Generalizing Auxeticity to Aperiodic Networks

SPUR Final Paper, Summer 2018

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August 1, 2018

Abstract

Materials with a negative Poisson's ratio are defined to be auxetic, causing them to contract along every axis or stretch along every axis when deformed. This property can be represented mathematically as an auxetic deformation of an underlying network. Traditionally, auxeticity defined in terms of a lattice structure has been used to classify deformations of a network. In this paper, we construct an equivalent lattice-independent definition of auxeticity, which generalizes well to arbitrary networks. Using this, we are able to methodically classify the auxeticity of aperiodic networks constructed by the cut-and-project method. Notably, in 2 dimensions, the set of infinitesimal auxetic deformations at the identity can be reduced to a linear algebra condition. We investigate the implications of our results on the Penrose tiling, constructing multiple auxetic deformations.

Contents

1 Introduction

Most materials when stretched in one direction, experience an contraction along a secondary direction. Such a property is self-evident in objects like putty and rope but is also present in most objects around us. This property is captured in the Poisson's ratio which relates the transverse strain to the longitudinal strain in the directional of a contraction/expansion. A few select materials have a negative Poisson's ratio, which allows them to do the reverse: when stretched in one direction, they stretch in all directions. These materials exhibit what is known as auxetic behavior [6].

Perhaps what makes auxetic materials counterintuitive is the fact that when an auxetic material is stretched, it undergoes a noticeable volume increase [6]. Auxetic materials also tend to have non-uniform density, allowing the inner workings of the network structure to move freely. As a result, auxetic materials have often been compared to foams, sharing many similarities [5]. The classic example of an auxetic material is paper. When a piece of paper is crumbled to form a 3-D object, stretching the paper along one direction causes it to expand in all directions [11].

Due to their odd physical properties, auxetic materials have seen a rise in use in recent years. Most notably, they are used practically in shock absorbing materials such as packing foam, sponges, and body armor [6]. As a result of their utility, it is important to have a purely mathematical foundation for defining and understanding the mechanics behind auxeticity. To do so, we consider the underlying network of a material:

We define a *network* as a graph[1] $G = (V, E)$ embedded in $\mathbb{R}^d$, with potentially infinitely many vertices but with finite degree at every vertex, with all vertices $v_1 \neq v_2 \in V$ satisfying $|v_1 - v_2| > \epsilon$ for some $\epsilon > 0$. We then define a *configuration space* $\mathcal{C}(G)$ consisting of all graphs $G' = (V', E')$ such that there exists an isomorphism $\phi : G' \to G$ with the property

$$\forall e' \in E', \quad |\phi(e')| = |e'|$$

We define a *one-parameter deformation* (e.g. a deformation that occurs over time) as a continuous function $f(t) : [a, b] \to \mathcal{C}(G)$ on the vertices of a graph G. Within a configuration space, there are also multiple *deformation spaces* each consisting of the space of networks that can be obtained from one other through one-parameter deformations.

We consider one-parameter deformations $f(t)$ in the deformation space of a graph G. One important question to ask is: when is an one-parameter deformation *auxetic*? Defined informally in physics, a material is *auxetic* if and only if an expansion along some axis causes an expansion in a perpendicular axis or if a contraction along some axis causes an contraction in a perpendicular axis.

The classic contrast of an auxetic network vs. a non-auxetic network is in Figure 1. There, both hexagons tiles tessellate the plane and create uniform networks. The convex hexagon is like most materials: when stretched horizontally, it contracts slightly in the vertical direction. The concave hexagon, on the other hand, is auxetic: its specialized network structure forces the tiles to expand in all directions when expanded horizontally. Although both networks are in the same deformation space, the concavity of the hexagon lends it auxetic properties.

One definition for auxeticity in literature [7] is known as *Expansive Auxeticity*:

[1] Throughout this paper, a graph generally refers to its embedding

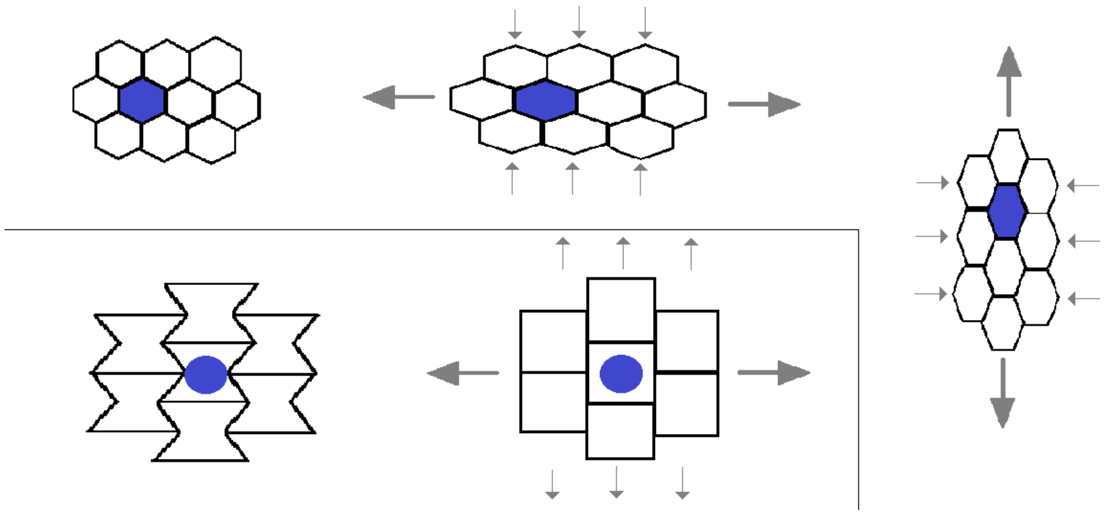


Figure 1: Non-auxetic hexagon (top, right) and auxetic hexagon (bottom-left). The peculiar network structure of the auxetic hexagon compels it to expand when horizontally stretched. Image from [12]

Definition 1. *Given graphs $G = (V, E)$ and $G' = (V', E')$ in a deformation space, the map $\phi : G \to G'$ is an **expansive auxetic deformation** if $\forall v, w \in V$ with $vw \notin E$*

$$|v - w| \leq |\phi(v) - \phi(w)|$$

Definition 2. *Given graphs $G = (V, E)$ and $G' = (V', E')$ in a deformation space, the map $\phi : G \to G'$ is a **strict expansive auxetic deformation** if there exists some epsilon $\epsilon > 0$, such that $\forall v, w \in V$ with $vw \notin E$*

$$|v - w|(1 + \epsilon) \leq |\phi(v) - \phi(w)|$$

Definition 3. *An one-parameter deformation $f(t)$ is considered an **expansive auxetic path** over the interval $[a, b]$ if, for $t_1 < t_2 \in [a, b]$, the map $\phi : f(t_1) \to f(t_2)$ is an expansive auxetic deformation. A **strict expansive auxetic path** is defined similarly.*

In other words, this definition of expansive auxeticity captures the idea that when a network stretches in all directions, pairwise distances generally increase in length. It can be checked that the one-parameter deformation of a concave hexagon is indeed expansive auxetic by the definitions provided.

Although we are only considering auxetic paths in which the network is stretched, by proceeding in the negative direction of a one-parameter deformation, we can also obtain an auxetic path in which the network is contracted instead. Therefore, any definition or theorem proven for an "auxetic stretch" can be re-expressed in an equivalent fashion for an "auxetic contraction".

2 Auxeticity in Periodic Networks

We consider periodic networks as defined below:

Definition 4. *A network of a graph G in $\mathbb{R}^d$ is* ***periodic*** *if and only if the group of translational symmetries $T(G)$ is isomorphic to $\mathbb{Z}^d$.*

Definition 5. *Let the matrix Λ be the matrix whose columns are the vectors $v_1, v_2, \ldots, v_d$ which comprise a d-dimensional lattice of G. Let $\pi(G)$ represent the corresponding lattice.*

It will be useful to split a periodic network into vertex orbits obtained by following lattice vectors.

Lemma 2.1. *A periodic network has finitely many vertex orbits.*

Proof. Assume for the sake of contradiction that there are infinitely many vertex orbits. Then inside a parallelepiped formed by the vectors of Λ, there is at least one representative of each vertex orbit. But having infinitely many vertices inside a parallelepiped contradicts the fact that our embedded graph G satisfies $|v_1 - v_2| > \epsilon$ for some $\epsilon > 0$. □

In literature [7], there is another notion of auxeticity tailored for periodic networks, which we will define as follows:

Definition 6. *Given graphs $G = (V, E)$ and $G' = (V', E')$ in a deformation space, the map $\phi : G \to G'$ is a* ***lattice auxetic deformation*** *if the linear operator $T : \pi(G) \to \pi(G')$ is* ***amplifying****.*

Definition 7. *A one-parameter deformation $f(t)$ is a* ***lattice auxetic path*** *on the interval $[a, b]$ if and only if for every two points $t_1 < t_2 \in [a, b]$, the linear operator $T_{t_1t_2} : \pi(f(t_1)) \to \pi(f(t_2))$ is* ***amplifying****.*

The term “amplifying” is defined as follows:

Definition 8. *A linear operator T is* ***amplifying*** *if and only if*

$$\inf_{|x|=1} |Tx| \geq 1$$

Similarly, a linear operator T is ***strict amplifying*** *if and only if*

$$\inf_{|x|=1} |Tx| > 1$$

One can similarly define notions of a *strict lattice auxetic deformation* and a *strict lattice auxetic path* from the previous definition. The canonical example of a periodic network that never allows for a strict lattice auxetic path is $\mathbb{Z}^d$ embedded in $\mathbb{R}^d$. Despite a large deformation space, all deformations of $\mathbb{Z}^d$ simply contract along one axis and stretch along another.

A one-parameter deformation is a continuous motion, thus empirically proving that there exists an auxetic deformation between every pair of networks may be computationally difficult. However, by looking at the derivative of the Gram matrix $w_t == \Lambda_t^T \Lambda_t$, it can be reduced to a condition of semi-definiteness:

Theorem 2.2. *A one-parameter deformation of a periodic network is lattice auxetic if and only if the derivative of the Gram matrix $w_t = \Lambda_t^T \Lambda_t$ is positive semi-definite.*

Proof. For $t \in [a, b]$, the Gram matrix is equivalent to

$$w_r = \Lambda_a^T T_{at}^T T_{at} \Lambda_a$$

$$(w_r)' = \Lambda_a^T (T_{at}^T T_{at})' \Lambda_a$$

Since Λ_a is non-singular, it is equivalent to show that $(T_{at}^T T_{at})'$ is positive semi-definite.

$$\begin{aligned} 0 &\leq x^T (T_{at}^T T_{at})' x \\ \Leftrightarrow |T_{at}x| &\text{ is non-decreasing} \\ \Leftrightarrow |T_{at_1}x| &\leq |T_{at_2}x| \qquad (\forall t_1 < t_2 \in [a, b]) \\ \Leftrightarrow |T_{at_1}x| &\leq |T_{t_1 t_2} T_{at_1} x| \end{aligned}$$

T_{at_1} is also non-singular so by substituting $y = T_{at_1}x$ this expression is equivalent to

$$|y| \leq |T_{t_1 t_2} y|$$

which is equivalent to the definition of lattice auxeticity. □

By replacing all inequalities with their strict versions, we can also arrive at the following corollary:

Corollary 2.2.1. *A one-parameter deformation of a periodic network is strict lattice auxetic if and only if the derivative of the Gram matrix $w_t = \Lambda_t^T \Lambda_t$ is positive definite.*

The above theorem also provides a simple and easy to use method of defining when an one-parameter deformation is lattice auxetic. In particular, we can use it prove that $\mathbb{Z}^d$ has no strict lattice auxetic paths:

Theorem 2.3. *The graph $\mathbb{Z}^d$ embedded in $\mathbb{R}^d$ has no strict auxetic paths.*

Proof. Consider a one-parameter deformation $f(t)$. The columns c_i of matrix Λ_t, which represent edges of the graph, are fixed to some constant length l_i throughout the deformation. Thus, in the product $\Lambda_t^T \Lambda_t$, the diagonal entries are fixed to the values l_i^2. This implies the derivative has zeroes along the diagonal, which can never be positive definite. □

Theorem 2.2 has seen use in more complex examples. In particular, it was used to classify the α to β transition of cristobalite to be auxetic [4].

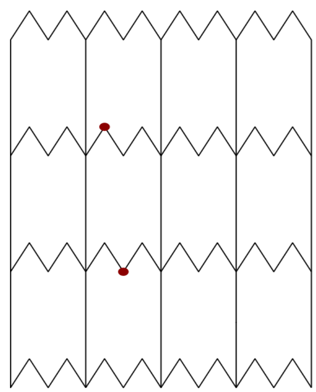
This definition of auxeticity for periodic networks is a compelling characterization since, intuitively, it correlates exactly with the notion of a material contracting or stretching along a particular axis: locally, vertices could move closer together, but, globally, the overall movement of the vertices is a stretch. The definition of expansive auxeticity is much stronger, asking for all pairwise distances to increase. Perhaps unsurprisingly, the following theorem is true:

Theorem 2.4. *Expansive Auxeticity implies Lattice Auxeticity, and Strict Expansive Auxeticity implies Strict Lattice Auxeticity*

Proof. It suffices to prove the strict case, since setting $\epsilon = 0$ is essentially equivalent to the non-strict case.

Now consider a particular vertex orbit of a vertex $p \in V$. The set of rational multiples of the lattice vectors of Λ forms a dense subset of the space outside of the unit ball centered at p. This set of vectors also correspond to edges between vertices of the vertex orbit of p. Therefore, after the map ϕ, by the expansive property, all of these vectors remain outside the ball of radius $1 + \epsilon$. However, since these vectors are also a linear combination of lattice vectors, the action of ϕ on these vectors is identical to the action of T. This is enough to conclude that $|Tv|$ is at least $(1+\epsilon)|v|$. $\square$

The converse is unfortunately not true. A clean example is the following:



As this lattice is stretched horizontally, the two red vertices will actually move closer together, despite the lack of a vertical contraction.

The fundamental reason that Theorem 2.4 works is that given a fixed vertex $v \in V$ and a direction vector $\vec{d}$, there exists a sequence $\{v_i\} \in V$ such that $\overrightarrow{vv_i}$ approaches the direction $\vec{d}$. This is a common theme that will continue to resonate throughout this paper.

3 Alternate Definition of Auxeticity

Unfortunately, since lattice auxeticity is defined in terms of the lattice, it does not generalize to aperiodic networks and other more general graphs. And yet, lattice auxeticity captures the intuitive notion of vertices generally moving further apart even when individual vertices may move closer together. This motivates defining an alternate definition of auxeticity that captures the essence of lattice auxeticity while generalizing well to arbitrary graphs.

Definition 9. *Given a embedded graph $G = (V, E)$, a subset $P \subset V$ is **relatively dense**[2] if there exists a real number $N > 0$ such that for every d-dimensional ball in $\mathbb{R}^d$ of radius N, at least one point of P is contained in it.*

Definition 10. *Given graphs $G = (V, E)$ and $G' = (V', E')$ in a deformation space, the map $\phi : G \to G'$ is a **P-auxetic deformation**, if there exists a real number $N > 0$ and a relatively dense vertex subset $P \subset V$, such that $\forall v, w \in P$*

[2]Nomenclature for a set of points in a metric space with finite covering radius

$$|v-w| \leq |\phi(v)-\phi(w)|$$

Similar to expansive auxeticity, it is useful to define a strict form of P-auxeticity with the alternate condition of

$$\exists\ \epsilon > 0,\ \text{s.t.}\ \forall v, w \in P,\ |v-w|(1+\epsilon) \leq |\phi(v)-\phi(w)|$$

In essence, P-auxeticity says that there exists some "abundant" subset of the vertices such that all pairwise distances increase among them. Such a subset can be quite sparse implying that this definition tells us something interesting about the global movement of vertices. In some ways, it is similar to that of lattice auxeticity, since a vertex orbit in itself is a representative of the overall network.

The crux of this definition is encoded in the following theorem:

Theorem 3.1. *For a periodic deformation $\phi : G \to G'$, the linear operator $T : \pi(G) \to \pi(G')$ satisfies $|Tv| \geq (1+\epsilon)|v|$ if and only if there exists a relatively dense subset $P \in V$ such that $\forall v, w \in P$*

$$|v-w|(1+\epsilon) \leq |\phi(v)-\phi(w)|$$

Proof. For the forward direction, take a vertex orbit as a subset P (which is clearly relatively dense), then the action of ϕ is identical to that of the operator T, so all pairwise distances increase by a factor of at least $1+\epsilon$.

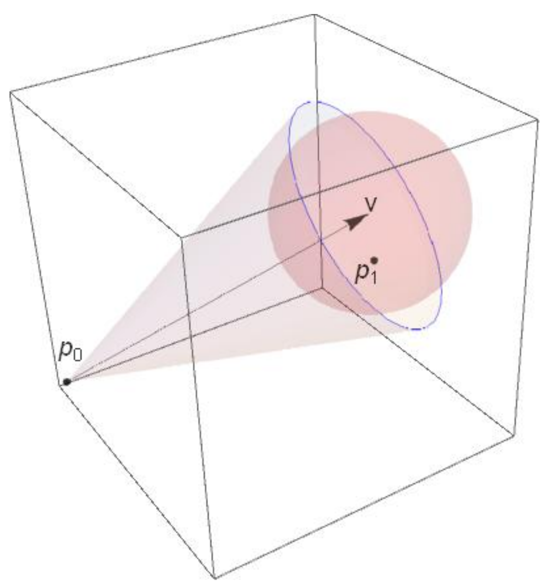


Figure 2: A cone of all vectors within θ_ϵ of v containing a ball with a vertex p_1

For the reverse direction, we wish to show that $|Tv| \geq (1+\epsilon)|v|$ given all pairwise distances among a relatively dense subset P increase by a factor of $1+\epsilon$. Fix a particular v and fix a vertex $p_0 \in P$. Now we choose another epsilon θ_ϵ and construct a cone originating at p_0 consisting all of points $q_0 \in \mathbb{R}^d$ such that the angle between $\overrightarrow{p_0q_0}$ and v is at most θ_ϵ. For every N, we can fit a ball of radius N inside the cone, as shown in Figure 2. In particular, by the relatively dense condition, there must exist another point $p_1 \in P$ inside the cone. Then construct another cone originating at p_1 consisting of all points q_1 such that the angle between $\overrightarrow{p_1q_1}$ and v is at most θ_ϵ to construct a second point p_2, and so on. Eventually, two of the points p_i and p_j will be of the same vertex orbit by Lemma 2.1. Because of this, ϕ acting on $\overrightarrow{p_ip_j}$ is identical to T acting on $\overrightarrow{p_ip_j}$. Thus,

$$|\phi(\overrightarrow{p_ip_j})| \geq (1+\epsilon)|\overrightarrow{p_ip_j}|$$
$$|T(\overrightarrow{p_ip_j})| \geq (1+\epsilon)|\overrightarrow{p_ip_j}|$$

As $\theta_\epsilon \to 0$, $\overrightarrow{p_ip_j} \to v$, so by continuity, $|Tv| \geq (1+\epsilon)|v|$ as desired. □

Corollary 3.1.1. *A periodic deformation ϕ is lattice auxetic iff it is P-auxetic; and is strict lattice auxetic iff it is strict P-auxetic.*

In particular, Theorem 3.1 implies more than just Corollary 3.1.1; it also states that the ϵ used in both definitions can be freely translated from one definition to the other without any "transactional loss". This suggests that it is well-defined to talk about the extent to which a deformation is auxetic. For the remainder of this paper, we will use the definition of P-auxetic interchangeably with auxetic.

4 Auxeticity in Aperiodic Networks

Quasicrystals are defined as seemingly crystalline structures that lack the periodic structure of actual crystals. In 1982, the first quasicrystal, a metal alloy sample, was discovered in the U.S. National Bureau of Standards by Dan Shechtman who noticed that the alloy had an odd 10-fold diffraction pattern. His discovery was meet with much skepticism and ridicule from the scientific community, including from colleagues and Nobel Laureate Linus Pauling [1]. In 2009, a mineralogical study discovered a naturally forming quasicrystal known as icosahedrite (chemical formula $Al_{63}Cu_{24}Fe_{13}$) [3]. In 2011, almost 30 years later, Dan Shechtman won the Nobel Prize in Chemistry for his discovery of quasicrystals. Quasicrystals, like auxetic materials, exhibit unusual properties, making them useful in certain applications such as in razor blades and heat insulation [1]. As more natural and artificial quasicrystals are inevitably discovered, it is important to generalize notions of auxeticity to them. We attempt to apply P-auxeticity to quasicrystals by studying a class of aperiodic networks (i.e. networks which are not periodic).

As a sidenote, there is some dissonance in literature for the mathematical definition of a quasicrystal. The cut-and-project construction proposed later in this section is one potential definition [8]. We will use the term "aperiodic network" to avoid confusion.

Perhaps the most famous aperiodic network ever discovered was the Penrose tiling using only 2 tiles, put forth by Sir Roger Penrose, seen in Figure 3. The Penrose tiling is made of rhombi whose angles are in convenient mutliples of $\frac{2\pi}{5}$ to neatly tessellate the plane. This gives the Penrose tiling the rare proprety of 5-fold symmetry, a property that periodic networks cannot exhibit due to the crystallographic restriction [2]. Most notably, despite being aperiodic, there are only 5 particular types of edges that are used in the graph. When viewed at a particular angle, it appears as the projection of 5-dimensional hypercubes onto 2 dimensions.

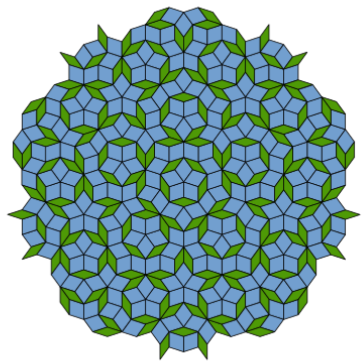


Figure 3: The Penrose tiling colored for contrast. Image from [13]

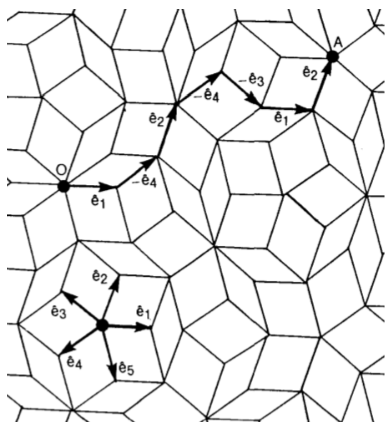


Figure 4: Every edge of the Penrose tiling is made up of one of the 5 vectors e_1, e_2, $\ldots$, e_5. A walk along the edges is depicted above. Image from [10]

De Bruijn showed that the Penrose tiling arises from the "cut-and-project" method [1]:

Definition 11. *Given a hyperplane S of dimension r embedded in $\mathbb{R}^d$ with a d-dimensional lattice structure Γ (not necessarily comprised of the elementary basis vectors), the **cut-and-project** method is as follows: there is a r-dimensional **slice** defined as*

$$\left\{x + \sum_{i=1}^{d} l_i e_i | x \in S, 0 \leq l_i \leq 1\right\}$$

where e_i are the elementary basis vectors of $\mathbb{R}^d$. A tiling is formed by orthogonally projecting all lattice points of Γ and edges between adjacent lattice points on or inside the slice down onto S.

Intuitively, one can think of the slice as moving a hypercube along the hyperplane S without rotations.

In general, it has been shown that if the hyperplane S is irrationally sloped (i.e. if it passes through the origin, it passes through no other points with integer coordinates), then the tiling produced is aperiodic by its relationship to Sturmian words [9]. The classic example is projecting from 2 dimensions to 1 dimension seen in Figure 5. There, the slice is defined to be the set of lattice points between two parallel 1-dimensional lines. The pattern of vertical and horizontal edges produces an aperiodic pattern if and only if the slope of the line is irrational.

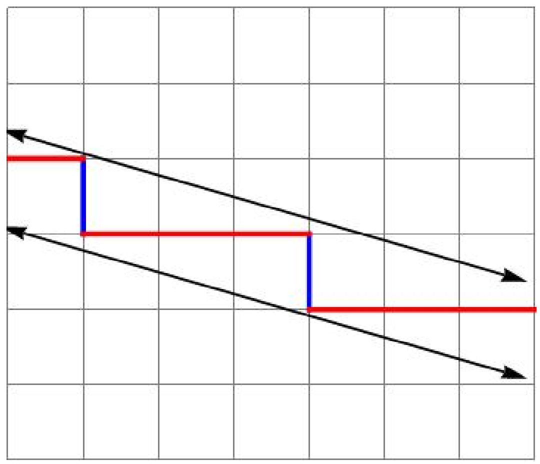


Figure 5: A slice of a 2-dimensional space onto a 1 dimensional plane (i.e. line). With a slope of $-\frac{1}{\sqrt{12}}$, the projections of the red and blue edges onto the line forms an aperiodic pattern.

For the case of the Penrose tiling, the basis vectors that span S are as follows:

$$V_{\parallel} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sqrt{\frac{2}{5}} \begin{bmatrix} 1 & 0 \\ \cos\left(\frac{2\pi}{5}\right) & \sin\left(\frac{2\pi}{5}\right) \\ \cos\left(\frac{4\pi}{5}\right) & \sin\left(\frac{4\pi}{5}\right) \\ \cos\left(\frac{6\pi}{5}\right) & \sin\left(\frac{6\pi}{5}\right) \\ \cos\left(\frac{8\pi}{5}\right) & \sin\left(\frac{8\pi}{5}\right) \end{bmatrix}$$

To make use of the symmetry, it makes sense to define the 3 vectors that span the space orthogonal to the plane:

$$V_{\perp} = \begin{bmatrix} v_3 & v_4 & v_5 \end{bmatrix} = \sqrt{\frac{2}{5}} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{4\pi}{5}\right) & \sin\left(\frac{4\pi}{5}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{8\pi}{5}\right) & \sin\left(\frac{8\pi}{5}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{12\pi}{5}\right) & \sin\left(\frac{12\pi}{5}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{16\pi}{5}\right) & \sin\left(\frac{16\pi}{5}\right) & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Most notably, these vectors are pairwise orthonormal, forming the useful relation:

$$V \cdot V^T = I \text{ for } V = \begin{bmatrix} V_{\parallel} & V_{\perp} \end{bmatrix} \tag{1}$$

Such matrices $V_{\|}$ and $V_{\perp}$ can be generated in general for arbitrary dimensions.

We now consider the lattice vectors that comprise Γ. Define E as the matrix whose columns are these lattice vectors. For the usual Penrose tiling, this matrix is the identity. Then the projection vectors $v_{\|}$ and $v_{\perp}$ of the lattice point $c = \langle c_1, c_2, \ldots, c_d \rangle \in \Gamma$ onto the hyperplane are equal to

$$v_{\|} = V_{\|}^T E c$$

$$v_{\perp} = V_{\perp}^T E c$$

While the $v_{\|}$ portion is free to populate the hyperplane, only lattice points c's that produce fairly small $v_{\perp}$ to remain in the slice are actually considered in our projections. Thus there is a need to classify the set of points c that produce suitable $v_{\perp}$ that lie within the slice.

Notice that in the matrix expression $V_{\perp}^T E$, the nullspace is exactly the dimension r space spanned by $V_{\|}$. This implies the following: If one starts at a lattice point c and adds a suitable multiple of a vector v on S to be arbitrarily close to another lattice point $c' \approx c + kv$, then the $v_{\perp}$ of both lattice points will be approximately the same. Therefore, if lattice point c is in the slice, lattice point c' must also be in the slice. Thus there is an approximate lattice that appears among the points in the slice, spanned by the column vectors of $V_{\|}$.

Motivated by the previous result, we look at the expression:

$$\left(V_{\|}^T E\right) V_{\|}$$

The result of this expression is a r by r matrix representing the distance moved along the hyperplane after a step along a column vector of $V_{\|}$. In particular, this is analogous to the matrix Λ of a periodic network if one treats the column vectors as a lattice. Using this, we can arrive at a definition of auxeticity for aperiodic networks. However, a bit more work is needed beforehand in defining a proper deformation of a cut-and-project construction:

Recall that the matrix E represents the lattice vectors of Γ. As we perform deformations on Γ, the matrix E will change, causing deformations in the Penrose tiling. However, taking an arbitrary deformation of Γ, does not guarantee a valid deformation of an aperiodic tiling for two reasons. One reason is that additional lattice points may enter the slice and others may exit. If this occurs, vertices may randomly appear and disappear on the hyperplane, which does not constitute a proper deformation.

To enforce this, the matrix product $V_{\perp}^T E$ must be fixed. By keeping this quantity fixed, all normal vectors $v_{\perp}$ are held fixed, thus exactly those lattice points who were in the slice to begin with remain in the slice. If not, over time, the error term $V_{\perp}^T (E_{t_1} - E_{t_2}) c$ between timesteps t_1 and t_2 will be large for some lattice point c in the slice to the point that c is present in the slice at time t_1 and not in t_2, causing it to "disappear" from the network.

To help fix the matrix product, we can rewrite the matrix E as VA for a matrix A. Then by the orthogonality condition (1), we have

$$\begin{aligned} &V_{\perp}^T V A \\ &= \begin{bmatrix} 0_{(d-r)\times r} & I_{(d-r)\times(d-r)} \end{bmatrix} A \end{aligned}$$

As we can see, only the bottom $d - r$ rows of A affect are enforced to be constant throughout a deformation. The top r rows are free to move at will.

The last thing that is needed of a deformation of Γ is to keep edge lengths in the hyperplane constant. Consider the expression:

$$V_{\parallel}^T E$$

The ith column represents the vector $v_{\parallel}$ of the lattice vector e_i projected onto the hyperplane.

$$V_{\parallel}^T E = V_{\parallel}^T V A = \begin{bmatrix} I_{r\times r} & 0_{r\times(d-r)} \end{bmatrix} A$$

This corresponds to the first r rows of A. In particular, every column of the first r rows of A, corresponds to the displacement (i.e. the vector of an edge) in the hyperplane after following one of the e_i's. The magnitude of each column must be constant throughout a deformation in order to keep edge lengths constant. In the case of the Penrose tiling, the magnitudes of each vector must be $\sqrt{\frac{2}{5}}$. This motivates defining the matrix B such that B corresponds to the first r rows of A. In particular, it is easy to verify that $V_{\parallel}^T E V_{\parallel}$ is equivalent to $BV_{\parallel}$. In general, it is quicker to compute and verify properties of $BV_{\parallel}$ rather than $V_{\parallel}^T E V_{\parallel}$.

Subject to these rules, the allowed deformations are clear: only rotations to the columns of matrix B. In 2 dimensions, methods of rotation are quite limited: namely, each vector rotates an angle θ_i around a fixed axis. We experiment with such rotations in Section 5.

Definition 12. *Given graphs G and G' corresponding to lattice vector matrices E and E' respectively, a map $\phi : G \to G'$ is a* ***cut-and-project deformation*** *if $V_{\perp}^T E = V_{\perp}^T E'$ and if edges distances are preserved after projection.*

Given the previous rules on allowed deformations, we can proceed to our main theorem:

Theorem 4.1. *A cut-and-project deformation $\phi : G \to G'$ is strict P-auxetic if and only if the matrix T equaling*

$$V_{\parallel}^T E_{G'} V_{\parallel} (V_{\parallel}^T E_G V_{\parallel})^{-1}$$

is strict amplifying.

Proof. We first prove the reverse direction: if $|Tx| \geq (1+\epsilon)|x|$, then there exists a relatively dense subset $P \subset V$ such that all pairwise distances increase by a factor of $1+\epsilon_1$ for some $\epsilon_1 < \epsilon$.

Fix a lattice point v in the slice. We define a subset P as the set of lattice points $p \in \Gamma$ such that there exists constants $k_1, \ldots, k_r$ such that

$$\left| p - v - \sum_{i=1}^{r} k_i v_i \right| \leq \epsilon_2$$

for a carefully chosen value of $\epsilon_2 > 0$. In particular, when computing the normal vector of each lattice point from the hyperplane (namely $V_{\perp}^T E p$), since $v_1, \ldots, v_r$ are in the nullspace of $V_{\perp}^T E$, the set of possible normal vectors is limited to a $(d-r)$-dimensional ball of size ϵ_2 around that of v. In particular, by choosing ϵ_2 small enough, the ball will be small enough to fit within the space of normal vectors allowed within the slice. Therefore, all points p can be made to be within the slice.

Consider points $p_1, p_2 \in P$ and the points $q_1 = v + \sum_{i=1}^{r} k_{i_1} v_i$ and $q_2 = v + \sum_{i=1}^{r} k_{i_2} v_i$ that correspond to them. The vector $\overrightarrow{q_1 q_2}$ lies on S, so $\phi(\overrightarrow{q_1 q_2})$ must also lie on on S, since all normal vectors $v_{\perp}$ are held fixed after the action of ϕ.

The matrix $V_{\parallel}^T E_G V_{\parallel}$ represents a bijective map that maps the subspace of points of the lattice Γ that lie on hyperplane S to their corresponding point $p \in \mathbb{R}^r$ in the network G. The matrix $V_{\parallel}^T E_{G'} V_{\parallel}$ performs a similar function for network G'.

Define $\operatorname{proj}_S \overrightarrow{q_1q_2} \in \mathbb{R}^r$ to be the projection of $\overrightarrow{q_1q_2}$ onto S for network G. Then $\phi(\operatorname{proj}_S \overrightarrow{q_1q_2})$ can be thought of as the projection of $\overrightarrow{q_1q_2}$ onto S for network G'. Therefore, by using the above two maps,

$$\phi(\operatorname{proj}_S \overrightarrow{q_1q_2}) = V_{\|}^T E_{G'} V_{\|} (V_{\|}^T E_G V_{\|})^{-1} \operatorname{proj}_S \overrightarrow{q_1q_2}$$

$$\phi(\operatorname{proj}_S \overrightarrow{q_1q_2}) = T \operatorname{proj}_S \overrightarrow{q_1q_2} \tag{2}$$

In particular, we have

$$|\phi(\operatorname{proj}_S \overrightarrow{q_1q_2})| \geq (1+\epsilon)|\operatorname{proj}_S \overrightarrow{q_1q_2}|$$

Now using the fact that $|p_1 - q_1| \leq \epsilon_2$ and $|p_2 - q_2| \leq \epsilon_2$,

$$|\phi(\operatorname{proj}_S \overrightarrow{p_1p_2})| + 2\epsilon_2 \geq |\phi(\operatorname{proj}_S \overrightarrow{q_1q_2})| \geq (1+\epsilon)|\operatorname{proj}_S \overrightarrow{q_1q_2}| \geq (1+\epsilon)(|\operatorname{proj}_S \overrightarrow{p_1p_2}| - 2\epsilon_2)$$

$$|\phi(\operatorname{proj}_S \overrightarrow{p_1p_2})| + 2\epsilon_2 \geq (1+\epsilon)(|\operatorname{proj}_S \overrightarrow{p_1p_2}| - 2\epsilon_2)$$

$$\frac{|\phi(\operatorname{proj}_S \overrightarrow{p_1p_2})|}{|\operatorname{proj}_S \overrightarrow{p_1p_2}|} \geq (1+\epsilon) - \frac{2\epsilon_2(1+\epsilon)}{|\operatorname{proj}_S \overrightarrow{p_1p_2}|} - 2\epsilon_2$$

By choosing an arbitrarily small ϵ_2, we can conclude that for any $\epsilon_1 < \epsilon$

$$\frac{|\phi(\operatorname{proj}_S \overrightarrow{p_1p_2})|}{|\operatorname{proj}_S \overrightarrow{p_1p_2}|} \geq (1+\epsilon_1)$$

as desired.

The reverse direction is similar to that of Theorem 3.1. We wish to show that $\frac{|Tv|}{|v|}$ is at least $1+\epsilon$ given that there exists some subset P of the vertices such that all pairwise distances increase by a factor of at least $1+\epsilon$. First fix a global constant k. Then fix a particular v and fix a particular vertex $p \in P$. For $\theta_\epsilon > 0$, consider the cone of lattice points p' such that the angle between $\overrightarrow{pp'}$ and v is at most θ_ϵ. Eventually, for every N, a dimension r ball with radius N will fit inside the cone sufficiently far away. Thus there exists a point $p^* \in P$ such that that p^* lies within the cone and such that $|\overrightarrow{pp}^*| \geq k$ using our fixed constant k. Repeating our work from the forward direction, we can define the points $q, q^* \in \Gamma$ to be the matrix $(V_{\|}^T E_G V_{\|})^{-1}$ acting on the points p and p^* respectively while embedded in network G. Also, let l and l^* refer to the lattice points of Γ corresponding to p and p^*.

$$|\phi(\overrightarrow{pp}^*)| \geq (1+\epsilon)|\overrightarrow{pp}^*|$$

Both points q and l orthogonally project to p in network G. This implies that q and l are within some distance m of each other, where m is the largest possible length of a normal vector from a point in the slice to the hyperplane S. Similarly, q^* and l^* are within m of each other. We have that

$$|\phi(\overrightarrow{qq}^*)| + 2m \geq |\phi(\overrightarrow{ll}^*)|$$

which implies

$$|\phi(\overrightarrow{qq}^*)| + 2m \geq |\phi(\overrightarrow{ll}^*)| \geq |\phi(\overrightarrow{pp}^*)| \geq (1+\epsilon)|\overrightarrow{pp}^*|$$

From our construction, $\overrightarrow{qq}^*$ lies on S, so following a similar argument to (2):

$$|T(\overrightarrow{pp}^*)| + 2m \geq (1+\epsilon)|\overrightarrow{pp}^*|$$

$$\frac{|T(\overrightarrow{pp}^*)|}{|\overrightarrow{pp}^*|} \geq (1+\epsilon) - \frac{2m}{|\overrightarrow{pp}^*|}$$

By $|\overrightarrow{pp}^*| \geq k$,

$$\frac{|T(\overrightarrow{pp}^*)|}{|\overrightarrow{pp}^*|} \geq (1+\epsilon) - \frac{2m}{k}$$

As $\theta_\epsilon \to 0$, $\overrightarrow{pp}^* \to v$, so by continuity we have

$$\frac{|Tv|}{|v|} \geq (1+\epsilon) - \frac{2m}{k}$$

As we raise the global constant k to be arbitrarily large, we arrive at

$$\frac{|Tv|}{|v|} \geq (1+\epsilon)$$

as desired. □

This theorem implies that there is a rough equivalence between the P-auxeticity of an aperiodic graph and the amplification of the matrix $V_\|^T E V_\| = BV_\|$. Our proof implies that the ϵ_P of P-auxeticity can be converted directly to the ϵ_T of matrix amplification. On the other hand, when transitioning from the ϵ_T to ϵ_P, our proof only shows that ϵ_P approaches ϵ_T from below. This implies that the matrix $B_{G'}V_\|(B_G V_\|)^{-1}$ being strict amplifying is necessary and sufficient for strict P-auxeticity, whereas $B_{G'}V_\|(B_G V_\|)^{-1}$ being amplifying is only necessary for P-auxeticity. *A priori*, this is usually not a problem, and P-auxeticity can be interchangeably used with matrix amplification.

It is also to be noted that the proof of Theorem 4.1 does not rely on the fact that edge lengths of the graph are held constant. Therefore, the theorem also defines a sense of auxeticity to a more general space of graphs than the configuration space $\mathcal{C}(G)$ we are working in.

5 Cut-and-Project Constructions onto 2 Dimensions

In 2 dimensions, the cut-and-project method allows for a quick classification of auxetic paths due to the limitations of the possible matrices $BV_\|$ when moving E infinitesimally away from the identity.

To classify such limitations, we attempt to construct a one parameter cut-and-project P-auxetic deformation $f(t)$ defined on some small neighborhood $[0, \epsilon)$ with Γ being aligned with the elementary basis vectors at $t = 0$. Suppose such an infinitesimal deformation is performed to move $BV_\|$ from the identity I to a new matrix $I + H$, for some matrix H. Then the norm of this new matrix is equal to

$$\inf_{|x|=1} x^T(I+H)^T(I+H)x = \inf_{|x|=1} x^T(I + H + H^T + H^T H)x$$

The $H^T H$ term is $o(H)$, so

$$= x^T(1)x + x^T(H + H^T)x = 1 + x^T(H + H^T)x$$

Therefore to show that $I + H$ is amplifying, it suffices to show that $H + H^T$ is positive semi-definite. Showing it is negative semi-definite also proves auxeticity in the reverse direction.

Since we are working in 2 dimensions, we can express the matrix product $BV_{\|}$ as:

$$\frac{1}{k}\begin{bmatrix} l_1 \cos a_1(t_1) & l_2 \cos a_2(t_2) & l_3 \cos a_3(t_3) & \dots & l_d \cos a_d(t_d) \\ l_1 \sin a_1(t_1) & l_2 \sin a_2(t_2) & l_3 \sin a_3(t_3) & \dots & l_d \sin a_d(t_d) \end{bmatrix} \begin{bmatrix} l_1 \cos\theta_1 & l_1 \sin\theta_1 \\ l_2 \cos\theta_2 & l_2 \sin\theta_2 \\ \dots & \dots \\ l_d \cos\theta_d & l_d \sin\theta_d \end{bmatrix} \tag{3}$$

with $a_i(t_i) = \theta_i + t_i$ and where l_i is the length of the ith lattice vector of Γ projected down to S. By multiplying the matrices in the reverse order, the trace of this product evaluated at the origin is equal to $\sum_{i=1}^{d} l_i^2$. Since this product should be the identity at the origin, we add a positive normalizing constant $k = \frac{\sum_{i=1}^{d} l_i^2}{2}$. We attempt to obtain the partial derivative of this matrix with respect to each variable t_i. To simplify notation, we define the matrix A to be the matrix $BV_{\|}$. Differentiating,

$$\frac{\partial A}{\partial t_i} = \frac{1}{k}\begin{bmatrix} 0 & \dots & -l_i \sin a_i(t_i) & \dots & 0 \\ 0 & \dots & l_i \cos a_i(t_i) & \dots & 0 \end{bmatrix} \begin{bmatrix} l_1 \cos\theta_1 & l_1 \sin\theta_1 \\ l_2 \cos\theta_2 & l_2 \sin\theta_2 \\ \dots & \dots \\ l_d \cos\theta_d & l_d \sin\theta_d \end{bmatrix}$$

$$\frac{\partial A}{\partial t_i} = \frac{l_i^2}{k}\begin{bmatrix} -\sin a_i(t_i) \\ \cos a_i(t_i) \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix}$$

Let us now consider a directional derivative of $BV_{\|}$ at the origin. It can be written as a linear combination of the partial derivatives of each t_i:

$$H = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -\sin a_i(t_i) \\ \cos a_i(t_i) \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix} \right)\Bigg|_{\langle 0,\dots,0\rangle}$$

$$H = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix} \right)$$

$$H = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -\sin\theta_i \cos\theta_i & -\sin^2\theta_i \\ \cos^2\theta_i & \sin\theta_i\cos\theta_i \end{bmatrix} \right) \tag{4}$$

We wish to show useful properties of H. Consider $H + H^T$:

$$H + H^T = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -2\sin\theta_i\cos\theta_i & \cos^2\theta_i - \sin^2\theta_i \\ \cos^2\theta_i - \sin^2\theta_i & 2\sin\theta_i\cos\theta_i \end{bmatrix} \right)$$

We have the magical reduction of:

$$H + H^T = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -\sin 2\theta_i & \cos 2\theta_i \\ \cos 2\theta_i & \sin 2\theta_i \end{bmatrix}\right) \tag{5}$$

$$H + H^T = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} -\sin 2\theta_i & \cos 2\theta_i \\ \cos 2\theta_i & \sin 2\theta_i \end{bmatrix}\right)$$

$$H + H^T = \frac{1}{k}\sum_{i=1}^{d}\left(c_i l_i^2 \begin{bmatrix} \cos 2\theta_i & -\sin 2\theta_i \\ \sin 2\theta_i & \cos 2\theta_i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

Thus, $H + H^T$ is a reflection over $y = x$ followed by a sum of rotation matrices of angle θ_i. Now we make use of the following surprisingly not-well-known lemma:

Lemma 5.1. *In $\mathbb{R}^2$, any linear combination of a finite set of rotation matrices is equal to a constant multiple of some rotation matrix.*

Proof. A constant times a 2-dimensional rotation matrix is equivalent to a complex number $a + bi$. A sum of two complex numbers is another complex number, so the lemma is proven. □

Using this lemma, we arrive at the result that $H + H^T$ is equal to $kR_\theta S_{y=x}$ for constants k and θ where $S_{y=x}$ is the matrix representing a reflection over the line $y = x$. Fortunately, it is quite easy to classify the positive definiteness of such a matrix. In particular, the composition of a rotation and a reflection matrix is another reflection matrix across a different axis. But reflection matrices have eigenvalues of 1 and -1, so the only way to have negative or positive semi-definiteness is through being the zero matrix. This implies the following:

Remark 5.1.1. *If there exists an auxetic path through the origin, then $H + H^T$ is the zero matrix.*

Thus, in order to allow an auxetic path through the origin, the first order term of norm must be the zero matrix. We examine the second order term by examining the second directional derivative of $A^T A$ at the origin. By a similar analysis, if the second order term of $A^T A$ is positive semi-definite or negative semi-definite given the first order term is the zero matrix, then we have an auxetic path through the origin.

Let our directional derivative be in the direction $\hat{c} = \langle c_1, c_2, \ldots, c_d\rangle$. Looking at the second derivative,

$$\partial_{\hat{c}}^2(A^T A) = \sum_{i=1}^{d} c_i^2 \frac{\partial^2(A^T A)}{\partial t_i^2} + \sum_{i\neq j} c_i c_j \frac{\partial^2(A^T A)}{\partial t_i \partial t_j}$$

Using the reduction $\frac{\partial^2 A}{\partial t_i \partial t_j} = 0$ for $i \neq j$ and evaluating,

$$\partial_{\hat{c}}^2(A^T A) = \sum_{i=1}^{d} c_i^2 \left(\frac{\partial^2 A}{\partial t_i^2} + \frac{\partial^2 A}{\partial t_i^2}^T + 2\frac{\partial A}{\partial t_i}^T \frac{\partial A}{\partial t_i}\right) + \sum_{i\neq j} c_i c_j \left(\frac{\partial A}{\partial t_i}^T \frac{\partial A}{\partial t_j} + \frac{\partial A}{\partial t_j}^T \frac{\partial A}{\partial t_i}\right)$$

$$= \sum_{i=1}^{d} c_i^2 \left(\frac{\partial^2 A}{\partial t_i^2} + \frac{\partial^2 A}{\partial t_i^2}^T\right) + \left(\sum_{i=1}^{d} 2\frac{\partial A}{\partial t_i}^T \frac{\partial A}{\partial t_i} + \sum_{i\neq j} c_i c_j \left(\frac{\partial A}{\partial t_i}^T \frac{\partial A}{\partial t_j} + \frac{\partial A}{\partial t_j}^T \frac{\partial A}{\partial t_i}\right)\right)$$

$$= \sum_{i=1}^{d} c_i^2 \left(\frac{\partial^2 A}{\partial t_i^2} + \frac{\partial^2 A}{\partial t_i^2}^T \right) + 2 \left(\sum_{i=1}^{d} c_i \frac{\partial A}{\partial t_i} \right)^T \left(\sum_{i=1}^{d} c_i \frac{\partial A}{\partial t_i} \right) \tag{6}$$

Looking at the second half of equation (6), we can notice that it is an expression in terms of H, which can be alternatively defined as $\sum_{i=1}^{d} c_i \frac{\partial A}{\partial t_i}$. We know that $H + H^T$ is the zero matrix, thus:

$$H = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$$

for some constant κ. We can also write down H explicitly using (4):

$$H = \frac{1}{k} \sum_{i=1}^{d} \left(c_i l_i^2 \begin{bmatrix} -\sin\theta_i \cos\theta_i & -\sin^2\theta_i \\ \cos^2\theta_i & \sin\theta_i \cos\theta_i \end{bmatrix} \right)$$

Combining the previous two expressions, we can arrive at the following deduction:

$$\kappa - (-\kappa) = \frac{1}{k} \sum_{i=1}^{d} c_i l_i^2 (\sin^2\theta_i - (-\cos^2\theta_i)) = \frac{1}{k} \sum_{i=1}^{d} c_i l_i^2$$

$$\kappa = \frac{1}{2k} \sum_{i=1}^{d} c_i l_i^2$$

We wish to compute $\frac{\partial^2 A}{\partial t_i^2}$.

$$\frac{\partial^2 A}{\partial t_i^2} = \frac{l_i^2}{k} \begin{bmatrix} -\cos\theta_i \\ -\sin\theta_i \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix}$$

Thus equation (6) becomes:

$$\partial_{\hat{c}}^2 (A^T A) = \frac{1}{2k^2} \begin{bmatrix} \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 & 0 \\ 0 & \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 \end{bmatrix} - \frac{1}{k} \sum_{i=1}^{d} 2 c_i^2 l_i^2 \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \end{bmatrix}$$

$$= \frac{1}{2k^2} \begin{bmatrix} \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 & 0 \\ 0 & \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 \end{bmatrix} - \frac{1}{k} \sum_{i=1}^{d} 2 c_i^2 l_i^2 \begin{bmatrix} \cos^2\theta_i & \sin\theta_i \cos\theta_i \\ \sin\theta_i \cos\theta_i & \sin^2\theta_i \end{bmatrix}$$

$$= \frac{1}{2k^2} \begin{bmatrix} \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 & 0 \\ 0 & \left(\sum_{i=1}^{d} c_i l_i^2\right)^2 \end{bmatrix} - \frac{1}{k} \sum_{i=1}^{d} c_i^2 l_i^2 \begin{bmatrix} 2\cos^2\theta_i & 2\sin\theta_i \cos\theta_i \\ 2\sin\theta_i \cos\theta_i & 2\sin^2\theta_i \end{bmatrix}$$

$$= \frac{1}{2k^2}\begin{bmatrix} \left(\sum_{i=1}^d c_i l_i^2\right)^2 & 0 \\ 0 & \left(\sum_{i=1}^d c_i l_i^2\right)^2 \end{bmatrix} - \frac{1}{k}\sum_{i=1}^d c_i^2 l_i^2 \begin{bmatrix} 1+\cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & 1-\cos 2\theta_i \end{bmatrix}$$

$$= \frac{1}{2k^2}\begin{bmatrix} \left(\sum_{i=1}^d c_i l_i^2\right)^2 & 0 \\ 0 & \left(\sum_{i=1}^d c_i l_i^2\right)^2 \end{bmatrix} - \frac{1}{k}\begin{bmatrix} \sum_{i=1}^d c_i^2 l_i^2 & 0 \\ 0 & \sum_{i=1}^d c_i^2 l_i^2 \end{bmatrix} - \frac{1}{k}\sum_{i=1}^d c_i^2 l_i^2 \begin{bmatrix} \cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & -\cos 2\theta_i \end{bmatrix} \tag{7}$$

The matrix $c_i^2 l_i^2 \begin{bmatrix} \cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & -\cos 2\theta_i \end{bmatrix}$ is clearly a constant times a reflection matrix. Summed over all i, the result is also a constant λ times a reflection matrix R by Lemma 5.1 with eigenvalues $\pm\lambda$. We wish to prove that (7) is negative semi-definite. It suffices to prove that all eigenvalues are less than or equal to 0. Without loss of generality, assume λ is the non-negative eigenvalue. We wish to show that:

$$\frac{1}{2k^2}\left(\sum_{i=1}^d c_i l_i^2\right)^2 - \frac{1}{k}\left(\sum_{i=1}^d c_i^2 l_i^2\right) + \frac{\lambda}{k} \leq 0$$

$$\lambda \leq \left(\sum_{i=1}^d c_i^2 l_i^2\right) - \frac{1}{2k}\left(\sum_{i=1}^d c_i l_i^2\right)^2$$

Using the fact that $k = \frac{\sum_{i=1}^d l_i^2}{2}$,

$$\lambda \leq \left(\sum_{i=1}^d c_i^2 l_i^2\right) - \frac{1}{\sum_{i=1}^d l_i^2}\left(\sum_{i=1}^d c_i l_i^2\right)^2 \tag{8}$$

We can write an explicit expression for λ as the square root of the negative determinant of the reflection matrix:

$$\lambda = \sqrt{\left(\sum_{i=1}^d c_i^2 l_i^2 \cos 2\theta_i\right)^2 + \left(\sum_{i=1}^d c_i^2 l_i^2 \sin 2\theta_i\right)^2} \tag{9}$$

Define the complex numbers x_i such that $x_i = l_i(\cos 2\theta_i + i \sin 2\theta_i)$. Then (9) becomes,

$$\lambda = \left|\sum_{i=1}^d c_i^2 x_i^2\right|$$

By (5), we also have the curious identity of $\sum_{i=1}^d c_i x_i^2 = 0$. By expanding the product $BV_\parallel$, we can also compute $\sum_{i=1}^d x_i^2$:

$$BV_\parallel = \frac{2}{d}\sum_{i=1}^{d} l_i^2 \begin{bmatrix} \cos^2\theta_i & \sin\theta_i\cos\theta_i \\ \sin\theta_i\cos\theta_i & \sin^2\theta_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{d}\sum_{i=1}^{d} l_i^2 \begin{bmatrix} 1+\cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & 1-\cos 2\theta_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{d}\sum_{i=1}^{d} l_i^2 \begin{bmatrix} \cos 2\theta_i & \sin 2\theta_i \\ \sin 2\theta_i & -\cos 2\theta_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sum_{i=1}^{d} l_i^2 \cos 2\theta_i = 0, \quad \sum_{i=1}^{d} l_i^2 \sin 2\theta_i = 0$$

Therefore $\sum_{i=1}^{d} x_i^2 = 0$. We define a weighted mean of c_i's defined as:

$$\overline{c} = \frac{\sum_{i=1}^{d} c_i l_i^2}{\sum_{i=1}^{d} l_i^2}$$

Applying $\sum_{i=1}^{d} c_i x_i^2 = 0$ and $\sum_{i=1}^{d} x_i^2 = 0$ to prove (8),

$$\lambda = \left|\sum_{i=1}^{d} c_i^2 x_i^2\right| = \left|\sum_{i=1}^{d} (c_i^2 - 2c_i\overline{c} + \overline{c}^2) x_i^2\right|$$

$$\lambda = \left|\sum_{i=1}^{d} (c_i - \overline{c})^2 x_i^2\right|$$

By the triangle inequality,

$$\lambda = \left|\sum_{i=1}^{d} (c_i - \overline{c})^2 x_i^2\right| \leq \sum_{i=1}^{d} (c_i - \overline{c})^2 l_i^2 \tag{10}$$

Expanding $\sum_{i=1}^{d}(c_i - \overline{c})^2 l_i^2$, we can see the following identity:

$$\sum_{i=1}^{d}(c_i - \overline{c})^2 l_i^2 = \left(\sum_{i=1}^{d} c_i^2 l_i^2\right) - \frac{1}{\sum_{i=1}^{d} l_i^2}\left(\sum_{i=1}^{d} c_i l_i^2\right)^2$$

Therefore the second derivative is always negative semi-definite given $H + H^T = 0$, and notably never positive semi-definite, unless it is the zero matrix.

To check when the second derivative is the zero matrix, we can look at the equality case of the eigenvalue equation (8). Since the x_i^2 all point in distinct directions in a non-trivial cut-and-project construction, the equality case only occurs when all of the c_i's are equal, which translates to a rotation of the plane which is not strict amplifying. Our work can be summed up in the following theorem:

Theorem 5.2. *When Γ of a non-trivial cut-and-project construction is aligned to the elementary basis vectors, any infinitesimal change H to the matrix $BV_{\parallel}$ causes it to be amplifying in the reverse direction if and only if $H+H^T$ is the zero matrix. There are no infinitesimal changes H such that the matrix $BV_{\parallel}$ is strict amplifying in the forward direction, implying no strict P-auxetic deformations in the forward direction.*

Our derivation of Theorem 5.2 provides a quick way of finding deformations that produce such matrices H. In particular, the condition

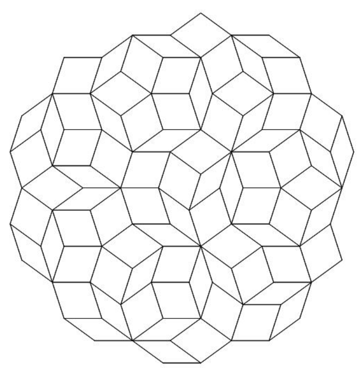
$$\sum_{i=1}^{d} c_i l_i^2(\cos 2\theta_i + i \sin 2\theta_i) = 0$$

$$\Leftrightarrow \sum_{i=1}^{d} c_i x_i^2 = 0 \text{ where } x_i = l_i(\cos\theta_i + i\sin\theta_i)$$

where c_i is a component of a directional derivative vector $\hat{c} = \langle c_1, \ldots, c_d \rangle$ of expression (3) is equivalent to the condition $H + H^T = 0$ by expression (5). As we will see in Section 6, the alternate condition is very useful in locating auxetic trajectories.

6 Case Study of the Penrose Tiling

Now that the framework for detecting and constructing auxeticity in aperiodic tilings has been laid out, we can now examine the auxetic deformations of the Penrose tiling. Below is a sample Penrose tiling:



After a deformation, the Penrose tiling is likely to expand notably in one direction and contract in another as seen in Figure 6. The exact axes of deformation are given roughly by the eigenvectors of the matrix $BV_{\parallel}$. Such deformations are non-auxetic.

We can apply the condition of $\sum_{i=1}^{5} c_i x_i^2$ from Section 5 to construct auxetic contractions of the Penrose tiling starting from the identity. Fortunately, the set of complex numbers x_i^2 for the Penrose tiling is the set of fifth roots of unity. Since we are dealing with five 2 dimensional vectors, the nullspace is a 3 dimensional, implying the existence of a 3 dimensional surface of auxetic deformations. One notable vector in the nullspace is the all 1 vector: $\hat{c} = \langle 1, \ldots, 1 \rangle$ The resulting

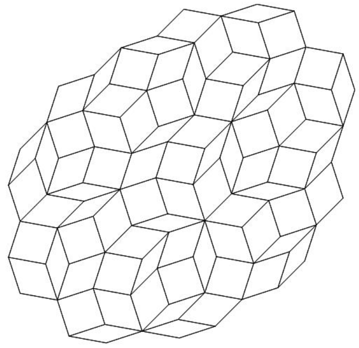


Figure 6: The circular shape of the original tiling deforming into an ellipsoidal shape by an non-auxetic deformation.

image after following such a vector is a rotation. This holds in general, since adding a fixed constant to the angle of every projection of the lattice vectors of Γ only serves to rotate the figure by that fixed constant. Ignoring the all one-vector, the remaining non-trivial auxetic deformations lie within a 2 dimensional surface. Figure 7 depicts two such auxetic deformations.

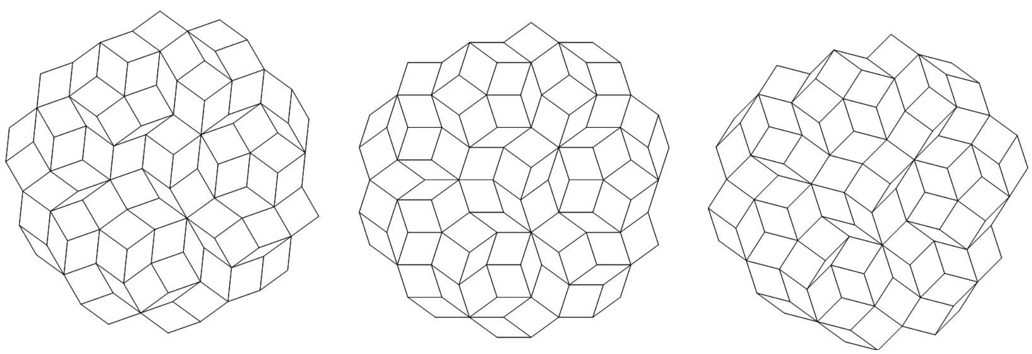


Figure 7: Two deformed auxetic networks (left, right) compared to the the original tiling (center)

Due to the second order nature of auxeticity at the identity, such contractions are quite small. In particular, for the two networks in Figure 7, the contraction ratios are equal to 0.9956 and 0.9909 respectively. As a result, the fact that they contract is not immediately visible.

Almost all (if not all) of the notable auxetic periodic networks make use of some concave tiles to grant the property of auxeticity. For example, a convex hexagon is not auxetic while a hexagon with two opposite concave angles is auxetic, despite the two being the same deformation space. Thus, the fact that the Penrose tiling allows for an auxetic deformation while only using convex tiles is perhaps surprising.

Another potentially surprising result is the fact that $\mathbb{Z}^5$ exhibits no auxetic deformations, while its orthogonal projection onto a lower dimensional hyperplane does exhibit auxetic deformations. This implies that notions of auxeticity are difficult to translate from a lower dimensional space to a higher dimensional space, possibly due to the increased dimension of potential axes in which a deformation must contract or stretch.

7 Future Work and Acknowledgements

Other possible venues of research include: generalizing the result from Section 5 to higher dimensions, possibly with the use of quaternions; finding the "most contracted" deformation of the Penrose tiling with use of semi-definite programming; and applying P-auxeticity to a more general case of networks such as randomized networks.

This paper was developed as a result of the MIT Summer Program in Undergraduate Research (SPUR). The author would like to thank Vishal Patil, for suggesting the project and providing guidance as a mentor for 6 weeks; Prof. Davesh Maulik and Prof. Ankur Moitra, for helpful comments and critiques; and the countless others who helped make SPUR and this research possible.

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