Model categories in equivariant rational homotopy theory

SPUR Final Paper, Summer 2017

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August 2, 2017

Abstract

Model categories are a useful formalization of homotopy theory, and the notion of Quillen equivalence between them expresses what it means for two homotopy theories to be equivalent. Bousfield and Gugenheim showed that the model categories of simplicial sets and of commutative differential graded algebras over $\mathbb{Q}$ are close to Quillen equivalent, in that there is a Quillen adjunction between them such that the induced adjunction on homotopy categories restricts to an equivalence on simply-connected rational objects of finite rational type. We extend this to the case of an action of a discrete group, showing that for any small category C the induced Quillen adjunction between functors from C to simplicial sets and functors from C to commutative differential graded algebras also restricts to an equivalence on appropriately well-behaved objects. We also explore equivariant rational homotopy theory from this perspective, for the action of a finite group.

1 Introduction

Rational homotopy theory is the study of spaces considered up to 'rational homotopy equivalence', where a rational homotopy equivalence is a map $f : X \to Y$ such that $f_* : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ is an isomorphism for all n. This is a coarser notion of equivalence than the usual (weak) homotopy equivalence, where f should induce an isomorphism on the homotopy groups themselves.[1] However, ignoring the torsion information allows for an excellent algebraic development of the resulting theory. In particular, under mild assumptions (for example, simple connectivity), to a space X can be associated a 'minimal' commutative differential graded algebra $\mathfrak{M}_X$, such that $\mathfrak{M}_X \cong \mathfrak{M}_Y$ if and only if X and Y are rationally equivalent. Better still, $\mathfrak{M}_X$ can be constructed — often fairly explicitly — using only the knowledge of X's rational cohomology algebra. For details, see [Hes07].

Given the great success of ordinary rational homotopy, it is natural to seek an extension to the equivariant case. Equivariant homotopy theory essentially replaces topological spaces, simplicial sets, and similar objects with functors from some small 'indexing' category into the category of such objects. The reasons for this are as follows. The equivariant Whitehead theorem indicates that an equivariant map $f : X \to Y$ of G-spaces should be considered to be a G-weak homotopy equivalence if the map $f^H : X^H \to Y^H$ induced on fixed-point spaces is an ordinary weak homotopy equivalence for any subgroup H of G. Elmendorf's theorem further explains that the resulting homotopy theory is captured by $\mathrm{Hom}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top})_{\mathrm{proj}}$.[2] Here, **Top** is the category of topological spaces; $\mathcal{O}_G$ is the *orbit category* of G, which can be defined as the full subcategory of G-spaces which are transitive; and proj refers to a particular *model category* structure placed on this category. Model categories are a useful formalism for homotopy theory; here, we use them to begin extending the results of rational homotopy theory to the equivariant case.

1.1 Acknowledgments

I thank Robert Burklund for suggesting thinking about these ideas and for many helpful conversations and clarifications. I also thank the MIT SPUR program, for enabling this work to take place; in particular, Slava Gerovitch, for organizing the program, and David Jerison and Ankur Moitra, for helpful general advice about mathematics research.

1.2 Preliminaries and Notations

We assume the reader is familiar with the basics of ordinary category theory. We write $L \dashv R : \mathcal{C} \rightleftarrows D$ to indicate that $L : \mathcal{C} \to \mathcal{D}$ is left adjoint to $R : \mathcal{D} \to \mathcal{C}$.

[1] A simple example of a rational homotopy equivalence which is not a homotopy equivalence is the n-fold cover of S^1 by S^1, for $n > 1$.

[2] For a more in-depth explanation of the significance of Elmendorf's theorem, see the introduction to [Ste10].

A category is *complete* if it has all small limits and *cocomplete* if it has all small colimits. We take $\mathbb{Q}$ to be our base field for all constructions, and all chain/cochain complexes are cohomologically graded.

2 Model Categories

2.1 Definition

Model categories were originally defined by Quillen in 1967. They provide a convenient way to manage the technical details of homotopy theory, especially those related to lifting and extension problems.

We recall the definition of a model category, following Riehl [Rie09], and introduce the examples relevant to rational equivariant homotopy theory.

Definition 2.1. Let $\mathcal{C}$ be a category and L, R two subsets of morphisms in $\mathcal{C}$. A *functorial factorization into* (L, R) is the following data: for each $f : X \to Y$ in $\mathcal{C}$, a choice of factorization $f = r(f)\ell(f)$ with $\ell(f) \in L$ and $r(f) \in R$, such that for a commuting square

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "x"'] & Y \arrow[d, "y"'] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

there are commuting squares

$$\begin{tikzcd} X \arrow[r, "\ell(f)"'] \arrow[d, "x"'] & F(f) \arrow[r, "r(f)"'] \arrow[d, "F(x,y)"'] & Y \arrow[d, "y"'] \\ X' \arrow[r, "\ell(f')"'] & F(f') \arrow[r, "r(f')"'] & Y' \end{tikzcd}$$

which are functorial in the appropriate sense, i.e. $F(\mathrm{id}, \mathrm{id}) = \mathrm{id}$ and $F(x'x, y'y) = F(x', y')F(x, y)$ for commuting rectangles

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, "x"'] & Y \arrow[d, "y"'] \\ X' \arrow[r, "f'"'] \arrow[d, "x'"'] & Y' \arrow[d, "y'"'] \\ X'' \arrow[r, "f''"'] & Y''. \end{tikzcd}$$

Definition 2.2. A functorial factorization into (L, R) is a *weak functorial factorization* if additionally L is the set of morphisms having the left lifting property with respect to R and R is the set of morphisms having the right lifting property with respect to L. That is to say, $r \in R$ if and only if for all $\ell \in L$ and commuting solid squares

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

a dashed arrow exists making both triangles commute, and $\ell \in L$ if and only if for all $r \in R$ and solid diagrams as above, a dashed arrow exists.

The original definition of a model category used weak factorization systems rather than weak functorial factorization systems, essentially requiring that factorizations existed but were not necessarily given by a functor. We follow Hovey [Hov91] in requiring the functorial factorization, which will exist in all the examples we work with.

Definition 2.3. A *model category* is a complete and cocomplete category $\mathcal{C}$ equipped with three classes of morphisms: cofibrations, fibrations, and weak equivalences; and equipped with two weak functorial factorizations:
into (cofibrations, fibrations $\cap$ weak equivalences),
and into (cofibrations $\cap$ weak equivalences, fibrations); such that, in addition, if any two of f, g, and fg are weak equivalences, so is the third (if defined).

The last property is also called '2-out-of-3'. This definition is deliberately terse, although even a more explicit definition would hide the large amount of homotopy theory that can be done working entirely within an arbitrary model category. For example, this definition implies that all three classes are closed under retracts, that is, in a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"'] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r] & B \end{tikzcd}$$

if f' is a weak equivalence, fibration, resp. cofibration, then f is a weak equivalence, fibration, resp. cofibration. (In this situation we say that f is a *retract* of f'.)

An *acyclic cofibration* is a cofibration that is also a weak equivalence; dually, an *acyclic fibration* is a fibration that is also a weak equivalence. If $A \to B$ is a weak equivalence, we may write $A \xrightarrow{\simeq} B$.

We think of the cofibrations as 'well-behaved injections', the fibrations as 'well-behaved surjections', and the weak equivalences as 'maps which are isomorphisms in homotopy'. This last notion is made precise by the homotopy category of a model category.

Definition 2.4. The *homotopy category* $\mathrm{Ho}\,\mathcal{C}$ of a model category $\mathcal{C}$ is the localization of $\mathcal{C}$ at its weak equivalences.

It is characterized by a universal property: for all categories $\mathcal{D}$, the category $\mathrm{Hom}(\mathrm{Ho}\,\mathcal{C}, \mathcal{D})$ of functors from the homotopy category must be naturally equivalent to the full subcategory of $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ consisting of those functors F such that whenever f is a weak equivalence in $\mathcal{C}$, $F(f)$ is an isomorphism in $\mathcal{D}$. The existence, and convenient ways to work with, the homotopy category can be established using model category theory; for now, we'll assume it exists and work with it via its universal property. Note that the universal property implies in particular a functor $\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ which sends weak equivalences to isomorphisms. We'll write this, when it needs to be written, as $c \mapsto [c]$. For convenience, we'll assume that this functor is an isomorphism on objects; if we construct a category with the objects of $\mathcal{C}$ where a map $c \to c'$ is, by definition, a map $[c] \to [c']$ in $\mathrm{Ho}\,\mathcal{C}$, this will again satisfy the universal property, so we may as well assume $\mathrm{Ho}\,\mathcal{C}$ is this to begin with.

2.2 Examples

Example (Topological spaces)**.** The category **Top** of topological spaces, with Serre fibrations as the fibrations and maps inducing isomorphisms on all homotopy groups as weak equivalences, can be given a model category structure. Note that from these two pieces of data, the cofibrations are determined: they are the maps having the left lifting property with respect to Serre fibrations that induce isomorphisms on homotopy groups.

Example (Simplicial sets)**.** The category **sSet** of simplicial sets can be given a model structure with weak equivalences consisting of those maps whose geometric realization is a weak equivalence of topological spaces and cofibrations consisting of monomorphisms. Dually to the previous example, the fibrations are determined by these two choices; the fibrations in this model structure are usually called Kan fibrations.

Example (Cochain complexes)**.** The category **Ch** of nonnegatively graded chain complexes over $\mathbb{Q}$ can be given a model structure where cofibrations are degreewise injections except possibly in degree zero, fibrations are degreewise surjections, and weak equivalences are maps which induce isomorphisms in homology.

2.2.1 Model structures on functor categories

Another important class of examples arise as follows: if $\mathcal{C}$ is a model category and A is any small category, the category $\mathrm{Hom}(A, \mathcal{C})$ of functors from A into $\mathcal{C}$ can be equipped with several model structures. We'll consider two, the injective and projective structures. In each of these, the weak equivalences are objectwise weak equivalences; that is, a natural transformation $\eta : F \Rightarrow G : A \to \mathcal{C}$ is a weak equivalence just when $\eta_a : Fa \to Ga$ is a weak equivalence for all a. In the projective model structure, the fibrations are the objectwise fibrations; in the injective model structure, the cofibrations are the objectwise cofibrations. These model structures do not always exist, but if they do exist are unique. The two model structures are related by the next lemma.

Lemma 2.5. *A fibration in the injective model structure is a fibration in the projective model structure; dually, a cofibration in the projective model structure is a cofibration in the injective model structure.*

Proof. It suffices to show the first half; the second half will then follow, since a projective cofibration has LLP with respect to all acyclic projective fibrations, hence in particular with respect to all acyclic injective fibrations, and so is an injective cofibration.

Given an injective fibration $\eta : F \to G$, we want to show that each component η_a is a fibration, i.e. that η_a has RLP with respect to all acyclic cofibrations. Given a lifting problem

$$\begin{tikzcd} c \arrow[r, "p"'] \arrow[d, "h"', "\simeq"] & Fa \arrow[d, "\eta_a"'] \\ c' \arrow[r, "q"'] & Ga \end{tikzcd}$$

with h an acyclic cofibration, we consider the corresponding diagram in $\mathrm{Hom}(A, \mathcal{C})$

$$\begin{tikzcd} \mathcal{C}(a,-) \otimes c \arrow[r, "\tilde{p}"'] \arrow[d, "\mathcal{C}(a,-)\otimes h"', "\simeq"] & F \arrow[d, "\eta"'] \\ \mathcal{C}(a,-) \otimes c' \arrow[r, "\tilde{q}"'] & G. \end{tikzcd}$$

Here, $S \otimes c$ is the coproduct of S-many copies of c. The map $\tilde{p}$ is defined so that $\tilde{p}_{a'} : \mathcal{C}(a, a') \otimes c \to Fa'$ is $F(f)p$ on the copy of c corresponding to some $f : a \to a'$; the map $\tilde{q}$ is defined similarly. Commutativity of the square is the requirement that for any a' and $f : a \to a'$, we have $\eta_{a'} F(f)p = G(f)qh : c \to Ga'$, which holds since $G(f)qh = G(f)\eta_a p = \eta_{a'} F(f)p$ by the previous square and naturality. In order to apply the lifting property of η, then, we need only that $\mathcal{C}(a, -) \otimes h$ is an acyclic injective cofibration. This in turn just means that each component $\mathcal{C}(a, a') \otimes h$ is an acyclic cofibration. But it's easy to show that an arbitrary coproduct of acyclic cofibrations is an acyclic cofibration; to show that $S \otimes h$ has the left lifting property with respect to all fibrations, for example, one need only observe that a lifting problem for $S \otimes h$ is essentially many lifting problems for h, indexed by elements of S, and that solving all these independently is possible (since h is an acyclic cofibration) and sufficient. Therefore there's a natural transformation $\tilde{f} : \mathcal{C}(a, -) \otimes c' \to F$ with $\eta \tilde{f} = \tilde{q}$ and $\tilde{f}(\mathcal{C}(a, -) \otimes h) = \tilde{p}$. Considering the components at a, we have $\eta_a \tilde{f}_a = \tilde{q}_a$ and $\tilde{f}_a(\mathcal{C}(a, a) \otimes h) = \tilde{p}_a$. Looking at this on the copy of c corresponding to the identity on a, we see that $\eta_a f = q$ and $fh = p$, where f is $\tilde{f}_a$ restricted to the copy of c' corresponding to the identity. This means f solves our original lifting problem, so we're done. □

3 Commutative differential graded algebras

3.1 Basic notions

The last example of a model category which will be important for us is the model category of commutative differential graded algebras. Before introducing it, however, we introduce commutative differential graded algebras themselves.

Definition 3.1. A commutative differential graded algebra, or CDGA, over $\mathbb{Q}$ is a commutative monoid in the category of nonnegatively graded chain complexes over $\mathbb{Q}$ (with the Koszul symmetric monoidal structure).

More explicitly, such an algebra is a graded unital $\mathbb{Q}$-algebra equipped with a differential of degree $+1$, required to obey the graded commutativity $xy = (-1)^{\deg(x)\deg(y)}yx$ and graded Leibniz identity $d(xy) = (dx)y + (-1)^{\deg(x)}x(dy)$. We write A^k for the degree-k piece of a CDGA A. We denote the category of CDGAs by **CDGA**.

There is a left adjoint, denoted Λ, to the forgetful functor from CDGAs to cochain complexes, which forms the free commutative differential algebra generated by a cochain complex. The category of CDGAs also has coproducts; the coproduct of A and B is written $A \otimes B$, and we have $(A \otimes B)^k = \oplus_{i+j=k} A^i \otimes_{\mathbb{Q}} B^j$, with multiplication $(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$ and differential $d(a \otimes b) = da \otimes b + (-1)^{\deg(a)} a \otimes db$. With these notions, we can introduce an important class of morphisms of CDGAs.

Definition 3.2. A map $f : A \to B$ of CDGAs is a *relative Sullivan algebra* if the underlying graded algebra of B is isomorphic to that of $A \otimes \Lambda V$, in a way that makes the map underlying f into the canonical inclusion $a \mapsto a \otimes 1$; and, further, there exists a basis v_i of V indexed by some well-ordered set I, such that for all i, we have that dv_i is in the subalgebra of B generated by A and v_j for $j < i$.

An algebra B is called a *Sullivan algebra* if the map $\mathbb{Q} \to B$ is a relative Sullivan algebra.

Intuitively, a relative Sullivan algebra is one that can be built up from A by successively attaching generators that make already-existing cocycles into coboundaries. An example of a non-Sullivan algebra is $\mathbb{Q}[x, y, z]$, a free algebra on three generators of degree one, where $dx = yz$, $dy = zx$, and $dz = xy$.

Definition 3.3. A relative Sullivan algebra is *minimal* if the basis v_i can be chosen so that $i < j$ implies $\deg(v_i) \leq \deg(v_j)$.

Intuitively, a minimal relative Sullivan algebra doesn't add add any cocycles which it then makes into coboundaries (thus leaving the cohomology as it was before). An example of a non-minimal Sullivan algebra is $\mathbb{Q}[x_k, dx_k]$, a free algebra on one generator of degree k, whose differential is a generator in degree $k + 1$.

3.2 Model structure

The model structure on CDGAs is obtained using a transfer theorem, which is stated as theorem 3.6 of [GS06]; given an adjunction $F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}$ between complete and cocomplete categories, it gives conditions under which a model category structure on $\mathcal{C}$ induces a model category structure on $\mathcal{D}$. In this 'promoted' model structure, a map h is a weak equivalence resp. fibration if Gh is a weak equivalence resp. fibration. Specifically, one must require that:

1. If a map c in $\mathcal{D}$ has the left lifting property with respect to all fibrations (that is, maps h with Gh a fibration in $\mathcal{C}$), then it is a weak equivalence (that is, Gc is a weak equivalence in $\mathcal{C}$). This certainly must hold if the model structure is to exist, because maps with the LLP with respect to all fibrations should be precisely the acyclic cofibrations of the model structure.

2. G commutes with colimits which are indexed by an ordinal. These are also called *sequential colimits*.

3. The model category structure on $\mathcal{C}$ is *cofibrantly generated*, which means there are sets I, J of cofibrations and acyclic cofibrations respectively, which generate the model structure in the sense that a map is a fibration resp. acyclic fibration if and only if they have the right lifting property with respect to all morphisms in J resp. I, and which satisfy the additional condition that the Hom-functor $\mathcal{C}(X, -)$ commutes with sequential colimits of cofibrations resp. acyclic cofibrations whenever X is the domain of a morphism in I resp. J. For this last condition, we also say that domains of maps in I are *small* with respect to cofibrations, and domains of maps in J are *small* with respect to acyclic cofibrations; intuitively, they're too small to be stretched across all of the layers of an infinite union at the same time.

That these hypotheses apply to the case of cochain complexes and CDGAs, with the free algebra/underlying cochain complex adjunction, is stated in [Hes07], along with a characterization of the resulting model structure on CDGAs: the weak equivalences are the maps which are isomorphisms on homology, the fibrations are the degreewise surjections, and the cofibrations are retracts of relative Sullivan algebras.

Sometimes we'll work with the category of *augmented* CDGAs, denoted $\mathbf{CDGA}_+$. This is the overcategory $\mathbf{CDGA}/\mathbb{Q}$ over the initial object. By the dual of ([Hov91], 1.1.8) this has a model category structure where a map is a cofibration, fibration, or weak equivalence if its underlying map in $\mathbf{CDGA}$ is such.

4 Quillen adjunctions

4.1 Definition

We return to the general theory of model categories. First, note that the axioms of a model category enjoy a duality, which extends categorical duality; if $\mathcal{C}$ is a model category, the opposite of its underlying category inherits a canonical model structure with fibrations equal to the cofibrations of $\mathcal{C}$ and vice versa, and the same weak equivalences. Therefore, having introduced some notion or proven some proposition valid for all model categories, its dual follows formally.

For example, we define an object X in a model category to be *cofibrant* if the unique map from the initial object of $\mathcal{C}$ to X is a cofibration. The dual notion is that an object X is *fibrant* if the unique map from X to the terminal object of $\mathcal{C}$ is a fibration. The functorial factorization into a cofibration followed by an acyclic fibration, applied to the map from the initial object to X, gives an object CX which depends functorially on X, and a map $\mathrm{cof}_X : CX \to X$ which is a (component of a) natural transformation. This is called *cofibrant replacement*; as the name suggests, CX is cofibrant, since the map from the initial object to CX is the cofibration part of the factorization of the map from the initial object to X; and $CX \to X$ is a weak equivalence (in fact, an acyclic fibration) again by properties of the factorization. Dually, we have *fibrant replacement*, $\mathrm{fib}_X : X \to FX$. A guiding principle is that, in order for the set of maps from X to Y to be 'correct', we ought to have X cofibrant and Y fibrant.

We now consider the most common notion of a morphism between model categories, a Quillen adjunction.

Definition 4.1. A *Quillen adjunction* from the model category $\mathcal{C}$ to the model category $\mathcal{D}$ is a pair of adjoint functors $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{D}$ such that L preserves cofibrations and R preserves fibrations.

From this, it follows that L preserves acyclic cofibrations and R preserves acyclic fibrations; each of these only requires some contemplation of the equivalence between lifting problems of the form

$$\begin{tikzcd} LX \arrow[r] \arrow[d] & Y \arrow[d] \\ LZ \arrow[r] & W \end{tikzcd}$$

and those of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & RY \arrow[d] \\ Z \arrow[r] & RW. \end{tikzcd}$$

Less obviously, but importantly, it follows from the above that L preserves weak equivalences between cofibrant objects and R preserves weak equivalences between fibrant objects.

Lemma 4.2 (Ken Brown's lemma)**.** *If a functor between model categories preserves acyclic cofibrations between cofibrant objects, it preserves all weak equivalences of cofibrant objects. Dually, if a functor between model categories preserves acyclic fibrations between fibrant objects, it preserves all weak equivalences of fibrant objects.*

Proof. See Lemma 1.1.12 in [Hov91]. $\square$

4.2 Derived adjunction

The previous fact allows the construction of the *derived adjunction* of a Quillen adjunction. This is an adjoint pair $\operatorname{Ho} L \dashv \operatorname{Ho} R : \operatorname{Ho}\mathcal{C} \rightleftarrows \operatorname{Ho}\mathcal{D}$ between homotopy categories, given as follows.

First, define a functor $\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ to be the composite

$$\mathcal{C} \xrightarrow[C]{} \mathcal{C} \xrightarrow[L]{} \mathcal{D} \xrightarrow[[-]]{} \operatorname{Ho}\mathcal{D},$$

and observe that this sends weak equivalences to isomorphisms. If $c \to c'$ is a weak equivalence, then $Cc \to Cc'$ is a weak equivalence by 2-out-of-3 applied to

$$\begin{tikzcd} Cc \arrow[r] \arrow[d, "\simeq"'] & Cc' \arrow[d, "\simeq"'] \\ c \arrow[r, "\simeq"'] & c' \end{tikzcd}$$

since the vertical maps are also weak equivalences. Then $Cc \to Cc'$ is a weak equivalence of cofibrant objects, so by Ken Brown's lemma (4.2) L sends it to a weak equivalence $LCc \to LCc'$; and by the definition of the homotopy category this implies that $[LCc] \to [LCc']$ is an isomorphism. Thus, our composite functor essentially factors through $\operatorname{Ho}\mathcal{C}$: there is a functor $\operatorname{Ho} L : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ with $(\operatorname{Ho} L)[c] \cong [LCc]$ naturally in c. Arguing dually, we see also the existence of $\operatorname{Ho} R : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$. Now for the adjunction between $\operatorname{Ho} L$ and $\operatorname{Ho} R$. We'll do this by constructing the unit and counit and showing they obey the appropriate identities. For the unit, we want for each $[c]$ a natural map $\eta_{[c]} : [c] \to \operatorname{Ho} R(\operatorname{Ho} L[c])$ in $\operatorname{Ho}\mathcal{C}$. By the natural isomorphisms above, this may as well be viewed as a map $[c] \to [RFLCc]$. We can take this to be the composite of

$$[c] \xrightarrow[[\mathrm{cof}_c]^{-1}]{} [Cc] \xrightarrow[[\eta_{Cc}]]{} [RLCc] \xrightarrow[[R(\mathrm{fib}_{LCc})]]{} [RFLCc].$$

Dually, the counit can be defined as $\epsilon_{[d]} = [\mathrm{fib}_d]^{-1}[\epsilon_{Fd} L(\mathrm{cof}_{R(Fd)})]$. Checking that the composites $(\operatorname{Ho} R)(\epsilon_{[d]})\eta_{\operatorname{Ho} R[d]}$ and $\epsilon_{\operatorname{Ho} L[c]}(\operatorname{Ho} L)(\eta_{[c]})$ are identities is a tedious but routine verification, using the above definitions and the corresponding identities of the original adjunction. For a presentation of the derived adjunction which focuses on the bijection between hom-sets rather than the unit and counit, see lemma 1.3.10 of [Hov91].

We'll be interested in cases where $\eta_{[c]}$ and $\epsilon_{[d]}$ are isomorphisms in the homotopy category, and in particular when $R(\mathrm{fib}_{LCc})\eta_{Cc}$ is a weak equivalence (and the dual for ϵ), which implies that $\eta_{[c]}$ is an isomorphism. The most extreme case is when this always holds; note that it suffices to state it for an arbitrary cofibrant object, since c does not directly appear in $R(\mathrm{fib}_{LCc})\eta_{Cc}$, only Cc.

Definition 4.3. A Quillen adjunction $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{D}$ with unit η and counit ϵ is a *Quillen equivalence* if for all cofibrant c, the map $R(\mathrm{fib}_{Lc})\eta_c : c \to RFLc$ is a weak equivalence, and for all fibrant d, the map $\epsilon_d L(\mathrm{cof}_{Rd}) : LCRd \to d$ is a weak equivalence.

If L and R form a Quillen equivalence, it easily follows that $\mathrm{Ho}\, L \dashv \mathrm{Ho}\, R$ is an adjoint equivalence of categories between $\mathrm{Ho}\, \mathcal{C}$ and $\mathrm{Ho}\, \mathcal{D}$.

4.3 Functoriality of model structures on functors

If $\mathcal{C}$ and $\mathcal{D}$ are model categories, and $L \dashv R$ a Quillen adjunction between them, then for any small category A, postcomposition with L and R induces a Quillen adjunction between $\mathrm{Hom}(A, \mathcal{C})_{\mathrm{inj}}$ and $\mathrm{Hom}(A, \mathcal{D})_{\mathrm{inj}}$, if both exist; similarly for the projective model structure. If $L \dashv R$ is indeed a Quillen equivalence, then the induced adjunction is also a Quillen equivalence. This is A.2.8.6 in [Lur09].

4.4 Examples

Example. There is a Quillen equivalence $\mathbf{sSet} \rightleftarrows \mathbf{Top}$, where the left adjoint is geometric realization and the right adjoint is the singular simplicial set functor.

Example. The content of lemma 2.5 is essentially that the identity functor on $\mathrm{Hom}(A, \mathcal{C})$ forms both halves of a Quillen equivalence $\mathrm{Hom}(A, \mathcal{C})_{\mathrm{proj}} \rightleftarrows \mathrm{Hom}(A, \mathcal{C})_{\mathrm{inj}}$.

The most important example for us is a Quillen adjunction $\mathbf{sSet} \rightleftarrows \mathbf{CDGA}^{\mathrm{op}}$. This is not a Quillen equivalence, but we can identify sets of objects for which the unit and counit are isomorphisms in the homotopy category. The adjunction is defined by a simplicial CDGA, which we'll denote $\mathcal{A}$. For each k in the simplicial indexing category Δ, define $\mathcal{A}_k$ to be the algebra which is free on generators $t_0, \ldots, t_k$ of degree zero and $dt_0, \ldots, dt_k$ of degree one, with the notationally-suggested differential, and the additional relations $\sum_i t_i = 1$ and $\sum_i dt_i = 0$. These are essentially the differential forms on the standard k-simplex which are given by rational polynomials in the standard coordinates on the simplex. Since the face and degeneracy maps between the standard simplices are also appropriately polynomial, pullback along them gives maps between the different $\mathcal{A}_k$ which make $\mathcal{A}$ into a simplicial CDGA; for an explicit description of these maps, see [Hes07], 1.19.

The fact that $\mathcal{A}$ carries two different, but compatible, structures means that it can be used to convert from one to the other; if X is a simplicial set, then we have a family of $\mathbb{Q}$-modules $\mathbf{sSet}(X, \mathcal{A}^i)$, which collectively form a CDGA which we'll denote $\mathbb{A}(X)$. Similarly, if B is a CDGA, then $\mathbf{CDGA}(B, \mathcal{A}_k)$ fit together into a simplicial set, which we'll denote $\mathbb{S}(B)$. We have isomorphisms

$$\mathbf{sSet}(X, \mathbf{CDGA}(B, \mathcal{A})) \cong \mathbf{sCDGA}(X \times B, \mathcal{A}) \cong \mathbf{CDGA}(B, \mathbf{sSet}(X, \mathcal{A}))$$

where $\mathbf{sCDGA}$ is the category of simplicial CDGAs, i.e. functors $\Delta^{\mathrm{op}} \to \mathbf{CDGA}$. This shows that $\mathbb{S}$ and $\mathbb{A}$ are adjoint on the right. We prefer to view this as an adjunction $\mathbb{A} \dashv \mathbb{S} : \mathbf{sSet} \rightleftarrows \mathbf{CDGA}^{\mathrm{op}}$. This is shown to be a Quillen adjunction in [BG76], but for it to be a Quillen equivalence is unfortunately too much to hope for; intuitively, CDGAs over $\mathbb{Q}$ can only capture 'rational' information, and the fact that the Eilenberg-MacLane space $K(G, n)$ is not equivalent to the point for a torsion group G is 'invisible' to the rational world.

However, Bousfield and Gugenheim do show (theorem 10.1) that the derived unit and counit are weak equivalences, with some assumptions on the object. They state this only for fibrant and cofibrant objects, but this is sufficient to show it for all objects, since in $\mathrm{Ho}\,\mathcal{C}$ any $[c]$ is isomorphic to some $[c']$ with c' fibrant and cofibrant, and the unit/counit are natural transformations. The specific conditions are these:

- If a simplicial set is connected, rational (meaning all of its homotopy groups are rational vector spaces), nilpotent, and of finite rational type (meaning its homology over $\mathbb{Q}$ is finite dimensional in each degree), then the derived unit is a weak equivalence in $\mathbf{sSet}$, hence an isomorphism in the homotopy category.
- If a CDGA is cohomologically connected (meaning that $H^0 \cong \mathbb{Q}$), finite type (meaning that it is weakly equivalent to a CDGA which is finite-dimensional in each degree), then the derived counit is a weak equivalence in $\mathbf{CDGA}$, hence an isomorphism in the homotopy category.

Nilpotence is a condition involving the fundamental group; for the rest of what follows, we replace it with the stronger simple connectivity, i.e. that the fundamental group vanishes. This means that we also require cohomological 1-connectivity of CDGAs, i.e. that $H^0 = \mathbb{Q}$ and $H^1 = 0$. Informally, what we have is a 'partial equivalence', meaning an equivalence between appropriately-chosen pieces of the homotopy categories. Another important fact about the situation is that a simplicial set obeying the above conditions is sent by the adjunction to a CDGA obeying the above conditions, and vice versa; this is also part of theorem 10.1 in [BG76].

This adjunction also extends to a Quillen adjunction $\mathbf{sSet}^+ \rightleftarrows \mathbf{CDGA}_+^{\mathrm{op}}$ between pointed simplicial sets and augmented CDGAs. A corresponding 'partial equivalence' result for these is also part of theorem 10.1 in [BG76].

5 Functoriality of partial equivalences

In the non-equivariant case, Bousfield and Gugenheim's result tells us that when working with spaces/simplicial sets obeying the appropriate conditions, applying

$\mathbb{A}$ and passing to CDGAs loses no information. Equivariantly, we know that the geometric information of a G-space X can be represented as a functor from $\mathcal{O}_G^{\mathrm{op}}$ to **Top** or **sSet** which records the fixed-point spaces of each subgroup; so we'd like to have a relation between $\mathrm{Hom}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{sSet})$ and $\mathrm{Hom}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{CDGA})$. We'll abstract the details of the situation away, and consider a general setup where a 'partial equivalence' type result is known.

Therefore suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories, $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{D}$ is a Quillen adjunction, and $\mathcal{C}_0$, $\mathcal{D}_0$ are full subcategories of $\mathcal{C}$, $\mathcal{D}$ respectively, such that the following hold.

- If an object of $\mathcal{C}$ is weakly equivalent to an object of $\mathcal{C}_0$, then it is already in $\mathcal{C}_0$; similarly, if an object of $\mathcal{D}$ is weakly equivalent to an object of $\mathcal{D}_0$, then it is already in $\mathcal{D}_0$.
- $L\mathcal{C}_0 \subseteq \mathcal{D}_0$ and $R\mathcal{D}_0 \subseteq \mathcal{C}_0$.
- If $c_0 \in \mathcal{C}_0$ is cofibrant, then the unit of the derived adjunction, $\eta : c_0 \to RFLc_0$, is a weak equivalence. Dually, if $d_0 \in \mathcal{D}_0$ is fibrant, then the counit of the derived adjunction, $\epsilon : LCRd_0 \to d_0$, is a weak equivalence.

The last condition says that the derived adjunction between homotopy categories restricts to an adjoint equivalence on the full subcategories defined by $\mathcal{C}_0$ and $\mathcal{D}_0$.

From all this, we conclude the following.

Lemma 5.1. *With the assumptions above, for any small category A, if the projective model structures on $\mathrm{Hom}(A, \mathcal{C})$ and $\mathrm{Hom}(A, \mathcal{D})$ exist, then the induced derived adjunction between their homotopy categories restricts to an adjoint equivalence on the full subcategories defined by $\mathrm{Hom}(A, \mathcal{C}_0)$ and $\mathrm{Hom}(A, \mathcal{D}_0)$; and the same holds if 'projective' is replaced by 'injective'.*

Proof. We treat the case of the projective model structure; the injective case follows by a dual argument.

As previously, it suffices to prove that the unit resp. counit are weak equivalences when the objects in question are cofibrant resp. fibrant, since the inclusion of cofibrant resp. fibrant objects into the homotopy category is an equivalence of categories.

First, given $d_0 : A \to \mathcal{D}_0$ which is projectively fibrant, we want to show that $L \circ C(Rd_0) \to d_0$ is a weak equivalence. Note that $C(Rd_0)$ is the cofibrant replacement of the functor $R \circ d_0$, not fibrant replacement composed with the functor $R \circ d_0$; that need not be a projectively cofibrant functor. The map is a weak equivalence just when for each a, $L(C(Rd_0)(a)) \to d_0(a)$ is a weak equivalence. Now, the map $C(Rd_0) \to Rd_0$ is a projective acyclic fibration, hence its components are all acyclic fibrations. Similarly, $C(Rd_0)$ is projectively cofibrant, hence all its components are cofibrant by lemma 2.5. Thus, in the diagram

$$\begin{tikzcd} Rd_0(a) & CRd_0(a) \arrow[l, "\simeq"'] \arrow[dl, dashed] \\ (C(Rd_0))(a) \arrow[u, "\simeq"'] & \end{tikzcd}$$

the dashed arrow exists and, by two-out-of-three, is a weak equivalence. Indeed, it is a weak equivalence between cofibrant objects, so by Ken Brown's lemma (4.2), L carries it to a weak equivalence. Thus we have a diagram

$$d_0(a) \longleftarrow L((C(Rd_0))(a)) \longleftarrow LCRd_0(a).$$

The composite is the counit of the ordinary derived adjunction at $d_0(a)$, which is a weak equivalence by assumption since $d_0(a)$ is fibrant. The second map is a weak equivalence, from what we've just said. Thus by two-out-of-three, the first map is a weak equivalence, which is what we wanted.

Now, given $c_0 : A \to \mathcal{C}_0$ which is projectively cofibrant, we want to show that $c_0 \to R \circ F(Lc_0)$ is a weak equivalence. Since fibrant replacement in the projective model structure may be done objectwise, we assume we've chosen it to be that way. Since weak equivalence is also defined objectwise, we just have to prove that for all a, $c_0(a) \to RFLc_0(a)$ is a weak equivalence, which is immediate by assumption since $c_0(a)$ is always cofibrant. $\square$

To complete the application of this to $\mathbb{A} \dashv \mathbb{S} : \mathbf{sSet} \rightleftarrows \mathbf{CDGA}^{\mathrm{op}}$, it only remains to show that the projective model structures on $\mathrm{Hom}(A, \mathbf{sSet})$ and $\mathrm{Hom}(A, \mathbf{CDGA}^{\mathrm{op}})$ exist, for a small category A.

The category of simplicial sets is a cofibrantly generated model category ([Hov91], 3.6.5), with generating cofibrations the inclusions of the boundary of the n-simplex into the n-simplex and generating acyclic cofibrations the inclusions of 'horns', that is the boundary of the n-simplex with an additional $(n-1)$-face removed, into the n simplex. Since all simplicial sets are small ([Hov91], 3.6.5), in particular all the domains of these generating morphisms, and hence by [Bay+14], prop. 4.5, the projective structure on $\mathrm{Hom}(A, \mathbf{sSet})$ exists for any small category A.

Although it does not directly bear on the adjunction of $\mathbb{A}$ and $\mathbb{S}$, we note here that **Top** is also cofibrantly generated ([Hov91], 2.4.19), with generating cofibrations the inclusion of n-spheres into $(n+1)$-disks, and generating acyclic cofibrations the inclusions of the n-disk D^n into one end of the cylinder $D^n \times I$, where I is the unit interval. Since all topological spaces are small with respect to inclusions ([Hov91], 2.4.1), in particular so are the domains of these generating morphisms, and so we can use the same proposition as before to conclude that the projective model structure on $\mathrm{Hom}(A, \mathbf{Top})$ exists for any small category A.

For the case of $\mathbf{CDGA}^{\mathrm{op}}$, things are less straightforward. The model category **CDGA** is cofibrantly generated, but its opposite cannot be expected to be. However, we notice the duality of functor categories

$$\mathrm{Hom}(A, B^{\mathrm{op}}) \cong \mathrm{Hom}(A^{\mathrm{op}}, B)^{\mathrm{op}}.$$

Suppose B is a model category and the injective model structure on $\mathrm{Hom}(A^{\mathrm{op}}, B)$ exists, and use it to define a model structure on $\mathrm{Hom}(A, B^{\mathrm{op}})$ using model-categorical duality and the above isomorphism. A natural transformation $\eta : F \Rightarrow G : A \to B^{\mathrm{op}}$ will be a fibration in this model structure if and only if it is a cofibration in $\mathrm{Hom}(A^{\mathrm{op}}, B)_{\mathrm{inj}}$, if and only if each component is a cofibration in B, if and only if each component is a fibration in B^{op}. Similarly, a natural transformation will be a weak equivalence if and only if each component is a weak equivalence in B^{op}. This means that, for B a model category, if the injective model structure on $\mathrm{Hom}(A^{\mathrm{op}}, B)$ exists, then the projective model structure on $\mathrm{Hom}(A, B^{\mathrm{op}})$ exists. We'll verify that the injective model structure indeed exists when $B = \mathbf{CDGA}$, which will complete our chain of categories connected by Quillen adjunctions (which we know how to restrict to equivalences): we can go from $\mathrm{Hom}(A, \mathbf{Top})_{\mathrm{proj}}$ to $\mathrm{Hom}(A, \mathbf{sSet})_{\mathrm{proj}}$ to $\mathrm{Hom}(A, \mathbf{CDGA}^{\mathrm{op}})_{\mathrm{proj}} \cong (\mathrm{Hom}(A^{\mathrm{op}}, \mathbf{CDGA})_{\mathrm{inj}})^{\mathrm{op}}$.

For the injective model structure on $\mathrm{Hom}(A^{\mathrm{op}}, \mathbf{CDGA})$ to exist, it suffices by proposition A.2.8.2 in Lurie [Lur09] that **CDGA** be a *combinatorial* model category. This means it is a cofibrantly generated model category which is *locally presentable* as a category. We know that **CDGA** is cofibrantly generated; see, for example, [Hes07]. Essentially, **Ch** is cofibrantly generated, and transferring a model structure across an adjunction preserves cofibrant generation.

To define local presentability, we need the notion of a λ-directed colimit, for a cardinal λ.

Definition 5.2. A λ-*directed poset* is a poset in which every subset of size strictly less than λ has an upper bound. A λ-*directed colimit* is a colimit indexed over a λ-directed poset.

Definition 5.3. A cocomplete category C is *locally presentable* if, for some regular cardinal λ, there is a set S of objects such that $s \in S$ implies that $C(s, -)$ preserves λ-directed colimits, and every object of C is a λ-directed colimit of objects of S.

This definition follows [AR94].

Lemma 5.4. *The category of CDGAs is locally presentable, with $\lambda = \aleph_0$ and S the set of finitely generated algebras.*[3]

Proof. First, a map from a finitely generated algebra F to $\mathrm{colim}_i A_i$ is specified by finitely many elements of $\mathrm{colim}_i A_i$ corresponding to the images of the generators (which may be required to satisfy some relations). By inspection, an element in a colimit of algebras is, possibly nonuniquely, a polynomial in elements which are in the images of the canonical maps $A_i \to \mathrm{colim}_i A_i$. But

[3]Technically, for this to be a set, we should choose one representative of each isomorphism class of finitely generated algebras, and let S be the set of all chosen representatives.

for an $\aleph_0$-directed colimit, this means that it comes from a single A_i — take i to be an upper bound, in the indexing poset, of the finitely many i's involved in the polynomial. Then, take an upper bound of these is for the finitely many generators of F. Our map will factor through this $A_i \to \operatorname{colim}_i A_i$. This means that $\operatorname{Hom}(F, -)$ commutes with $\aleph_0$-directed colimits.

Second, an algebra is the colimit of its finitely generated subalgebras; it is easy to see that this is a $\aleph_0$-directed colimit, and the statement itself amounts to saying that a map $A \to B$ of algebras is equivalently a choice of maps $F \to B$ for all finitely generated subalgebras F of A, such that these choices agree on intersections. This is certainly true. $\square$

All of the arguments above go through just as easily for pointed simplicial sets and augmented CDGAs.

6 Actions of discrete groups

Here, we look at actions of discrete groups from the model categorical perspective. We'll be working with systems of augmented CDGAs, that is, functors $\mathcal{O}_G \to \mathbf{CDGA}_+$, and systems of cochain complexes, that is, functors $\mathcal{O}_G \to \mathbf{Ch}$. We often consider the objectwise cohomology of such objects; if A is a system of cochain complexes or augmented CDGAs, then $H^*(A)$ is a system of graded rational vector spaces. (In the latter case, it is also a graded algebra.) We consider these functor categories to be equipped with the injective model structure; there is a Quillen adjunction between the two, the left adjoint Λ_+ of which is the free algebra functor, with the augmentation that kills all the generators (but not the unit) and the right adjoint U of which is the underlying cochain complex except in degree zero, where it is the submodule consisting of those elements whose augmentation is zero.

If V is a system of vector spaces, we denote V considered as a cochain complex of systems concentrated in degree k by $K(V, k)$.

Definition 6.1. A functor I from $\mathcal{O}_G$ to vector spaces is *injective* if for all inclusions of functors $F \to G$ and natural transformations $f : F \to I$, there is a natural transformation $g : G \to I$ which extends f.

Definition 6.2. Suppose $f : A \to B$ is a map of cochain complexes. Following Griffiths and Morgan [GM13], define the cochain complex M_f by $M_f^n = A^n \oplus B^{n-1}$ with differential $d(a, b) = (da, f(a) - db)$. Its cohomology is, by definition, the relative cohomology of f. This cohomology is usually written $H^*(A, B)$ rather than $H^*(f)$, even though it depends strongly on f, because $H^*(f)$ could also refer to the induced map $H^*(A) \to H^*(B)$.

Since this definition is functorial, it also makes sense for maps of functors to cochain complexes. The maps $b \mapsto (0, -b)$ and $p : (a, b) \mapsto a$ both commute with coboundaries, and they induce a long exact sequence. Again, all this is functorial.

Also, note that if A and B are both injective in each degree, so is M_f, being a direct sum of those two; the differential is irrelevant. Finally, we write $M(A)$ for M_{id_A}.

Definition 6.3. A *dual Postnikov tower* for a 1-connected system of augmented CDGAs A is a collection of maps $A_i \to A$ for $i \geq 0$, together with maps of maps

$$\begin{tikzcd} A_{i+1} \arrow[dr] & \\ & A \\ A_i \arrow[uu] \arrow[ur] & \end{tikzcd},$$

such that $A_0 = A_1 = \mathbb{Q}$, the induced map $\operatorname{colim} A_i \to A$ induces an isomorphism in objectwise cohomology, and for all i, the map $A_i \to A_{i+1}$ is given by a pushout of the form of the right hand square in

$$\begin{tikzcd} \Lambda_+(K(V, i+2)) \arrow[r] \arrow[d] & \Lambda_+(V') \arrow[r] \arrow[d] & A_i \arrow[d] \\ \Lambda_+(\Sigma^{-1}M(K(V, i+2))) \arrow[r] & \Lambda_+(\Sigma^{-1}M(V')) \arrow[r] & A_{i+1}. \end{tikzcd}$$

Here, V' is a fibrant replacement for $K(V, i+2)$ in systems of cochain complexes, and $\Sigma^{-1}X$ is defined by $(\Sigma^{-1}X)^i = X^{i+1}$.

Note that $\operatorname{colim} A_i \to A$ being a quasi-isomorphism is equivalent to $A_i \to A$ being an isomorphism in degrees $\leq i$ for all i, since these cohomologies cannot be changed by higher-degree additions. Also, it implies that $H^{i+1}(A_i) \to H^{i+1}(A)$ is an injection, since $H^{i+1}(A_i) \to H^{i+1}(A_{i+1})$ is. Considering the long exact sequence in relative cohomology, in turn, reveals that $H^k(A_i, A) = 0$ if $k \leq i+1$.

To see that $\Lambda_+(V') \to \Lambda_+(\Sigma^{-1}M(V'))$ is always a cofibration, note that one can first attach the cocycles, and then attach everything else; this shows it to be a relative Sullivan algebra. So $A_i \to A_{i+1}$ is the pushout of a cofibration, hence a cofibration; hence each A_i is cofibrant, as is their colimit.

Lemma 6.4. *Suppose that A is such that A^k is an injective functor for all $k < n$, and $H^k(A) = 0$ for $k < n$. Then $B^n(A)$ is injective, and so the exact sequence $0 \to B^n(A) \to Z^n(A) \to H^n(A) \to 0$ is split.*

Proof. Suppose $k < n$; we'll show that $B^k(A)$ injective implies $B^{k+1}(A)$ injective. This will suffice, since $B^0(A) = 0$ is clearly injective. Since $H^k(A) = 0$, we have $B^k(A) \cong Z^k(A)$, so the exact sequence $0 \to Z^k(A) \to A^k \to B^{k+1}(A) \to 0$ expresses $B^{k+1}(A)$ as a quotient of an injective by an injective. This implies it's injective. (To show this, note that the sequence is split.) $\square$

Note that A being injective, as a functor to graded vector spaces, means that each A^k is injective as a functor to vector spaces. This is especially useful in light of the following.

Lemma 6.5. *Suppose $A \to B$ is a fibration (in the injective model structure on systems of CDGAs). Then its kernel, considered as a functor to graded vector spaces, is injective.*

Proof. Write K for the kernel, and suppose we're given $V \hookrightarrow W$ and a map $V \to K$ of functors to graded vector spaces. By free-forgetful adjunction, this gives a diagram

$$\begin{tikzcd} \Lambda_+(V, dV) \arrow[r] \arrow[d, hook] & A \arrow[d] \\ \Lambda_+(W, dW) \arrow[r, "0"'] & B \end{tikzcd}$$

of functors to CDGAs. Here, $\Lambda_+(V, dV)$ is the free augmented CDGA on the cochain complex which is $V^i \oplus V^{i-1}$ in degree i, with the obvious differential. We want to show the left-hand arrow is an acyclic cofibration. Both of these conditions are checked objectwise, since we're using the injective model structure; it's clear that it's a weak equivalence, since the cohomology of $\mathbb{Q}(V, dV)$ vanishes no matter what V is. To show that it is a cofibration at some object G/H of $\mathcal{O}_G$, that is, that $\Lambda_+(V(G/H), dV(G/H)) \hookrightarrow \Lambda_+(W(G/H), dW(G/H))$ is a relative Sullivan algebra, pick a complement of $V(G/H)$ in $W(G/H)$ and a well-ordered basis $\{w_\alpha\}$ for it; then add, first all the dw_α, then all the w_α. It's clear that this is a relative Sullivan algebra. Note that none of this needs to respect the action of automorphism groups in $\mathcal{O}_G$, or anything like that; the definition of Sullivan algebra requires only the existence of a basis of a certain form, not a choice of one. Since the map is an acyclic cofibration, there's a map solving the lifting/extension problem; by free-forgetful adjunction again, one sees that it corresponds to a map $W \to A$ which extends $V \to K \to A$ and is zero when composed with $A \to B$; that is, it's a map $W \to K$ which extends $V \to K$. This is as desired. □

Lemma 6.6. *Suppose a map of (systems of) cochain complexes is injective in positive degrees and a quasi-isomorphism. Then it is injective in degree zero.*

Proof. Say $f : A \to B$ is such a map, and $a_0 \in A^0$ is such that $f(a_0) = 0$. Then $df(a_0) = f(da_0) = 0$, and so $da_0 = 0$ since f is injective in degree 1. That is, $a_0 \in Z^0 A$ is a cocycle. But f is an isomorphism on cocycles, so $f(a_0) = 0$ implies $a_0 = 0$. □

Proposition 6.7. *Every fibrant, cohomologically 1-connected system of algebras has a dual Postnikov tower.*

Proof. We'll build it inductively, preserving the invariant that each stage A_i has an underlying cochain complex UA_i which is degreewise injective. The constant functor $\mathbb{Q}$ itself suffices for stages 0 and 1, and is cofibrant. Its underlying cochain complex $U\mathbb{Q}$ is zero, hence degreewise injective.

Suppose it's already been successfully constructed up to stage n. Then Lemma 6.4 applies to the relative system of cochain complexes M of $A_n \to A$; the relative object M is degreewise injective since in each degree, it's a direct sum of a degree of A_n and one of A, each of which is injective, the latter by lemma 6.5. Choose a splitting s, and consider the resulting map $V = H^{n+2}(M) \to Z^{n+2}(M) \to Z^{n+2}(A_n)$. Also, choose a fibrant replacement V' of $K(V, n+2)$ which is zero in degrees less than $n+2$; this is possible by desuspending $n+2$ times to obtain $K(V, 0)$, fibrantly replacing that, and then resuspending the result $n+2$ times (i.e. applying the functor Σ^{n+2} defined by $(\Sigma^{n+2}X)^i = X^{i-(n+2)}$). Our data so far fits into the following diagram.

$$\begin{tikzcd}
\Lambda_+(K(V,n+2)) \arrow[r] \arrow[d] \arrow[rr, bend left, "s"] & \Lambda_+(V') \arrow[d] & M \arrow[r] & A_n \arrow[ddr] & \\
\Lambda_+(\Sigma^{-1}M(K(V,n+2))) \arrow[r] \arrow[drrrr, bend right] & \Lambda_+(\Sigma^{-1}M(V')) & & & \\
& & & & A
\end{tikzcd}$$

where the left square is in the image of Λ_+ and is comprised of cofibrations. Now we need to find a map $s' : V' \to M$ making

$$\begin{tikzcd}
K(V,n+2) \arrow[r, "s"'] \arrow[d] & M \\
V' \arrow[ur, "s'"'] &
\end{tikzcd}$$

commute. We can do this degreewise using M's degreewise injectivity (which follows from that of A_n and A); in degree $n+2$, this is just the fact that $V \to V'^{n+2}$ is an inclusion (by 6.6), while in higher degrees the inclusion is that of $\operatorname{im} d$ into V'^i. (The fact that $K(V, n+2) \to V'$ is a quasi-isomorphism is also needed, to make sure the maps $\operatorname{im} d \to M^i$ are well-defined.) This gives us a diagram

$$\begin{tikzcd}
\Lambda_+(K(V,n+2)) \arrow[r, "\simeq"'] \arrow[d] & \Lambda_+(V') \arrow[r, "s'"'] \arrow[d] & M \arrow[r] & A_n \arrow[d] \arrow[ddr] & \\
\Lambda_+(\Sigma^{-1}M(K(V,n+2))) \arrow[r, "\simeq"'] \arrow[drrrr, bend right] & \Lambda_+(\Sigma^{-1}M(V')) \arrow[rr] \arrow[drrr] & & A_{n+1} \arrow[dr] & \\
& & & & A
\end{tikzcd}$$

where the right square is a pushout. Note that in each degree, A_{n+1} is a direct sum of tensor products of components of A_n and V', both of which are injective; hence A_{n+1} is injective in each degree, by proposition 7.36 in Scull [Scu01].

For $k \leq n$, we know by assumption that $H^k(i_n)$ is an isomorphism, and it's clear by degree considerations that $H^k(A_n \to A_{n+1})$ is an isomorphism, so $H^k(A_{n+1} \to A)$ is an isomorphism. Now, for the rest, we have overlapping commutative diagrams with exact rows. (This follows the argument from Griffiths and Morgan [GM13].)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^{n+1}(A_n) \arrow[r, hook] \arrow[d, "\cong"'] & H^{n+1}(A_{n+1}) \arrow[r] \arrow[d] & H^{n+2}(A_n, A_{n+1}) \arrow[r] \arrow[d] & H^{n+2}(A_n) \arrow[d, "\cong"'] \\
0 \arrow[r] & H^{n+1}(A_n) \arrow[r, hook] & H^{n+1}(A) \arrow[r] & H^{n+2}(A_n, A) \arrow[r] & H^{n+2}(A_n)
\end{tikzcd}$$

By the 5-lemma, to show the middle map is an isomorphism, it suffices to show the fourth map is an isomorphism. This fourth map is the map which sends the class of (a_{n+2}, a_{n+1}) to the class of $(a_{n+2}, i_{n+1}a_{n+1})$. Let's look at the domain more closely: for (a_{n+2}, a_{n+1}) to be a relative cocycle means precisely that $da_{n+1} = a_{n+2}$. Considering the pushout square above, we can see that $A_{n+1}^{n+1} = A_n^{n+1} \oplus V'^{n+2}$, since there's nothing in degrees 1 to $n+1$ of V'. So, writing $a_{n+1} = a'_{n+1} + v$, we have $a_{n+2} = da'_{n+1} + dv$. If we add the coboundary $d(a'_{n+1}, 0)$ to this cocycle, the result is of the form (dv, v). One can check that no coboundary is of this form. However, not all v are suitable; in order for the coboundary of v, when considered as an element of A_{n+1}^{n+1}, to land in A_n^{n+2}, we must have that v is a cocycle, when considered as an element of V'^{n+2}. These cocycles are precisely V, however, so the cohomology under consideration is isomorphic to V itself. Starting from V and following, first this isomorphism, then the vertical map, we take v to the class of (dv, v) to the class of $(dv, i_{n+1}v) = sv$. But the class of sv is v itself, by definition of s! Thus this map is an isomorphism and hence so is the induced map on H^{n+1}.

Finally, we want the map on H^{n+2} to be an injection.

$$\begin{tikzcd}
H^{n+2}(A_n, A_{n+1}) \arrow[r] \arrow[d, "\cong"'] & H^{n+2}(A_n) \arrow[r] \arrow[d, "\cong"'] & H^{n+2}(A_{n+1}) \arrow[r] \arrow[d] & H^{n+3}(A_n, A_{n+1}) \arrow[d] \\
H^{n+2}(A_n, A) \arrow[r] & H^{n+2}(A_n) \arrow[r] & H^{n+2}(A) \arrow[r] & H^{n+3}(A_n, A)
\end{tikzcd}$$

By the 4-lemma, it suffices for the fourth map in the above diagram to be an injection. Again, let's look closely at its domain; it's comprised of classes of pairs (a_{n+3}, a_{n+2}) with $a_{n+3} \in A_n^{n+3}$ and $a_{n+2} \in A_{n+1}^{n+2}$, with the relation $a_{n+3} = da_{n+2}$. Now, A_{n+1}^{n+2} is $A_n^{n+2} \oplus V'^{n+3}$, since $A_n^1 = 0$. As before, we can add a coboundary of the form $d(x, 0)$ to eliminate the first component, so we essentially have $v \in V'^{n+3}$ with $dv = 0$ in V'. Now, since $H^{n+3}(V') = 0$, we get that $v = dv'$, and this implies that in fact our arbitrary cocycle is a coboundary. So the domain of the fourth map is just zero, and it is an injection. □

Note that the arguments with elements above are valid, even in the functor category; one can use generalized elements, or perhaps appeal to the Freyd-Mitchell embedding theorem.

This construction is essentially the same as that in Scull [Scu01]; we've used the injective structure on the model category to provide us with injective resolutions 'for free'. Therefore, in the finite case, the properties that Scull proves hold of the colimit; for example, if A_i and B_i are towers for systems of algebras A and B, any weak equivalence $\operatorname{colim} A_i \to \operatorname{colim} B_i$ is homotopic to an isomorphism. (Note that homotopy between maps, as considered in Scull, is an instance of the general notion of homotopy in a model category; for which, see e.g. [Hov91], 1.2.4.) This implies that this colimit, called the *minimal model* of A, is an algebraic structure whose isomorphism classes correspond to homotopy classes of spaces; such structures are, in a sense, the ultimate goal of homotopy theory. It is interesting to consider whether the use of model categories could simplify or clarify the proofs of these important properties of minimal models.

7 General group actions

The material of the previous section had very little dependence on the group G; indeed, we could repeat essentially the same discussion for $\mathrm{Hom}(A, \mathbf{CDGA})_{\mathrm{inj}}$ for any small category A. Unfortunately, for a non-discrete topological group, one must consider categories of topologically enriched functors rather than functors.

To illustrate this, consider the 'naive' equivariant homotopy theory, by which we mean the homotopy theory in which an equivariant map is considered to be an equivalence if it is an ordinary weak equivalence, with no condition that it induce weak equivalences on fixed-point spaces. In the discrete case, this means we are talking about the model category $\mathrm{Hom}(\mathbf{B}G, \mathbf{Top})_{\mathrm{proj}}$, where $\mathbf{B}G$ is the category with a single object $*$ and morphisms G; it's easy to verify that a functor in this category is precisely a G-space, and the notion of equivalence is as previously stated. However, for a general group G, functors $\mathbf{B}G \to \mathbf{Top}$ correspond to a space X and a not-necessarily-continuous monoid homomorphism $G \to \mathbf{Top}(X, X)$. Otherwise put, the action map $G \times X \to X$ is not necessarily continuous in the first variable. In order to correct this, we can recognize that $\mathbf{Top}$ and $\mathbf{B}G$ are both *topologically enriched* categories, that is, categories where each hom-set is a topological space and composition is a continuous function between spaces. Considering topologically enriched functors then recovers the correct notion of a G-space. However, the relationships that we know between the categories $\mathbf{Top}$, $\mathbf{sSet}$, and $\mathbf{CDGA}^{\mathrm{op}}$ only hold on the level of ordinary categories, not enriched ones; therefore, in order to exploit them, we want to find a description of the category of enriched functors $\mathbf{B}G \to \mathbf{Top}$ that only uses $\mathbf{Top}$ as an ordinary category. For the naive homotopy theory, this is in fact possible: the category of enriched functors, which we'll denote $\mathrm{Hom}_{\mathbf{Top}}(\mathbf{B}G, \mathbf{Top})_{\mathrm{proj}}$ since the enrichment is over $\mathbf{Top}$, is Quillen equivalent to the overcategory $\mathbf{Top}/\mathrm{B}G$ (which becomes a model category by declaring a morphism to be a weak equivalence, fibration, or cofibration if its underlying morphism in **Top** is). This non-bolded $\mathrm{B}G$ is the base space of the universal bundle $\mathrm{E}G \to \mathrm{B}G$. The right Quillen functor, from $\mathrm{Hom}_{\mathbf{Top}}(\mathbf{B}G, \mathbf{Top})$ to $\mathbf{Top}/\mathrm{B}G$, sends X to $X \otimes_G \mathrm{E}G$, the quotient of $X \times \mathrm{E}G$ by the diagonal action $g(x, e) = (gx, ge)$, equipped with the map that collapses X to a point, $X \otimes_G \mathrm{E}G \to * \otimes_G \mathrm{E}G = \mathrm{E}G/G = \mathrm{B}G$. The left Quillen functor, in the other direction, sends $X \to \mathrm{B}G$ to the pullback $X \times_{\mathrm{B}G} \mathrm{E}G$, with the action of G on $\mathrm{E}G$. Verifying that these are an adjunction of ordinary categories is a straightforward exercise, using the fact that $\mathrm{E}G \to \mathrm{B}G$ is a principal bundle.

Lemma 7.1. *The functor $- \otimes_G \mathrm{E}G$ preserves fibrations.*

Proof. Suppose we have an equivariant Serre fibration $f : X \to Y$; we want $X \otimes_G \mathrm{E}G \to Y \otimes_G \mathrm{E}G$ to be a Serre fibration. By definition of a Serre fibration, this means that every lifting problem of the form

$$\begin{tikzcd} D^n \arrow[r, "g"'] \arrow[d, "{(\mathrm{id},0)}"'] & X \otimes_G \mathrm{E}G \arrow[d, "f \otimes_G \mathrm{E}G"'] \\ D^n \times I \arrow[r, "h"'] & Y \otimes_G \mathrm{E}G \end{tikzcd}$$

should be solvable. (Here D^n is the n-dimensional disk and I is the closed unit interval.) We can translate this into a diagram in $\mathbf{Top}/\mathrm{B}G$ by equipping D^n with the map $p_X g : D^n \to X \otimes_G \mathrm{E}G \to \mathrm{B}G$ and equipping $D^n \times I$ with the map $p_Y h : D^n \times I \to Y \otimes_G \mathrm{E}G \to \mathrm{B}G$. Then, by the usual adjunction, it corresponds to an equivariant problem

$$\begin{tikzcd} D^n \times_{\mathrm{B}G} \mathrm{E}G \arrow[r] \arrow[d, "{(\mathrm{id},0,\mathrm{id})}"'] & X \arrow[d, "f"'] \\ (D^n \times I) \times_{\mathrm{B}G} \mathrm{E}G \arrow[r] & Y \end{tikzcd}$$

However, of course, $p_X g$ and $p_Y h$ are null-homotopic, since D^n and $D^n \times I$ are contractible. Since homotopic maps define isomorphic bundles when pulled back, there are equivariant homeomorphisms $D^n \times_{\mathrm{B}G} \mathrm{E}G \cong D^n \times G$ and $(D^n \times I) \times_{\mathrm{B}G} \mathrm{E}G \cong D^n \times I \times G$. These can indeed be chosen so that the induced map is the inclusion at 0. Since equivariant maps $A \times G \to X$ correspond with non-equivariant maps $A \to X$, this means that it suffices to solve a non-equivariant lifting problem

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, "{(\mathrm{id},0)}"'] & X \arrow[d, "f"'] \\ D^n \times I \arrow[r] & Y \end{tikzcd}$$

which is possible since f is a Serre fibration. ∎

It is shown in lemma 4.1 of [KW17] that both of these functors preserve weak equivalences, hence that they form a Quillen adjunction (we already know the right member preserves fibrations and weak equivalences, which means that it in particular preserves fibrations and acyclic fibrations, which is sufficient). The cited lemma also shows that this induces an equivalence on homotopy categories; that is, it is a Quillen equivalence. (Note that $X \times EG$ is a cofibrant replacement for the G-space X.)

This description, as the homotopy theory of $\mathbf{Top}/\mathrm{B}G$, allows us to apply our prior sequence of Quillen adjunctions, after proving some lemmas about overcategories. By [Hir05], if $\mathcal{C}$ is a model category, there is a model structure on $\mathcal{C}/X$ for any object X in which a map is a fibration, cofibration, or weak equivalence if its underlying map in $\mathcal{C}$ is so.

Lemma 7.2. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories, with Y a fibrant object in $\mathcal{D}$, and suppose $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{D}$ is a Quillen adjunction. Then there is a Quillen adjunction $\mathcal{C}/RY \rightleftarrows \mathcal{D}/Y$, which is a Quillen equivalence if the original adjunction is.*

Proof. The left adjoint sends $c \to RY$ to the composition $Lc \to LRY \to Y$, while the right adjoint simply sends $d \to Y$ to $Rd \to RY$. That these functors are adjoint is a simple check. It's clear that the right adjoint preserves fibrations and acyclic fibrations; hence the pair is a Quillen adjunction.

For the part about Quillen equivalences, first suppose $c \to RY$ is cofibrant, that is, c is cofibrant as an object of $\mathcal{C}$. We want to conclude that the unit of the derived adjunction is a weak equivalence. Consider the fibrant replacement of $Lc \to LRY \to Y$ in $\mathcal{D}/Y$. It amounts to a choice of object d and maps making the diagram

$$\begin{tikzcd}
Lc \arrow[rr] \arrow[d] & & d \arrow[ddl] \\
LRY \arrow[dr] & & \\
& Y &
\end{tikzcd}$$

commute, such that $Lc \to d$ is an acyclic cofibration and $d \to Y$ is a fibration. Since Y is fibrant, this implies that d is also fibrant, so d is another choice for the fibrant replacement of Lc in $\mathcal{D}$. Arguing as in lemma 5.1, $Lc \to d$ factors through FLc, and the map $FLc \to d$ realizing this factorization is a weak equivalence of fibrant objects. The map $c \to RLc \to Rd$, which is what we want to conclude is a weak equivalence, therefore factors as $c \to RLc \to RFLc \to Rd$. We know $c \to RLc \to RFLc$ is a weak equivalence, by assumption, since it's the unit of the ordinary derived adjunction; and we know $RFLc \to Rd$ is a weak equivalence, because R preserves weak equivalences between fibrant objects.

Finally, suppose that $d \to Y$ is fibrant, that is, $d \to Y$ is a fibration in $\mathcal{D}$. Since Y is fibrant, this also implies that d is fibrant. We want the counit of the derived adjunction to be a weak equivalence. Consider the cofibrant replacement of $Rd \to RY$ in $\mathcal{C}/RY$. It amounts to a choice of object c and maps making the diagram

$$\begin{tikzcd} c \arrow[rr] \arrow[dr] & & Rd \arrow[dl] \\ & RY & \end{tikzcd}$$

commute, such that c is cofibrant in $\mathcal{C}$ and $c \to Rd$ is an acyclic fibration. Again, it's clear that CRd, with the obvious maps, defines a possible choice of cofibrant replacement; so the derived counit map $Lc \to LRd \to d$ factors as $Lc \to LCRd \to LRd \to d$, the first piece of which is a weak equivalence by Ken Brown's lemma (4.2), and the second piece of which is the ordinary derived counit, which we know to be a weak equivalence. □

This proof makes it clear that, in a situation where the derived unit and counit are only sometimes weak equivalences, a 'partial equivalence' result analogous to 5.1 still holds.

Definition 7.3. A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence.

Lemma 7.4. *Suppose $\mathcal{C}$ is a model category and X, Y are two objects of $\mathcal{C}$, and $f : X \to Y$ a map. Then there is a Quillen adjunction $\mathcal{C}/X \rightleftarrows \mathcal{C}/Y$, where $\mathcal{C}/X \to \mathcal{C}/Y$ is composition with f and $\mathcal{C}/Y \to \mathcal{C}/X$ is pullback along f. Further, if f is a weak equivalence and $\mathcal{C}$ is right proper, this pair is a Quillen equivalence.*

Proof. It's clear that the two functors are adjoint and that composition preserves cofibrations and acyclic cofibrations, so there is indeed a Quillen adjunction.

For the statement about Quillen equivalence, it will be more convenient for us to use the following alternate characterization: a Quillen adjunction is a Quillen equivalence if whenever c is cofibrant and d is fibrant, $Lc \to d$ is a weak equivalence if and only if $c \to Rd$ is a weak equivalence. In this case, a map p corresponds with a map q in this way precisely when they fit into the diagram

$$\begin{tikzcd} c \arrow[r, "q"'] \arrow[rr, bend left, "p"] \arrow[dr] & P \arrow[r] \arrow[d] & d \arrow[d] \\ & X \arrow[r, "\simeq"'] & Y \end{tikzcd}$$

where the right square is a pullback. If $d \to Y$ is fibrant, that is, is a fibration, then by right properness $P \to d$ is a weak equivalence, and 2-out-of-3 shows that p is a weak equivalence if and only if q is. □

To apply this to our case, we start with the fibrant $\mathrm{B}G$ in **Top**, and consider a fibrant replacement $\mathfrak{M}_{\mathrm{B}G} \leftarrow \mathbb{A}(\mathrm{Sing}(\mathrm{B}G))$ in $\mathbf{CDGA}^{\mathrm{op}}$.[4] We may in particular

[4]So, the arrow in this sentence goes the opposite direction that the actual map of algebras does.

take this to be a minimal model. By the results of Bousfield and Gugenheim, the adjunct $\mathrm{Sing}(\mathrm{B}G) \to \mathbb{S}(\mathfrak{M}_{\mathrm{B}G})$ of this map is also a weak equivalence, if we knew that $\mathrm{Sing}(\mathrm{B}G)$ is rational, 1-connected, and of finite rational type. These last two conditions are true for many connected Lie groups, but BG is not usually rational; however, we can replace it for this purpose with its rationalization, which is a rational space rationally equivalent to it, and for rational spaces X, maps to the rationalization correspond with maps to $\mathrm{B}G$ itself. Thus, we ultimately get a duality between 1-connected finite type G-spaces considered up to 'naive' equivariant rational homotopy equivalence and 1-connected finite type CDGAs equipped with a map from $\mathfrak{M}_{\mathrm{B}G}$.

For the true, non-'naive' homotopy, the situation is more complicated still: $\mathcal{O}_G$ not only carries a natural topology on its hom-sets, but on its object set as well. (Intuitively, in this topology, the object G/aHa^{-1} should be close to G/H whenever a is close to the identity.) Multiple objects also means that one needs to consider 'actions' which go from one space to another rather than staying in the same space, and we do not yet know how to capture as we did for the naive homotopy. Recording this information in a way that uses only the ordinary categorical structure on **Top** is an interesting challenge which, if accomplished, would yield a category of algebraic objects — likely, functors from some category into CDGAs, or an undercategory thereof — together with a functor from G-spaces into this category, which remembers all rational homotopy-theoretic information about 1-connected spaces. Once this algebraic category is identified, we can hope to identify 'minimal' objects whose isomorphism types correspond to homotopy types, as in the non-equivariant case.

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