DEFORMED AIRY KERNELS IN HEIGHT FUNCTION FLUCTUATIONS OF THE STOCHASTIC SIX VERTEX MODEL

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Abstract. In this paper, we consider the stochastic six vertex model for a special class of initial data, known as generalized step Bernoulli initial data, that were first introduced by Aggarwal and Borodin. In this setting, we study the asymptotic behavior of the height function, whose fluctuations are known to exhibit a phase transition along a critical line. In particular, we exploit the asymptotic equivalence between the height function and a sequence of Schur measures, to show that on the critical line, the height function has fluctuations of order $N^{\frac{1}{3}}$, whose distribution is a generalization of the Tracy-Widom distribution previously found in random matrix and percolation models by Borodin and Peche.

Contents

1. Introduction

The stochastic six vertex model is a probability distribution on the collection of non-intersecting paths on the grid $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, where, in this setting, non-intersecting means that any two paths are not allowed to share an edge, but they are allowed to share a vertex. The paths start from a subset of the points on the vertical axis $\{(1, n) : n \in \mathbb{Z}_{>0}\}$, and they only move upwards and to the right. The long term behavior of the model can vary greatly depending on the initial data of the model, i.e. the subset of points on the vertical axis that have outgoing paths. In this paper, we will be considering a class of initial data that were first introduced by Aggarwal and Borodin in [1], known as generalized step Bernoulli initial data, which are defined with the help of the stochastic higher spin six vertex model. We will see their precise definition in the following section.

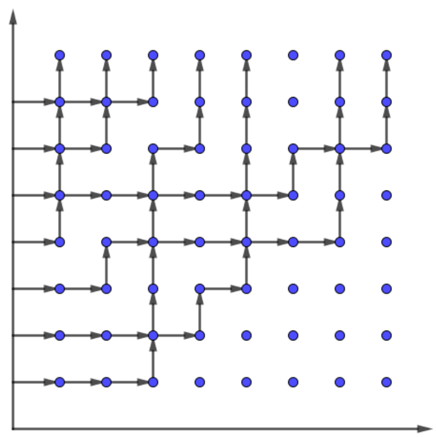

FIGURE 1. An instance of the stochastic six vertex model

It is easy to see that the information of a configuration of the six vertex model can be captured using the height function

$$h : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0},$$

which counts the number of paths that are below and to the right of each vertex $(x, y) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, namely in the half-strip $[x, \infty) \times [0, y]$.

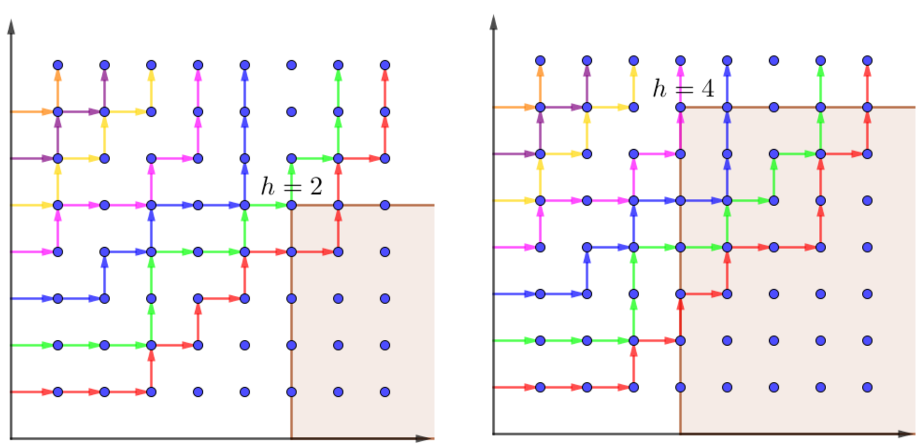

FIGURE 2. The Height Function at the points $(6, 5)$ and $(4, 7)$ respectively. At each point, we count the number of paths that go through the shaded area, including its boundary.

In this paper, we are interested in studying the asymptotic behavior of the height function at a point $h(xN, yN)$, as $N \to \infty$, for any $(x, y) \in (0, 1)^2$. It has been shown by Aggarwal and Borodin in [1], that the fluctuations of the height function of the stochastic six vertex model with generalized step Bernoulli initial data, exhibit a phase transition along a critical line through the origin. Above the line, the fluctuations are Gaussian of order $N^{\frac{1}{2}}$, whereas below the line the fluctuations are Tracy-Widom of order $N^{\frac{1}{3}}$. In the same work, it was also shown that along the critical line, the fluctuations are of order $N^{\frac{1}{3}}$, and follow the Baik-Ben Arous-Peche (BBP) distribution.

It is interesting to note that all distributions that were mentioned above, play a prominent role in Random Matrix Theory. In particular, the Tracy-Widom distribution is the law of the largest eigenvalue of a large Gaussian Hermitian random matrix (GUE), and the BBP distribution is the law of the largest eigenvalue of a large Gaussian Hermitian random matrix that is spiked by a deterministic, finite rank matrix. In the main section of this paper, we will be reproving the fact that the height function along the critical line has BBP fluctuations of order $N^{\frac{1}{3}}$, using a different technique, that we briefly explain in the following paragraph. Moreover, we explain how this technique is expected to be extendable to other types of initial data, that would allow us to observe a wider class of distributions, which generalize BBP, in the stochastic six vertex model. These distributions, which also appear in random matrix theory as the distributions of perturbations of random matrices, were first introduced by Borodin and Peche in **??**, and they are parametrized by pairs of non-negative specializations.

Our proof will follow a technique that was first outlined by Borodin in [2], and was also outlined in [1] for a special case of the BBP distribution. In this technique, a matching is established between the height function of the stochastic higher spin six vertex model at a sequence of points, and a sequence of Schur measures on partitions. This matching allows us to exploit the determinantal nature of Schur measures, to reduce the problem of asymptotically analyzing the height function, to the asymptotic analysis of a double contour integral.

Acknowledgements. This paper is a result of the SPUR program of the MIT Mathematics Department under the guidance of mentor Roger Van Peski, and Professor Alexei Borodin. At this point, I would like to thank Roger for all the guiding discussions and advice and for helping me build the necessary background and intuition to be able to understand and make progress on this problem. I would also like to thank Professor Alexei Borodin for his insightful answers to my questions, and for suggesting this interesting problem that helped me get a better understanding of Integrable Probability. Finally, I am grateful to Professor David Jerison and Professor Ankur Moitra for their guidance, as well as Dr. Slava Gerovitch and the MIT Mathematics Department for organizing the SPUR program.

2. Preliminaries

In this section, we will formally define our model and the class of initial data that we will be considering, and give precise definitions for the various distributions that appear in the asymptotic analysis of the height function. Moreover, we will give a brief overview of Macdonald and Schur measures, and state the relevant results from [2] that will allow us to establish the matching that we described in the Introduction.

2.1. **Stochastic Six Vertex Model.** Let $\mathcal{C}$ be the collection of all non-intersecting paths in the quadrant, as described in the Introduction. Our goal is to define a probability measure $\mathcal{P}$ on $\mathcal{C}$. We can do so by constructing $\mathcal{P}$ as an infinite volume limit of probability measures $\mathcal{P}_n$, on the collection $\mathcal{C}_n$ of non-intersecting paths on the part of the quadrant that lies between the axes and the line $x + y = n$.

Given a configuration $\mathcal{C}_n$, we obtain the distribution of $\mathcal{C}_{n+1}$ in a Markovian way. For a point (x, y) such that $x + y = n$, let $i_v, i'_v \in \{0, 1\}$ be the number of vertical incoming and outgoing paths respectively, and $i_h, i'_h \in \{0, 1\}$ the number of horizontal incoming and outgoing paths respectively. Due to the conservation of incoming and outgoing paths on each vertex, there are six possibilities (hence the name of the model) for i_h, i'_h, i_v, i'_v. They are given by the following probabilities:

$$\begin{aligned}
&\mathbb{P}\big[(i'_v, i'_h) = (0,0)\big|(i_v, i_h) = (0,0)\big] = 1\\
&\mathbb{P}\big[(i'_v, i'_h) = (1,0)\big|(i_v, i_h) = (1,0)\big] = 1\\
&\mathbb{P}\big[(i'_v, i'_h) = (0,0)\big|(i_v, i_h) = (0,0)\big] = \delta_1\\
&\mathbb{P}\big[(i'_v, i'_h) = (0,1)\big|(i_v, i_h) = (1,0)\big] = 1 - \delta_1\\
&\mathbb{P}\big[(i'_v, i'_h) = (0,1)\big|(i_v, i_h) = (0,1)\big] = \delta_2\\
&\mathbb{P}\big[(i'_v, i'_h) = (1,0)\big|(i_v, i_h) = (0,1)\big] = 1 - \delta_2
\end{aligned} \tag{1}$$

Before we make any further definitions, we would like to note that it is often convenient to think of the horizontal axis of the quadrant as representing time, and each path of the stochastic six vertex model as representing the position of a particle that only moves up along the vertical axis. In what follows, we will liberally adopt this perspective when convenient, and will refer to the stochastic six vertex model "run for time T".

2.2. **Stochastic Higher Spin Six Vertex Model.** This model generalizes the stochastic six vertex model, by allowing paths to also share vertical edges apart from vertices. It is defined very similarly to the original model, but with modified transition probabilities. Again, let $\mathcal{C}_n$, (x, y), i_v, i'_v, i_h, i'_h as above. This time, i_v, i'_v need not be in $\{0, 1\}$.

Note that a necessary condition for i_v, i'_v, i_h, i'_h is

$$i_h + i_v = i'_h + i'_v.$$

Now, given that the number of vertical and horizontal incoming paths is i_v, i_h, the probabilities of the various possibilities for the outgoing paths are given by:

$$
\begin{aligned}
\mathbb{P}\big[(i_v', i_h') = (i_v, 0)\big|(i_v, i_h) = (i_v, 0)\big] &= \frac{1 - q^{i_v} s_x \xi_x u_y}{1 - s_x \xi_x u_y} \\
\mathbb{P}\big[(i_v', i_h') = (i_v - 1, 1)\big|(i_v, i_h) = (i_v, 0)\big] &= \frac{(q^{i_v} - 1) s_x \xi_x u_y}{1 - s_x \xi_x u_y} \\
\mathbb{P}\big[(i_v', i_h') = (i_v, 1)\big|(i_v, i_h) = (i_v, 1)\big] &= \frac{q^{i_v} s_x^2 - s_x \xi_x u_y}{1 - s_x \xi_x u_y} \\
\mathbb{P}\big[(i_v', i_h') = (i_v + 1, 0)\big|(i_v, i_h) = (i_v, 1)\big] &= \frac{1 - q^{i_v} s_x^2}{1 - s_x \xi_x u_y}
\end{aligned}
\tag{2}
$$

where q, s_x, u_y, ξ_x are the parameters of the model, and are chosen in a way so that the above probabilities are nonnegative. This immediately implies that they are in $[0, 1]$, since the sum of the first two probabilities is 1, and so is the sum of the last two.

There are a number of ways to ensure that the nonnegativity condition is satisfied, but we will focus on a particular case. Namely, the case when there exists a positive integer m such that:

- $q \in (0, 1)$,
- $s_x = q^{-\frac{m}{2}}$ for all $x \geq 1$,
- $0 < \xi_x, u_y$ for all $x, y \geq 1$,
- $\xi_x u_y < q^{-\frac{m}{2}}$ for all $x, y \geq 1$.

In this setting, $\frac{m}{2}$ is called the ***spin*** of the vertex model. Note that in this case, if a vertex (x, y) has m incoming vertical paths, and 1 horizontal path, the probability of then having $m + 1$ outgoing vertical paths is 0, based on the above. Thus, in the spin $\frac{m}{2}$ model, at most m paths can share a vertical edge.

The **stochastic six vertex model** (non higher spin), is a special case of the spin $\frac{1}{2}$ case, in which $\xi_x = 1$, and $u_x = u$ is constant for all x. The results that we will state in the following section, will ultimately be for this model. However, we need the definition of the higher spin six vertex model in order to define the class of boundary data that we will be considering and establish the matching between the height function and the Schur measures that we mentioned above.

2.3. **Boundary Data.** Now that we have seen how to obtain the distribution of $\mathcal{C}_{n+1}$ given that of $\mathcal{C}_n$, it remains to discuss the initial data. As we mentioned in the Introduction, all the paths will be starting from the vertical axis. In the simplest case, a path starts from every point on the vertical axis with probability 1. This is what is known as ***step initial data***.

However [1] introduces a more general class of initial data, known as ***generalized step Bernoulli initial data***. For some positive integer m and $b_1, \ldots, b_m \in [0, 1]$, these are obtained by running the higher spin six vertex model with modified transition probabilities and step initial data, on the vertical strip $[1, m+1] \times [1, \infty)$. The modified transition probabilities at a point (x, y) depend on b_x, and are given by:

$$(3)\qquad \begin{aligned} \mathbb{P}\big[(i'_v,i'_h)=(i_v,0)\big|(i_v,i_h)=(i_v,0)\big] &= 1-(1-q^{i_v})b_x \\ \mathbb{P}\big[(i'_v,i'_h)=(i_v-1,1)\big|(i_v,i_h)=(i_v,0)\big] &= (1-q^{i_v})b_x \\ \mathbb{P}\big[(i'_v,i'_h)=(i_v,1)\big|(i_v,i_h)=(i_v,1)\big] &= b_x \\ \mathbb{P}\big[(i'_v,i'_h)=(i_v+1,0)\big|(i_v,i_h)=(i_v,1)\big] &= 1-b_x \end{aligned}$$

After we run the higher spin model on the strip, we use the horizontal paths that are outgoing from the vertical line $x = m$, as our initial data. This can be thought of as shifting the $y-$axis by m and then starting a path ensemble on the new axes (Figure 3). Since in the higher spin six vertex model, paths are not allowed to share horizontal edges, every vertex on the line $x = m$ will have at most 1 outgoing path, so these are valid initial data.

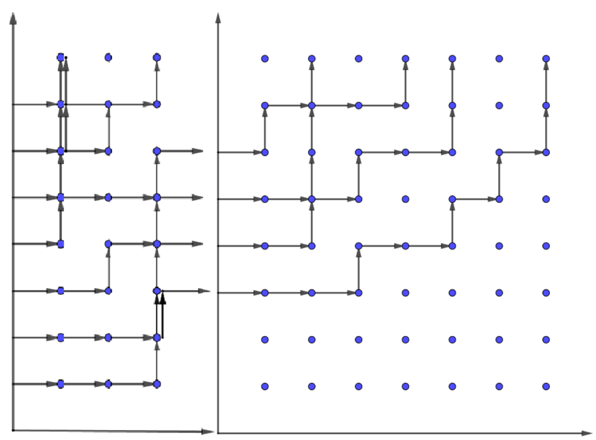

FIGURE 3. Obtaining generalized step Bernoulli initial data, by running the stochastic higher spin six vertex model for $m = 3$ columns, and using the outcome as initial data for the stochastic six vertex model.

Remark 2.1. It is important to note that we can view the stochastic six vertex model with generalized step Bernoulli i.d., as an instance of the stochastic higher spin six vertex model, where the first m columns evolve using equation 3, and the remaining evolve using equation 1. This is explained in paragraph 4.1.1 of [1], and we will give a brief overview here.

Suppose that we are running the stochastic six vertex model with transition probabilities given by 1, and generalized step $(b_1, \ldots, b_m)-$Bernoulli initial data, as in 3. We want to apropriately set the parameters of 2. We can do so by setting:

$$q = \frac{\delta_1}{\delta_2}, \qquad \kappa = \frac{1-\delta_1}{1-\delta_2}, \qquad s = q^{-\frac{1}{2}}, \qquad u_j = u = \kappa s, \qquad \beta_j = \frac{b_j}{1-b_j},$$

and then assigning the remaining parameters of the stochastic higher spin six vertex model as follows:

- For $j \in [m]$, $s_j \to 0$, and $\xi_j = -\frac{\beta_j}{s_j u}$.
- For $j > m$, $s_j = s$, and $\xi_j = 1$.

By substituting these into 2, it is easy to see that we obtain 1 and 3 in the desired regions of the quadrant.

The significance of being able to view the stochastic six vertex model with this class of initial data as a special case of the Higher Spin version, is that this will allow us to exploit the matching that exists between the latter and Schur measures.

2.4. **Macdonald and Schur Measures.** As mentioned in the introduction, the asymptotic analysis of the height function is done via a comparison to Schur measures, which are a special case of Macdonald measures. For the sake of brevity, we will not expand on this topic here, beyond introducing the required notation. For a detailed overview, the reader is referred to [3], or Section 3 of [2]. Here, we will be following the notation of the latter.

Let $\mathbb{Y}$ denote the set of all integer partitions, and Sym the symmetric functions on $X = (x_1, x_2, \dots)$, with coefficients in $\mathbb{C}(q,t)$ for some $0 \leq q, t < 1$. Let $P_\lambda(X; q, t)$ for $\lambda \in \mathbb{Y}$ denote the Macdonald symmetric functions parametrized by partitions, which form a basis for Sym. The restriction of a Macdonald symmetric function to finitely many variables $x_1, \dots, x_n$, gives rise to the Macdonald polynomials, which will also be denoted by P_λ. In the case when $q = t$, P_λ become the Schur symmetric polynomials s_λ, which are independent of q, t.

Definition 2.2. A *specialization* ρ, is an algebra homomorphism Sym $\to \mathbb{C}$. Furthermore, ρ is said to be *Macdonald nonnegative*, if the image under ρ of every skew Macdonald symmetric function is nonnegative.

A simple example of a Macdonald nonnegative specialization, is the substitution $x_i = \alpha_i \in [0, \infty)$, where finitely many of the α_i are non-zero. Although, such examples do not fully classify Macdonald nonnegative specializations, there is a simple classification of them, which we describe below.

Let $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ and $\beta = \{\beta_i\}_{i=1}^{\infty}$ be two sequences of nonnegative numbers such that $\sum^{\infty}(\alpha_i + \beta_i) < \infty$, and let γ be a nonnegative number. Given these, we can define a nonnegative specialization ρ as follows.

Let $g_n,\ n \geq 0$ be an algebraically independent system of generators of Sym (these are explicitly defined in e.g. Section 3 of [2], but their exact form is not important to us). Then, the following expression uniquely defines a Macdonald nonnegative specialization, and in fact all such specializations can be written in this form.

$$\sum_{n=0}^{\infty} g_n(\rho) u^n = \exp(u) \prod_{i=1}^{\infty} \frac{(t\alpha_i u; q)_\infty}{(\alpha_i u; q)_\infty} (1 + \beta_i u),$$

where $(a; q)_\infty$ denotes the $q-$*Pochhammer* symbol that is defined as

$$(a; q)_\infty = \prod_{k=1}^{\infty} (1 - aq^k)$$

. Given this classification, we will use the shorthand notation $\rho = (\alpha, \beta, \gamma)$ to refer to nonnegative specializations.

Now, we are ready to define the Macdonald and Schur measures.

Definition 2.3. Let $\rho^1 = (\{\alpha_i^{(1)}\}_{i\geq 1}, \{\beta_i^{(1)}\}_{i\geq 1}, \gamma^{(1)})$ and $\rho^2 = (\{\alpha_i^{(2)}\}_{i\geq 1}, \{\beta_i^{(2)}\}_{i\geq 1}, \gamma^{(2)})$ be nonnegative specializations such that $\alpha_i^{(1)}\alpha_j^{(2)} < 1$ for all i, j. We definte the *Macdonald Measure* $\mathbf{MM}(\rho^1, \rho^2)$, to be the probability measure on the set $\mathbb{Y}$ of all partitions, such that the weight of a partition λ is proportional to the value of the Macdonald function P_λ specialized at ρ_1 and ρ_2.

Remark 2.4. The condition on the products of the α variables is necessary and sufficient for the total measure over all partitions to be finite.

Remark 2.5. The Schur measure arises from the Macdonald measure in the case when $q = t$.

2.5. **Matching and Asymptotic Equivalence.** The following Definition is necessary for us to state Corollary 5.11 of [2]. It gives us the precise matching between the height function of the stochastic higher spin six vertex model evaluated at a point and a Macdonald measure.

Definition 2.6 ([2], Def. 4.1)**.** Consider an instance of the stochastic higher spin six vertex model with parameters Q, s_x, ξ_x, u_y as in Equation 2, and a Macdonald measure with parameters (q, t) and specializations $\rho_1 = (X, 0, 0)$, $\rho_2 = (\alpha, \beta, 0)$, where $X = (x_1, \ldots, x_n)$, $\alpha = \{\alpha_i\}_{i\geq 1}$ and $\beta = \{\beta_i\}_{i\geq 1}$. Look at a point (M, N) of the six vertex model. We say that the Macdonald measure *matches* the six vertex model at the given point if the following hold:

- $Q = t$, $N = n$, $\{u_1, \ldots, u_N\} = \{x_1, \ldots, x_N\}$.
- The α-variables can be partitioned into clusters of geometric progressions $C_{k_j}(\tilde{\alpha}_j)$ with starting term $\tilde{\alpha}_j$, k_j total terms, and ratio t.
- The β-variables can be partitioned into clusters of geometric progressions $C_{l_j}(\tilde{\beta}_j)$ with starting term $\tilde{\beta}_j$, l_j total terms, and ratio q.
- There is a bijection between the s_x, $1 \leq x \leq M-1$ and the α and β clusters, with αs corresponding to positive s_x and βs to negative.
- In particular, if $C_k(\tilde{\alpha})$ corresponds to s_x, then $s_x = t^{-k/2}$ and $\xi_x = \frac{s_x}{\tilde{\alpha}}$.
- Finally, if $C_l(\tilde{\beta})$ corresponds to s_x, then $s_x = -q^{l/2}$ and $\xi_x = -\frac{1}{s_x\tilde{\beta}}$.

Now, we are ready to state, without proof, the result about asymptotic equivalence.

Proposition 2.7 ([2], Cor. 5.11)**.** *Suppose that we are given a sequence of points (X_T, Y_T) of the stochastic higher spin six vertex model, as well as a sequence of Macdonald measures, both intexed by T. Suppose moreover that for large enough T, the Macdonald measure matches the six vertex model at the corresponding point. Then, the random variables $\mathfrak{H}(X_T, Y_T)$ and $n - \ell(\lambda)$ are asymptotically equivalent, where $\ell(\lambda)$ is the length of a random partition sampled using the corresponding Macdonald measure, and n is as in definition 2.6.*

We will not give a precise definition of asymptotic equivalence here, for that the reader is referred to Section 5 of [2]. For our purposes, the asymptotic equivalence of two random variables will allow us to reduce the asymptotic analysis of one random variable to that of the other - we will not go into further technical details.

2.6. **Kernels and Distributions.** We will now define the various distributions that are relevant to our discussion, namely the Tracy-Widom distribution, the Baik-Ben Arous-Peche (BBP) family of distributions that were first intorduced in [4], as well as a family of distributions that generalizes both and was first introduced by Borodin and Peche in [5]. We will first define correlation kernels for each family of distributions, and then make use of Fredholm determinants to define the corresponding CDFs. A brief overview of Fredholm determinants can be found in Appendix A of [1]. Here, we just note that formally, Fredholm determinants are defined as follows.

Definition 2.8. Given $K : \mathbb{R}^2 \to \mathbb{C}$ and $s \in \mathbb{R}$, we define the *Fredholm determinant*

$$\det(\mathrm{Id} + K)_{L^2(s,\infty)} := 1 + \sum_{k=1}^{\infty} \int_s^{\infty} \cdots \int_s^{\infty} \det[K(x_i, x_j)]_{i,j=1}^k \prod_{\ell=1}^k dx_\ell$$

Definition 2.9. The *Airy kernel*, K_{Ai}, is defined by the double contour integral

$$K_{Ai}(x,y) = \frac{1}{4\pi^2} \oint \oint \exp\big(\frac{w^3}{3} - \frac{v^3}{3} - xv + yw\big) \frac{dwdv}{w-v},$$

where the contours for w and v are shown in Figure 2.6 below.

Definition 2.10. The *Baik-Ben Arous-Peche (BBP) kernel* with parameter $c \in \mathbb{R}^m$, $K_{\mathrm{BBP};c}$, is defined by the double contour integral

$$K_{\mathrm{BBP};c}(x,y) = \frac{1}{4\pi^2} \oint \oint \exp\big(\frac{w^3}{3} - \frac{v^3}{3} - xv + yw\big) \prod_{j=1}^m \frac{v + c_j}{w + c_j} \frac{dwdv}{w-v},$$

where the contours for w and v are shown in Figure 2.6 below.

The following family of kernels are deformations of the Airy kernel, and were first introduced by Borodin and Peche in [5].

Definition 2.11. Let $\alpha = \{\alpha_i^\pm\}_{i=1}^\infty$, $\beta = \{\beta_i^\pm\}_{i=1}^\infty$ be four sequences of nonnegative real numbers such that $\sum \alpha_i^\pm + \beta_i^\pm \leq \infty$, and $\gamma = \{\gamma^\pm\}$ two nonnegative numbers. The *Deformed Airy kernel* or *Borodin-Peche kernel* with parameters α, β, γ, $K_{\mathrm{BP};\alpha,\beta,\gamma}$, is defined by the double contour integral

$$K_{\mathrm{BP};\alpha,\beta,\gamma}(x,y) = \frac{1}{4\pi^2} \oint \oint \exp\big(\frac{w^3}{3} - \frac{v^3}{3} - xv + yw\big) \exp(\gamma^+(w-v) + \gamma^-(w^{-1} - v^{-1})) \cdot \prod_{i=1}^m \left[\frac{(1 - \alpha_i^+ v)(1 - \alpha_i^- v^{-1})}{(1 - \alpha_i^+ w)(1 - \alpha_i^- w^{-1})} \cdot \frac{(1 + \beta_i^+ w)(1 + \beta_i^- w^{-1})}{(1 + \beta_i^+ v)(1 + \beta_i^- v^{-1})} \right] \frac{dwdv}{w-v}$$

Remark 2.12. Note that a natural way to think about $(\alpha^+, \beta^+, \gamma^+)$, and $(\alpha^-, \beta^-, \gamma^-)$ in the above definition, is as a pair of Macdonald nonnegative specializations ρ^+ and ρ^-. In other words, the Borodin-Peche kernels defined above, are parametrized by a pair of nonnegative specializations, just like Macdonald measures are.

The above three definitions are successive generalizations of the previous ones. In particular, the Airy kernel is obtained from the BBP family of kernels by setting $m = 0$, and the BBP kernel is obtained from the BP family of kernels (at least for the case when the c_i are negative) by setting $\alpha_i^- = \beta_i^\pm = \gamma_i^\pm = 0$, and then setting $\alpha_i^+ = -\frac{1}{c_i}$, for $i \leq m$, and $\alpha_i^+ = 0$ otherwise.

We are now ready to define the distributions related to these kernels.

Definition 2.13. The Tracy-Widom distribution, F_{TW}, is defined by

$$F_{\mathrm{TW}}(s) = \det(\mathrm{Id} - K_{Ai})_{L^2(s,\infty)},$$

for $s \in \mathbb{R}$.

In a similar fashion, we define the BBP family of distributions $F_{\mathrm{BBP};c}(s)$, and the BP family of distributions $F_{\mathrm{BP};\alpha,\beta,\gamma}(s)$.

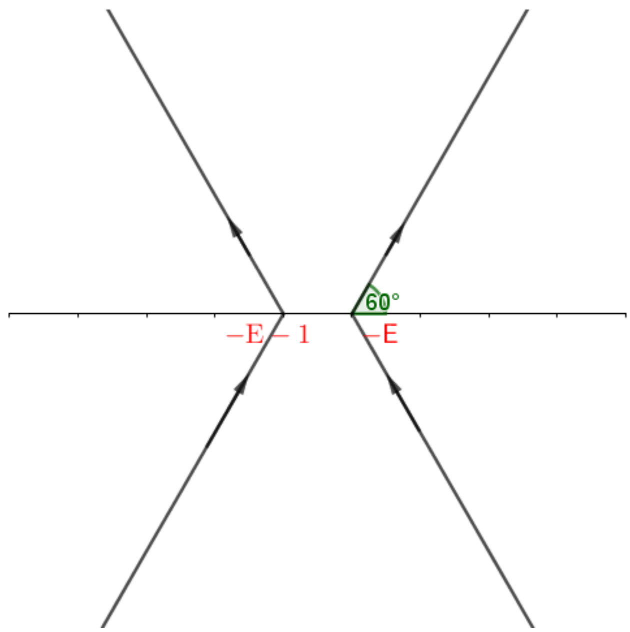

FIGURE 4. The contours for the Airy and BBP kernels. The $v-$contour is on the left, and the $w-$contour on the right. For the TW kernel, $E = 0$, whereas for the general BBP kernel, $E > \max_{i\in[m]} c_i$.

3. Theorem Statement and Proof Via Schur Measures

In this section, we restate a Theorem that was first proven by Aggarwal and Borodin in [1], and then present an alternative proof, which was also outlined in the same paper.

Theorem 3.1 ([1] Theorem 1.6, Part 2)**.** *Fix two positive real numbers $\delta_1 < \delta_2 < 1$, a real number $b \in (0,1)$, an integer $m \geq 1$, and infinite families of real numbers*

$$\{b_{1,T}\}_{T\in\mathbb{N}}, \{b_{2,T}\}_{T\in\mathbb{N}}, \dots, \{b_{m,T}\}_{T\in\mathbb{N}} \subseteq (0,1).$$

Assume moreover that there exist real numbers $d_1, \dots, d_m$ such that $\lim_{T\to\infty} T^{\frac{1}{3}}(b_{i,T} - b) = d_i$, for each $i \in [1,m]$.
Let $T > 0$ be a positive integer. Consider the stochastic six vertex model, run for time

T, with δ_1, δ_2 as in Equation (1), and generalized $(b_{1,T}, \dots, b_{m,T})-$Bernoulli initial data. Set

$$\chi = b(1-b), \quad \kappa = \frac{1-\delta_1}{1-\delta_2}, \quad \Lambda = b + \kappa(1-b), \quad \theta = k^{-1}\Lambda^2, \quad f = \left(\frac{1-\theta^2}{4}\right)^{\frac{2}{3}}.$$

Assume that $\{x_T\}_{T\in\mathbb{N}}, \{y_T\}_{T\in\mathbb{N}}$ are sequences of positive real numbers such that $\eta_T = \frac{x_T}{y_T}$ satisfies $\lim_{T\to\infty} T^{\frac{1}{3}}(\eta_T - \theta) = d$, for some real number d. Set

$$\mathcal{H}(x,y) = \frac{(\sqrt{(1-\delta_1)y} - \sqrt{(1-\delta_2)x})^2}{\delta_2 - \delta_1},$$

$$\mathcal{F}(x,y) = \frac{\kappa^{\frac{1}{6}}(\sqrt{\kappa x} - \sqrt{y})^{\frac{2}{3}}(\sqrt{\kappa y} - \sqrt{x})^{\frac{2}{3}}}{(\kappa - 1)(xy)^{\frac{1}{6}}}.$$

Define $c = (c_1, \dots, c_m)$, where for each index $j \in [m]$,

$$c_j = -\frac{f}{\chi} \cdot \left(d_j + \frac{\kappa d}{2(\kappa-1)\Lambda}\right)$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{\mathcal{H}(x_T, y_T)T - \mathfrak{H}(x_T T, y_T T)}{\mathcal{F}(x_T, y_T)T^{\frac{1}{3}}} \le s\right] = F_{BBP;\, c}(s).$$

The proof below follows a similar structure to what is outlined in Appendix B of [1] for the special case when $c = 0^m$.

proof sketch. Using the notation of the Theorem statement, our goal is to show that:

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{\mathcal{H}(x_T, y_T)T - \mathfrak{H}(x_T T, y_T T)}{\mathcal{F}(x_T, y_T)T^{\frac{1}{3}}} \le s\right] = F_{\mathrm{BBP};\, c}(s)$$

For some fixed integer $T > 0$, let $X_T = x_T T$ and $Y_T = y_T T$. Let $\mathcal{A} := (s, \dots, s)$ be the tuple consisting of $X_T - 1$ copies of $s = q^{\frac{-1}{2}}$, where $q = \frac{\delta_1}{\delta_2}$, and let $\mathcal{B} := \bigcup_{j=1}^{m} \{u\beta_j^{-1}, qu\beta_j^{-1}, q^2 u\beta_j^{-1}, \dots\}$, where $\beta_j = \frac{b_j}{1-b_j}$. Finally, let $Y = (u^{-1}, \dots, u^{-1})$ be the tuple consisting of Y_T copies of u^{-1}, where $u = \kappa s$. Then, by Proposition 2.7, we have that the quantity $T - \ell(\lambda)$, where λ is a random partition sampled using the Schur measure **SM** defined by specializations $\rho := \rho_{\mathcal{A},\mathcal{B}}$ and Y, is asymptotically equivalent with $\mathfrak{H}(X_T, Y_T)$. Given this equivalence, our problem is reduced to showing that:

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{(\mathcal{H}(x_T, y_T) - 1)T + \ell(\lambda)}{\mathcal{F}(x_T, y_T)T^{\frac{1}{3}}} \le s\right] = F_{BBP;c}(s),$$

where λ at time T is sampled according to the Schur measure described above.

The benefit of this reduction is that asymptotic analysis of Schur measures is a well studied topic. In particular, it has been shown that for a random partition λ with respect to a Schur measure, $X(\lambda) := \{\lambda_i - i\}_{i\in\mathbb{Z}_{>0}}$ forms a determinantal point process, whose correlation kernel $K(i,j)$ can be expressed as a double contour integral (see, for example, the section on Macdonald processes in [3]). In our specific case, this integral is given by:

$$K(i,j) = \frac{1}{4\pi^2} \oint\oint \left(\frac{1-s^{-1}\kappa^{-1}v^{-1}}{1-s^{-1}\kappa^{-1}w^{-1}}\right)^{Y_T} \left(\frac{1-sw}{1-sv}\right)^{X_T-1} \frac{w^j}{v^{i+1}} \prod_{k=1}^{m} \frac{(-s\kappa\beta_k^{-1}v;q)_\infty}{(-s\kappa\beta_k^{-1}w;q)_\infty} \frac{dvdw}{v-w}$$

Here, the contours of v, w are positively oriented closed loops such that the contour of w is contained in that of v. Both contours contain 0 and $s^{-1}\kappa$, but leave outside s^{-1} and $-q^{-j}s^{-1}\kappa^{-1}\beta_k$ for each k, j. Note that for large enough T, the β_j are close enough to each other, since they all converge to $\beta = \frac{b}{1-b}$. Changing variables to $\tilde{v} = -qs\kappa v$ and $\tilde{w} = -qs\kappa w$, we get that:

$$K(i,j) = \frac{(qs\kappa)^{i-j}}{4\pi^2} \oint\oint \left(\frac{1+q\tilde{v}^{-1}}{1+q\tilde{w}^{-1}}\right)^{Y_T} \left(\frac{q+\kappa^{-1}\tilde{w}}{q+\kappa^{-1}\tilde{v}}\right)^{X_T-1} \frac{\tilde{w}^j}{\tilde{v}^{i+1}} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}\tilde{v};q)_\infty}{(q^{-1}\beta_k^{-1}\tilde{w};q)_\infty} \frac{d\tilde{v}d\tilde{w}}{\tilde{v}-\tilde{w}}$$

Now, note that for a partition λ, $-\ell(\lambda)$ can be easily seen to be the minimum of the set $Y(\lambda) := \mathbb{Z} \setminus X(\lambda)$. By a complementation argument which can be found in the Appendix of [6], known as Kerov's Complementation Principle, it is then shown that Y is also a determinantal point process, whose correlation kernel is given by

$$\tilde{K}(i,j) = \mathbf{1}_{i=j} - K(i,j)$$

and can also be expressed as a contour integral (see [1] Appendix B for more details). Denoting $\mathcal{H} = \mathcal{H}(x_T, y_T)$, we get the following:

$$\begin{aligned}
&\tilde{K}(i+(\mathcal{H}-1)T, j+(\mathcal{H}-1)T) \\
=&\frac{(qs\kappa)^{i-j}}{4\pi^2} \oint\oint \left(\frac{1+q\tilde{v}^{-1}}{1+q\tilde{w}^{-1}}\right)^{y_T T} \left(\frac{q+\kappa^{-1}\tilde{w}}{q+\kappa^{-1}\tilde{v}}\right)^{x_T T-1} \frac{\tilde{w}^{(\mathcal{H}-1)T}}{\tilde{v}^{(\mathcal{H}-1)T}} \frac{\tilde{w}^j}{\tilde{v}^{i+1}} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}\tilde{v};q)_\infty}{(q^{-1}\beta_k^{-1}\tilde{w};q)_\infty} \frac{d\tilde{v}d\tilde{w}}{\tilde{v}-\tilde{w}} \\
=&\frac{(qs\kappa)^{i-j}}{4\pi^2} \oint\oint \left[\left(\frac{1+q\tilde{v}^{-1}}{1+q\tilde{w}^{-1}}\right)^{y_T} \left(\frac{q+\kappa^{-1}\tilde{w}}{q+\kappa^{-1}\tilde{v}}\right)^{x_T-\frac{1}{T}} \frac{\tilde{w}^{\mathcal{H}-1}}{\tilde{v}^{\mathcal{H}-1}}\right]^T \frac{\tilde{w}^j}{\tilde{v}^{i+1}} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}\tilde{v};q)_\infty}{(q^{-1}\beta_k^{-1}\tilde{w};q)_\infty} \frac{d\tilde{v}d\tilde{w}}{\tilde{v}-\tilde{w}} \\
=&\frac{(qs\kappa)^{i-j}}{4\pi^2} \oint\oint \exp\Big(T\big(G_T(\tilde{w}) - G_T(\tilde{v})\big)\Big) \frac{\tilde{w}^j}{\tilde{v}^{i+1}} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}\tilde{v};q)_\infty}{(q^{-1}\beta_k^{-1}\tilde{w};q)_\infty} \frac{d\tilde{v}d\tilde{w}}{\tilde{v}-\tilde{w}},
\end{aligned}$$

where $G_T(z) = y_T\left(\frac{x_t-1}{y_T}\log(q+\kappa^{-1}z) - \log(z+q) + (1+\frac{\mathcal{H}-1}{y_T})\log(z)\right)$. Note that for large T, the function $G(z) = \theta\log(q+\kappa^{-1}z) - \log(z+q) + \mathcal{H}\log(z)$ only differ by $O(1/T^{\frac{1}{3}})$, as $\frac{x_t-1}{y_T} \to \theta$ with speed $T^{\frac{1}{3}}$. Thus, asymptotically, G_T and G exhibit similar behavior, which has been analyzed in Section 6 of [1]. Here, the contour of $\tilde{v}$

is contained in that of $\tilde{w}$, both contours contain 0 and $-q$, and both leave out $q\kappa$ and $q^{1-j}\beta_k$ for all k, j. Now, for large enough T, we localize around $q\beta$, and set:

$$\sigma = \frac{1}{\mathcal{F}T^{\frac{1}{3}}}, \qquad \tilde{v} = q\beta(1+\sigma\hat{v}), \qquad \tilde{w} = q\beta(1+\sigma\hat{w}).$$

For two new parameters r, r', we now compute:

$$\begin{aligned}
&\tilde{K}((\mathcal{H}-1)T - \frac{r}{\sigma}, (\mathcal{H}-1)T - \frac{r'}{\sigma}) \\
=&\frac{(qs\kappa)^{-\frac{r-r'}{\sigma}}}{4\pi^2} \oint\oint \exp\Big(T\big(G_T(\tilde{w}) - G_T(\tilde{v})\big)\Big) \frac{\tilde{w}^{-\frac{r'}{\sigma}}}{\tilde{v}^{-\frac{r}{\sigma}+1}} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}\tilde{v};q)_\infty}{(q^{-1}\beta_k^{-1}\tilde{w};q)_\infty} \frac{d\tilde{v}d\tilde{w}}{\tilde{v}-\tilde{w}} \\
=&\frac{(qs\kappa)^{\frac{r'-r}{\sigma}}}{4\pi^2} \oint\oint \exp\Big(T\big(G_T(q\beta(1+\sigma\hat{w})) - G_T(q\beta(1+\sigma\hat{v}))\big)\Big) \frac{(q\beta(1+\sigma\hat{w}))^{-\frac{r'}{\sigma}}}{(q\beta(1+\sigma\hat{v}))^{-\frac{r}{\sigma}+1}} \cdot \\
&\prod_{k=1}^{m} \frac{(\beta\beta_k^{-1}(1+\sigma\hat{v});q)_\infty}{(\beta\beta_k^{-1}(1+\sigma\hat{w});q)_\infty} \frac{(q\beta\sigma)^2 d\hat{v}d\hat{w}}{(q\beta\sigma)(\hat{v}-\hat{w})} \\
=&\frac{\sigma(s\kappa\beta^{-1})^{\frac{r'-r}{\sigma}}}{4\pi^2} \oint\oint \exp\Big(T\big(G_T(q\beta(1+\sigma\hat{w})) - G_T(q\beta(1+\sigma\hat{v}))\big)\Big) \frac{(1+\sigma\hat{w})^{-\frac{r'}{\sigma}}}{(1+\sigma\hat{v})^{-\frac{r}{\sigma}+1}} \cdot \\
&\prod_{k=1}^{m} \frac{(\beta\beta_k^{-1}(1+\sigma\hat{v});q)_\infty}{(\beta\beta_k^{-1}(1+\sigma\hat{w});q)_\infty} \frac{d\hat{v}d\hat{w}}{\hat{v}-\hat{w}}
\end{aligned}$$

Let $\bar{K}_T(r,r') := \big(\sigma(s\kappa\beta^{-1})^{\frac{r'-r}{\sigma}}\big)^{-1} \tilde{K}((\mathcal{H}-1)T - \frac{r}{\sigma}, (\mathcal{H}-1)T - \frac{r'}{\sigma})$. We want to show that as $T \to \infty$,

$$\bar{K}_T(r,r') \to K_{\mathrm{BBP};c}(r,r').$$

To do this, we will asymptotically analyze the various factors of our integrand separately.

Firstly, note that due to the fact that G_T and G have similar behavior asymptotically, we get that:

$$\Big(T\big(G_T(q\beta(1+\sigma\hat{w})) - G_T(q\beta(1+\sigma\hat{v}))\big)\Big) \cong \frac{\hat{w}^3}{3} - \frac{\hat{v}^3}{3} + o(1),$$

as is explained in Appendix B of [1]. Moreover,it is simple to see that:

$$\frac{(1+\sigma\hat{w})^{-\frac{r'}{\sigma}}}{(1+\sigma\hat{v})^{-\frac{r}{\sigma}+1}} \cong \exp(r\hat{v} - r'\hat{w}) + o(1).$$

Finally,

$$\prod_{k=1}^{m} \frac{(\beta\beta_k^{-1}(1+\sigma\hat{v});q)_\infty}{(\beta\beta_k^{-1}(1+\sigma\hat{w});q)_\infty} = \prod_{k=1}^{m} \frac{\hat{v}+c_k}{\hat{w}+c_k} + o(1),$$

which is exactly the product term that appears in the BBP kernel. Combining these together gives us the desired result. □

Remark 3.2. Note that in the above proof, we gave a formal description of the steepest descent argument for the asymptotic analysis of the kernels. For a rigorous proof of the asymptotics, one needs to take more care with explicitly defining the integration contours and proving the decay of the integrand.

4. Generalizations and Future Work

As mentioned previously, the preceding section provides an alternative proof to a Theorem that had already been proven in [1]. What is interesting about this method of proof, is that it can potentially allow us to observe a wider range of distributions in the asymptotics of the stochastic six vertex model. In particular, the BBP family of distributions that were observed in Theorem 3.1, are just a special case of the Borodin-Peche distributions that were defined in Section 2.6, and are parametrized by a pair of Macdonald nonnegative specializations. It is possible that the entire range of these distributions can be observed in the asymptotics of the stochastic six vertex model, given a general enough class of initial data. In fact, with a few modifications to the above proof, we should be able to establish the case when one of the two specializations of the Borodin-Peche distribution is arbitrary and the other is trivial, although the details of this argument have not yet been worked out.

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