Biclique Partitions and Off-diagonal Ordered Ramsey Numbers

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Abstract

We study two separate combinatorial problems. First, we look at the problem of partitioning the edges of the complete graph K_n into the minimum possible number of complete bipartite graphs (bicliques) such that every edge is covered exactly k times. It is known that at least $n-1$ bicliques are required. The previously best-known upper bound was $O(kn)$, but de Caen, Gregory, and Pritikin showed that for $k \leq 18$, merely $n-1$ bicliques are sufficient for all sufficiently large n, and conjectured that the same holds for every k. We make progress towards this conjecture by showing that $n + o(n)$ bicliques suffice for every fixed k.

We then turn to the Ramsey theory of ordered graphs. For ordered graphs G and H, the ordered Ramsey number $r_<(G, H)$ is the smallest N such that every bicoloring of the complete graph on $[N]$ contains either a blue copy of G or a red copy of H, where the embedding must preserve the relative order of vertices. One number of interest, first studied by Conlon, Fox, Lee, and Sudakov, is the "off-diagonal" ordered Ramsey number $r_<(M, K_3)$, where M is an ordered matching on N vertices. The best-known upper bound is a trivial bound $O(N^2/\log N)$, but there is no known family of arbitrarily large matchings with $r_<(M, K_3) = \omega(N^{4/3})$, and Conlon et al. hypothesize that $r_<(M, K_3) = O(N^{2-\epsilon})$ for every ordered matching M. We resolve two special cases of this conjecture. We show that the off-diagonal ordered Ramsey numbers for matchings in which edges do not cross are nearly linear. We also prove a truly sub-quadratic upper bound for random matchings with interval chromatic number 2.

1 Introduction

1.1 Biclique Partitions

Let L be a list of positive integers, and let G be a graph. The *L-biclique covering number* of a graph G, which we refer to as $\text{bp}_L(G)$, is the size of the smallest collection of complete bipartite subgraphs of G so that every edge of G is covered exactly l_i times by this collection, for some $l_i \in L$. The special cases which have received the most attention are $\text{bp}_{\{1\}}(G)$, known as the *biclique partition number*, and $\text{bp}_{\{1,2,3,\dots\}}(G)$, known as the *biclique covering number* of graph G, or alternatively the *bipartite dimension*. From an algorithmic standpoint, even approximating the biclique partition/covering number of general graphs is NP hard. Both problems have connections to many other areas of computer science such as communication complexity, and we refer the reader to [9] for more on these connections.

Since Graham and Pollak [7] showed in 1971 that

$$\mathrm{bp}_{\{1\}}(K_n) = n - 1,$$

determining and estimating the L-biclique covering numbers of specific graphs has been an active area of combinatorics. In particular there has been interest in determining the L-biclique covering numbers of the complete graph K_n for various lists L [5, 1, 11]. See [5] for a more comprehensive survey on this line of work.

In this paper, we look at the list $L = \{k\}$. By a generalization of the Graham-Pollak theorem, it is known that $\mathrm{bp}_{\{k\}}(K_n) \geq n - 1$ for all n and k. It has been conjectured that this is essentially tight:

Conjecture 1.1 (de Caen et al. [3])**.** *For every positive integer k,*

$$bp_{\{k\}}(K_n) = n - 1$$

for all sufficiently large n.

The same authors resolve the conjecture for $k \leq 18$ by constructions related to design theory [3]. However, to our knowledge, the previously best-known upper bound for general k is $\mathrm{bp}_{\{k\}}(K_n) = O(kn)$, obtained by compounding a small-k construction.

We will show that Conjecture 1.1 is true to leading order. More precisely, we construct a family of designs (inspired by classical ideas of Nisan and Wigderson [10]), which in turn yields a $\{k\}$-covering of K_n by at most $n + k\sqrt{n} + 2kn^{3/4}$ bicliques.

Theorem 2.5. *Let k be a positive integer. For all sufficiently large n,*

$$bp_{\{k\}}(K_n) \leq n + k\sqrt{n} + 2kn^{3/4}.$$

1.2 Ordered Ramsey numbers

A classical area of extremal combinatorics is Ramsey theory. Introduced by Ramsey [12] and popularized by Erdos and Szekeres [6], the Ramsey number of a graph G, commonly denoted by $r(G)$, is the smallest n so that every edge bicoloring of the complete graph K_n contains a monochromatic copy of G. Shrinking the sizable gap between the asymptotic upper/lower bounds on $r(K_n)$ has been a major open problem for decades, spurring extensive work on a plethora of related questions in Ramsey theory.

One variant of Ramsey numbers which has recently received attention is the analogue for ordered graphs. An *ordered graph* on $[n]$ is a graph on n vertices which are given distinct labels in $\{1, \ldots, n\}$. Given an ordered graph G, the *ordered Ramsey number* of G, denoted by $r_<(G)$, is the smallest n so that every edge bicoloring of the ordered complete graph on n vertices contains a monochromatic copy of G which preserves the relative vertex ordering of G. As with the unordered case, one can define the *off-diagonal* ordered Ramsey number of two graphs G and H, denoted by $r_<(G, H)$, as the smallest n so that every edge bicoloring of the ordered complete graph on n vertices contains either an order preserving red copy of G or an order preserving blue copy of H.

The first systematic studies of ordered Ramsey numbers were conducted by Conlon, Fox, Lee, and Sudakov [4] and by Balko, Cibulka, Král, and Kynčl [2]. However, as pointed out by the authors of [4], a number of classic results in extremal combinatorics can be reinterpreted as statements about ordered Ramsey numbers. For instance, Erdos and Szekeres proved [6] that every sequence of at least $(n-1)^2+1$ distinct numbers contains either an increasing subsequence of length n or a decreasing subsequence of length n. This result is implied by the bound $r_<(P_n, K_n) \le (n-1)^2+1$, where P_n is the n-vertex path imbued with the natural monotonic ordering: for any sequence of n distinct numbers $x_1, \ldots, x_n$, color (i, j) red if $x_i < x_j$ and blue otherwise.

Perhaps the simplest nontrivial family of ordered graphs from the perspective of ordered Ramsey theory is *matchings*, in which every vertex has degree 1. Conlon, Fox, Lee, and Sudakov provide a number of bounds for general matchings, for matchings satisfying certain properties, and for off-diagonal ordered Ramsey numbers involving matchings. Relevant to this paper is their work on bounding the largest possible value of $r_<(M, K_3)$, where M is a matching. They have the following result:

Theorem 1.2 (Conlon, Fox, Lee, and Sudakov [4]). *There are positive constants c_1 and c_2 such that for all positive integers n,*

$$c_1\left(\frac{n}{\log n}\right)^{4/3} \le \max_M r_<(M, K_3) \le c_2\frac{n^2}{\log n}$$

where the maximum is taken over all ordered matchings M on n vertices.

The upper bound in this theorem is in some sense trivial, in that it does not make use of any properties of matchings; rather, it bounds $r_<(M, K_3)$ by the well-studied unordered Ramsey number $r(K_n, K_3)$ (whose asymptotics are known), only making use of the fact that every graph on n vertices can be embedded in K_n. For this reason and perhaps other reasons, Conlon, Fox, Lee, and Sudakov hypothesize [4] that the upper bound can be improved to $r_<(M, K_3) \le n^{2-\epsilon}$ for some $\epsilon > 0$.

We contribute two results in the direction of this conjecture. We first look at the special case of ordered matchings where the edges do not cross. That is, for any two edges (i, j) and (k, l) with $i < j$ and $k < l$, the intervals $[i, j]$ and $[k, l]$ are either disjoint or nested one inside the other. We call the matchings which satisfy this condition "parenthesis matchings", after the useful fact that these matchings correspond with balanced parenthesis sequences. Indeed, it is this correspondence which partially motivates our proof of the following theorem.

Theorem 3.8. *For any $\epsilon > 0$ there is a constant c such that every parenthesis matching M on n vertices has*

$$r_<(M, K_3) \le cn^{1+\epsilon}.$$

To state our second result, we must define the *interval chromatic number* of an ordered graph. Analogous to the chromatic number of an unordered graph, the interval chromatic number $\chi_<(G)$ of a graph G is the minimum number of contiguous intervals into which the vertex set must be split so that each interval is an independent set in G.

Conlon, Fox, Lee, and Sudakov present a number of general results accompanied by much stronger specific results for matchings with small interval chromatic number [4]. In a similar spirit, we prove a sub-quadratic bound on $r_<(M, K_3)$ for random matchings with interval chromatic number 2.

Theorem 3.15. *There is a constant c such that, if an ordered matching M on n vertices with interval chromatic number 2 is picked uniformly at random, then*

$$r_<(M, K_3) \le cn^{\frac{24}{13}}$$

with high probability.

Observe that the statement is not probabilistic over bicolorings; rather, it is a true Ramsey-type result which applies to almost all matchings.

1.3 Roadmap

We outline the remainder of this paper. In Section 2, we describe and prove our results on $\{k\}$-biclique partitions of the complete graph. Subsequently, in Section 3, we move on to ordered Ramsey numbers, achieving new upper bounds for the off-diagonal ordered Ramsey numbers of parenthesis matchings and random matchings with interval chromatic number 2. Finally, in Section 4 we outline possible directions for future research, describing a few of the many interesting questions about biclique partitions and ordered Ramsey numbers which remain open.

2 A $\{k\}$-biclique partition of K_n

Our biclique construction is by means of a *combinatorial design* in the sense used by Nisan and Wigderson [10] in their classical paper on pseudorandom generators.

Definition 2.1. A family of sets $\{S_1, \dots, S_n\}$ with $S_1, \dots, S_n \subseteq [d]$ is a (n, d, t, m)-design if:

1. $|S_i| = m$ for all $i \in [n]$;
2. $|S_i \cap S_j| \le t$ for all $i, j \in [n]$ with $i \neq j$.

We construct our designs differently, though, in order to achieve better bounds for our specific choices of parameters:

Lemma 2.2. *For any positive integers m and t, there exists some N such that an (n, d, t, m)-design with $d \le 2mn^{1/(t+1)}$ exists for all $n \ge N$.*

Proof. Let N be large enough that there are at least m prime numbers in the interval $[n^{1/(t+1)}, 2n^{1/(t+1)}]$ for every $n \ge N$; this is possible by the Prime Number Theorem. Fix some $n \ge N$, and choose distinct primes $p_1, \dots, p_m \in [n^{1/(t+1)}, 2n^{1/(t+1)}]$. We will pick sets $S_1, \dots, S_n$ from the disjoint union

$$U = \bigcup_{j=1}^{m} \mathbb{Z}/p_j\mathbb{Z}.$$

For $i \in [n]$, let S_i consist of m elements from U, one from each group. Specifically, for $j \in [m]$, pick element $i \pmod{p_j}$ from group $\mathbb{Z}/p_j\mathbb{Z}$.

It is clear that $|S_i| = m$ for all i. Furthermore, suppose that $|S_i \cap S_j| > t$ for some distinct $i, j \in [n]$. Then among the chosen primes, there are $t+1$ primes $p_{i_1}, \dots, p_{i_{t+1}}$ with $i \equiv j \pmod{p_{i_k}}$ for each $k \in [t+1]$. But then

$$\prod_{k=1}^{t+1} p_{i_k} \mid (i - j).$$

Since $i \neq j$, it follows that

$$|i - j| \ge \prod_{k=1}^{t+1} p_{i_k} \ge n,$$

a contradiction.

To complete the proof, note that the sets are chosen from a universe of size $|U| = \sum_{j=1}^{m} p_j$, which does not exceed $2mn^{1/(t+1)}$. $\square$

Remark 2.3. The above design is in fact optimal up to constant factors. Consider any (n, d, t, m)-design, where the sets are contained in a universe U of size d. For every $(t+1)$-element subset of U, there is at most one set among $S_1, \ldots, S_n$ which contains the subset; since each S_i contains $\binom{m}{t+1}$ subsets of size $t+1$, we must have $\binom{d}{t+1} \geq n\binom{m}{t+1}$, so

$$d \geq \left(n\binom{m}{t+1}(t+1)!\right)^{1/(t+1)} \geq \frac{1}{e}n^{1/(t+1)}m.$$

We will only use the special case $(n, k\sqrt{n}, 1, \lfloor k/2 \rfloor)$ of Lemma 2.2, which we state explicitly below as a corollary.

Corollary 2.4. *For any integer k and all n sufficiently large, there are sets $S_1, \ldots, S_n \in [k\sqrt{n}]$ with $|S_i| = \lfloor k/2 \rfloor$ for all i and $|S_i \cap S_j| \leq 1$ for all $i \neq j$.*

Now we make use of this design to construct a $\{k\}$-biclique covering of K_n.

Theorem 2.5. *Let k be a positive integer. Then for all sufficiently large n,*

$$bp_{\{k\}}(K_n) \leq n + k\sqrt{n} + 2kn^{3/4}.$$

Proof. Let $S_1, \ldots, S_n$ be the sets from Corollary 2.4. Define bicliques $B_1, \ldots, B_{k\sqrt{n}}$ on K_n by letting B_i be the biclique between $\{j \in [n] \mid i \in S_j\}$ and $\{j \in [n] \mid i \notin S_j\}$. Then any edge (i, j) is covered exactly $|S_i|+|S_j|-2|S_i \cap S_j|$ times, and this number is equal to either $2\lfloor k/2 \rfloor - 2$ or $2\lfloor k/2 \rfloor$ (depending on whether $|S_i \cap S_j| = 0$ or 1).

If k is odd, every edge needs to be covered 1 or 3 more times. Call the latter edges "triple-edges". An edge (i, j) is a triple-edge if and only if there exists some index l and remainder r such that $i \equiv j \equiv r \pmod{p_l}$. We can define a clique $C_{l,r}$ consisting of all vertices i with $i \equiv r \pmod{p_l}$. Observe that every triple-edge is contained in exactly one clique, and every clique contains only triple-edges. To make progress, we'll construct a $\{1, 2\}$-biclique covering of each clique. It is known [5] that $\mathrm{bp}_{\{1,2\}}(K_n) \leq 2\sqrt{n}$ for any n. The number of cliques $C_{l,r}$ is at most $k\sqrt{n}$, and each clique has size at most $\sqrt{n}$, so $2kn^{3/4}$ biclique are needed to $\{1, 2\}$-cover every clique. Now every edge needs to be covered 1 or 2 more times.

If k is even, every edge needs to be covered only 0 or 2 more times, so we skip the above step. Finally, in either case, we "pad" the covering so that every edge is covered exactly k times. Define bicliques $D_1, \ldots, D_n$ where D_i is the star centered at vertex i and containing edges to all vertices $j < i$ such that (i, j) needs to be covered 1 or 2 more times, and all vertices $j > i$ such that (i, j) needs to be covered 2 more times.

This completes the construction, and the total number of bicliques used is at most $n + k\sqrt{n} + 2kn^{3/4}$. □

Remark 2.6. The main ingredient of the above theorem is the $\{2k-2, 2k\}$-biclique covering of K_n using $2k\sqrt{n}$ bicliques. A bound in [5] shows that $\sqrt{n/2}$ bicliques are necessary for this list covering, so the asymptotic dependence on n cannot be decreased.

3 Off-diagonal ordered Ramsey numbers

In this section we leave behind biclique partitions, and investigate the ordered Ramsey number $r_<(M, K_3)$, where M is a matching. Since every graph on N vertices embeds in the complete graph

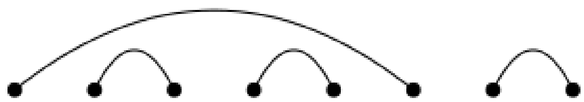

Figure 1: The parenthesis matching corresponding to the parenthesis sequence (()())().

K_N, and the ordered Ramsey number $r_<(K_N, K_3)$ is equal to the Ramsey number $r(N,3)$, which has been asymptotically determined to be $\Theta(N^2/\log N)$, it follows (as pointed out in [4]) that $r_<(M, K_3) = O(N^2/\log N)$ for a matching M on N vertices. However, this bound does not make use of any properties of matching graphs.

In Section 3.1, we achieve a nearly linear bound for matchings whose edges do not cross. In Section 3.2, we obtain a slightly sub-quadratic bound for random matchings with interval chromatic number 2.

Throughout the remainder of this paper, we make no serious attempts to optimize constants.

3.1 Parenthesis Matchings

Earlier we defined "parenthesis matchings" as matchings for which the edges do not cross. We claim without proof that every parenthesis matching corresponds uniquely with a balanced parenthesis sequence—that is, a sequence of correctly matched open and close parentheses. The bijection is straightforward; each matched pair of parentheses corresponds with an edge in the matching. See Figure 1 for an example.

We start with perhaps the simplest nontrivial parenthesis matching, and work our way up to general parenthesis matchings. Define the nested matching graph NM_k of size k to be the graph on $[2k]$ where (i,j) is an edge if and only if $i+j = 2k+1$. We establish the off-diagonal ordered Ramsey number of NM_k up to constant factors:

Proposition 3.1. *For any positive integer k,*

$$4k - 2 < r_<(NM_k, K_3) \leq 6k.$$

Proof. The lower bound follows from a simple construction: color the ordered complete graph K_{4k-2} such that $\{1, \ldots, 2k-1\}$ and $\{2k, \ldots, 4k-2\}$ form two red cliques, and all remaining edges are blue. Then there are no blue triangles, and no red edge (i,j) has $|i-j| > 2k-2$, so there cannot be a red matching on $2k$ vertices.

For the upper bound, pick an arbitrary bicoloring of K_{6k}. Suppose the graph contains no blue copies of K_3. If any vertex has blue degree at least $2k$, then there is a red clique of size $2k$, which must contain M. Otherwise, the number of blue edges is at most $6k^2$. Hence, the number of red edges is at least $12k^2 - 3k$. Let E_R be the set of red edges, and define a strict partial order on E_R by $(i,j) < (l,m)$ if $l < i < j < m$. We wish to show that there is a "chain" of edges $e_1, \ldots, e_k$ with $e_1 < \cdots < e_k$. Suppose not; define a function $L : E_R \to \{1, \ldots, k-1\}$ where $L(e)$ is the longest chain ending at e. Observe that $L^{-1}(n)$ is an "anti-chain" for each $n \in [k-1]$. That is, for any $e_1, e_2 \in L^{-1}(n)$, we cannot have $e_1 < e_2$ nor $e_2 < e_1$.

Applying the pigeonhole principle, fix some n such that $|L^{-1}(n)| \geq 12k$. For $1 \leq i \leq 6k$ let a_i be the minimum index j such that $(i,j) \in |L^{-1}(n)|$, and let b_i be the maximum such index. Then $\sum_{i=1}^{6k}(b_i + 1 - a_i) \geq 12k$, so $\sum_{i=1}^{6k}(b_i - a_i) \geq 6k$. It follows that there exist indices $i < j$ with $b_i > a_j$. But then $i < j < a_j < b_j$, so edges (i, a_i) and (j, b_j) are comparable. This contradicts our claim that $L^{-1}(n)$ is an anti-chain, so there must be a chain of length at least k. The edges in the chain comprise the red embedding of NM_k into the graph. □

We believe that the upper bound is far from optimal. In particular, we make the following conjecture.

Conjecture 3.2. *For any positive integer k,*

$$r_<(NM_k, K_3) = 4k - 1.$$

The nested matching can be used to bound the corresponding ordered Ramsey numbers for a more general class of matchings. As we will build up more complex parenthesis matchings from simpler ones, we need a way to keep track of the growth of $r_<(M, K_3)$. One approach is the following lemma:

Lemma 3.3. *Let $A_1, \ldots, A_{2k-1}$ be (possibly empty) balanced parenthesis sequences inducing matchings $M_1, \ldots, M_{2k-1}$. Then*

$$(A_1(A_2(\cdots(A_{k-1}(A_k)A_{k+1})\cdots)A_{2k-2})A_{2k-1})$$

is a balanced parenthesis sequence which induces some matching M, with

$$r_<(M, K_3) \leq r_<(NM_{k+t}, K_3),$$

where $t = \sum_{i=1}^{k} \max(r_<(M_i, K_3), r_<(M_{2k-i}, K_3))$.

Proof. Pick an arbitrary bicoloring of the complete graph on $r_<(NM_{k+t}, K_3)$ vertices. Assume that there is no blue copy of K_3. Then there is a red copy of NM_{k+t}. Starting with the innermost edge of the matching and working outwards, delete as many matched pairs as necessary until there is space for a red copy of M_k. Every deletion increases the number of inner vertices by at least one, so there will be space after at most $r_<(M_k, K_3)$ steps. Save the current innermost matched pair (which will correspond to the parentheses around A_k), and continue deleting subsequent matches until there is space for a red copy of M_{k-1} (to the left of the saved match) and a red copy of M_{k+1} (to the right of the saved match). The number of deletions is at most $\max(r_<(M_{k-1}, K_3), r_<(M_{k+1}, K_3))$; save the new innermost match.

Repeating the above process $k-2$ more times yields a complete red copy of M. Note that the process does not run out of matches, since only k matches are saved, and at most t matches are deleted. □

In the above lemma, the Ramsey number of each matching M_i is multiplied by a constant factor arising from the Ramsey number of a nested matching NM_n. It is possible to decrease the dependence on the central matching M_k, in exchange for larger constants on the remaining matchings and on the length of the matching.

Lemma 3.4. *Let $A_1, \ldots, A_{2k-1}$ be balanced parenthesis sequences inducing matchings $M_1, \ldots, M_{2k-1}$. Let M be the parenthesis matching induced by the expression*

$$(A_1(A_2(\cdots(A_{k-1}(A_k)A_{k+1})\cdots)A_{2k-2})A_{2k-1}).$$

If $l = \sum_{i \neq k} r_<(M_i, K_3)$ and $t = r_<(M_k, K_3)$, then

$$r_<(M, K_3) \leq t + 20(k + l + |A_k|).$$

Proof. Pick an arbitrary bicoloring of the ordered complete graph on $t + 20(k + l + |A_k|)$ vertices. Assume that there is no blue copy of K_3. Let X denote the first $10(k + l + |A_k|)$ vertices; let Y denote the next t vertices; and let Z denote the remaining $10(k+l+|A_k|)$ vertices. Observe that Y contains a red copy of A_k.

Suppose that there is a red copy of NM_{k+l} in $X \cup Z$, where the first $k + l$ vertices are in X and the remaining $k + l$ vertices are in Z. Then, just as in Lemma 3.3, we can start with the innermost matching and work outwards, deleting matchings to make space for red copies of $A_1, \ldots, A_{k-1}$ and $A_{k+1}, \ldots, A_{2k}$. Only l matchings need be deleted, and by the end, the graph $X \cup Y \cup Z$ contains a red copy of M.

Now suppose the converse, so the maximum number of nested matchings from X to Z is less than $k + l$. As in Proposition 3.1, define the natural strict partial order on the red edges between X and Z. A set of nested edges forms a "chain", and the largest anti-chain contains no more than $|X| + |Z| = 20(k + l + |A_k|)$ red edges. We know that the red edges can be partitioned into less than $k + l$ anti-chains, so the number of red edges between X and Z is at most $20(k + l + |A_k|)(k + l)$, which we upper bound by $20(k + l + |A_k|)^2$.

Thus, the number of blue edges between X and Z is at least $80(k + l + |A_k|)^2$. Hence there must be a vertex $v \in X$ with at least $8(k + l + |A_k|)$ blue edges into Z. Since the graph was assumed to be blue K_3-free, it follows that the set of blue neighbors of v forms a red clique of size $8(k + l + |A_k|)$. As $|M| \leq 8(k + l + |A_k|)$, we conclude that the bicoloring contains a red copy of M. □

Every parenthesis matching is in a bijection with an ordered, rooted tree. The above lemma allows us to bound the off-diagonal Ramsey number of the tree by the Ramsey numbers of all the branches off any path. Intuitively (and we will formalize the intuition later), this bound is strong on unbalanced trees and weak on well-balanced trees. For the latter case, we have the following simple lemma. While it is a special case of the above lemma aside from unimportant constant factors, we will use it for a different purpose (namely, well-balanced trees), so we state it separately for clarity.

Lemma 3.5. *Let A be a balanced parenthesis sequence inducing the matching M. Then (A) is a balanced parenthesis sequence inducing some matching M', and*

$$r_<(M', K_3) \leq r_<(M, K_3) + |M'| + 1.$$

Proof. Let $t = r_<(M, K_3)$ and let n be the number of vertices in matching M'. Pick an arbitrary bicoloring of the ordered complete graph on $[t + n + 1]$. Suppose there are no blue triangles. Then there is a red copy of M in $\{2, \ldots, t+1\}$. So if there is a red edge from 1 to any of $\{t+2, \ldots, t+n+1\}$, we have found a red copy of M'. Otherwise, every edge from 1 to $\{t + 2, \ldots, t + n + 1\}$ is blue, so $\{t + 2, \ldots, t + n + 1\}$ form a red clique of size n, which must contain the matching M'. □

With the above lemmas, we can prove a subquadratic bound on the Ramsey numbers of all balanced parenthesis matchings. Two convexity results are needed; we postpone their proofs to Appendix A.

Lemma 3.6. *Let $a_0, a_1, a_2, \ldots, a_k \geq 0$ and $\delta > 1$ and $m > 0$ be real numbers. Let $r = m^{-1/(\delta-1)}$. If $s = \sum_{i=0}^{k} a_i \geq 1$ and $a_i \leq rs$ for all $1 \leq i \leq k$, then*

$$m(a_0 + ca_1^\delta + \cdots + ca_k^\delta) \leq cs^\delta$$

for any $c \geq m$.

Lemma 3.7. *Let $a_1, \ldots a_k \geq 0$ and $\delta \geq 1$ be real numbers. Let $r \in (0, 1)$. If $s = \sum_{i=1}^{k} a_i$ and $a_i \leq rs$ for all $1 \leq i \leq k$, then*

$$a_1^\delta + \cdots + a_k^\delta \leq r^{\delta-1} s^\delta.$$

In the following proof we'll use the bijection between parenthesis matchings on n vertices and ordered rooted trees of size $s = n/2 + 1$. The basic idea is to induct on tree size and decompose the tree into smaller trees by one of two methods, depending on the relative weights of the root's child subtrees.

Call an edge r*-heavy* if $s_{\text{child}} \geq r \cdot s_{\text{parent}}$, where s_{child} is the size of the child subtree and s_{parent} is the size of the parent subtree. If the inequality does not hold, call the edge r*-light*. Similarly call a vertex r-heavy or r-light if its parent edge is r-heavy or r-light, respectively.

If all children of the root are r-light for an appropriate choice of r (slightly less than 1), we apply the inductive hypothesis to each child separately, and use Lemma 3.5 to obtain a bound for the entire tree. Since every child subtree is a constant factor smaller than the entire tree, the lemma intuitively yields a sufficiently good recurrence.

If however the root has an r-heavy child, Lemma 3.5 does not suffice. Instead we trace a path of heavy edges from the root down, decomposing the tree into a number of branches, as well as possibly some subtrees at the tail end of the path. Here we use Lemma 3.4. We know that every branch is $(1-r)$-light, so can afford to multiply the sum of Ramsey numbers of the branches by 20 in the lemma. We only know the tail subtrees to be r-light, which is why they are treated differently in the lemma.

Formalizing the above proof sketch requires some manipulation of inequalities and applications of Lemma 3.6 and Lemma 3.7. We work through these below.

Theorem 3.8. *For any $\epsilon > 0$ there is a constant c such that every parenthesis matching M on n vertices has $r_<(M, K_3) \leq cn^{1+\epsilon}$.*

Proof. Let $\epsilon > 0$. Set $r = 1 - 23^{-2/\epsilon}$, and set $c = 23/(1 - r^\epsilon)$. A parenthesis matching on n vertices uniquely corresponds with an ordered rooted tree of size $s = n/2 + 1$. We induct on the tree size s. If $s = 1$, the corresponding matching is the empty matching on 0 vertices, for which the claim is trivially true. Fix an ordered rooted tree of size $s > 1$, corresponding to a matching M. There are two cases which we will treat separately; either the tree root has an r-heavy child, or not.

Suppose that the tree root does not have an r-heavy child. Let $s_1, \ldots, s_k$ be the sizes of the child subtrees of the root. Let $M_1, \ldots, M_k$ be the matchings corresponding to the respective subtrees, and let $t_i = r_<(M_i, K_3)$ for each $i \in [k]$. With a slight abuse of notation, identifying the matchings with their parenthesis sequences, we have

$$M = (M_1)(M_2)\ldots(M_k).$$

Lemma 3.5 provides the bound $r_<((M_i), K_3) \leq t_i + 2s_i + 1$. Since the Ramsey number of a union of ordered graphs on disjoint intervals of vertices is subadditive, it follows that

$$r_<(M, K_3) \leq \sum_{i=1}^{k}(t_i + 2s_i + 1) \leq 3s + \sum_{i=1}^{k} t_i.$$

By the inductive hypothesis and Lemma 3.7 (using the assumption that every subtree is r-light), we have

$$r_<(M, K_3) \leq 3s + \sum_{i=1}^{k} cs_i^{1+\epsilon} \leq 3s + cr^\epsilon s^{1+\epsilon} \leq cs^{1+\epsilon}.$$

The last step follows since c was chosen to be sufficiently large.

The remaining case to consider is if the tree root has a heavy child. Then there is some path which starts at the root and consists entirely of heavy edges (possibly only one edge, or possibly more). Let $s_1^b, \ldots, s_k^b$ be the sizes of all subtrees which branch off the heavy path, and let s^h be the (vertex) size of the heavy path. Let $M_1^b, \ldots, M_k^b$ be the corresponding matchings, and let $t_i^b = r_<(M_i^b, K_3)$ for each $i \in [k]$. For ease of notation, suppose that the deepest vertex in the heavy path has k' children, and its child subtrees are indexed $1 \ldots k'$. The whole matching M can be decomposed into a nested matching along with embedded matchings $(M_1^b), \ldots, (M_k^b)$. For instance, if $k = 3$ and $k' = 1$ then one possibility is $M = ((M_2^b)(()(M_1^b))(M_3^b))$. By Lemma 3.5, the following bound holds for every matching M_i^b:

$$r_<((M_i^b), K_3) \leq t_i^b + 3s_i^b.$$

So by Lemma 3.4, we have

$$r_<(M, K_3) \leq \sum_{i=1}^{k'} \left(t_i^b + 3s_i^b\right) + 20\left(s^h + \sum_{i=k'+1}^{k} \left(t_i^b + 3s_i^b\right) + \sum_{i=1}^{k'} s_i^b\right).$$

By the inductive hypothesis, it follows that

$$\begin{aligned} r_<(M, K_3) \quad &\leq \quad \sum_{i=1}^{k'} \left(c\left(s_i^b\right)^{1+\epsilon} + 3s_i^b\right) \\ &\quad + 20\left(s^h + \sum_{i=k'+1}^{k} \left(c\left(s_i^b\right)^{1+\epsilon} + 3s_i^b\right) + \sum_{i=1}^{k'} s_i^b\right). \end{aligned}$$

Reordering terms and absorbing the term $3\sum_{i=k'+1}^{k} s_i^b$ into the outer constant factor through the bound $c \geq 20$, we get

$$\begin{aligned} r_<(M, K_3) \quad &\leq \quad c\sum_{i=1}^{k'} \left(s_i^b\right)^{1+\epsilon} + 23\sum_{i=1}^{k'} s_i^b \\ &\quad + 23\left(s^h + c\sum_{i=k'+1}^{k} \left(s_i^b\right)^{1+\epsilon}\right). \end{aligned} \tag{1}$$

To bound the first two terms of Equation 1, we observe that for each $i \leq k'$, subtree i is r-light, and therefore $s_i^b \leq rs$. An application of Lemma 3.7, along with the bound $cr^\epsilon + 23 \leq c$, gives

$$\begin{aligned} c\sum_{i=1}^{k'} \left(s_i^b\right)^{1+\epsilon} + 23\sum_{i=1}^{k'} s_i^b &\leq cr^\epsilon \left(\sum_{i=1}^{k'} s_i^b\right)^{1+\epsilon} + 23\sum_{i=1}^{k'} s_i^b \\ &\leq c\left(\sum_{i=1}^{k'} s_i^b\right)^{1+\epsilon}. \end{aligned} \tag{2}$$

For the remaining terms of Equation 1, observe that for any $i > k'$, subtree i has an r-heavy sibling, so s_i^b is at most $1 - r$ times the parent's subtree size, and therefore at most $(1 - r)s$. We will use one of two approaches (below, **A** and **B**) depending on the cumulative weight of these subtrees.

A. If $s^h + \sum_{j=k'+1}^{k} s_j^b \geq 23^{-1/\epsilon}s$, then we can bound $s_i^b \leq 23^{1/\epsilon}(1-r)\left(s^h + \sum_{j=k'}^{k} s_j^b\right)$ for all $i > k'$. We know that $c \geq 23$ and $23^{1/\epsilon}(1 - r) \leq 23^{-1/\epsilon}$, so an application of Lemma 3.6 yields

$$23\left(s^h + c\sum_{i=k'+1}^{k} \left(s_i^b\right)^{1+\epsilon}\right) \leq c\left(s^h + \sum_{i=k'+1}^{k} s_i^b\right)^{1+\epsilon}. \tag{3}$$

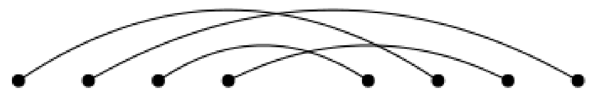

Figure 2: A matching M with interval chromatic number 2, and corresponding permutation $\pi(M) = (2, 4, 1, 3)$.

Summing together the bounds from Equation 2 and Equation 3 and applying the most basic convexity bound, we get the desired bound

$$\begin{aligned} r_<(M, K_3) &\le c\left(\sum_{i=1}^{k'} s_i^b\right)^{1+\epsilon} + c\left(s^h + \sum_{i=k'+1}^{k} s_i^b\right)^{1+\epsilon} \\ &\le cs^{1+\epsilon}. \end{aligned}$$

B. If $s^h + \sum_{j=k'+1}^{k} s_j^b < 23^{-1/\epsilon}s$, then we are unable to bound s_i^b against $s^h + \sum_{j=k'+1}^{k} s_j^b$, but we know that the latter quantity is much smaller than s. So we instead use the weak bound

$$23\left(s^h + c\sum_{i=k'+1}^{k}\left(s_i^b\right)^{1+\epsilon}\right) \le 23c\left(s^h + \sum_{i=k'+1}^{k} s_i^b\right)^{1+\epsilon}. \tag{4}$$

Now we combine Equation 2 with Equation 4, using the simple inequality $(1-x)^{1+\epsilon} + 23x^{1+\epsilon} \le 1$ for $x \in (0, 23^{-1/\epsilon})$, and obtain

$$\begin{aligned} r_<(M, K_3) &\le c\left(\sum_{i=1}^{k'} s_i^b\right)^{1+\epsilon} + 23c\left(s^h + \sum_{i=k'+1}^{k} s_i^b\right)^{1+\epsilon} \\ &\le cs^{1+\epsilon}. \end{aligned}$$

This completes the induction. □

3.2 Random Matchings with $\chi_<(M) = 2$

Recall that the *interval chromatic number* $\chi_<(G)$ of an ordered graph G is the minimum number of contiguous intervals into which the vertex set must be split so that each interval is an independent set in G.

In this section, we show that for almost every matching M with interval chromatic number 2, the bound of $\widetilde{O}(n^2)$ on $r_<(M, K_3)$ can be beaten. More specifically, we exhibit a condition on M which is sufficient to guarantee an improved bound on $r_<(M, K_3)$, and then prove that a random matching with interval chromatic number 2 satisfies this condition with high probability.

The set of matchings on $2n$ vertices with interval chromatic number 2 is in bijection with the permutation group S_n, and it is often notationally convenient to examine the permutation corresponding to a given matching. See Figure 2 for an example.

Definition 3.9. Let M be an ordered matching on $[2n]$ with interval chromatic number 2. Then its "corresponding permutation" $\pi(M)$ is the permutation on $[n]$ which maps i to $j - n$ for every edge $(i, j) \in M$.

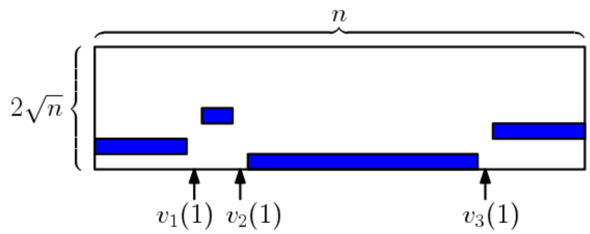

Figure 3: One possibility for the set of segments $F(1)$ in the blue adjacency matrix, if $\pi(m) = (2, 4, 1, 3)$.

Definition 3.10. We say that a given permutation $\pi \in S_n$ contains an "exact pattern" ρ if ρ is an ordered subset of $[n]$ and there are indices $1 \leq i_1 < \cdots < i_k \leq n$, where $k = |\rho|$, such that $\pi(i_j) = \rho(j)$ for all j. For instance, the permutation $\pi = (3, 5, 6, 1, 2, 4)$ contains the exact pattern $(6, 1, 4)$ but does not contain the exact pattern $(1, 2, 3)$.

We are interested in using exact patterns as a metric for the "intersection" of two permutations. Specifically, we make the following definition.

Definition 3.11. Let $\pi, \sigma \in S_n$ be permutations. Define the "ordered intersection" of π and σ, denoted $\mathrm{Int}(\pi, \sigma)$, to be the largest k such that both π and σ share an exact pattern of length k.

In the theorem below, we do something slightly stronger than bounding on the Ramsey number $r_<(M, K_3)$ for certain matchings M. Rather, we show that in a blue K_3-free graph on $2n$ vertices, there is a tradeoff between finding a red copy of the matching M in the bipartite subgraph $[1, n] \cup [n+1, 2n]$ and finding a large red clique (which of course contains every matching of that size) in $[1, n]$ or symmetrically in $[n+1, 2n]$.

Theorem 3.12. *Fix $\epsilon \in (0, 1)$ and $\alpha, \beta > 0$ with $\alpha + \beta \leq \epsilon/4$. Let M be an ordered matching on $2n^{1/2+\alpha}$ vertices with interval chromatic number 2, such that the corresponding permutation $\pi = \pi(M)$ satisfies $\mathrm{Int}(\pi(M), \pi(M) + h) \leq n^{(1-\epsilon)(1/2+\alpha)}$ for every $h \in [n^{1/2+\alpha}]$. Then every red/blue coloring of the ordered complete graph on $[2n]$ contains either:*

- *a blue copy of K_3,*
- *a red copy of $K_{n^{1/2+\beta}/4 - n^{\epsilon/4}/2}$, or*
- *a red copy of M within the bipartite subgraph $[1, n] \cup [n+1, 2n]$.*

Proof. Fix a bicoloring C of the ordered complete graph on $[2n]$, and suppose that it contains none of the hypothesized blue or red structures. Then in particular, for $1 \leq i \leq n + 1 - n^{1/2+\alpha}$, we know that there are no red copies of M between $[1, n]$ and $[n + i, n + i + n^{1/2+\alpha})$.

Fix some $i \leq n^{\epsilon/4}$. Let $v_1(i)$ be the first vertex in $[n]$ such that $C(v_1(i), i + \pi(1))$ is red (or $v_1(i) = \infty$ if no such vertex exists). Let $b_2(i)$ be the first vertex in $[n]$ after $v_1(i)$ such that $C(v_2(i), i + \pi(2))$ is red (or, again, $v_2(i) = \infty$ if no such vertex exists). Iteratively define $v_3(i), \ldots, v_{n^{1/2+\alpha}}(i)$ in the same way. Also let $f(i)$ be the first index at which $v_{f(i)}(i) = \infty$. By our assumption that $[1, n] \cup [n + i, n + i + n^{1/2+\alpha})$ is red M-free, this index exists.

The vertices $v_1(i), \ldots, v_{f(i)-1}(i)$ demarcate $f(i)$ blue segments in the adjacency matrix of $[1, n] \cup [n+1, n+2n^{1/2+\alpha}]$. That is, for $1 \leq j \leq f(i)$ we have $C(k, i+\pi(j))$ is blue for all $v_{j-1}(i) < k < v_j(i)$ (where for convenience we set $v_0(i) = 0$ and $v_{f(i)}(i) = n+1$). Treating C as an $n \times 2n^{1/2+\alpha}$ matrix, each segment is in a distinct row, and the segments occupy distinct intervals of columns, covering a total of at least $n - n^{1/2+\alpha}$ columns. If any segment had length at least $n^{1/2+\beta}$, then some vertex would have $n^{1/2+\beta}$ blue edges, so the coloring would contain either a blue triangle or a red $K_{n^{1/2+\beta}}$. So henceforth we assume that every segment has length at most $n^{1/2+\beta}$.

For each $i \leq n^{\epsilon/4}$ let $F(i)$ be the set of $f(i)$ blue segments as defined above (see Figure 3 for an example). We seek to lower bound the number of blue edges in $F(i)$ which are not contained in any $F(i')$ for $i' < i$. So fix $i' < i \leq n^{\epsilon/4}$. Suppose that there are k segments in $F(i)$ which intersect with segments in $F(i')$.

Since each segment in $F(i)$ is in a different row, as is each segment of $F(i')$, each of the k intersecting segments in $F(i)$ intersects with a unique segment in $F(i')$. Suppose that $s_1, s_2 \in F(i)$ and $t_1, t_2 \in F(i')$ where s_1 intersects t_1 and s_2 intersects t_2. Then $\text{row}(s_1) = \text{row}(t_1)$, and $\text{row}(s_2) = \text{row}(t_2)$. And since the segments $F(i)$ hit disjoint intervals of columns, as do the segments $F(i')$, we have $\text{columns}(s_1)$ is "left" of $\text{columns}(s_2)$ in the adjacency matrix if and only if $\text{columns}(t_1)$ is "left" of $\text{columns}(t_2)$. So the k intersecting segments define an exact pattern in both $\pi + i'$, which describes the row indices of the segments $F(i')$, and $\pi + i$, which describes the row indices of the segments $F(i)$. It follows that k is at most $\text{Int}(\pi, \pi + i - i')$, which is by assumption at most $n^{(1-\epsilon)(1/2+\alpha)}$. Summing over all $i' < i$, at most $in^{(1-\epsilon)(1/2+\alpha)}$ segments in $F(i)$ intersect with previous segments.

Every segment has length at most $n^{1/2+\beta}$ by assumption. Thus, for each $i \leq n^{\epsilon/4}$, the blue segments in $F(i)$ contribute at least

$$n - n^{1/2+\alpha} - in^{(1-\epsilon)(1/2+\alpha)}n^{1/2+\beta} = n - n^{1/2+\alpha} - in^{1-\epsilon/2+\beta+(1-\epsilon)\alpha}$$

new blue edges. When $i = 1$ the contribution is $n - n^{1/2+\alpha}$; when $i = n^{\epsilon/4}$, the contribution is at least $-n^{1/2+\alpha}$. The contributions decrease linearly, so in total there are at least $n^{1+\epsilon/4}/2 - n^{1/2+\alpha+\epsilon/4}$ blue edges in the bipartite graph $[1, n] \cup [n + 1, n + 2n^{\alpha+1/2}]$. So some vertex has blue degree at least $n^{1/2+\beta}/4 - n^{\epsilon/4}/2$, implying that there is either a blue triangle or a red clique of size $n^{1/2+\beta}/4 - n^{\epsilon/4}/2$. □

We seek to show that for random permutations π and for any integer h, the intersection of π with the shifted permutation $\pi + h$ is sublinear in the length of π with high probability. The general outline of the proof is as follows. We bound the expected number of long exact patterns contained in both π and $\pi + h$. To do so, we of course sum over all long exact patterns, splitting into two cases. If the exact pattern ρ has small intersection with $\rho + h$, we can straightforwardly obtain a good bound on the probability that ρ embeds into both permutations. However, if ρ has large intersection with $\rho + h$, we cannot do so. Instead we show that the number of such exact patterns is extremely small.

The following lemma formalizes the last step of the above outline.

Lemma 3.13. *Fix positive integers n, $k \leq n$, and h. Pick an exact pattern ρ of length k from $[n]$ uniformly at random. Then the probability that the set intersection $\rho \cap (\rho + h)$ has size at least t, and there exists some permutation $\pi \in S_n$ such that ρ and $\rho + h$ are both exact patterns in π, does not exceed*

$$\frac{2^{2k-t}k^{k-t}}{k!}.$$

Proof. Observe that it is possible to pick an exact pattern uniformly at random by two independent choices: first, pick an unordered subset of $[n]$ with size k. Second, pick some ordering for the subset. We will show that for any unordered subset $U \subseteq [n]$ with size k such that $|U \cap (U + h)| \geq t$, if we pick an ordering on U uniformly at random and thereby induce an exact pattern ρ, then the probability that ρ and $\rho + h$ are both exact patterns in some permutation π does not exceed $2^{2k-t}k^{k-t}/k!$. This will prove the lemma.

Fix any $U \subseteq [n]$ with $|U| = k$ and $|U \cap (U+h)| \geq t$. The number of elements $a \in U$ such that $a - h \notin U$ does not exceed $k - t$, so U can be partitioned into $k - t$ arithmetic progressions, each with common difference h.

Pick some permutation $\sigma \in S_k$. This yields an ordering of U, in which the smallest element of U is placed in position $\sigma(1)$, and so forth. Hence, an exact pattern ρ is induced. Suppose that the ordering is "compatible": that is, ρ and $\rho + h$ are both exact patterns in some permutation π. Since ρ and $\rho + h$ fix the order in π of the sets of elements U and $U + h$ respectively, it must hold that $U \cap (U + h)$ has the same order in ρ and $\rho + h$. Pick any arithmetic progression $\{a + ih\}_{i=0}^m \subseteq U$. We have that $a + ih$ precedes $a + (i+1)h$ in ρ if and only if $a + ih$ precedes $a + (i+1)h$ in $\rho + h$, or equivalently $a + (i-1)h$ precedes $a + ih$ in ρ. So the arithmetic progression must either have a monotone increasing order or a monotone decreasing order in ρ.

The key observation was that for any $a, b \in U$ where neither a nor b is the first term in its arithmetic progression, a precedes b in ρ if and only if $a - h$ precedes $b - h$. We use this observation to bound the total number of compatible orderings. There are 2^{k-t} ways to assign a direction to each progression, either monotone increasing or monotone decreasing. Fix one such assignment, and suppose that m_{inc} progressions are monotone increasing. There are at most 2^k ways to pick the subset of locations $L_{\text{inc}} \subseteq [k]$ to which the increasing-ordered progressions are assigned. It remains to pick an embedding of the increasing-ordered progressions in L_{inc}, and an embedding of the decreasing-ordered progressions in $[k] \setminus L_{\text{inc}}$. The two cases are symmetric, so we consider the increasing-ordered progressions.

For notational convenience, arbitrarily index the increasing-ordered progressions $A_1, \ldots, A_{m_{\text{inc}}}$. Now define a map $\Phi : S_{|L_{\text{inc}}|} \to L_{\text{inc}}^{m_{\text{inc}}}$ from embeddings of the increasing-ordered progressions into L_{inc} (which are in bijection with the permutations $S_{|L_{\text{inc}}|}$) to tuples $(v_1, \ldots, v_{m_{\text{inc}}})$, where v_i is the index assigned to the first element of progression A_i.

We claim that the restriction of Φ to compatible embeddings is injective. Pick two different compatible orderings of U, inducing exact patterns ρ_1 and ρ_2, and assume for the sake of contradiction that $\Phi(\rho_1) = \Phi(\rho_2)$. Suppose that j is the first index at which ρ_1 and ρ_2 differ. By assumption, the first term of each arithmetic progression has the same index in ρ_1 and ρ_2. Therefore neither $\rho_1(j)$ nor $\rho_2(j)$ is a first term in its progression. Now observe that $\rho_1(j)$ precedes $\rho_2(j)$ in ρ_1, but $\rho_2(j)$ precedes $\rho_1(j)$ in ρ_2. Hence, $\rho_1(j) - h$ precedes $\rho_2(j) - h$ in ρ_1, and in ρ_2 the opposite holds. However, $\rho_1(j) - h$ and $\rho_2(j) - h$ are both in the first $j - 1$ terms of ρ_1, which are equal to the first $j - 1$ terms of ρ_2. So one of the relative orderings is impossible! Contradiction, so the restriction of Φ is injective.

Thus, there are at most $|L_{\text{inc}}|^{m_{\text{inc}}}$ ways to compatibly embed the increasing-ordered progressions into L_{inc}, and similarly there are at most $(k - |L_{\text{inc}}|)^{k-t-m_{\text{inc}}}$ ways to embed the decreasing-ordered progressions into $[k] \setminus L_{\text{inc}}$. So the total number of compatible orderings is at most $2^{k-t} 2^k k^{k-t}$. Since the total number of orderings is $k!$, the result follows. ☐

Now we can prove our desired result on random permutations.

Lemma 3.14. *Fix some $\alpha > 0$ and some positive integers n and h. If $\pi \in S_n$ is a permutation chosen uniformly at random, then*

$$\Pr\left[\operatorname{Int}(\pi, \pi + h) \geq n^{2/3+\alpha}\right] \leq \left(e^5 n^{-3\alpha/2}\right)^{n^{2/3+\alpha}}.$$

Proof. We proceed by bounding the expected value of $\operatorname{Int}(\pi, \pi + h)$. Let $k = n^{3/4}$. Pick any exact pattern ρ of size k in $[n]$. Then ρ is contained in both π and $\pi + h$, for any permutation $\pi \in S_n$, if and only if ρ and $\rho - h$ are both contained in π. If the smallest element of ρ is less than $h+1$, then $\rho - h$ cannot be contained in any permutation, so assume the contrary.

The probability that ρ and $\rho - h$ are both exact patterns in a random permutation $\pi \in S_n$ is at most the probability that $\rho \cap (\rho - h)$ and $\rho \setminus (\rho - h)$ and $(\rho - h) \setminus \rho$ are all exact patterns in π. Here, the intersection/difference of two exact patterns is taken to be the set-theoretic intersection/difference, ordered according to whichever exact pattern contains the set (and picking either pattern if both contain the set). But these three exact patterns are disjoint, so the corresponding events are independent. Suppose that $m(\rho) = |\rho \cap (\rho - h)|$. Since an exact pattern of length r is contained in a random permutation with probability $1/r!$, we have that ρ and $\rho - h$ are contained in a random $\pi \in S_n$ with probability at most

$$\frac{1}{m(\rho)!} \cdot \frac{1}{(k - m(\rho))!^2}.$$

Observe that as a function of m, the above fraction is largest when $m(\rho) \approx k - \sqrt{k}$, and is increasing on $[1, k - \sqrt{k}]$ and decreasing on $[k - \sqrt{k}, k]$. Hence, the bound is strong for $m(\rho)$ small. Summing over all exact patterns ρ with $m(\rho) \leq k/2$, and using the trivial bound that the number of exact patterns is $n!/(n-k)!$, we have that

$$\mathbb{E}\left[\# \text{ contained patterns } \rho \text{ with } m(\rho) \leq k/2\right] \leq \frac{n!}{(n-k)!(k/2)!^3}.$$

The expectation is taken over permutations $\pi \in S_n$, and a "contained pattern" is an exact pattern ρ such that ρ and $\rho - h$ are contained in π.

To bound the expectation for patterns ρ with $m(\rho) > k/2$, we first discard the patterns ρ for which there is no permutation π containing both ρ and $\rho - h$. Now Lemma 3.13 gives that the number of remaining patterns is only

$$\frac{2^k k^{k/2}}{k!} \frac{n!}{(n-k)!} = \binom{n}{k} 2^k k^{k/2}.$$

Using this result and assuming the worst case that $m(\rho) = k - \sqrt{k}$, we get

$$\mathbb{E}\left[\# \text{ contained patterns } \rho \text{ with } m(\rho) > k/2\right] \leq \binom{n}{k} \frac{2^k k^{k/2}}{(k - \sqrt{k})!(\sqrt{k})!^2}.$$

Putting everything together, simplifying, and substituting $k = n^{2/3+\alpha}$,

$$\begin{aligned}
\mathbb{E}\left[\# \text{ contained patterns}\right] &\leq \frac{n!}{(n-k)!(k/2)!^3} + \binom{n}{k} \frac{2^k k^{k/2}}{(k - \sqrt{k})!} \\
&\leq \frac{n^k (2e)^{3k/2}}{k^{3k/2}} + \frac{n^k 2^k e^{2k}}{k^{k/2} (k - \sqrt{k})^{k - \sqrt{k}} k^{\sqrt{k}}} \\
&\leq \frac{n^k (2e)^{3k/2}}{k^{3k/2}} + \frac{n^k 2^{k + 2\sqrt{k}} e^{2k}}{k^{3k/2}} \\
&\leq \left(e^5 n^{-3\alpha/2}\right)^{n^{2/3+\alpha}}.
\end{aligned}$$

□

The above lemma and Theorem 3.12 imply the main result of this section—a subquadratic bound on $r_<(M, K_3)$ for random matchings with interval chromatic number 2—as a corollary.

Theorem 3.15. *Let M be an ordered matching on $2m$ vertices with interval chromatic number 2, picked uniformly at random. Then there is a constant c such that*

$$r_<(M, K_3) \leq c m^{24/13}$$

with high probability.

Proof. Setting $\delta = 4/\log m$ the statement of Lemma 3.14 becomes

$$\Pr_{\pi \in S_m}\left[\mathrm{Int}(\pi, \pi + h) \geq e^4 m^{2/3}\right] \leq e^{-e^4 m^{2/3}}.$$

Picking a matching M on $2m$ vertices with interval chromatic number 2 uniformly at random, we have $\mathrm{Int}(\pi(M), \pi(M) + h) \leq m^{2/3+4/\log m}$ for all $h \in [m]$ with high probability. Thus we can apply Theorem 3.12 with parameters $\epsilon = 1/3 - 4/\log m$ and $\alpha = \beta = 1/24 - 1/(2\log m)$ and $n = cm^{24/13}$, where c is chosen sufficiently large that

$$n^{13/24-1/(2\log m)}/4 - n^{1/12-1/\log m}/2 \geq 2m$$

and

$$2n^{13/24-1/(2\log m)} \geq 2m.$$

So with high probability, every bicoloring of $[2n]$ contains either a blue triangle or a red copy of M or a red clique of size at least $2m$. □

4 Future Work

Our work on biclique partitions shows that $\mathrm{bp}_{\{k\}}(K_n) = n + o(n)$ for all k. This bound is asymptotically tight, since there is a lower bound of $\mathrm{bp}_{\{k\}}(K_n) \geq n - 1$. However, the original conjecture of de Caen, Gregory, and Pritikin remains open: does $\mathrm{bp}_{\{k\}}(K_n) = n - 1$ for all sufficiently large n, for each k? We suspect that a very different approach is necessary, since our proof seems to incur unavoidable losses through its use of the design.

From our study of ordered Ramsey numbers, many open questions remain. Most significant, perhaps, is the original question posed by Conlon, Fox, Lee, and Sudakov: does there exist some $\epsilon > 0$ such that $r_<(M, K_3) \leq n^{2-\epsilon}$ for every ordered matching M on n vertices? Based on our Theorem 3.15, a number of natural intermediate questions arise. In particular, a reasonably modest step beyond random matchings with $\chi_<(M) = 2$ would be the following:

Conjecture 4.1. *For every χ, there is a constant $\epsilon(\chi) > 0$ such that*

$$r_<(M, K_3) \leq O(n^{2-\epsilon(\chi)})$$

for almost every ordered matching M on n vertices with interval chromatic number $\chi_<(M) = \chi$.

Conversely, we are curious how far from the truth the exponent $\frac{24}{13}$ in our Theorem 3.15 is. It seems plausible that our argument can be optimized to produce a significantly better bound, and we do not know of any lower bounds for this class of matchings that come anywhere near this bound.

Regarding parenthesis matchings, we were unable to find a family for which $r_<(M, K_3)$ is superlinear, leaving a slight gap beneath our upper bound. Such a construction would be quite interesting to us.

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A Convexity Inequalities

We provide here proofs of Lemma 3.6 and Lemma 3.7.

Lemma 3.6. *Let $a_0, a_1, a_2, \ldots, a_k \geq 0$ and $\delta > 1$ and $m > 0$ be real numbers. Let $r = m^{-1/(\delta-1)}$. If $s = \sum_{i=0}^{k} a_i \geq 1$ and $a_i \leq rs$ for all $1 \leq i \leq k$, then*

$$m(a_0 + ca_1^\delta + \cdots + ca_k^\delta) \leq cs^\delta$$

for any $c \geq m$.

Proof. Suppose that $0 < a_i \leq a_j < 2^{-1/(\delta-1)}s$ for some distinct indices $1 \leq i, j \leq k$. Since $f(x) = x^\delta$ is a convex function, if we decrease a_i and increase a_j by a common amount $\min(a_i, rs - a_j)$, the left-hand side of the inequality increases, while the right-hand side remains constant. Furthermore, the number of values a_i which are equal to neither 0 nor rs decreases. Hence, it suffices to prove the inequality in the case where no two such values exist. Without loss of generality, we have $a_1 = \cdots = a_{n-1} = rs$ and $a_{n+1} = \cdots = a_k = 0$. Observe that $n - 1 = (s - a_0 - a_n)/(rs)$.

Now we have

$$\begin{aligned} m(a_0 + ca_1^\delta + \cdots + ca_k^\delta) &= ma_0 + mc(n-1)(rs)^\delta + mca_n^\delta \\ &= ma_0 + mc(s - a_0 - a_n)(rs)^{\delta-1} + mca_n^\delta \\ &\le ma_0 + mc(s - a_0)(rs)^{\delta-1} \\ &\le ca_0 + c(s - a_0)s^{\delta-1} \\ &\le cs^\delta \end{aligned}$$

where the first inequality holds since $a_n \le rs$, so $mca_n^\delta \le mca_n(rs)^{\delta-1}$; the second inequality holds by the assumption $r = m^{-1/(\delta-1)}$; and the third inequality holds since $s^{\delta-1} \ge 1$. □

Lemma 3.7. *Let $a_1, \ldots a_k \ge 0$ and $\delta \ge 1$ be real numbers. Let $r \in (0, 1)$. If $s = \sum_{i=1}^k a_i$ and $a_i \le rs$ for all $1 \le i \le k$, then*

$$a_1^\delta + \cdots + a_k^\delta \le r^{\delta-1}s^\delta.$$

Proof. As in the previous lemma, we only need to prove the case where $a_1 = \cdots + a_{n-1} = rs$ and $a_{n+1} = \cdots = a_k = 0$, since all other cases can be "sharpened" into this one. As before but dropping the a_0-term, $n - 1 = (s - a_n)/(rs)$. The bound is now simple:

$$\begin{aligned} a_1^\delta + \cdots + a_k^\delta &= (n-1)(rs)^\delta + a_n^\delta \\ &= (s - a_n)(rs)^{\delta-1} + a_n^\delta \\ &\le r^{\delta-1}s^\delta. \end{aligned}$$

□