Lifting to $\mathbb{Z}/p^2$ and Splittings of the de Rham Complex

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Abstract

This paper works with a scheme in characteristic p admitting a smooth lifting to $\mathbb{Z}/p^2$. We wish to study whether the de Rham complex splits into the direct sum of its cohomology sheaves. Deligne and Illusie proved that if the dimension is less than p then the de Rham complex splits. Although there are known examples of schemes that do not admit a lifting and for which there is no splitting of the de Rham complex, there are no known examples of schemes admitting a lifting for which its de Rham complex does not split. In this paper we look at schemes of higher dimensions that admit a lifting to $\mathbb{Z}/p^2$ and measure how close the de Rham complex is to splitting by studying the Hodge-de Rham spectral sequence.

1 Introduction

The de Rham complex is an object of study in different areas of algebraic geometry as well as complex differential geometry. One of the main properties that has allowed Hodge theory to be studied in complex differential geometry is the Hodge decomposition. This result can be obtained by the splitting of the de Rham complex by way of the Hodge-de Rham spectral sequence. In characteristic 0 this decomposition has led to theorems like the Kodaira-Akizuki-Nakano vanishing which are important for the study of birational geometry.

In 1987, Pierre Deligne and Luc Illusie published *"Rélevements Modulo p^2 et Décomposition du Complexe de de Rham"* [3] which contains the first proof by purely algebraic methods of the splitting of the de Rham complex in characteristic 0. By comparison, previous proofs relied on Serre's GAGA to go from analytic methods to algebraic results.

Deligne and Illusie proof used work in characteristic p schemes in an essential way. In particular they used schemes that admitted a smooth lifting to $\mathbb{Z}/p^2$. They proved that for such schemes if the dimension was less than p then the de Rham complex splits. They were able to use this result to lift up to characteristic 0 and prove that the de Rham complex splits for any scheme in characteristic 0. Even though their result led to splitting in the characteristic 0 setting, the situation on prime characteristic was still open. In fact various examples of schemes in prime characteristic where the de Rham complex does not split were constructed. Some of these examples can be found in Lang [6] and Schröer [13]. On the other hand, in [4] Illusie states that there is a family of schemes, called ordinary schemes, for which we have a splitting of the de Rham Complex regardless of dimension. Nevertheless, there is a class of schemes for which not much is known, this would be schemes that admit a smooth lifting to $\mathbb{Z}/p^2$ but are not ordinary. In this paper we study these schemes. One of the main results in this paper is that for schemes that admit a smooth lift the Hodge-de Rham spectral sequence has no short differentials.

We now give an outline of this paper. In the following section we will give a more technical description of Deligne and Illusie's work. We do this because in subsequent sections we will build upon the techniques Deligne and Illusie used. In the next section we study in detail the situation of what happens in dimension p which is where Deligne and Illusie's work stops. Next, inspired by the work done in dimension p we work on schemes of all dimensions and prove that there are no short differentials in the conjugate sequence of the Hodge-de Rham Spectral sequence. Finally, in the last section we give some homological criteria for the splitting of the de Rham Complex.

1.1 Aknowledgements

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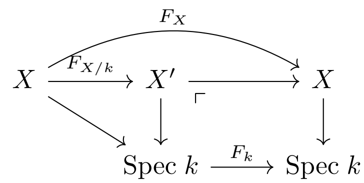
1.2 Notation and Conventions

Throughout the paper we will work with a scheme X lying over a perfect field k of characteristic p. In particular with those that admit a lift to $\mathbb{Z}/p^2$. Such a lift, denoted $\tilde{X}$, is a scheme lying over the truncated Witt vectors $W_2(k)$ such that $\tilde{X} \times_{W_2(k)} k \simeq X$. Throughout the paper we also use the relative Frobenius morphism which is defined as the map $F_{X/k}$ that makes the following diagram commute.



Where $X' = X \times_k k$ and F_k is the Frobenius morphism in k and the composition of the maps on the top row is the absolute Frobenius morphism F_X.

When we refer to the complex $\Omega^\bullet_X$ we mean the de Rham complex of a scheme X. We denote its hypercohomology as $H^{dR}(X)$ the de Rham cohomology of the scheme X.

When working with the homology of groups we will use the groups C_p, C_{p-1} and Σ_n often. In general we will let γ be the generator of C_p, α the generator of C_{p-1} and σ an element of Σ_n. When considering a C_p module $M[C_p]$ we let $M[C_p] = M[g]$ where g will be the generator of C_p inside the module. We will also use the Wreath product $\wr$ which is defined as follows. If we have Γ a subgroup of Σ_n and we let G be any group we consider the wreath product $G \wr \Gamma$ as the group $G^{\times n}$ with an action given by Γ that permutes the n copies of G by permutations.

Finally, we will let $D(X)$ be the derived category of X. We will consider this as an infinity category so that the homotopy colimit $(\Omega^1_{X'}[-1])^{\otimes n}_{h\Sigma_n}$ is defined, where the action of Σ_n is given by permutation of the elements of the tensor product twisted by the sign representation.

2 Theorem of Deligne and Illusie

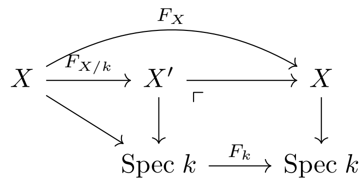
In this section we review the technical details of Deligne and Illusie's theorem on the splitting of the de Rham complex. We do this because in the following sections we will build upon some of the results and techniques used in this paper.

We first need to introduce the de Rham complex which is the main object of study. The **de Rham Complex** $\Omega^\bullet_X$ is a complex where the objects are the sheaves Ω^n_X of a scheme X. The cotangent sheaf Ω^1 consists of the differential forms of the scheme X, and each Ω^n contains the n-forms that are defined as the exterior power $\Omega^n := \Lambda^n \Omega^1$. The **de Rham Cohomology** $H^{dR}(X)$ is defined as the hypercohomology of this complex.

Even though this is a complex of $\mathscr{O}_X$ modules, it is not $\mathscr{O}_X$ linear and is one of the reasons we cannot do much algebraically on this complex. However, in characteristic p there are ways to go around this as Deligne and Illusie did. We instead consider the complex $F_*\Omega^\bullet_X$, the pushforward of the de Rham complex through the Frobenius morphism. The complex $F_*\Omega^\bullet_X$ is $\mathscr{O}_{X'}$ linear.

To understand this we introduce the relative Frobenius morphism of a scheme is.

Definition 2.1. Let X be a scheme lying over a perfect field k of characteristic p. We define the relative Frobenius map $F_{X/k}$ as the map that makes the following diagram commute.



Where $X' = X \times_k k$ and F_k is the Frobenius morphism in k and the composition of the maps on the top row is the absolute Frobenius morphism F_X.

For brevity we will denote $F_{X/k}$ as F.

Deligne and Illusie worked with the complex $F_*\Omega^\bullet_X$ which is the pushforward of the de Rham complex through the Frobenius morphism which is $\mathscr{O}_{X'}$ linear. This would allow more cohomological tools to be used to understand this complex. The next trick that they used involved considering the smooth lift of the scheme X, called $\tilde{X}$. There does not always exist a lifting, but if it exists then it can be used together with the Cartier isomorphism C^{-1}. This isomorphism plays a crucial part in Deligne and Illusie's proof.

Definition 2.2. We define a smooth **lift** $\tilde{X}$ of X from k to $W_2(k)$, where W_2 are the Witt vectors, as a scheme $\tilde{X}$ smooth over $W_2(k)$ such that $\tilde{X} \times_{W_2(k)} k \simeq X$.
If such a lift $\tilde{X}$ of X exists we say that $\tilde{X}$ has a lift to $\mathbb{Z}/p^2$.

Theorem 2.3 (Cartier)**.** *Let X be a smooth scheme over a perfect field k of characteristic p. Then there exists an isomorphism,*

$$C^{-1} : \bigoplus \Omega^i_{X'} \to \bigoplus \mathcal{H}^i F_* \Omega^\bullet.$$

Here $\mathcal{H}$ represents the cohomology sheaves of the complex. With this result due to Cartier, Deligne and Illusie are able to reduce the question of finding a quasi-isomorphism $\bigoplus \mathcal{H}^i F_* \Omega^\bullet_X[-i] \to F_*\Omega^\bullet$ to a quasi-isomorphism $\bigoplus \Omega^i_{X'}[-i] \to F_*\Omega^\bullet_X$. And the way they are able to do so is through lifting X to $\tilde{X}$ and locally lifting the Frobenius map $\tilde{F}$. This way they got their result.

Theorem 2.4 (Deligne-Illusie)**.** *Let X be a smooth scheme over a perfect field k of characteristic p that admits a smooth lift $\tilde{X}$ over $W_2(k)$, then there exists a quasi-isomorphism*

$$\varphi : \bigoplus_{i<p} \Omega^i[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet$$

Furthermore, if F lifts to a morphism $\tilde{F} : \tilde{X} \to \tilde{X}'$ then we have that

$$\varphi : \bigoplus \Omega^i[-i] \xrightarrow{\sim} F_* \Omega^\bullet$$

Where the isomorphisms are in the derived category $D(X')$.

The proof of this theorem can be found in [3] and [4]. In Illusie's section of [4] he explains how to conclude that the existence of a global Frobenius lift induces a decomposition. We want to show a small part of the proof. We present this argument as the work done in the following section is based on the ideas used in this fragment of the proof.

Lemma 2.5 (Delligne-Illusie)**.** *Let X be a smooth scheme over a perfect field k of characteristic p that admits a smooth lift $\tilde{X}$ over $W_2(k)$, then there exists a map*

$$\varphi^1 : \Omega^1_{X'}[-1] \to F_*\Omega^\bullet_X$$

in the derived category $D(X')$ that induces a quasi-isomorphism

$$\varphi : \mathscr{O}_{X'} \oplus \Omega^1_{X'}[-1] \xrightarrow{\sim} \tau_{\leq 1} F_*\Omega^\bullet_X.$$

Proof of Theorem 2.4. By the lemma above and the fact that $\Omega^1_{X'}$ is a flat $\mathscr{O}_{X'}$ module then the functor $\overset{L}{\otimes}\Omega^1_{X'}$ is equivalent to the functor $\otimes\Omega^1_{X'}$ this lets us see that $(\Omega^1_{X'}[-1])^{\otimes n} \simeq (\Omega^1_{X'})^{\otimes n}[-n]$. From this we see that we have a diagram

$$\begin{array}{ccc} (\Omega^1_{X'})^{\otimes n}[-n] & \xrightarrow{(\varphi^1)^{\otimes n}} & (F_*\Omega^\bullet_X)^{\otimes n} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi} \\ \Omega^n_{X'}[-n] & & F_*\Omega^\bullet_X \end{array}$$

Where q is the projection and π is the product morphism.

If $n < p$ then we see that the projection map $(\Omega^1_{X'})^{\otimes n} \to \Omega^n$ admits a section $s : \Omega^n_{X'} \to (\Omega^1_{X'})^{\otimes n}$ given by

$$s(a_1 \wedge a_2 \wedge \cdots \wedge a_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sign(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

so that we actually have a diagram

$$\begin{array}{ccc} (\Omega^1_{X'})^{\otimes n}[-n] & \xrightarrow{(\varphi^1)^{\otimes n}} & (F_*\Omega^\bullet_X)^{\otimes n} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \pi} \\ \Omega^n_{X'}[-n] & \xrightarrow{\varphi^n} & F_*\Omega^\bullet_X \end{array}$$

Finally by the multiplicative property of the Cartier isomorphism we have that the map

$$\varphi = \bigoplus_{n<p} \varphi^n : \bigoplus_{n<p} \Omega^n_{X'}[-n] \to F_*\Omega^\bullet_X$$

induces the Cartier isomorphism in cohomology sheaves. □

Now that we have seen the proof of theorem 2.4 assuming Lemma 2.5 we can see why the proof breaks at dimension p. We need a natural map $\varphi^n : \Omega^n_{X'}[-n] \to F_*\Omega^\bullet$ but Deligne and Illusie can only do so for $n < p$ using the section s described above. The reason why this section is needed is that in the derived category there is no functor for exterior powers. Nevertheless, when a global lift of Frobenius exists then the map φ^1 lifts to a morphism of chain complexes (this is actually an if and only if) and in this category we are able to take the exterior powers.

Inspired by this proof and the observations above, we will define a natural map $\varphi^p := (\varphi^1)^{\otimes p}_{h\Sigma_p}$ which is the homotopy colimit of $(\varphi^1)^{\otimes p}$ with respect to the action of Σ_p by permutation and the sign homomorphism. The object $(\Omega^1_{X'})^{\otimes n}_{h\Sigma_n}$ can be easily computed and it agrees with $\Omega^n[-n]$ for $n < p$, this leads us to believe that this object and map can be used to continue Deligne and Illusie's argument.

3 The map φ^p

As explained in the last section we will study the map $\varphi^p := (\varphi^1)^{\otimes p}_{h\Sigma_p} : (\Omega^1_{X'}[-1])^{\otimes p}_{h\Sigma_p} \to F_*\Omega^\bullet_X$. We now notice that because Ω^1 is a flat $\mathscr{O}_X$ module then $\otimes^L \Omega^1$ in $D(X')$ agrees with tensoring with Ω^i in the category $Sh(X)$. Thus we will have that $(\Omega^1[-1])^{\otimes p} \simeq (\Omega^1)^{\otimes p}[-p]$. To get the exterior power Ω^p we will take the homotopy colimit $((\Omega^1)^{\otimes p}[-p])_{h\Sigma_p}$, this will define a map which we will call $\varphi^p : (\Omega^1_{X'})^{\otimes p} \to F_*\Omega^\bullet_X$.

We wish to compute this homotopy colimit, and in fact it agrees with the group homology of Σ_p acting on $(\Omega^1_{X'})^{\otimes p}[-p]$, which we will now simply write as $(\Omega^1_{X'})^{\otimes p}_{h\Sigma_p}$. So the next step is to compute this homology.

Proposition 3.1. *We have the following*

$$H_n(\Sigma_p, (\Omega^1_{X'})^{\otimes p}) \simeq \begin{cases} \Omega^p_{X'} & n = 0 \\ F^*_{X'}\Omega^1_{X'} & n \equiv p-2, p-1 \ mod\ 2(p-1) \\ 0 & otherwise \end{cases}$$

where H_n is the group homology of Σ_p where it acts in $(\Omega^1_{X'})^{\otimes p}$ by permuting the factors of the tensor power and also has a twist by the sign of the permutation.

Before proving proposition 3.1, we will introduce some notation and then give a lemma. We will call γ the generator of C_p and α the generator of C_{p-1}, furthermore if we have a C_p module $M[C_p]$ we will call g the generator of C_p inside the module so that $M[C_p] = M[g]$. Unless stated otherwise we will assume that any C_p, C_{p-1} or $C_p \rtimes C_{p-1}$ module of the form $M^{\otimes p}$ has the action given by their inclusion in Σ_p twisted by the sign homomorphism. Now we present the following lemma.

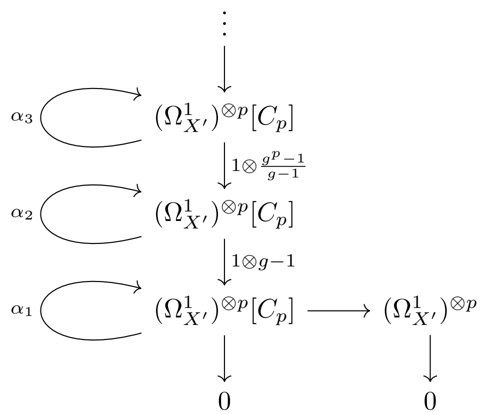
Lemma 3.2. *The following is a projective resolution for $k^{\otimes p}$ with the action by C_p given by permutations.*

$$\begin{array}{ccccccccccccc} & & & & & & & & & & k^{\otimes p} & \longrightarrow & 0 \\ & & & & & & & & & & {\scriptstyle\pi}\big\uparrow & & \\ \cdots & \longrightarrow & k[C_p] & \xrightarrow{\frac{\gamma^p-1}{\gamma-1}} & k[C_p] & \xrightarrow{\gamma-1} & k[C_p] & \xrightarrow{\frac{\gamma^p-1}{\gamma-1}} & k[C_p] & \xrightarrow{\gamma-1} & k[C_p] & \longrightarrow & 0 \end{array}$$

The verification that this is exact is straightforward. We have used the fact that $k^{\otimes p} \simeq k$ to write on the resolution $k[C_p]$ instead of $k^{\otimes p}[C_p]$ with the trivial action on $k^{\otimes p}$. We also used and will use $\frac{\gamma^r-1}{\gamma-1}$ as a formal symbol for $1 + \cdots + \gamma^{r-1}$.

Proof of Proposition 3.1. By [2] Theorem 10.1 we are be able to compute the higher homology groups as the group homology of $C_p \rtimes C_{p-1}$ lying inside Σ_p. This follows as C_p is a $p - Sylow$ subgroup of Σ_p and C_{p-1} is its normalizer.

To compute the homology we will tensor $(\Omega^1_{X'})^{\otimes p}$ with the resolution given in the lemma above, we may do so as Ω^1 is a flat module. Furthermore we will have a C_{p-1} action in each $(\Omega^1_{X'})^{\otimes p}[C_p]$ which we will describe inductively and will describe them as α_i to distinguish in which level this action is given. By our assumptions we have the action given as a permutation, but we will also need to describe how it acts on $g^i \in (\Omega^1_{X'})^{\otimes p}[C_p]$ (where g is the generator of C_p inside $(\Omega^1_{X'})^{\otimes p}$), for all i. For the first copy of $(\Omega^1_{X'})^{\otimes p}[C_p]$ the action on g^i will be the trivial action.



The picture above describes the complex and the actions each module has. As a remark we see that we would compute the same cohomology if we let the objects of the complex have the trivial action but we make the maps include the permutation. So we could do the computation on any of these complexes.

We will now define the actions α_i. We begin by defining α_1 and α_2. For this we will first let n be an integer that is a multiplicative generator modulo p. In particular we will choose n such that $n^{p-1} - 1$ is divided by p^2, we can always do so because $p | n^{p-1} - 1$ and so $p^2 | n^{p(p-1)} - 1$ so we just change our representative to be n^p if needed. With this, we define α_1 as acting on g^i as $\alpha_1(g^i) = g^{ni}$ and for α_2 we let $\alpha_2(g^i) = g^{ni}\frac{g^n-1}{g-1}$, this is a valid C_{p-1} action as we verify that

$$\alpha_2^p(g^i) = \alpha_2^{p-1}(g^{ni}\frac{g^n - 1}{g-1}) = \alpha_2^{p-2}(g^{n^2 i}\frac{g^{n^2} - 1}{g^n - 1}\frac{g^n - 1}{g-1}) = \cdots$$

$$= g^{n^p i}\frac{g^{n^p} - 1}{g^{n^{p-1}} - 1} \cdots \frac{g^n - 1}{g-1} = g^i \frac{g^{n^p} - 1}{g-1} = g^i$$

the last equality follows from the fact that $p^2 | n^p - 1$, as $\frac{g^{n^p}-1}{g-1} = 1 + a + ag + \cdots + ag^{p-1}$ where a is $\frac{n^{p-1}-1}{p}$ and so it is a multiple of p and thus 0.

We now give the action of α_3. This will be $\alpha_3(g^i) = g^{ni}\frac{g^{np}-1}{g^p-1}$. To see this is a valid C_{p-1} action we just repeat a similar argument as done for α_2.

In similar ways we will be able to define α_{2i} based on α_{2i-1} and α_{2i+1} from α_{2i}. We see that if α_{2i-1} is acting by sending $g^i \to g^{ni} f(g^i)$ then α_{2i} acts by sending $g^i \to g^{ni} f(g^i) \frac{g^n-1}{g-1}$ and α_{2i+1} acts by $g^i \to g^{ni} f(g^i) \frac{g^{np}-1}{g^p-1}$. It is straightforward to see that these actions act equivariantly and thus they define a $C_p \rtimes C_{p-1}$ action on the complex.

Now, we want to compute the cohomology of the resolution. We start by first taking the quotient by the C_p action. We see that after this if we look at a map that was $1 \otimes (g-1)$ then the kernel is still given by elements that are the quotient of elements in the original kernel and these come from quotients of elements in the image of the previous object of the complex. Thus to see where the failure of exactness comes we need to see new elements in the kernel. It is straightforward to verify that these elements are $a \otimes a \otimes \cdots \otimes a$ and something similar happens for the map $1 \otimes \frac{g^p-1}{g-1}$ where only the elements $a \otimes a \cdots \otimes a$ are in the kernel.

If we now quotient by the C_{p-1} action we will see that there are no new nonzero elements mapped to 0 as the maps didn't change and thus the kernel can only come from elements whose quotient agrees with the quotient of another element in the kernel. It is thus straightforward to see that the homology groups are generated by the elements $a \otimes \cdots \otimes a$ in every level. But we see that the action of $\alpha_{2i}, \alpha_{2i+1}$ in these elements send $a \otimes \cdots \otimes a \to -1 n^i (a \otimes \cdots \otimes a)$ the -1 comes from the sign of α which we can easily verify is -1 because it acts transitively as a permutation (it is equivalent to sending $a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \to a_0 \otimes a_{ni} \cdots \otimes a_{n(p-1)}$) and as $p-1$ is even or $p=2$ then the sign is -1. And so we must have $a \otimes \cdots \otimes a = -1 n^i (a \otimes \cdots \otimes a)$ which only happens if $-n^i = 1$ if and only if $i \equiv \frac{p-1}{2} \mod p-1$.

So we conclude that H^0 is Ω^p as desired and that the only non vanishing elements are in H^{p-2}, H^{p-1} and they are identified as the sheaf that has sections $a \otimes \cdots \otimes a$ for every section $a \in \Omega^1$. We see that this is equivalent to a vector bundle where we take the p^{th} powers of the transition functions used for Ω^1. This follows because if we take affine charts and wish to go from a basis $e_1 \cdots e_d$ to a basis $a_1 \cdots a_d$ then we have transition functions given by a matrix f_{ij} and then we see that if $a_1 = \sum f_{1i} e_i$ then $a_1 \otimes \cdots \otimes a_1 = \sum (f_{1i} e_i \otimes \cdots \otimes f_{1i} e_i) = \sum f_{1i}^p e_1 \otimes \cdots \otimes e_1$ and this follows because any element that does not have all the sections of the tensor product equal vanishes. And this sheaf agrees with the pullback $F^*_{X'} \Omega^1_{X'}$ as in this case the transition functions would be $a_1^p = F^*(a_1) = F^*(\sum f_{1i} e_i) = \sum f_{1i}^p e_i^p$ and we see they agree.

□

Having proved this, we first notice that a similar argument follows to compute $(\Omega^1_{X'})^{\otimes n}_{h\Sigma_n}$ for $p < n < 2p$ and the difference in the cohomology sheaves is that instead of $F^* \Omega^1_{X'}$ we have $\Omega^r_{X'} \otimes F^* \Omega^1_{X'}$. One might then ask the question of what happens when we try to compute the cohomology for higher $n (= p + r)$. And in the next section we will study this situation.

4 φ^n and the Hodge-de Rham Spectral Sequence

As stated at the end of the previous section it is straightforward to see that we will have similar homology sheaves for $(\Omega^1_{X'})^{\otimes n}$ as for $(\Omega^1_{X'})^{\otimes p}$ when $n < 2p$. This motivates us to see what happens for even higher n. However, the computations are too complicated after $n = p^2$, as the p-Sylow group is not simply a direct sum of $C_p's$. Despite this, we may recognize that we have a gap in between the 0 and $p-2$ levels which we may expect to be there for all n. This is an expected result by looking at the homological stability results for Σ_n proved by Nakaoka in [8]. By homological stability we mean that $H_i(\Sigma_n, C_p) \simeq H_i(\Sigma_n, C_p)$ for $i < c_n$ where c_n is a constant depending on n and $c_n \to \infty$. There is also a similar homological stability result for oriented configuration spaces done by Palmer in [12] that suggests that for the alternating group and the action of Σ_n twisted by the sign representation there is a similar stability result. Although in this paper we are not able to prove a homological stability result for the group homology of Σ_n with its action twisted by the sign representation, we will prove that the expected gap is there for all n.

Theorem 4.1. *Let X be a smooth scheme over a perfect field k of characteristic $p > 3$ that has a smooth lifting $\tilde{X}$. Then for all n, the cohomology sheaves from 1 to $p-3$ of the object $(\Omega^1_{X'})^{\otimes n}_{h\Sigma_n}$ are 0. Where $(\Omega^1_{X'})^{\otimes n}$ has a Σ_n action by permutation twisted by the sign representation.*

We see that the statement above only makes sense for prime numbers $p > 3$. And this makes sense because for $p = 2, 3$ we do not expect there to be a gap in the cohomology. Having said this, throughout this section we assume $p > 3$.

Before giving the proof of theorem 4.1 we will present some results from group homology that will be needed. Most of the results about the homology of symmetric groups we use are due to Nakaoka in [8] and [9]. The book on group cohomology [1] of Adem and Milgram also has a good section on this topic.

We will now give a few results on the homology of groups and particularly for Σ_n which we will use. But first, we recall the definition of the wreath product. If we have Γ a subgroup of Σ_n and we let G be any group we consider the wreath product $G \wr \Gamma$ as the group $G^{\times n}$ with an action given by Γ that permutes the n copies of G by permutations.

One important fact we will need is that the p-Sylow subgroups of Σ_n look all like direct sums of $C_p \wr C_p \wr \cdots \wr C_p$ where we would have a_m copies of $\underbrace{C_p \wr C_p \wr \cdots \wr C_p}_{m \text{ copies}}$ where $n = \sum a_i p^i$ is its expansion base p.

Another fact that we will use is that the homology $H_*(\Sigma_n, M)$ has a gap between 0 and $2p-3$ for any trivial module M.

Proposition 4.2 (Lyndon-Hochschild-Serre Spectral Sequence)**.** *Let G be a group, N a normal subgroup and A a G-module. Then we have a spectral sequence*

$$H_p(G/N, H_q(N, A)) \Rightarrow H_{p+q}(G, A)$$

With these homological tools we can now proceed to the proof of theorem 4.1.

Proof of theorem 4.1. We will prove this by induction. We will assume that we have the desired result for p, p^2, $\cdots$, p^m and we will prove the result for all integers from $p^m + 1$ to p^{m+1}.

First we notice that because the pullback of the inclusion of a point $i_x : x \to X$ commutes with tensor product and with colimits we can compute the dimension of the cohomology sheaves (which are vector bundles) of $(\Omega^1_{X'})^{\otimes n}_{h\Sigma_n}$ as follows. We can detect the rank of a vector bundle V by computing the dimension of the vector space $i_x^* V$. Thus we can see that $i_x^*(\Omega^1_{X'}[-1])^{\otimes n} \simeq (i_x^* \Omega_{X'})^{\otimes n}_{h\Sigma_p}$ and so it is enough to compute cohomology for the vector space $(i_x^* \Omega_{X'})^{\otimes n}$. This tells us that if we prove the vanishing statement for homology of vector spaces we would get the desired result for vector bundles. So let us work with a vector space V and as a module we have $V^{\otimes n}$ with the Σ_n action given by permutations twisted by the sign representation.

Because we know the structure of the p-Sylow subgroup of Σ_n we have that the group $\prod(\prod_1^{a_m} \Sigma_{p^m})$ contains a p-Sylow subgroup of Σ_n and thus we have a surjective map

$$H_r(\prod(\prod_1^{a_m} \Sigma_{p^m}), V^{\otimes n}) \to H_r(\Sigma_n, V^{\otimes n})$$

then we see that we can use the Lyndon-Hochschild-Serre spectral sequence to get the desired vanishing. To do so we do the computation one by one so that we let $G = G' \oplus \Sigma_{p^m}$ for some m and our module will be $V^{\otimes n}$ with the action as subgroups of Σ_n. Then we see that

$$H_i(\Sigma_{p^s}, H_j(G, V^{\otimes n})) \Rightarrow H_{i+j}(G \times \Sigma_{p^s}, V^{\otimes n})$$

then we notice that $V^{\otimes n} \simeq \bigoplus V^{\otimes |G|}$ as G modules then we have

$$H_i(\Sigma_{p^m}, H_j(G, \bigoplus V^{\otimes |G|})) \simeq H_i(\Sigma_{p^m}, \bigoplus H_j(G, V^{\otimes |G|}))$$

and by induction we know that $H_j(G, V^{\otimes |G|})$ vanishes for $0 < j < p-2$ and thus also does $H_j(G, V^{\otimes n})$. So we need to understand $H_i(\Sigma_{p^m}, H_0(G, V^{\otimes n}))$ to do so we first notice that $H_0(G, V^{\otimes n})$ is by definition the quotient of $V^{\otimes n}$ by its G action and we see it is isomorphic to $\wedge^{|G|} V \otimes V^{n-|G|}$. Then decomposing this as $\bigoplus V^{\otimes p^m}$, which we can do as the G and Σ_{p^m} actions are independent, we similarly get that for $0 < i < p-2$ $H_i(\Sigma_{p^m}, H_0(G, V^{\otimes n}))$ vanishes. From this we can conclude that $H(G \times \Sigma_{p^m}, V^{\otimes n})$ has the same gap.

Now we are left with proving we have the desired gap in the homology of $\Sigma_{p^{m+1}}$. For this we notice that the wreath product $\Sigma_{p^m} \wr \Sigma_p$ contains the p-Sylow group of $\Sigma_{p^{m+1}}$ so we wish to compute the group homology of this group. To do this we use the Lyndon-Hochschild-Serre spectral sequence with $(\Sigma_{p^m})^{\times p}$ as a normal subgroup of $\Sigma_{p^m} \wr \Sigma_p$ with $V^{\otimes p^{m+1}}$ as a module that is acted by permutations twisted by the sign representation. Then we get

$$\bigoplus_{i+j=n} H_i(\Sigma_p, H_j((\Sigma_{p^m})^{\times p}, V^{\otimes p^{m+1}})) \Rightarrow H_{i+j}(\Sigma_{p^m} \wr \Sigma_p, V^{\otimes p^{m+1}})$$

and similarly as above we can see that the left hand side has no nonzero elements except when $i = j = 0$. As we know the gap exists for $H_j((\Sigma_{p^m})^p, V^{\otimes p^{m+1}})$ as we can use the same argument as in the previous paragraph. And then we only need to know what $H_i(\Sigma_p, H_0((\Sigma_{p^m})^{\times p}, V^{\otimes p^{m+1}}))$ is. But similarly as above we can see that $H_0((\Sigma_{p^m})^{\times p}, V^{\otimes p^{m+1}}) \simeq (\wedge^{p^m} V)^{\otimes p}$ and as a Σ_p module it is acted by permutations twisted by the sign representation. From this we can see that $H_i(\Sigma_p, H_0((\Sigma_{p^m})^{\times p}, V^{\otimes p^{m+1}}))$ vanishes for $0 < i < p-2$. And so as desired we have that $H_{i+j}(\Sigma_{p^m} \wr \Sigma_p, V^{\otimes p^{m+1}})$ has the same gap.

□

Now that we have this vanishing result for the homology groups of $(\Omega^1_{X'})^{\otimes n}$ we want to see what information we get from considering the map $\varphi : \bigoplus(\Omega^1_{X'}[-1])^{\otimes i} \to F_* \Omega^\bullet_X$ we have in the same way that Deligne and Illusie had that this map induces the Cartier isomorphism between the cohomology sheaves $\Omega^i_{X'}$ and $\mathcal{H}^i F_* \Omega^\bullet_X$. With this we see that there is a surjective map between the spectral sequences that compute the hypercohomology of both objects. And we will use this fact to prove the following proposition.

Proposition 4.3. *Let X be a smooth scheme over a perfect field k of characteristic $p > 3$ that admits a smooth lifting. Then the conjugate spectral sequence of the Hodge-de Rham spectral sequence*

$$H^i(X', \Omega^j_{X'}) \Rightarrow H^{i+j}(X', F_* \Omega^\bullet_X)$$

has no nonzero differentials of size smaller than $p-2$.

Proof. Let us call the spectral sequences of $\bigoplus(\Omega^1_{X'}[-1])^{\otimes i}$ and $F_*\Omega^\bullet_X$ E and E' so that they have pages E_r and E'_r with differentials d_r and d'_r respectively. Let us call the map induced in each page as f_r. We will prove by induction that if E_r has no nonzero differentials and if f_r is surjective then the same is true for $r+1$. We have that the map f_{r+1} is constructed from f_r by setting $E_{r+1} = \ker\, d_r/\text{im}\, d_r$ and similarly $E'r+1 = \ker\, d'_r/\text{im}\, d'_r$ as all d_r are zero then $E_{r+1} = \ker\, d_r$ and then the induced map to $\ker\, d'_r$ is surjective and its projection to $\ker\, d'_r/\text{im}\, d'_r$ is also surjective.

Then we have in particular that for $r < p-2$ the r^{th} page E_r, has differentials $d_r = 0$ due to theorem 4.1. Because f is a morphism of spectral sequences we must have that $d_r \circ f_r = f_r \circ d'_r$ and if $d_r = 0$ and f_r is surjective then we must also have d'_r is 0 as otherwise the left hand side is always 0 but the right hand side could be nonzero.

□

5 Criteria for Degeneration of the Hodge-de Rham Spectral Sequence

In this section we will first relate the results we got in the previous section to the Hodge-de Rham spectral sequence. At the end of the section we also give some cohomological conditions that will result in the degeneration of the Hodge-de Rham spectral sequence at the E_1 page for schemes of dimension $p+1$. An important fact that we will use that appears in [3] is the following.

Proposition 5.1 (Deligne-Illusie)**.** *Let X be a smooth proper scheme over a perfect field k of characteristic p. Then we have degeneration of the Hodge-de Rham spectral sequence at the E_1 page if and only if we have degeneration of the conjugate spectral sequence at the E_2 page.*

By the map of spectral sequences that we gave in proposition 4.3 we see that we could understand the conjugate spectral sequence better if we studied the cohomology groups $H^i(X', F^*\Omega^1_{X'})$. In particular we would expect to have degeneration of the spectral sequence with the vanishing of some of these groups. Working in this direction makes us notice that as Ω^n is a flat $\mathscr{O}_X$ module then

$$H^i(X', F^*\Omega^1_{X'} \otimes \Omega^n) \simeq H^i(X', F^*\Omega^1_(X')) \otimes \Omega^n_{X'},$$

from this we get the following proposition.

Proposition 5.2. *Let X be a smooth proper scheme over a perfect field k of characteristic p that admits a smooth lift $\tilde{X}$. If the dimension of X $n < 2p$ and the cohomology groups $H^i(X', F^*\Omega^1_{X'})$ for $i < \dim X - p + 1$ vanish then the de Rham complex splits.*

Proof. We see that if we consider the spectral sequence for $(\Omega^1_{X'}[-1])^{\otimes r}$ for $r \geq p$ then by the discussion and proof of proposition 3.1 we have that the cohomology sheaves of $(\Omega^1_{X'}[-1])^{\otimes r}$ are Ω^r at level $-r$ and $F^*\Omega^1_{X'} \otimes \Omega^{r-p}$ in levels $-r+p-2$ and $-r+p-1$. This gives us a spectral sequence that looks like following at the E_2 page.

$$\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
H^0(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) & H^1(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) & \cdots & H^n(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) \\
H^0(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) & H^1(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) & \cdots & H^n(X', F^*\Omega^1_{X'} \otimes \Omega^{r-p}) \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H^0(X', \Omega^r_{X'}) & H^1(X', \Omega^r_{X'}) & \cdots & H^n(X', \Omega^r_{X'})
\end{array}$$

But we see that the only differentials that could affect the bottom row are in the pages $p-2$ and $p-1$. These would be maps $H^i(X', \Omega^i_{X'} \otimes F^*\Omega^1_{X'}) \to H^{i+p-1}(X', \Omega^r)$ or $H^i(X', \Omega^i_{X'} \otimes F^*\Omega^1_{X'}) \to H^{i+p}(X', \Omega^r)$. But then we have that $H^i(X', \Omega^i_{X'} \otimes F^*\Omega^1_{X'}) \simeq H^i(X', F^*\Omega^1_{X'}) \otimes \Omega^i_{X'} \simeq 0$ for j. So no differentials affect the lower level. Knowing this we are able to repeat the argument of proposition 4.3 and see that both spectral sequences must vanish as the differentials we would be interested in are 0 and the surjectiveness is preserved in each page. This proves that the conjugate spectral sequence degenerates and thus the Hodge-de Rham spectral sequence does. This proves that the de Rham complex splits. □

Now we will develop another criterion which is based only on theorem 2.4 of Deligne and Illusie. To get the criterion we will need Poincaré duality in the Hodge-de Rham spectral sequence.

Proposition 5.3. *Let X be a smooth proper scheme of dimension n over a perfect field k of characteristic p. The pushforward of the de Rham complex has Poincaré Duality. This means that*

$$\tau_{\leq i} F_*\Omega^\bullet_X \simeq \tau_{\geq n-i} F_*\Omega^\bullet_X$$

for all integers $0 \leq i \leq n$.

Sketch of Proof. To see this duality we follow the argument given by Deligne and Illusie [3]. This duality follows from Serre's Duality for Ω^i_X and Ω^{n-i}_X, which has a perfect pairing $\Omega^i_X \times \Omega^{n-i}_X \to \Omega^i_X$ given by $\alpha \times \beta \to \alpha \wedge \beta$. Because F_* is finite and flat this perfect pairing gives rise to a perfect pairing $F_*\Omega^i_X \times F_*\Omega^{n-i}_X \to F_*\Omega^n_X$. These pairings are then respected when taking homology. A more detailed account can be found in [3] and a similar proof that shows the duality for the homology groups of the spectral sequence can be found in [14]. So we can use duality in terms of the truncation functors but also in the form of $E^{i,j} \simeq E^{p-i,p-j}$ in the spectral sequence.

□

We now present a criterion for the degeneration of the Hodge-de Rham spectral sequence.

Proposition 5.4. *If X is a smooth proper scheme over a perfect field k of characteristic p that admits a smooth lifting $\tilde{X}$. If $\dim X = p+1$ and either of the following conditions is satisfied*
(a) $H^{p+1}(X, \Omega^1) \simeq H^0(X, \Omega^p) \simeq 0$ or
(b) $H^p(X, \Omega^1) \simeq H^1(X, \Omega^p) \simeq 0$ and $H^{p+1}(X, \mathscr{O}_X) \simeq H^0(X, \Omega^{p+1}) \simeq 0$.
then the Hodge-de Rham spectral sequence degenerates in the E_1 page.

Proof. As X has a smooth lift then we know that $\tau_{<p}F_*\Omega^\bullet_X$ decomposes and by Poincaré Duality that is also true for $\tau_{\geq 2}F_*\Omega^\bullet_{X'}$. Then we want to see the maps going from

$$(\Omega^p[-p] \to \Omega^{p+1}[-p-1]) \to \bigoplus \Omega^i[-i]$$

then we can see that we have an exact triangle

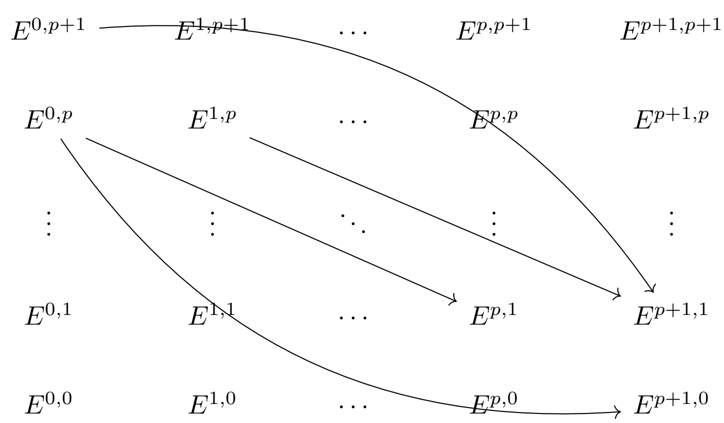
$$\tau_{<p}F_*\Omega^\bullet_X \to F_*\Omega^\bullet_X \to (\Omega^p[-p] \to \Omega^{p+1}[-p-1])$$

and then applying $\tau_{\geq 2}$ we get

$$\tau_{[2,p-1]}F_*\Omega^\bullet_X \to \tau_{\geq 2}F_*\Omega^\bullet_X \to (\Omega^p[-p] \to \Omega^{p+1}[-p-1])$$

and from this we see that the only possible nonzero differentials are from the first two rows to the last two rows.

Writing down the possible differentials there are only 4 possible maps that avoid the middle part which we know splits off the rest of the complex. We show thi maps below



as we can see the maps go from $E^{0,p+1} \to E^{p+1,1}$, $E^{0,p} \to E^{p+1,0}$, $E^{0,p} \to E^{p,1}$ and $E^{1,p} \to E^{p+1,1}$. From this we see that from Poincaré duality $E^{0,p} \simeq E^{p+1,1}$, $E^{0,p+1} \simeq E^{p+1,0}$ and $E^{1,p} \simeq E^{p,1}$ from which we get the desired conclusion, as we cannot have differentials from or onto 0.

□

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