vee$. We'll call this the *double line dual*. Lines lying on the double line dual are exactly the lines tangent to a conic, thus the double line dual has the same properties (and equations) of the dual conic in other characteristics. As a result, the structure of Γ_F is somewhat different in characteristic 2 than in other characteristics, as encapsulated in the following proposition.

Proposition 3.12. When F is a field of characteristic 2, Γ_F has two (geometrically) irreducible components. The reduced structure of each component is isomorphic to the blowup of $\mathbb{P}^5_F$ along $\mathcal{L}_F$, and we call these $\Gamma_F^{(1)}$ and $\Gamma_F^{(2)}$. Furthermore, the reduced structure at the intersection of the two components is $\mathcal{L}_F \times \mathcal{L}_F^\vee$. Γ_F is reduced outside of this intersection.

Proof. We verify this by using Sage to calculate the embedded points of Γ_F. Our code can be found in the appendix. ☐

The component, $\Gamma_F^{(1)}$ represents pairs $\left(C, D^{(\vee)}\right)$ where C is a conic and $D^{(\vee)}$ is the double line dual (i.e. the strange point considered as a double line in the dual space) to C. Similarly, $\Gamma_F^{(2)}$ represents pairs $(C, D^{(\vee)})$ where $D^{(\vee)}$ is some conic and C is the double line dual to $D^{(\vee)}$. Using Proposition 3.12, we can now prove that Γ_f and $\Gamma_{\mathbb{F}_2}$ have the same reduced structure.

Lemma 3.13. $(\Gamma_{\mathbb{F}_2})_{\text{red}} \subset \overline{\Gamma_{\mathbb{Q}}}$ inside $\mathbb{P}^5_{\mathbb{Z}} \times \mathbb{P}^{5\vee}_{\mathbb{Z}}$. In particular, Γ_{red} is flat over $\operatorname{Spec}(\mathbb{Z})$.

Proof. It follows from Example 3.10 and [Hol17, 2.1] that $\Gamma_{\mathbb{Z}[\frac{1}{2}]}$ is flat over $\operatorname{Spec}(\mathbb{Z}[\frac{1}{2}])$. So, we only need to show that $(\Gamma_{\mathbb{F}_2})_{\text{red}} \subset \overline{\Gamma_{\mathbb{Q}}}$. Let $\mathcal{S}'$ be the open subset of Γ that is isomorphic to $\mathcal{S}$. Then $\mathcal{S}'_{\mathbb{F}_2}$ is a dense open subset of the reduced $\Gamma^{(1)}_{\mathbb{F}_2}$. Since $\mathcal{S}' \cong \mathcal{S}$ is flat over $\operatorname{Spec}\mathbb{Z}$, its generic points are in $\mathcal{S}'_{\mathbb{Q}} \subset \Gamma_{\mathbb{Q}}$, which implies that

$$\Gamma^{(1)}_{\mathbb{F}_2} = \overline{\mathcal{S}'_{\mathbb{F}_2}} \subset \overline{\Gamma_{\mathbb{Q}}}.$$

Any isomorphism $\mathbb{P}^2 \cong (\mathbb{P}^2)^\vee$ induces an automorphism of Γ which swaps $\Gamma^{(1)}_{\mathbb{F}_2}$ and $\Gamma^{(2)}_{\mathbb{F}_2}$ and sends $\Gamma_{\mathbb{Q}}$ to itself. The result follows. ☐

We now define the main scheme we will consider that encapsulates our tangency conditions.

Definition 3.14. The *scheme of flags* is the scheme $\mathcal{F} \subset \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^{2\vee}_{\mathbb{Z}}$ of lines passing through a point; in other words, it is the closed subscheme given by the equation

$$x_0\ell_0 + x_1\ell_1 + x_2\ell_2 = 0.$$

A point in $\mathcal{F}$ is written as (p, ℓ), where ℓ is a line through p.

To define the scheme that we'd like to consider, we first define what we mean by conics being tangent to a point-line flag.

Definition 3.15. The *universal tangency flag* (over any ring) is the scheme

$$\mathcal{T} \subset \Gamma \times \mathcal{F}$$

defined by the equations

$$\left\{ \begin{array}{c|l} \left((C, C^\vee), (p, \ell)\right) & p \in C, \ell \in C^\vee \\ & \ell \text{ tangent to } C \text{ at } p, \\ & p \text{ tangent to } C^\vee \text{ at } \ell \end{array} \right\}$$

where a line ℓ is tangent to a conic C with equation f at a point p if all 2×2 minors of the matrix

$$\begin{pmatrix} \nabla f(p)_0 & \nabla f(p)_1 & \nabla f(p)_2 \\ \ell_0 & \ell_1 & \ell_2 \end{pmatrix} \tag{3.15.1}$$

are zero.

Over smooth conics, our definition is scheme-theoretically equivalent to the usual definition which does not invoke the dual.

Proposition 3.16. The subscheme of $\mathcal{S} \times \mathcal{F}$ defined by the equations

$$\left\{ \begin{array}{c|l} (C, (p, \ell)) & p \in C \\ & \ell \text{ tangent to } C \text{ at } p \end{array} \right\}$$

is isomorphic to $\mathcal{T} \cap \mathcal{S}' \times \mathcal{F}$ through the canonical isomorphism $\mathcal{S} \cong \mathcal{S}'$.

Proof. It suffices to check that the tangency conditions are equivalent. This is verified by Sage code, which can be found in the appendix.

□

Definition 3.17. The *universal tangency scheme* over a ring R is the scheme

$$\Lambda_R \subset \Gamma_R^6 \times_R \mathcal{F}_R^5$$

defined as intersection of the base change of the relevant tangency flags. More specifically, we label the 6 factors of Γ_R as $\Gamma_{R,i}, 1 \leq i \leq 6$ and the 5 factors of $\mathcal{F}_R$ as $\mathcal{F}_{R,i}, 1 \leq i \leq 5$. For each $1 \leq i \leq 5$, we consider the tangency flags $\mathcal{T}_{i,1} \subset \Gamma_{R,i} \times \mathcal{F}_{R,i}$ and $\mathcal{T}_{i,2} \subset \Gamma_{R,6} \times \mathcal{F}_{R,i}$. Then we may use a fibered product to base change each $\mathcal{T}_{i,j}$ to be a closed subset of $\Gamma_R^6 \times_R \mathcal{F}_R^5$ (for example, the base change of $\mathcal{T}_{1,1}$ would be $\mathcal{T}_{1,1} \times \Gamma_{R,2} \times \ldots \times \Gamma_{R,6} \times \mathcal{F}_{R,2} \times \ldots$). We define Λ_R is the scheme-theoretic intersection of all these base changes of tangency flags.

We have a canonical morphism $\pi_{\Lambda,R} : \Lambda_R \to \Gamma_R^5$ defined by the projection to the space of the first 5 conics. When R is an algebraically closed field (e.g. $\overline{\mathbb{F}_p}$ or $\overline{\mathbb{Q}}$), the fibers of $\pi_{\Lambda,R}$ correspond set-theoretically to the complete conics tangent to 5 given conics. When $R = \overline{\mathbb{Q}}$, the fiber generally lies in the dense open set $\mathcal{S}_R \subset \mathbb{P}_R^5$, in which case we have the usual count of smooth conics tangent to 5 given smooth conics (which is 3264). However, with $R = \overline{\mathbb{F}_2}$, the image $\mathbb{P}_R^5$ is only dense in the irreducible component $\Gamma_R^{(1)}$ (Proposition 3.12), and we will see that there are, when counted with appropriate multiplicity, only 1632 smooth conics tangent to 5 given smooth conics in characteristic 2.

4 An Example

To simplify our future computations, we introduce some notations which will simplify our formula for the tangency conditions. We will be working with multilinear algebra, in the vector space of homogeneous polynomials over projective space or of polynomials over affine space.

Notation 4.1. When we work with a closed subscheme $\mathrm{Spec}(R/I)$ of an affine space $\mathrm{Spec}(R)$ where R is a polynomial ring, equations of the form $f = g$ where $f, g \in R$ are some polynomial will mean that $f \in I$, where I is assumed.

Notation 4.2. Let k be a field. Most of the definitions in this section are defined for vectors up to nonzero/invertible scaling (i.e. elements of $k^n/k^\times$ for some n). They may also be defined analogously for vectors in k^n.

i) If p, q are two points in $k^3/k^\times$, we let the cross product $p \times q$ be the triple

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \in k^3/k^\times.$$

We can see that, if $p \times q = 0$ and both p and q are nonzero, then $p = q$ in $k^3/k^\times$.

ii) Similarly, we define the dot product $p \cdot q = p_0q_0 + p_1q_1 + p_2q_2$. A point p lies on the line with coordinates q if and only if $p \cdot q = 0$.

iii) For a point $p \in k^3/k^\times$, define

$$\mathrm{sq}(p) = (p_0^2, p_1^2, p_2^2, p_0p_1, p_0p_2, p_1p_2) \in k^6/k^\times.$$

Given a conic C with 6 coordinates representing its coefficients, we can express the relation $p \in C$ as $\mathrm{sq}(p) \cdot C = 0$.

iv) Similarly to the cross product, if we have any six vectors $p_1, \ldots, p_6$ in $k^6/k^\times$, we let $p_1 \wedge \cdots \wedge p_5$ and $p_1 \wedge \cdots \wedge p_6 = (p_1 \wedge \cdots \wedge p_5) \cdot p_6$ denote exterior products.

v) If C is any conic over $\mathbb{P}^2_k$, we let $\mathrm{st}(C) \in k^3/k^\times$ denote the strange 'point' of C. If C is not a double line, its strange point is not 0, and can thus be interpreted as a geometric point of $\mathbb{P}^2_k$. The equations for a line ℓ to be tangent to C (Definition 3.15) are equivalent to $\ell \cdot \mathrm{st}(C) = 0$.

vi) Given a line $\ell = (\ell_0, \ell_1, \ell_2)$ over $\mathbb{P}^2$, we let $s(\ell) = (0, 0, 0, \ell_2, \ell_1, \ell_0)$. The above tangency condition can also be written as $s(\ell) \cdot C = 0$.

vii) If $p \in k^3/k^\times$, we let

$$t(p) = \begin{pmatrix} 0 & 0 & 0 & p_1 & p_2 & 0 \\ 0 & 0 & 0 & p_0 & 0 & p_2 \\ 0 & 0 & 0 & 0 & p_0 & p_1 \end{pmatrix} \in k^{3\times 6}/k^\times,$$

so that $t(p)C = p \times \mathrm{st}(C)$ is the tangent line of C at p. We have the identities $pt(p) = 0$ and $\nabla_p \mathrm{sq}(p) = t(p)$.

We now introduce the following proposition, which will be useful for our computations to deal with nonreduced points.

Proposition 4.3. Let R be a ring. Given $p_1, p_2 \in R$ and $q_1, q_2 \in R[q_1, q_2]$, write $p = (1, p_1, p_2), q = (1, q_1, q_2)$. Suppose all terms in $(p \times q)(p \times q)^\intercal$ are contained in some ideal I of $R[q_1, q_2]$. Then, for any homogeneous polynomial f of degree n with 3 variables (say x_0, x_1, x_2), we have

$$f(q) - ((1-n)f(p) + q \cdot \nabla_x f(p)) \in I.$$

Proof. We have $(p_1 - q_1)^2 = (p_2 - q_2)^2 = (p_1 - q_1)(p_2 - q_2) = 0$ in $\mathrm{Spec}(R/I)$, so within our ideal I,

$$f(p) = f(q) + p \cdot \nabla_{p_1, p_2} f(q) = (1-n)f(q) + p \cdot \nabla_p f(q)$$

as desired. □

Now, we may move on to our example.

Example 4.4. Consider five general singular conics $C_1, \dots, C_5 \in \Gamma^{(1)}_{\overline{\mathbb{F}_2}}$ over $\overline{\mathbb{F}_2}$ so that C_i is composed of the two lines $\ell_i^{(1)}$ and $\ell_i^{(2)}$ which intersect at p_i. The fiber of $\pi_{\Lambda, \overline{\mathbb{F}_2}}$ over these 5 singular conics will corresponds to complete conics that are tangent to these 5 given conics. Then p_i is the strange point $\mathrm{st}(C_i)$ of C_i. We will count, in this specific example, that there are exactly 1632 conics in the fiber whose tangent conic lie inside $\Gamma^{(1)}_{\overline{\mathbb{F}_2}}$, and furthermore this count generally arise from 51 smooth tangent conics with multiplicity 32 each. In other words, we partly determine the scheme-theoretic structure of the fiber of $\Lambda_{\overline{\mathbb{F}_2}}$ over the five given singular conics. We'll use this example, along with the upper-semicontinuity from Remark 3.6, for our final proof.

We were unfortunately unable to determine the scheme-theoretic structure of the part of the fiber whose tangent conic lies inside $\Gamma^{(2)}_{\overline{\mathbb{F}_2}}$, and thus were unable to independently prove that there are generally 51 smooth conics tangent to 5 given smooth conics. We note that to show this count independently, we only needed to compute an upper bound on the multiplicities at each point, due to semi-uppercontinuity arguments.

We do show that, set-theoretically, the part of the fiber in $\Gamma^{(2)}_{\overline{\mathbb{F}_2}}$ consists of $32 + 80 + 80 = 192$ conics in general, and that 32 conics in the fiber have length 1. We believe that the remaining sets of 80 conics have lengths 4 and 16, which would add up to our $1632 = 31 + 80 \times 4 + 80 \times 16$ conics in the fiber inside $\Gamma^{(1)}_{\overline{\mathbb{F}_2}}$. Due to the upper-semicontinuity of multiplicity Remark 3.6, this implies that the multiplicity of conics in the fiber in $\Gamma^{(2)}_{\overline{\mathbb{F}_2}}$ is generally 1. In particular, the two parts of the fiber, although having the same count, have non-isomorphic scheme-theoretic structures.

First, we'll count the case that the tangent conic is smooth (i.e. inside $\Gamma^{(1)}_{\overline{\mathbb{F}_2}}$). We let the C be the coordinates for the tangent conic. Let q_i, k_i be the coordinates for the tangency points and lines, for each

$1 \leq i \leq 5$. Then the ideal cutting out the the fiber of the tangency scheme over $C_1, \ldots, C_5$ is generated by the following equations, for all $1 \leq i \leq 5$:

$$q_i \cdot k_i = 0 \qquad \text{(Tangent point lies on tangency line)} \tag{4.4.1}$$

$$\left(q_i \cdot \ell_i^{(1)}\right)\left(q_i \cdot \ell_i^{(2)}\right) = 0 \qquad \text{(Tangent point lies on singular conic)} \tag{4.4.2}$$

$$p_i \cdot k_i = 0 \qquad \text{(Tangent line tangent to singular conic)} \tag{4.4.3}$$

$$\operatorname{sq}(q_i) \cdot C = 0 \qquad \text{(Tangent point lies on the special conic)} \tag{4.4.4}$$

$$s(k_i) \cdot C = 0 \qquad \text{(Tangent line tangent to the special conic)} \tag{4.4.5}$$

To find the multiplicity at some conic and flags $(\overline{C}, (\overline{q_i}, \overline{k_i}))$ in the fiber, we work with the local ring of this scheme at this point. In the rest of the calculations with this example, we implicitly work in the open set where the first coordinate of each of p_i and $\overline{q_i}$ is 1 (i.e. $p_{i,0} = 1$ where $p_i = (p_{i,0}, p_{i,1}, p_{i,2})$). Because of this, $q_{i,0}$ is invertible in the local ring for all i, and we assume it equals 1. As a result, it will make sense to talk about non-homogeneous equations in q_i.

In the next part, we'll show that the multiplicity of each point in the scheme determined by (4.4.1)—(4.4.5) is 32. We do this by first analyzing conditions (4.4.2) and (4.4.3) for the tangency to the singular conics. Then we solve for the tangent conics in terms of the tangency flag, and use linear algebra to finish.

To avoid complications with dehomogenizing the coordinates, we'll leave k_i, C to be determined only up to an invertible scalar. Thus our equations such as $k_i = \ell_i^{(1)}$ will actually mean that $k_i = c\ell_i^{(1)}$ for some invertible scalar c in the local ring (for example, we might have $c = \frac{k_{i,0}}{\ell_{i,0}^{(1)}}$).

Begin by working with some fixed i. We note that, when the singular conics are in general position, we must have that either

- **a)** $\overline{q_i} \cdot \ell_i^{(j)} \neq 0$ for some j, or
- **b)** $\overline{k_i} \neq \ell_i^{(1)}, \ell_i^{(2)}$.

The first case gives that $q_i \cdot \ell_i^{(j)}$ is invertible in the local ring, so $q_i \neq p_i$, and thus happens when $\overline{C}$ is tangent to one of the lines. The second happens when $\overline{C}$ passes through the singular point p_i.

Case a). Here, we assume without loss of generality that $j = 2$. Then (4.4.2) gives that $q_i \cdot \ell_i^{(1)} = 0$, which implies that

$$0 = \left(\ell_i^{(1)} \cdot q_i\right) p_i - \left(\ell_i^{(1)} \cdot p_i\right) q_i = \ell_i^{(1)} \times (p_i \times q_i),$$

and thus

$$\ell_i^{(1)} = p_i \times q_i. \tag{4.4.6}$$

In addition, (4.4.1), (4.4.3), and (4.4.5) combine to give that

$$k_i = \ell_i^{(1)} \implies s\left(\ell_i^{(1)}\right) \cdot C = 0. \tag{4.4.7}$$

Case b). Here, we have from (4.4.3) that

$$p_i = \ell_i^{(1)} \times k_i = \ell_i^{(2)} \times k_i,$$

which implies via (4.4.1) and (4.4.2) that

$$(p_i \times q_i)(p_i \times q_i)^\intercal = \mathbf{0}_{3\times 3}, \tag{4.4.8}$$

where $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix. This equation enables us to utilize Proposition 4.3.

Now, we must have case a) for at most two values of i, since otherwise the point $\mathrm{st}(\overline{C})$ would lie on 3 general lines. We will move forward in the example when case b) occurs for $1 \leq i \leq 4$ and case a) occurs for $i = 5$; the computation for other conics are completely analogous.

First, we solve for the conic C. By (4.4.4) and (4.4.7),

$$C = \mathrm{sq}\,(q_1) \wedge \cdots \wedge \mathrm{sq}\,(q_4) \wedge s\big(\ell_5^{(1)}\big). \tag{4.4.9}$$

In addition, for each $1 \leq i \leq 4$, (4.4.1), (4.4.3), and (4.4.5) give that

$$0 = p_i \cdot k_i = p_i \cdot (q_i \times \mathrm{st}(C)) = q_i \cdot (p_i \times \mathrm{st}(C)) = q_i t(p_i) \left(\mathrm{sq}\,(q_1) \wedge \cdots \wedge \mathrm{sq}\,(q_4) \wedge s\big(\ell_5^{(1)}\big)\right). \tag{4.4.10}$$

For simplicity, we'll assume $i = 1$; the other indices are symmetric. Then since $\nabla_{q_1} \mathrm{sq}(q_1) = t(q_1)$, we can simplify using Proposition 4.3 for q_1, using that (4.4.10) is homogeneous of degree 3 in q_1, to get

$$\begin{aligned}
0 &= q_1 \cdot \nabla_{q_1} \left(q_1 t(p_1)(\mathrm{sq}(q_1) \wedge \cdots \wedge \mathrm{sq}(q_4) \wedge s\big(\ell_5^{(1)}\big))\right)\Big|_{q_1 = p_1} \\
&= q_1 t(p_1) \left(\mathrm{sq}(p_1) \wedge \mathrm{sq}(q_2) \wedge \mathrm{sq}(q_3) \wedge \mathrm{sq}(q_4) \wedge s\big(\ell_5^{(1)}\big)\right) \\
&+ \left(\big(q_1 t(p_1)\big) \wedge \big(p_1 t(p_1)\big) \wedge \mathrm{sq}(q_2) \wedge \mathrm{sq}(q_3) \wedge \mathrm{sq}(q_4) \wedge s\big(\ell_5^{(1)}\big)\right) \\
&= \mathrm{sq}(p_1) \wedge \big(q_1 t(p_1) + \mathrm{sq}(p_1)\big) \wedge \mathrm{sq}(q_2) \wedge \mathrm{sq}(q_3) \wedge \mathrm{sq}(q_4) \wedge s\big(\ell_5^{(1)}\big)
\end{aligned} \tag{4.4.11}$$

For simplicity, we let $f_i = q_i t(p_i) + \mathrm{sq}(p_i)$ and $f = f_1 \wedge \cdots \wedge f_4 \wedge s\big(\ell_5^{(1)}\big)$. Then again simplifying using Proposition 4.3 for q_2, q_3, q_4, we get

$$0 = \mathrm{sq}(p_i) \wedge f_1 \wedge \cdots \wedge f_4 \wedge s\big(\ell_5^{(1)}\big) = \mathrm{sq}(p_i) \cdot f \tag{4.4.12}$$

which is true for all $1 \leq i \leq 4$ by symmetry. We also have, for all $1 \leq i \leq 4$,

$$s\big(\ell_5^{(1)}\big) \cdot f = f_i \cdot f = 0 \tag{4.4.13}$$

To make equations (4.4.12) and (4.4.13) useful, we only need that f is nonzero. If we substitute $q_i = \overline{q_i} = p_i$ for $1 \leq i \leq 4$, we have

$$f|_{q_i = \overline{q_i}} = \mathrm{sq}(p_1) \wedge \cdots \wedge \mathrm{sq}(p_4) \wedge s\big(\ell_5^{(1)}\big) = \overline{C} \neq 0,$$

so at least one coefficient of f is invertible in the local ring. So, (4.4.12) and (4.4.13) imply that

$$f\left(f_i \wedge \mathrm{sq}(p_1) \wedge \cdots \wedge \mathrm{sq}(p_4) \wedge s\big(\ell_5^{(1)}\big)\right) = (f \cdot f_i)\left(\mathrm{sq}(p_1) \wedge \cdots \wedge \mathrm{sq}(p_4) \wedge s\big(\ell_5^{(1)}\big)\right) + \cdots = 0,$$

which gives that

$$0 = f_i \wedge \mathrm{sq}(p_1) \wedge \cdots \wedge \mathrm{sq}(p_4) \wedge s\big(\ell_5^{(1)}\big) = (q_i t(p_i) + \mathrm{sq}(p_i)) \cdot \overline{C} = q_i \cdot \overline{k_i}. \tag{4.4.14}$$

Since we also know that $\overline{k}_i \cdot p_i = 0$, we have for each $1 \leq i \leq 5$ that

$$k_i = p_i \times q_i = \overline{k}_i. \tag{4.4.15}$$

We see from equations (4.4.8) and (4.4.14) that each q_i has multiplicity 2 for $1 \leq i \leq 4$. Then by (4.4.3), (4.4.5), and (4.4.10), we may express C and k_i for $1 \leq i \leq 4$ as polynomials in our $q_1, \ldots, q_5$. Finally, $k_5 = \ell_5^{(1)}$, and q_5 is given by equations (4.4.1) and (4.4.4) only and has multiplicity 2. Hence our local ring is generated by independent generators $q_1, ..., q_5$, each with multiplicity 2, which gives a total multiplicity of 32, as desired.

We now consider the case of a smooth dual conic $D^{(\vee)} \in (\mathbb{P}^5)^\vee$. We'll use the same notation for the singular conic and tangent flag, and let $D^{(\vee)}$ be the coordinates of the dual conic. Again, we use $\overline{D^{(\vee)}}, \overline{q_i}, \overline{k_i}$ to denote the actual values at which we will compute the multiplicity.

We still have equations (4.4.1), (4.4.2), and (4.4.3) in this setting, but the analogues of (4.4.4) and (4.4.5) are slightly different. We see that

$$\mathrm{sq}(k_i) \cdot D^{(\vee)} = 0 \qquad \text{(Tangent line lies on dual conic)} \qquad (4.4.16)$$

$$s(q_i) \cdot D^{(\vee)} = 0 \qquad \text{(Tangent point tangent to dual conic)} \qquad (4.4.17)$$

So, our classification into cases still applies, and so we have either (4.4.6) or (4.4.8), whence we may use Proposition 4.3. Since we have a smooth dual conic, and a double line conic over $\mathbb{P}^2$, we can only have the second case for at most 2 of the 5 singular conics, as otherwise the double line would pass through 3 of the singular points.

We aim to show that the multiplicity is 4^m when we have the second case for m singular conics. We again pick a value of m; here we assume case a) for $1 \le i \le 3$, and case b) for $i = 4, 5$. We expect a multiplicity of $4^2 = 16$ here, and, again, the other cases are analogous.

We note that the other variables can be solved by the given equations: (4.4.16) and (4.4.17)

$$D^{(\vee)} = \mathrm{sq}(l_1^{(1)}) \wedge \mathrm{sq}(l_2^{(1)}) \wedge \mathrm{sq}(l_3^{(1)}) \wedge s(q_4) \wedge s(q_5) \qquad (4.4.18)$$

Again, we were unable to finish these calculations. The calculations for $m = 1$ can be done essentially "by hand" by restricting to an affine subset and setting $p_5 = \overline{q_5} = (1, 0, 0)$, and $\overline{k_5} = (0, 1, 0)$. It seems that similar calculations can be done in the $m = 2$ case, but we were unable to complete them.

5 Main Result

Let F be an algebraically closed field of characteristic 2. Let

$$\Lambda_F^{(1)} = \Lambda_F \cap ((\Gamma_F^{(1)})^6 \times \mathcal{F}_F^5)$$

be the scheme Λ on the first component of each complete conic. We will first calculate the length of $\pi_{\Lambda,F}$ at points $x \in \Lambda_F^{(1)}$.

First, we see that the morphism $\pi_{\Lambda,F}^{(1)} = \pi_{\Lambda,F}|_{\Lambda_F^{(1)}}$ factors through the morphism $\sigma_{\Lambda,F} : \Lambda_F^{(1)} \to (\Gamma_F^{(1)})^6$ of projection to the first factor. To calculate the length of $\sigma_{\Lambda,F}$, we write it as the fibered product of simpler morphisms. The following remark is motivated by arguments of Pacini and Testa [PT20, 3.2].

Remark 5.1. Define for each $1 \le i \le 5$ the scheme $\Lambda_{F,i} \subset (\Gamma_F^{(1)})^2 \times \mathcal{F}_F$ by setting the flag to be tangent to both smooth conics. Then the equations for $\Lambda_{F,i}$ are given by Proposition 3.16. We let the morphisms $\sigma_{\Lambda,i,F}$ be the projections onto $(\Gamma_F^{(1)})^2$. Then from Definition 3.17, we have that

$$\Lambda_F^{(1)} = \bigcap_{i=1}^{5} (\Lambda_{F,i} \times (\Gamma_F^{(1)})^4 \times \mathcal{F}_F^4)$$

is the intersection (or fibered product) of the closed subschemes cut out by the tangency conditions for each of the 5 flags. Furthermore, the projection morphism $\sigma_{\Lambda,F}$ is the canonical fibered product morphism from the projection morphisms $\sigma_{\Lambda,i,F}$.

Since length is multiplicative with fibered products and preserved under pullbacks (Proposition 3.4), it suffices to calculate the length of $\sigma_{\Lambda,i,F}$.

Proposition 5.2. The length of $\sigma_{\Lambda,i,F}$ is 2 over a dense open set in $(\Gamma_F^{(1)})^2$.

Proof. We have 2 fixed smooth conics C_1, C_2, and the scheme

$$\mathcal{U} = \left\{ (p, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} : \begin{array}{l} p \in \ell \\ p \in C_i \\ \ell \text{ tangent to } C_i \text{ at } p \end{array} \right\},$$

where we precisely define each of our equations below. We want to calculate the length of each point in this scheme.

Take one such point $(\overline{p}, \overline{\ell})$, and choose coordinates so $\overline{p} = (1, 0, 0)$, $\overline{\ell} = (0, 1, 0)$. In addition, we may work in the affine open subset of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ given by $p_0 = \ell_1 = 1$. We now calculate what must be true about the conics for this $(\overline{p}, \overline{\ell})$ to be in $\mathcal{U}$. The conics C_i have equations

$$a_{00}^{(i)} + a_{11}^{(i)} p_1^2 + a_{22}^{(i)} p_2^2 + a_{01}^{(i)} p_1 + a_{02}^{(i)} p_2 + a_{12}^{(i)} p_1 p_2 = 0. \tag{5.2.1}$$

As $\overline{p} \in C_i$, each must be 0 at $\overline{p}$, so $a_{00}^{(i)} = 0$ for each $i \in \{1, 2\}$. In addition, for $\overline{\ell}$ to be tangent to C_i at $\overline{p}$, the gradient $\left(a_{01}^{(i)}, a_{02}^{(i)}\right)$ must be parallel to $(1, 0)$. This gives that $a_{02}^{(i)} = 0$ for each i. These are the only conditions necessary for $\overline{p} \in C_i$ and $\overline{\ell}$ to be tangent to C_i.

Now, we calculate the scheme $\mathcal{U}$ in (p, ℓ). In addition to (5.2.1), and the condition

$$\ell_0 + p_1 + p_2 \ell_2 = 0, \tag{5.2.2}$$

which says that $p \in \ell$, we have the tangency conditions: all 2×2 minors of

$$\begin{pmatrix} \ell_0 & 1 & \ell_2 \\ a_{01}^{(i)} p_1 & a_{01}^{(i)} + a_{12}^{(i)} p_2 & a_{12}^{(i)} p_1 \end{pmatrix} \tag{5.2.3}$$

must be 0. The last minor of this is

$$a_{12}^{(i)} p_1 + a_{01}^{(i)} \ell_2 + a_{12}^{(i)} p_2 \ell_2 = 0;$$

we may use (5.2.2) to reduce this to

$$a_{01}^{(i)} \ell_2 + a_{12}^{(i)} \ell_0 = 0. \tag{5.2.4}$$

In general,

$$\det \begin{pmatrix} a_{01}^{(1)} & a_{12}^{(1)} \\ a_{01}^{(2)} & a_{12}^{(2)} \end{pmatrix} \neq 0,$$

since the strange points of C_1 and C_2 should be in general distinct (we can verify this using the example of 5 singular conic, in which case forcing the strange point to equal a singular point results in too many linear conditions on the coefficients of the conic), so some linear combination of (5.2.4) gives that $\ell_0 = \ell_2 = 0$, which gives via (5.2.2) that $p_1 = 0$. Finally, we have in general that $a_{22}^{(1)} \neq 0$, so (5.2.1) at $i = 1$ gives that $p_2^2 = 0$. We can check that the ideal $(\ell_0, \ell_2, p_1, p_2^2)$ generates the equations (5.2.1), (5.2.2), and (5.2.3) and defines a scheme of length 2, so we are done. ☐

We are now ready to present the final proof. This proof partially explains why $3264 = 51 \times 64$, and is, assuming the existence of an example, independent of Vainsencher's result [Vai78].

Theorem 5.3. There are generally at most 51 smooth conics tangent to 5 given smooth conics in characteristic 2.

Proof. Throughout this proof, let y denote a tuple of 5 smooth conics over $\mathbb{F}_2$ each inside the first component $\Gamma_{\mathbb{F}_2}^{(1)}$ of complete conics over $\mathbb{F}_2$, and let x be a point of $\Lambda_{\mathbb{F}_2}$ that projects down to y via $\pi_{\Lambda, \mathbb{F}_2}$.

We know from [EH16, 8.9] that $\pi_{\Lambda, \mathbb{Q}}$ generally has total fiber length 3264. It follows from Remark 3.6 that, on some open set $U \subset (\Gamma_{\mathbb{F}_2}^{(1)})^5$, we have

$$\mu_{\pi_{\Lambda, \mathbb{F}_2}}(y) = 3264$$

for all $y \in U$. By Example 4.4, this open set is nonempty, and thus it is dense. Furthermore, from this example, we know that the total fiber length of $\pi_{\Lambda, \mathbb{F}_2}^{(1)}$ is generally at most 1632. Now, by uppersemicontinuity of π-lengths (Remark 3.6) and Example 4.4, we have in general that

$$\lambda_{\pi_{\Lambda, \mathbb{F}_2}^{(1)}}(x) \leq 32.$$

Since $\pi_{\Lambda, \mathbb{F}_2}^{(1)}$ is projective and thus closed, this is true for the fiber of general y. By Remark 5.1, Proposition 5.2 and Proposition 3.5, however, the above length is always at least 32. So, for a general y, we have

i) the total fiber length of $\pi^{(1)}_{\Lambda,\mathbb{F}_2}$ is ≤ 1632, and

ii) the length at each point in the fiber is 32,

so we must have at most 51 geometric points in the fiber $\left(\pi^{(1)}_{\Lambda,\mathbb{F}_2}\right)^{-1}(y)$. By base change, this result holds over all fields. $\square$

Note that in the above proof, we would have shown that the total fiber length of $\pi^{(1)}_{\Lambda,\mathbb{F}_2}$ is generally equal to 1632 if we had an example where the total fiber length inside the component $\Gamma^{(2)}_{\mathbb{F}_2}$ is 1632. This would give the exact count of 51 smooth tangent conics, as opposed to our weaker result.

Using Vainsencher's count [Vai78], we can determine more specifically the structure of the fiber of $\pi_{\Lambda,\mathbb{F}_2}$.

Corollary 5.4. The fiber of $\pi_{\Lambda,\mathbb{F}_2}$ splits into 2 parts of equal total lengths by intersecting with the two components of $\Gamma_{\mathbb{F}_2}$. Its intersection with $\Gamma^{(1)}_{\mathbb{F}_2}$ generally consists of 51 distinct points, each with multiplicity 32, while its intersection with $\Gamma^{(2)}_{\mathbb{F}_2}$ generally consists of 1632 distinct reduced points.

It is an interesting question to explain why the count splits in half exactly, particularly as the two parts of the fiber are non-isomorphic. While our method of proof provides some insight to explain this phenomenon, more work is needed to find a more intuitive explanation.

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Appendix 1

For Proposition 3.12, we use Sagemath's *.irreducible_components()* function to calculate the embedded points of $\Gamma_{\mathbb{F}_2}$. Since the two irreducible components are isomorphic to the blow-up of $\mathbb{P}^5$ along the integral subscheme of double lines, they are also geometrically irreducible. Similarly, the embedded point at the intersection is isomorphic to $\mathbb{P}_2 \times \mathbb{P}_2$ and also geometrically irreducible. Thus we have also computed the embedded points of $\Gamma_{\overline{\mathbb{F}_2}}$.

We use the standard covering of $\mathbb{P}^5 \times \mathbb{P}^{5^\vee}$ by 36 affine open subsets, and calculate the embedded points of $\Gamma_{\mathbb{F}_2}$ on each affine open subset.

```
Space.<a00,a11,a22,a01,a02,a12,b00,b11,b22,b01,b02,b12> = AffineSpace(12, GF(2))
PolyRing = Space.coordinate_ring()

eq_conics = [a01*b12, a01*b02, a12*b01, a12*b02, a02*b01, a02*b12, a01*b01 - a02*b02,
    ↪ a01*b01 - a12*b12,
          a01*b01^2, a01^2*b01, a12*b12^2, a12^2*b12, a02*b02^2, a02^2*b02,
          b00*a01^2-b22*a12^2, b11*a01^2-b22*a02^2, b00*a02^2-b11*a12^2,
          a00*b01^2-a22*b12^2, a11*b01^2-a22*b02^2, a00*b02^2-a11*b12^2]

list1 = [a00,a11,a22,a01,a02,a12]
list2 = [b00,b11,b22,b01,b02,b12]

for var1 in list1:
    for var2 in list2:
        extra_eq = [var1-1, var2-1]
        ideal_conics = PolyRing.ideal(eq_conics + extra_eq)
        subscheme = Space.subscheme(ideal_conics)
        print(subscheme.irreducible_components())
```

For Proposition 3.16, we check over all standard affine coverings that the ideals generated by the equations in Definition 3.15 and Proposition 3.16 are equal by showing that they contain the generators of each other. Note that some affine coverings were excluded using symmetry.

```
import numpy as np
ConRing.<c00,c01,c02,c11,c12,c22,p0,p1,p2,l0,l1,l2> = PolynomialRing(QQ)
conic_eq = c00*p0*p0+c11*p1*p1+c22*p2*p2 + c01*p0*p1 + c12*p1*p2 + c02*p0*p2
grad = conic_eq.gradient()
matrix = np.array([[l0,l1,l2],
          grad[6:9]])
minors = [matrix[:,:2].tolist(), matrix[:,1:].tolist(), matrix[:,[0,2]].tolist()]
dets = [Matrix(minor).determinant() for minor in minors]

dual = [c12^2-4*c11*c22, c02^2-4*c00*c22, c01^2-4*c00*c11, 4*c01*c22-2*c02*c12, 4*c02*
    ↪ c11-2*c01*c12, 4*c12*c00-2*c02*c01] #d00,d11,d22,d01,d02,d12
make_dual = {p0:l0, p1:l1, p2:l2, l0:p0, l1:p1, l2:p2, c00: dual[0], c11:dual[1], c22:
    ↪ dual[2], c01: dual[3], c02:dual[4], c12:dual[5]}
duals_eq = conic_eq.subs(make_dual)
dual_dets = [dets[i].subs(make_dual) for i in range(3)]


for varl in [l0,l1]:
    for varc in [c00,c01,c02,c11,c12,c22]:
        givens = [p0*l0+p1*l1+p2*l2,p0-1,varl - 1,varc-1]
        van_ideal = ConRing.ideal(givens + dets + [conic_eq])
```

```
for u in range(3):
    if dual_dets[u] not in van_ideal:
        print(varl, varc, u)
    if duals_eq not in van_ideal:
        print(varl, varc)
```