Symplectic representation theory and the Weyl algebra in positive characteristic

SPUR Final Paper, Summer 2016
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3 August 2016

Abstract

Given a symplectic representation $V \oplus V^*$ of a finite group G over a field k with characteristic $p > 0$, we can extend the G-action in a natural way to an action on the Weyl algebra W in $\dim(V)$ variables. This allows us to form the smash algebra $W\#G$ as a tensor product of the Weyl algebra and the group algebra kG. In this paper, we explore the problem of whether the ideal generated by the trivial idempotent of the group algebra contains 1. We are able to give an explicit condition that is both necessary and sufficient in the case where G is abelian, and extend these techniques to tackle the case where G is solvable. In addition, we shed some insight on the problem for a general group G.

1 Introduction

The problem treated in this paper was motivated by the proof of a theorem in symplectic geometry, the details of which may be found in [1]. In the course of proving Lemma 6.1, the authors make use of the fact that the ideal generated by the trivial idempotent in the smash algebra $W\#G$ contains 1 when considered over a field of characteristic 0, which raises the question of whether this is always the case for an arbitrary field of characteristic p and an arbitrary symplectic representation. This paper aims to shed light on this problem, both for its own sake and because of the potential for this problem to relate to questions in algebraic group theory over characteristic 0.

In Section 2, we will provide clear definitions of the objects that will be under study, as well as introducing some additional tools from representation theory that will be of use in attacking the problem. In Section 3, we will prove some basic results applicable to every symplectic representation; among other things, we will make use of the theory of algebraic groups to state and prove results about the structure of the representations for which this question is nontrivial.

In Section 4, we will examine the simplest case: the case where G is abelian. We will combine facts about the structure of the symplectic group with results from the theory of multiplicative characters to give an explicit criterion for the trivial idempotent of a group G to generate the entire smash algebra, fully solving this case. Starting in Section 5, we will turn our attention to the so-called "symplectic block group", a particular natural subgroup of the symplectic group with some useful properties that make the problem more tractable. We will prove a result about solvable subgroups of the symplectic Weyl group that closely parallels the corresponding result about abelian groups, although we will no longer be able to explicitly give a criterion for whether the ideal AeA contains 1.

We will discuss the case where G is a general subgroup of the symplectic Weyl group in section 6 by making use of a more formal approach, with the goal of shedding some insight on this more difficult case. We will be able to reformulate the problem in a simpler fashion by making use of the decomposition of $\mathrm{k}G$ into simple modules.

2 Definitions

Let k be a field of characteristic $p > 2$, and let G be a finite subgroup of the symplectic automorphism group $\mathrm{Sp}(2n)$ that acts on the vector space $V \oplus V^* = k^n \oplus (k^n)^*$. In other words, we equip the vector space $V \oplus V^*$ with a faithful representation of some finite group G. We also require that this representation preserves some nondegenerate alternating bilinear form ω, which we can write as the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ by choosing a basis for V and extending it in the canonical way to a basis for $V \oplus V^*$.

We note that the action of G on V extends to an action of G on the Weyl algebra W; i.e. the quotient of the free unital associative algebra over k in $2n$ indeterminates $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ by the relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$. This action is defined as follows: G acts on the free associative unital algebra generated by our basis for $V \oplus V^*$. It is easy to verify that due to the preservation of the symplectic form, this action descends to an action on the quotient algebra; that is, G acts in a well-defined way on the Weyl algebra.

Let A denote the smash product $W\#G$ of the Weyl algebra W with the group algebra kG. This algebra is isomorphic as a k-vector space to k$G \otimes W$, and multiplication is given by the rule $gPg^{-1} = P^g$ for any $P \in W$; that is, we should think of the smash product as a semidirect tensor product. (Note that P^g denotes the group action on W.) We can now consider the two-sided ideal in this algebra generated by the idempotent of the trivial representation $e_G = \frac{1}{|G|}\sum_g g$. Note that we require here that p does not divide $|G|$, which we will assume is the case for the remainder of this paper.

Definition We say that a group G, along with its associated symplectic representation, is **W-potent** if the ideal AeA contains 1.

We note that the ideal defined above is generated by linear combinations of the form PeQ for elements $P, Q \in W$, since group elements can be commuted past elements of the Weyl algebra (modulo some invertible action) and are absorbed by the idempotent.

It will also be useful to consider the other primitive central idempotents of the group algebra kG. It is a well-known result in basic group representation theory that the idempotent corresponding to the irreducible representation V is given by $e_V = \frac{\dim V}{|G|}\sum_{g\in G}\chi_V(g^{-1})g$. (Note that $e_{\mathbf{1}} = e$.) These idempotents commute with kG and sum to 1; in addition, the product of any two distinct primitive idempotents is 0. The careful reader will note that the decomposition into irreducibles may well depend on k, which we have not assumed is algebraically closed. In fact, we will see later that we may in fact take k to be algebraically closed without loss of generality. Because of this, the reader may assume that whenever a one-dimensional irreducible character is referred to, we are working over a field extension containing all relevant roots of unity such that the characteristic polynomial of each group element splits. In fact, without loss of generality we may take k to be the algebraic closure of $\mathbb{F}_p$, since we are only concerned with semisimple elements of $\mathrm{Sp}(2n)$ of finite order.

Definition We say that an element of the Weyl algebra P generates the representation V if the left kG-module spanned by $\{P^g : g \in G\}$ under the conjugation action is isomorphic to V.

This definition is very important, as we can decompose any polynomial P into a sum of components P_i such that the representation generated by P_i is irreducible. This follows from Maschke's theorem.

3 Basic properties

Just using the definitions, we can establish some basic facts about W-potency.

Proposition 1.

- *Let $G < \mathrm{Sp}(2n)$ be a W-potent group and let $H < G$ be a subgroup. Then H is also W-potent.*
- *Let $G < \mathrm{Sp}(2n)$ and $H < \mathrm{Sp}(2m)$ be groups with symplectic representations. The direct product $G \times H$ embeds naturally as a subgroup of $\mathrm{Sp}(2(n+m))$. Then $G \times H$ is W-potent if and only if both of its factors are.*
- *Let $G < \mathrm{Sp}(2n)$ be a group, and let $M \in \mathrm{Sp}(2n)$ be arbitrary. Then MGM^{-1} is W-potent if and only if G is.*
- *Let $G < \mathrm{Sp}(2n)$ be a group. Let L be a field extension of k. G embeds naturally as a subgroup of the symplectic group acting on $L^n \oplus (L^n)^*$. Then G is W-potent over k if and only if it is W-potent over L.*

Proof.

- By supposition, $1 = \sum PeQ$ for some collection of elements of the Weyl algebra P, Q. Applying the linear map $\mathrm{k}G \to \mathrm{k}H$ that sends elements in $G \setminus H$ to 0 to this linear combination recovers 1 as a member of the ideal $Pe'Q$, where e' is the trival idempotent of $\mathrm{k}H$.
- Denote by e_1 and e_2 the trivial idempotents of $\mathrm{k}G$ and $\mathrm{k}H$ respectively. For one direction, suppose that G and H are both W-potent. Then $1 = \sum P_G e_1 Q_G = \sum P_H e_2 Q_H$, where the P_G and Q_G are contained in W_G and similarly for W_H. Then $1 = (\sum P_G e_1 Q_G)(\sum P_H e_2 Q_H)$. We also observe that P_G commutes with Q_H, P_G commutes with e_2, and Q_H commutes with e_1. Finally, note that $e_1 e_2 = e$, the trivial idempotent of the product group. This allows us to rewrite the above product as the sum of terms of the form $P_H P_G e Q_H Q_G$, which are in the ideal AeA. For the other direction, note that any element of $W_{G\times H}$ can be written as the product of an element of W_G and an element of W_H which commute with each other. Therefor, the same factorization trick applied in the other direction allows us to conclude that $Ae_{G\times H}A$ is contained in the product of the ideals $Ae_G A$ and $Ae_H A$.
- This follows because the isomorphism of smash algebras defined by $g \mapsto MgM^{-1}$ and $P \mapsto P^M$ preserves the element e and hence the ideal generated by it.

- The map between the smash algebra over k and the smash algebra over L preserves the dimensions and codimensions of ideals, if the ideals in the domain are taken as vector spaces over k and the ideals in the image are taken as vector spaces over L. The result follows by considering the codimension of AeA.

□

We will now discuss briefly the structure of the symplectic group over $\overline{\mathbb{F}_p}$. We will make use of the following two facts about algebraic groups (including the symplectic group):

- Every element in the symplectic group Sp(2n) can be written in the form $x = x_s x_u$, where x_s is semisimple, x_u is unipotent, and x_s and x_u commute.

- Every semisimple element of Sp(2n) is conjugate to a diagonal matrix by an element of the symplectic group; furthermore, any collection of commuting semisimple elements may be so diagonalized by a common element of the symplectic group.

We will also make use of the following fact from elementary ring theory: Any unipotent element in a ring of characteristic p has order equal to a power of p. This follows by writing the element as $1 + x$ where x is nilpotent, then using the identity $(1 + x)^{p^n} = 1 + x^{p^n}$. Combined with the first fact above, this implies that any element of the symplectic group with a finite order coprime to p — i.e., any member of a subgroup we are considering — is semisimple. The second fact implies that such an element may be diagonalized via an element of the symplectic group. Since we proved above that W-potency is conjugation invariant, this allows us to take abelian groups to be subgroups of the maximal torus generated by the diagonal elements with no loss of generality.

4 Abelian groups

We will start with the case where G is abelian. By the conjugacy result above, we may take any individual generator g to have the form $\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$. (Note that any diagonal element of the symplectic group has this form because for each i it must preserve the identity $\partial_i x_i - x_i \partial_i = 1$.)

Lemma 1. *A group G with associated symplectic representation is W-potent if and only if $AeA \cap W$ contains some monomial of the form $\prod_i \partial_i^{a_i} x_i^{b_i}$ where each of the a_i and b_i are strictly less than p.*

Proof. One direction is obvious. The other follows from the fact that $[\partial_i, f] = f_{x_i}$; i.e. taking the commutator of an element of the polynomial subalgebra with a differential operator differentiates with respect to that variable. A similar relation allows us to "differentiate" the operators; i.e. reduce their exponents. The condition on the size of the exponents allows us to avoid running into problems with the positive characteristic. □

The previous result is motivated by the fact that the Weyl algebra over a field of characteristic p has a nontrivial center. The ideal generated by $(x_i^p, \partial_i^p)_i$ is central, and this allows us to form the quotient algebra W_0 of W by this ideal. The conjugation action of G on W restricts to an action on W_0 if and only if G preserves the positive part of the center, which is indeed the case for many important groups. Because W_0 is finite-dimensional, it is a useful means to prove that certain groups are not W-potent.

Let $G < \mathrm{Sp}(2n)$ be an abelian group with associated symplectic representation; without loss of generality we may write G as a subgroup of the diagonal group $\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$. Hence, for any $g \in G$, its action on the Weyl algebra is given by $x_i^g = g_{ii}x_i$, and $\partial_i^g = g_{ii}^{-1}\partial_i$. We may define multiplicative characters $\chi_1, \ldots, \chi_n \in \hat{G}$ by $\chi_i(g) = g_{ii}^{-1}$.

Theorem 1. *G is W-potent if and only if any character in the character group $\hat{G}$ can be written in the form $\prod_i \chi_i^{a_i}$ where $0 \leq a_i < p$.*

Proof. We will make use of a convenient abuse of notation and use χ to mean both a one-dimensional representation and its associated character. Similarly, we will denote by $\chi_1 \cdot \chi_2$ both their product in the character group and the tensor representation $\chi_1 \otimes \chi_2$. We first make the observation that for any arbitrary $\chi \in \hat{G}$, $e_\chi x_i = x_i e_{\chi \cdot \chi_i}$. Similarly, $e_\chi \partial_i = \partial_i e_{\chi \cdot \chi_i^{-1}}$.

To prove the forward direction, it suffices to show that the following element is in AeA for each $\chi \in \hat{G}$: $\left(\prod_i x_i^{p-1}\right) e_\chi$. The result then follows because the sum of these elements is a scalar multiple of the polynomial $\prod_i x_i^{p-1}$ (by character orthogonality relations), which we can then differentiate using the lemma above to obtain 1. Let $\chi = \prod_i \chi_i^{a_i}$. Then $\prod_i x_i^{a_i} \cdot e \cdot \prod_i x_i^{p-1-a_i} = \prod_i x_i^{p-1} e_\chi$ as desired.

For the other direction, consider the quotient map ϕ that sends W to W_0 as defined above. We notice that the action of G on the quotient algebra is well defined, because the diagonal group preserves the positive part of the center. It will therefore suffice to show that the two-sided ideal generated by e in the quotient algebra does not contain 1, because if this is the case then certainly its preimage does not. We will show this fact by constructing explicitly an element in W_0 that is not contained in the image of the ideal AeA. This element is $\left(\prod_i x_i^{p-1}\right) e_\chi$, where χ is a character that does not have the form $\prod_i \chi_i^{a_i}$ with $0 \leq a_i < p$.

To show that this element indeed cannot be generated, we observe that the image of the ideal is a finite dimensional vector space spanned by $\prod_i \partial_i^{a_i} x_i^{b_i} \cdot$

$e \cdot \prod_i \partial_i^{c_i} x_i^{d_i}$ where $0 \leq a_i, b_i, c_i, d_i < p$. Commuting e past these elements using the group operation allows us to rewrite this as $\left(\prod_i \partial_i^{a_i} x_i^{b_i} \partial_i^{c_i} x_i^{d_i}\right) e_{\chi_0}$. Here $\chi_0 = \prod_i \chi_i^{b_i - a_i}$.

The next step is to put this expression in a canoncial form by commuting $x_i^{b_i} \partial_i^{c_i}$ past one another. It is not necessary to do this explicitly; it suffices to note that the result is contained in the span of vectors of the form $\partial_i^{c_i - k_i} x_i^{b_i - k_i}$, where $0 \leq k_i \leq \min(b_i, c_i)$. Therefore, if the element $\left(\prod_i x_i^{p-1}\right) e_\chi$ was contained in the ideal, then it would certainly be contained in the span of vectors of the form $\left(\prod_i \partial_i^{a_i + c_i - k_i} x_i^{b_i + d_i - k_i}\right) e_{\chi_0}$. Hence we must have $\chi_0 = \chi$, $a_i + c_i - k_i = 0$, and $b_i + d_i - k_i = p - 1$ for each $1 \leq i \leq n$. We note that χ_0 is a product of the χ_i; thus by assumption there must exist some i_0 for which $b_{i_0} - a_{i_0}$ is negative (since not all of the exponents can be in the range $0 \leq b_i - a_i < p$, and $b_i - a_i \geq p$ is absurd since $b_i < p$ and $a_i > 0$.)

However, subtracting the equations $a_i + c_i - k_i = 0$ and $b_i + d_i - k_i = p - 1$ gives $(b_i - a_i) + (d_i - c_i) = p - 1$. The quantity $b_i - a_i$ is negative, and the quantity $d_i - c_i$ must be strictly less than p. Hence we have reached a contradiction. □

5 Solvable subgroups of the symplectic Weyl group

We saw above that considering the intersection of AeA with the Weyl algebra is a useful way to approach the problem, and for sufficiently large p the problem of showing that AeA contains 1 reduces to showing that its intersection with W is nontrivial (by the differentiation lemma).

Definition We say that a group G is **W-regular** if there exists some element P in the Weyl algebra for which the conjugates $\{P^g\}_g \in G$ are linearly independent.

Definition The **symplectic block group** B_{2n} in $2n$ dimensions is defined as the group of block matrices

$$\begin{pmatrix} M & 0 \\ 0 & (M^{-1})^{\intercal} \end{pmatrix}.$$

The symplectic block group has two useful properties that are of use in studying its action on the Weyl algebra. Firstly, the action of the symplectic block group preserves the positive part of the center, and hence acts in a well-defined manner on W_0. Secondly, as we will see in the following lemma, every subgroup of the symplectic block group is W-regular.

Lemma 2. *Let G be a subgroup of the symplectic block group. Then G is W-regular; in fact we can take the Weyl element generating the regular representation to be a polynomial in the x_i.*

Proof. We observe that the action of G on the monomials $x_1, \ldots, x_n$ defines a faithful representation. It is clear that the action of G on the polynomial algebra generated by the x_i decomposes into a sum of the symmetric powers of this faithful representation. It is a well-known theorem in representation theory that the sum of the symmetric powers of a faithful G-representation, under the right conditions, contains a free kG-submodule (see [2], page 45). □

The W-regularity condition is useful because it guarantees that any irreducible representation of G or one of its subgroups occurs as the representation generated by some polynomial in W. In particular, all one-dimensional representations occur as the representation generated by some polynomial. As we saw earlier, if P generates the one-dimensional representation with idempotent e_χ, then $Pe = e_\chi P$. This fact is the key step in the proof of the following theorem:

Theorem 2. *Let G be a solvable subgroup of the symplectic block group. Then $AeA \cap W$ is nontrivial.*

Saying that $AeA \cap W$ is nontrivial is not the same as saying that it contains 1, although the two are equivalent in characteristic 0. The difficulty in characteristic p arises because W is not a simple algebra. Nonetheless, this is still a natural way to attack the problem that builds on our approach towards dealing with abelian groups. Our proof will once again make use of one-dimensional characters in an essential way.

Proof. Since G is solvable, it necessarily has some normal subgroup N with an abelian quotient G/N. We will first show that there is some element in AeA of the form Pe', where e' is the trivial idempotent of N and P is a nonzero member of the polynomial algebra generated by the x_i. We will then argue by induction that this implies the result.

First, we note that there are $[G : N]$ one-dimensional characters χ of G that satisfy $\chi(N) = 1$. Furthermore, the sum of the kG-idempotents corresponding to these characters is precisely e'. By W-regularity, we may choose some polynomials P_i with $1 \leq i \leq [G : N]$ such that $eP_i = P_i e_{V_i}$, where the V_i are the one-dimensional representations mentioned above. It follows that by taking the product $\prod_i P_i$ and placing e to the right of every polynomial except P_j (bearing in mind that the polynomials commute, since they are contained in the subalgebra generated by the x_i) allows us to realize $\left(\prod_i P_i\right) e_{V_j}$ for each j as a member of AeA. The sum of these elements is Qe' for some polynomial Q (in fact $Q = \prod_i P_i$).

Now notice that the argument above can be repeated with the presence of some arbitrary polynomial Q in the product, assuming Q commutes with all of the Weyl elements. Hence we can keep quotienting out and multiplying by additional polynomials until we have reached the end of the subnormal series of G, at which point we have shown that some nonzero polynomial (possibly of a very high degree) is present in $AeA \cap W$. □

Note that we make no attempt to control the degree of the polynomials or whether their product is contained in the positive part of the center. In particular, it would suffice to take a generating set χ_i for the character group $\hat{G}$ and find corresponding polynomials Q_i that generate these one-dimensional representations. Then, as above, we may place e at all the possible points inside the word $\prod_i Q_i^{\mathrm{ord}(\chi_i)-1}$ in order to generate e' multiplied by a polynomial. Doing so may produce an effective condition on the nature of the solvable groups that we can show are W-potent by this method.

6 The general case: A decomposition of A into $\mathrm{k}G$ bimodules

We turn now to the case of a general group G. Without the use of one-dimensional representations as a means to approach the problem, concrete results are much harder to come by. First, we will state and prove some basic results about the structure of the smash algebra A.

Lemma 3. *The smash algebra A is equal to the direct sum of independent linear subspaces $\bigoplus_{U,V} e_U A e_V$. Furthermore, any (two-sided) ideal I of the smash algebra decomposes in the same way, and $e_U I e_V = I \cap e_U A e_V$.*

Proof. Define the linear operator $\mathcal{F}_{U,V} \in End(A)$ as $a \mapsto e_U a e_V$. We observe that each $\mathcal{F}$ is idempotent, and that the product of any two distinct $\mathcal{F}, \mathcal{F}'$ is zero. Therefore the $\mathcal{F}$ project onto their images, which are linearly independent. Furthermore, the sum of all the $\mathcal{F}$ is the identity map due to the fact that $\sum_V e_V = 1$, so the direct sum of their images is equal to A.

The first claim about ideals follows from the second. Because I is an ideal, $\mathcal{F}_{U,V} \circ I \subset I$, and obviously $\mathcal{F}_{U,V} \circ I \subset e_U A e_V$. Finally, if $Q = e_U P e_V \in I$, then $\mathcal{F}_{U,V} \circ Q = Q$ and hence $Q \in \mathcal{F}_{U,V} \circ I$. The result follows by combining these assertions. □

Since A is a $\mathrm{k}G$-bimodule under multiplication, this motivates us to consider the $\mathrm{k}G$-bimodule structure of the projections $e_U A e_V$. We start by considering the case where either U or V is the trivial representation; this case is easier to handle and we can be more explicit.

Lemma 4. *The linear subspace $e_V A e$ is spanned by elements of the form Pe, where the representation generated by P is isomorphic to V. Similarly, the linear subspace eAe_V is spanned by elements of the form eP, where the representation generated by P is isomorphic to V^*.*

Proof. To prove the first assertion, it suffices to show that $e_V P e = P^{e_V} e$. This follows as a result of the fact that $ge = e$. The second assertion can be proved analogously to the first by commuting in the other direction. □

In the general case, we can still describe the kG-bimodule structure by using a more formal approach.

Lemma 5. *The left multiplication action of* kG *on* $e_U A e_V$ *is isomorphic to its action on* $U \otimes (U^* \otimes W \otimes V)^G \otimes V^*$*, where the action on* U *is the action of the irreducible representation* U *and its action on the other tensor components is trivial (i.e. they are multiplicity space). The right multiplication action is given by the right* G*-action on the same space, except now the action on* V^* *is the nontrivial action and* U *is now part of the multiplicity space.*

Proof. We will prove the statement for the left action; the right follows analogously. We first write $A = W \otimes \mathrm{k}G$; since e_V is central in the group algebra, we may write the module as $e_U \otimes W \otimes e_V \mathrm{k}G$. Left multiplication by the central primitive idempotent e_U corresponds to projection into the U-isotypic submodule. In addition, it is well known that $e_V \mathrm{k}G$ has a kG-bimodule structure given by $V \otimes V^*$; the V^* is multiplicity space for the left action and vice versa for the right action. (Note that this follows from the fact that the regular representation of a group G contains each irreducible representation V with multiplicity $\dim V$.)

Therefore, the left module structure is given by the U-isotypic summand of $W \otimes V$, all tensored by the multiplicity space V^*. By Schur's lemma we may rewrite this in the form $U \otimes (U^* \otimes W \otimes V)^G \otimes V^*$, where all but the first U is multiplicity space. □

Note that we are implicitly making use of the canoncial isomorphism $U^* \otimes V \cong \mathrm{Hom}(U, V)$. In addition, it is important to remember that the idempotent e_V in fact projects onto V^* instead of V when acting on the right.

We can now reformulate our question in the following way: A group G is W-potent if and only if the multiplicative maps $e_V Ae \otimes eAe_V \to e_V A e_V$ are surjective for each irreducible representation V. In terms of left kG modules using the description above, this map becomes $V \otimes (V^* \otimes W)^G \otimes (W \otimes V)^G \otimes V^* \to V \otimes (V^* \otimes W \otimes V)^G \otimes V^*$; ignoring factors that appear with the same action on the right and the left, the problem reduces to showing that the map $(V^* \otimes W)^G \otimes (W \otimes V)^G \to (V^* \otimes W \otimes V)^G$ is surjective. This suggests that we should seek to approach this problem by understanding this structure of the spherical subalgebra $e_V A e_V$.

Note that the above statement can be stated more explicitly as follows: Because $1 = \sum_V e_V$, it follows that the projection of 1 using the $e_U A e_V$ decomposition is simply as the sum of the identity elements of the spherical subalgebras. Therefore, if $1 \in AeA$ and we write $1 = \sum_i P_i e Q_i$, we can project into V to get $e_V = \sum_i e_V P_i e Q_i e_V$. By Lemma 4, we may rewrite this as $e_V = \sum_i P_i^{e_V} e Q_i^{e_{V^*}}$. Therefore the problem reduces to showing that for each irreducible representation V, we may write e_V as a linear combination of elements of the form PeQ, where P generates V and Q generates V^*. Right away this tells us that G must be close to W-regular in the sense that every irreducible representation must occur as a summand in the decomposition of W as a left kG-module under the conjugation action; in fact, if G preserves the positive part of the center we may replace W above with W_0.

We now make the following conjecture that we hope will lead to a resolution of the general case (of a subgroup of the block group) in a similar fashion as the solvable case; i.e. a proof that the interesction $AeA \cap W$ contains a nonzero element.

Conjecture 1. *Let G be a subgroup of the symplectic block group in $2n$ dimensions, and let V be an irreducible representation. Let P be a member of the polynomial algebra spanned by the x_i that generates V; let Q be a member of the same algebra that generates V^*. Schur's lemma tells us that $V \otimes V^*$ has a trivial isotypic summand of dimension 1; the representation defined by the conjugation action on the vector space spanned by $\{P^g Q^h : g, h \in G\}$ is isomorphic to a quotient of $V \otimes V^*$. We conjecture that the trivial summand of this quotient is nontrivial; i.e. there is some linear combination of $P^g Q^h$ that is nonzero and G-invariant. Furthermore, this element, possibly multiplied by some nontrivial member of the group algebra, is in the span of $P^g e Q^h$.*

If true, this conjecture would allow us to produce an element of the form $T\ell e_V$ in the linear span of the $P^g e Q^h$, where $\ell \in \mathrm{k}G$ and T is G-invariant. Since T commutes with kG and the submodule of kG generated by e_V is simple, some kG-linear combination of $T\ell e_V$ would therefore equal Te_V; producing such a G-invariant in each spherical subalgebra $e_V A e_V$ would allow us to realize their product as a member of W (mimicking our proof of the solvable case). A possible route to proving this conjecture lies in considering $e_V A e_V$ as a left kG module under conjugation and acting via the trivial idempotent, since this projects the subalgebra into the centralizer of kG; of course, any G-invariant polynomial multiplied by a primitive central idempotent is in this centralizer.
We close this section by providing a concrete bound on $|G|$ in the case that G is W-potent and acts on W_0.

Proposition 2. *Let G be a W-potent subgroup of the symplectic group* $\mathrm{Sp}(2n)$ *that preserves the positive degree part of the center of W. Then $|G| \leq p^{2n}$.*

Proof. By assumption G acts on W_0, allowing us to form the smash algebra $A_0 = W_0 \# G$. The ideal $A_0 e A_0$ is spanned by a basis for $W_0 \otimes W_0$ under the map $P \otimes Q \mapsto PeQ$. Also note that $\dim A_0 = \dim \mathrm{k}G \cdot \dim W_0 = |G| \cdot \dim W_0$. Hence if $A_0 e A_0 = A_0$, then necessarily $(\dim W_0)^2 \geq |G| \cdot \dim W_0$. Finally, note that $\dim W_0 = p^{2n}$, since a basis is given by $\prod_i \partial_i^{a_i} x_i^{b_i}$ with $0 \leq a_i, b_i < p$. □

7 Further directions

There are several directions for further research that present themselves at this juncture. The first is to provide an explicit description of those solvable groups for which the argument presented in this paper holds; this might lead to a condition in the solvable case that is both necessary and sufficient. In addition, there is much work to be done in the general case. For instance, in the case of the simple group A_5 (which acts naturally on $k^5 \oplus (k^5)^*$ via permutation block matrices), very little is known about the ideal AeA. One approach might be to consider the same group over $\mathbb{C}$; by the simplicity of the smash algebra over fields of characteristic zero, there should be some $\mathbb{C}$-linear combination of Weyl elements PeQ that equals 1, and perhaps there is a way to lift this linear combination into characteristic p.

Another important direction is to consider groups that are not conjugate to subgroups of the symplectic block group. One good place to start might be the "symplectic Weyl group" of elements in the normalizer of the diagonal group; this group preserves the positive part of the center, but in addition to block matrices it also contains elements that swap x_i and ∂_i. This has the advantage of still preserving the positive part of the center.

Finally, the hardest case of this problem seems to be those groups that do not act on W_0 in a well-defined way. Since making use of the finite dimensional algebra W_0 is the only technique in this paper capable of proving that a particular group is not W-potent, a new approach will have to be found for this case; perhaps there is some useful method short of giving an explicit description of the ideal AeA.

In terms of approaches that might be useful, it might be fruitful to attempt to unify the idempotent projections employed in Section 6 with the differentiation idea used in previous sections.

8 Acknowledgements

This project would not have been possible without the funding and research environment provided by the SPUR program, with special thanks to Professors Moitra, Jerison, and Gerovitch for organizing and furthering this summer's research. The author would also like to thank Augustus Longergan for his indispensible mentorship; without his assistance this project would not have gotten off the ground.

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