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Higher Bruhat orders in Type B

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Abstract. Authors Yu. I. Manin and V. V. Schechtman developed the theory of "higher Bruhat orders," presenting a family of combinatorial objects closely related to the symmetric group. In particular, the two authors define a series of ranked posets which generalize the weak left Bruhat order; their construction has found applications in geometry and representation theory. In this paper, we define a similar combinatorial system for the group B_n of hypercube symmetries. We prove the analogue of Manin and Schechtman's theorem for the cases $k = 1$ and $k = 2$, and relate these orders to the structure of the group B_n.

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1. Introduction & Preliminaries

In [5], authors Manin and Schechtman developed the theory of "higher Bruhat orders" for the Weyl groups of type A. These consisted of a sequence of ranked posets, $B(I_n, k)$, such that $B(I_n, 1)$ is isomorphic to the weak left Bruhat order on the Weyl group A_{n-1}. The posets $B(I_n, k)$ exhibit a number of nice structural and combinatorial properties, which will be outlined below.

These posets, and their relationship to the Weyl groups A_n, have found a number of useful applications. Notably, they were important in Elias and Williamson's development of Soergel Calculus (cf. [1]). In that setting, generalizations of the orders $B(I_n, k)$ to arbitrary Weyl groups are highly desirable. For more work which either extends or utilizes the theory of Manin and Schechtman, see [1, 2, 7].

Our project's aim is to generalize the notion of a "higher Bruhat order" to Weyl groups of type B. In particular, we hope to define a sequence of posets B_B with analagous properties.

To elaborate on the properties of $B(I_n, k)$ we wish to extend, we proceed into a discussion of the theory presented by Manin and Schechtman.

1.1. Higher Bruhat orders in type A (cf. [5]).

Let I_n denote the set $\{1 \dots n\}$, ordered in the usual way, and let $C(I_n, k)$ denote the set of k-element subsets of I_n. That is to say,

$$C(I_n, k) := \{S \subset I_n \,:\, \text{card}\,(S) = k\}\,.$$

There is a natural, *lexigographic* total ordering on $C(I_n, k)$, as follows. For any S and T in $C(I_n, k)$, with smallest elements s and t, respectively, define $<_k$ recursively, such that

$$S <_k T \text{ if } s < t, \text{ or if } s = t \text{ and } S \setminus \{s\} <_{k-1} T \setminus \{t\}\,,$$

until the standard ordering on $C(I_n, 1) \approx I_n$ is reached. The *anti-lexicographic* ordering may be obtained by reversing this order.

For each $K \in C(I_n, k+1)$, define $P(K)$ to be the set $\{S \in C(I_n, k) \,:\, S \subset K\}$. We refer to $P(K)$, viewed as a subset of $C(I_n, k)$, as a *k-packet.* Call a total ordering of $C(I_n, k)$ *admissible* if it restricts to either the lexicographic or anti-lexicographic order on each k-packet.

We will denote by $A(I_n, k)$ the set of admissible total orderings of $C(I_n, k)$. Some facts about $A(I_n, k)$ are outlined below:

- For any admissible ordering ρ, the reverse ordering, ρ^t, is also admissible.
- The lexicographic (resp. anti-lexicographic) ordering, denoted ρ_{min} (resp. ρ_{max}) are admissible.

- There is a function Inv $: A(I_n, k) \to \mathcal{P}(C(I_n, k+1))$ which sends ρ to the set of $K \in C(I_n, k+1)$ such that ρ restricts to the anti-lexicographic order on $P(K)$. For example, Inv $(\rho_{min}) = \emptyset$ and Inv $(\rho_{max}) = C(I_n, k+1)$.
- There is a function $N : A(I_n, k) \to \mathcal{P}(C(I_n, k+1))$ which sends ρ to the set of elements in $C(I_n, k+1)$ whose packet forms a chain in ρ.

(We use $\mathcal{P}(S)$ to denote the power set of S.)

We will now construct the set $B(I_n, k)$. Two admissible orderings, ρ and ρ', are called *elementarily equivalent* if one can be obtained from the other by the exchange of two neighboring elements who do not belong to a common packet. It is evident that for such pairs (ρ, ρ'), we have Inv $(\rho) =$ Inv (ρ'). We write $\rho \sim \rho'$ if there exists a finite sequence $\{\rho_i\}_1^m \subset A(I_n, k)$ such that $\rho_1 = \rho$, $\rho_m = \rho'$, and each pair (ρ_i, ρ_{i+1}) posesses elementary equivalence. It can be checked that '$\sim$' is an equivalence relation.

The set $B(I_n, k)$ is obtained by taking the quotient of $A(I_n, k)$ with respect to this equivalence relation; we will write $[\rho]$ to denote the equivalence class containing ρ. It follows from the above discussion that Inv $(r) =$ Inv (ρ) for all $\rho \in r$, hence the map Inv (r) is well defined for $r \in B(I_n, k)$. It is also convenient to write $N(r) = \cup_{\rho \in r} N(\rho)$.

Finally, we introduce the notion of a *packet-flip*, and the induced partial ordering of $B(I_n, k)$. If, for some $K \in C(I_n, k+1)$, the elements of $P(K)$ form a chain with respect to some ordering $\rho \in r$ (that is to say, if $K \in N(r)$), then we may form a new admissible ordering ρ' be reversing the order of this chain while fixing the position of each other element. If this is the case, we write $[\rho'] = p_K(r)$. This operation is called a *packet-flip*.

The resulting order is admissible as the size of the intersection of $P(K)$ with any other k-packet is at most 1, by construction; this guarantees the packet flip does not affect the order of any other k-packet.

We write $r <_{MS} r'$ if there exists a finite sequence $\{K_i\}_1^m \subset C(I_n, k+1)$ such that $r = r_1$, $r' = r_m$, $r_i = p_{K_i}(r_{i-1})$, and each K_i lies in $N(r_i) \setminus$ Inv (r_i). The following theorem was proven by Manin and Schechtman about the relation $<_{MS}$:

- $<_{MS}$ defines a partial order on $B(I_n, k)$.
- Under $<_{MS}$, $B(I_n, k)$ is a ranked poset with a unique minimal element, r_{min}, and a unique maximal element, r_{max}. The rank is given by $r \mapsto$ card (Inv (r)).
- The map $r_{min} < p_{K_1}(r_{min}) < \cdots < p_{K_m} \cdots p_{K_1}(r_{min}) \mapsto \rho : K_1 \prec \cdots \prec K_m$ defines a bijection from the set of maximal chains in $B(I_n, k)$ to the set $A(I_n, k+1)$.
- The map Inv $: B(I_n, k) \to \mathcal{P}(C(I_n, k+1))$ is injective.

This is the central result of Manin and Schechtman that we wish to generalize.

1.2. **Coxeter systems and reflection groups.** In this section we present necessary background information regarding reflection groups, Coxeter groups, and Bruhat orders.

Definition 1. Let $G \cong \langle S \mid R \rangle$ be a finite group presented by generators and relations, and $\mathcal{F}_S$ the free group over S. For some $g \in G$, consider the set $W_g \subset \mathcal{F}_S$ of words which evaluate to g. By well ordering, we can choose some $w_g \in W_g$ of minimal length.

We define the function $\ell : G \to \mathbb{N}$ which associates to each $g \in G$ the length of w_g. By convention, $\ell(1) = 0$. Often, $\ell(g)$ is referred to as the *length* of g with respect to the presentation $\langle S \mid R \rangle$.

From this point forward, the presentation $G = \langle S \mid R \rangle$ is assumed, so that we may speak freely about the length of an element in G.

Definition 2. Let $\mathrm{R}(g)$ denote the set of minimal-length words which evaluate to some $g \in G$. A word in $\mathrm{R}(g)$ is called a *reduced expression* for g.

Then, the *weak left (resp. right) Bruhat order* is defined by the following ordering relation:

- $g \leq h$ if there exists $w \in \mathrm{R}(g)$ and $w' \in \mathrm{R}(h)$ such that w is a suffix (resp. prefix) of w'.

These partial orders have been studied extensively in the context of reflection groups, and, more generally, Coxeter systems.

Definition 3. Let $S = \{s_i\}$ be a finite, indexed set, and m a symmetric function associating each element of $S \times S$ to an integer greater than 1, and such that $m(s_i,\, s_i) = 2$. The *Coxeter system* given by S and m is denoted (W, S), where W is the group generated by S with the relations

$$(s_i s_j)^{m(i,\,j)} = 1.$$

The group W is called a *Coxeter group.*

Theorem (cf. [3])**.** For each pair of distinct indices $i,\, j \in I$, the order of $s_i s_j$ in W is exactly $m(i,\, j)$.

Definition 4. Fixing $m = m(i, j)$, we see that the relations can be equivalently formulated $s_i s_j s_i \ldots (m \text{ factors}) = s_j s_i s_j \ldots (m \text{ factors})$. For $m > 2$, these will be called *braid relations.* If $m = 2$, they will be called *commutations.*

Given a Coxeter system (W, S), and an element $g \in W$, one can define a graph structure on the set $\mathrm{R}\,(g)$ by including an edge between two words that differ by the substitution of a braid relation or commutation. This graph is denoted $\tilde{\Gamma}(g)$.

Theorem (Matsumoto, cf. [4])**.** For any $g \in W$, $\tilde{\Gamma}(g)$ is connected.

Definition 5 (Reflection groups)**.** Let V be a Euclidean space, and define a *reflection* to be some $T \in O(V)$ whose (-1)-eigenspace has dimension 1, and whose 1-eigenspace has dimension $n-1$. With this definition, a *reflection group* is a matrix group $G \subset O(\mathbb{R}^n)$ generated by reflections. For a reflection group, one can find $\Phi_+ \subset V$ such that G permutes the set $\Phi_+ \cup \{-x : x \in \Phi_+\}$. The following facts about reflection groups will be used in our paper.

Theorem (cf. [3])**.**

- A choice of Φ_+ uniquely determines a set of reflections $S(\Phi_+)$ that generate G, which are called *simple reflections.*
  (This generating set is assumed for the next two statements.)
- The length of some $x \in G$ is equal to $\mathrm{card}\,(x(\Phi_+) \setminus \Phi_+)$.
- G has a unique longest element, w_0, which is the unique maximal element in the weak left (or right) Bruhat order.

We will end our discussion of Coxeter groups and reflection groups by stating the following theorem.

Theorem (Coxeter, cf. [3])**.** Any reflection group G with choice of simple roots S may be presented as a Coxeter system (G, S). Conversely, any finite Coxeter group W may be realized as a reflection group.

1.3. **The group A_n.** Some of the key components of the relationship between the construction of Manin and Schectman and the Weyl groups of type A are outlined below.

- There is a correspondence $\varphi : B(n, 1) \xrightarrow{\sim} A_{n-1}$.
- If A_{n-1} is partially ordered by the weak left Bruhat order, then φ is an isomorphism of ranked posets.
- $A(n, 2)$, when viewed as an undirected graph (where edges are given by the packet-flip operation or elementary equivalence), is isomorphic to $\tilde{\Gamma}(w_0)$.
- Two elements ρ and ρ' in $A(n, 2)$ possess elementary equivalence exactly when the corresponding vertices of $\tilde{\Gamma}(w_0)$ differ by a commutation.

An appropriate generalization of the "Higher Bruhat Order" to other Weyl groups should maintain these properties, which are related to Manin and Schectman's theorem for $k = 1, 2$. We are now in a position to define the general construction in Type B.

2. General Construction

Let E be any finite, totally ordered subset of $\mathbb{Z}$. Furthermore, we require that E has even cardinality, and is stable under negation. Let E_+ denote those elements of E which are positive.

Definition 6. We define $C_B(1, E)$ to be the set E. For $k > 1$, the set $C_B(E, k)$ is defined as the union of the sets $C_B^1(E, k)$ and $C_B^2(E, k)$, which are defined below.

$$\widetilde{C_B^1(E,k)} := \{S \subset E \,:\, \mathrm{card}\,(S) = k, \text{and the elements of } S \text{ have distinct absolute value}\},$$
$$C_B^2(E,k) := \{T \cup \{\star\} \,:\, T \subset E_+,\, \mathrm{card}\,(T) = k-1\}$$

$C_B^1(E, k)$ is constructed by partitioning $\widetilde{C_B^1(E, k)}$ (which are sets) into orbits under negation.

For the remainder of this paper, we will work with the set $J_n := \{-n \ldots n\} \setminus \{0\}$. For brevity, we will define σ to be the map sending $x \mapsto -x$.

Definition 7. In general, the *packet operation* P_B will identify with every element of $C_B(J_n, k+1)$ a subset of $C_B(J_n, k)$. For some $K \in C_B(J_n, k+1)$, the set $P_B(K)$ is referred to as a *k-packet*.

P_B is constructed differently for elements of $C_B^1(J_n, k)$ and $C_B^2(J_n, k)$.

- For $K \in C_B^1(J_n, k+1)$, let $R \in K$ be a representative. We define the set $\widetilde{P}_B(K)$ to consist of the σ-orbits of every k element subset of R. It is noted here that these σ-orbits do not coincide, as the elements of R belong to distinct σ-orbits. Moreover, it is clear that different choices of the representative R produce the same set $\widetilde{P}_B(K)$, so our map is well defined.
- For $K \in C_B^2(J_n, k+1)$, fix $K' = K \setminus \{\star\}$, and consider the set $S = K \cup \sigma(K)$. We define $\widetilde{P}_B(K)$ to be $C_B(S, k)$, noting that S is finite, totally ordered, has even cardinality, and is stable under negation.

Note that the elements of $C_B(J, 1)$ are not themselves σ-orbits. For this reason, we define

$$P_B(K) = \begin{cases} \cup_{T \in \widetilde{P}_B(K)} T & \text{if } K \in C_B^1(J_n, 2) \\ \widetilde{P}_B(K) & \text{otherwise} \end{cases}$$

Notation. Since $C_B^1(J_n, k)$ is a set of equivalence classes, we encounter the problem of choosing a good way to represent its elements. Wherever possible, we will choose the representative for which the element with the greatest magnitude is negative. For some $T \in C_B^1(J_n, k)$, if R is such a representative, we will denote T by the bracketed list $[a_1 \ldots a_k]$, where $t = \{a_1 \ldots a_k\}$. **Such a representative will be referred to as an *ideal* representative, and we will indicate where this choice of representative is assumed.**

For consistency, an element $s \in C_B^2(J_n, k)$ will be denoted as a bracketed list $[b_1 \ldots b_{k-1}, \star]$, as well.

We list elements with negative elements first, in increasing order, followed by positive elements, in decreasing order, e.g. $[-5, -2, 3, 1]$.

Definition 8. The *standard order* of the set $C_B(J_n, 2)$, is defined as follows. It is defined with respect to ideal representatives. For the purposes of comparison, elements $[a_1, \star]$ will assume the value $[-a_1, a_1]$.

- Elements represented by two negative indices occur first, in lexicographic order.

– Elements with a single negative index occur afterwards. If the elements are listed in increasing order, a_1, a_2, then the ordering is lexicographic in the following sense:

$$[a_1,\, a_2] < [b_1,\, b_2] \text{ if } a_1 < b_1, \text{ or if } a_1 = b_1 \text{ and } a_2 > b_2$$

Similarly, we have the following standard order for the set $C_B(J_n, 3)$. As before, elements of the form $[a_1,\, a_2,\, \star]$ will assume the value $[-a_1,\, -a_2,\, a_2]$ in comparison with other elements.

– Elements represented by three negative indices occur first, in lexicographic order.
– Elements represented by two negative indices occur second. For two elements represented by negative indices, we have the comparison

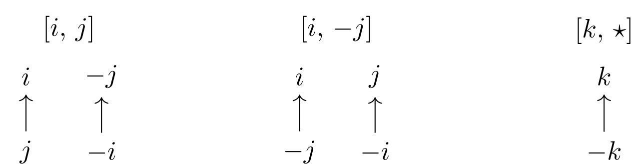
$$[a_1,\, a_2,\, a_3] < [b_1,\, b_2,\, b_3] \text{ if } [a_1,\, a_2] < [b_1,\, b_2], \text{ or if } [a_1,\, a_2] = [b_1,\, b_2] \text{ and } a_3 > b_3 \quad \text{where } a_3,\, b_3 > 0.$$

– Elements represented by a single negative index occur third. For two elements of this form, we have the comparison

$$[a_1,\, a_2,\, a_3] < [b_1,\, b_2,\, b_3] \text{ if } a_1 > b_1, \text{ or if } a_1 = b_1 \text{ and } [-a_2,\, -a_3] < [-b_2\, -b_3].$$

Lastly, the standard order for $C_B(J_n, 1)$ is given by the usual ordering of the set J_n. The standard ordering of a given set is denoted ρ_{min} wherever this notation is unambiguous. For these sets, the *reverse standard* ordering, ρ_{max} is obtained by reversing the standard ordering.

Definition 9. For a 2- or 3-packet P, the ordering of P is given by the restriction of the standard ordering to P. For a 1-packet, orderings are given by the following Hasse diagrams.



The *reverse ordering* of a 1-, 2-, or 3-packet T is given by reversing the direction of each inequality. By standard abuse of notation, T denotes both the set and the ordering relation, and $\operatorname{Rev} T$ denotes both the set and the reverse ordering.

Definition 10. A *comparable component* of a poset is defined to be a connected component of the poset's Hasse diagram.

Definition 11. We will now define the sets $B_B(J_n, k)$, for $k \leq 3$.

– We call a total ordering ρ of $C_B(J_n, k)$ *admissible* if for each k-packet P, ρ extends either P or $\operatorname{Rev} P$.
– The standard ordering, ρ_{min}, and reverse-standard ordering, ρ_{max}, are admissible.
– We define $A_B(J_n, k)$ to be the set of all admissible orderings of $C_B(J_n, k)$.
– For an admissible ordering ρ, $\operatorname{Inv}(\rho)$ is defined as the set of elements $K \in C_B(J_n, k+1)$ such that ρ extends $\operatorname{Rev} P_B(K)$.
– If for $\rho \in A_B(J_n, k)$ the set $N(\rho)$ consists of all elements $K \in C_B(J_n, k+1)$ such that for each comparable component $C \subset P_B(K)$, C forms a chain in ρ.
– Two elements of $C_B(J_n, k)$ *commute* if they are incomparable in each k-packet to which they both belong.
– Two orderings ρ, $\rho' \in A_B(J_n, k)$ are *elementarily equivalent* if ρ' can be obtained from ρ by exchanging the order of two adjacent, commuting elements.
– We define an equivalence relation on $A_B(J_n, k)$ by taking the transitive closure of this relation.

– The set $B_B(J_n, k)$ is constructed by taking the quotient of $A_B(J_n, k)$ with respect to the aforementioned equivalence relation. We write $[\rho]$ to denote the equivalence class containing a given admissible ordering ρ.
– We will define $N([\rho])$ to be $\cup_{\rho' \in [\rho]} N(\rho')$.

Noting that the order on each k-packet must not be trivial, it is clear that for any two equivalent orderings ρ, ρ', we have $\text{Inv}\,(\rho) = \text{Inv}\,(\rho')$. As such, the function Inv is well defined on the set $B_B(J_n, k)$. Moreover, we have the following result.

Proposition 12. The function Inv is injective on the set $B_B(J_n, k)$.

Proof. Consider two orderings ρ and ρ', such that $\text{Inv}\,(\rho) = \text{Inv}\,(\rho') = S$. Note that the transitive closure of the union over the ordering relations Rev P for $P \in S$ and Q for $Q \in C(J_n, k+1) \setminus S$ defines a poset structure on $C_B(J_n, k)$, and both ρ and ρ' must extend this poset. Furthermore, two elements are incomparable in this poset, then they must be incomparable in every packet to which they both belong.

Finally, we cite the well known theorem that any two linear extensions of a given finite poset differ by a finite sequence of transpositions of adjacent, incomparable elements. This coincides directly with our definition of elementary equivalence. So $[\rho] = [\rho']$ is uniquely determined by $\text{Inv}\,(\rho)$, as desired. □

Definition 13. If we consider some total ordering ρ in $A(J_n, k)$, together which some k-packet $P_B(K)$ for $K \in N(\rho)$, we can construct a new admissible order ρ' by reversing the order of each comparable component of $P_B(K)$ in ρ, while all other elements conserve their positions. This operation is called a *packet-flip*, and is written $\rho' = p_K(\rho)$. Clearly

$$\text{Inv}\,(\rho') = \begin{cases} \text{Inv}\,(\rho) \setminus K & \text{if } K \in \text{Inv}\,(\rho) \\ \text{Inv}\,(\rho) \cup \{K\} & \text{otherwise.} \end{cases}$$

For any $[\rho]$, $[\rho'] \in B_B(J_n, k)$, we write $[\rho] < [\rho']$ if there exists a finite sequence $\{\rho_i\}_1^m \subset A_B(J_n, k)$ such that $\rho_1 = \rho$, $\rho_m = \rho'$, and for each pair $(\rho_i,\, \rho_{i+1})$ there exists some $K_i \in N([\rho_i]) \setminus \text{Inv}\,([\rho_i])$ such that for some $\rho_i' \in [\rho_i]$, $\rho_{i+1} = p_{K_i}(\rho_i')$. This relation defines a partial ordering on the set $B_B(J_n, k)$.

Theorem 14. For the cases $k = 1, 2$, $B_B(J_n, k)$ has a unique maximal (respectively, minimal) element, given by $[\rho_{max}]$ (resp. $[\rho_{min}]$).

Our proof will make use of the following lemmas, which will be proved after the general argument is given.

Notation. For some admissible ordering $\rho \in A_B(J_n, k)$, and some $S \subset C_B(J_n, k)$, we write $\overline{S}(\rho)$ to denote the minimal chain containing S in ρ. Furthermore, we write S_K to denote the set $P_B(K)$.

Definition 15. Let $\rho \in A_B(J_n, 2)$ be given. We say that an element x *blocks* S in ρ when $x \in \overline{S}(\rho')$, for all $\rho' \in [\rho]$.

Lemma 16. Let a set $S \subset C_B(J_n, k)$ be given. If x does not block S in ρ, then there exists some $\rho' \in [\rho]$ such that $\overline{S}(\rho') \subset \overline{S}(\rho)$, and $x \notin \overline{S}(\rho')$.

Lemma 17. Suppose $K \notin \text{Inv}\,([\rho])$. Then $K \notin N([\rho])$ if and only if there exists some x which blocks S_K in ρ.

Lemma 18. Suppose $K \notin N([\rho]) \cup \text{Inv}\,([\rho])$. Then at least one of the following seven cases holds for all $\rho' \in [\rho]$. (An ideal representative is assumed.)

– If $K \in C_B^1(J_n, 3)$, fix $K = [i,\, j,\, k]$. Then, we have either:
  (1) $[i,\, j] < [i,\, x] < [i,\, k]$,
  (2) $[i,\, k] < [k,\, x] < [j,\, k]$, or
  (3) $[i,\, j] < [j,\, x] < [j,\, k]$,

for $x \in J_n \setminus \{i,\, j,\, k\}$.

– If $K \in C_B^2(J_n, 3)$, fix $K = [i,\, j,\, \star]$. Then, we have either:
  (1) $[i,\, j] < [i,\, x] < [i,\, \star]$,
  (2) $[i,\, \star] < [i,\, x] < [i,\, -j]$,

(3) $[i,\,-j] < [j,\,x] < [j,\,\star]$, or
(4) $[i,\,j] < [j,\,x] < [i,\,-j]$,

for $x \in J_n \setminus \{i,\,j,\,-i,\,-j\}$.

Lemma 19. Let $\rho \in A_B(J_n, 2)$ be given so that $\rho \notin [\rho_{max}]$. In all of the above cases, there exists some $K' \in C_B(J_n, 3) \setminus \mathrm{Inv}\,([\rho])$ such that either $\overline{S_{K'}}(\rho) \subsetneq \overline{S}_K(\rho)$ or the smallest element in $S_{K'}$ is greater than the smallest element in S_K.

Proof of theorem. Firstly note that any class of orderings $[\rho]$ satisfying $\mathrm{Inv}\,([\rho]) = C_B(J_n, k+1)$ must be maximal. By definition, ρ_{max}, has this property, so $[\rho_{max}]$ is a maximal element. By injectivity of the function Inv on $B_B(J_n, k)$, $[\rho_{max}]$ is the unique class of orderings for which $\mathrm{Inv}\,([\rho]) = C_B(J_n, k+1)$. Therefore, we need only to prove that an equivalence class $[\rho]$ for which $\mathrm{Inv}\,([\rho]) \subsetneq C_B(J_n, k+1)$ is not maximal.

Let $[\rho]$ be such an ordering, so we have some $K \in C_B(J_n, k+1) \setminus \mathrm{Inv}\,([\rho])$ such that each $P_B(K) \notin \mathrm{Inv}\,([\rho])$.

We will prove by induction that the existence of such a K implies the existence of some $K' \in N([\rho]) \setminus \mathrm{Inv}\,([\rho])$, and hence that $[\rho]$ is not maximal.

Case k = 1:

Since elements of $C_B(J_n, 1)$ are themselves indices, we use the notation "$\prec$" to denote the ordering in some admissible order ρ, and "$<$" refers to the ordering inherited from $\mathbb{Z}$.

Suppose there exists some $K \notin \mathrm{Inv}\,(\rho)$. Fix $k = [i,\,j]$ for $i,\,j \in J$, and suppose without loss of generality that i is the maximal element in $P_B(K)$ with respect to ρ. Let C denote the chain in ρ whose smallest element is j and whose greatest element is i.

Since $K \notin \mathrm{Inv}\,(\rho)$, $i > j$. If i and j are neighbors in ρ, then we're done. If not, there exists some k such that $j \prec k \prec i$ in ρ.

If $k > j$ then $[k,\,j] \in N(\rho)$ and we're done. If not, then we have $k < j$. This implies $k < i$, and $[i,\,k] \notin \mathrm{Inv}\,(\rho)$. By induction on the length of C, we conclude that ρ is not maximal. Minimality of ρ_{min} $k = 1$ follows by symmetry. □

Case k = 2:

Let $\rho \in A_B(J_n, 2)$ be given. Fix $S = P_B(K)$ for some $K \in C_B(J_n, 3)$, and let $\overline{S}$ denote the minimal chain containing S.

Note that if some ρ satisfying $\mathrm{Inv}\,([\rho]) \neq C_B(J_n, 3)$ is given, there exists some $K \in C_B(J_n, 3) \setminus \mathrm{Inv}\,([\rho])$.

If $K \in N([\rho])$, we have nothing left to prove. If not, by Lemma 17, we may conclude that some element blocks S_K in ρ. Then, by Lemma 18 and Lemma 19, we may assume the existence of some $K' \notin \mathrm{Inv}\,([\rho])$ such that the smallest element in $S_{K'}$ is greater than the smallest element in S_K, or the minimal chain containing $S_{K'}$ is a strict subset of $\overline{S}(\rho')$ for all $\rho' \in [\rho]$.

Note that in any possible ordering ρ', the number of elements greater than the smallest element in S is finite. Likewise, the length of $\overline{S}(\rho')$ is finite. By induction on these two parameters, we may conclude the existence of some $K^* \in N([\rho]) \setminus \mathrm{Inv}\,([\rho])$. This shows that $[\rho]$ is not maximal. Once again, minimality of $[\rho_{min}]$ follows by symmetry.

□

Proof of Lemma 16. Let ρ and S be given, and let x be some element which does not block S in ρ. By our assumption, we can find some $\hat{\rho} \in [\rho]$ such that $x \notin \overline{S}(\hat{\rho})$. Without loss of generality, $x < \min S$ in $\hat{\rho}$. Then let T be the subset of $\overline{S}(\rho)$ which is less than or equal to x in ρ. Clearly T forms a chain in ρ.

Let $\mathbf{t} = t_1 \dots t_r$ be some sequence of pairs of adjacent, commuting elements which are transposed to carry ρ to $\hat{\rho}$, and $\mathbf{t}' = t'_1 \dots t'_s$ be the subsequence obtained by deleting all pairs for which one or both of the elements do not belong to T. We will show that $\mathbf{t}'$ is a *valid* sequence of adjacent transpositions, namely there exist ρ'_i such that:

– $\rho'_1 = \rho$,
– t'_i is adjacent in ρ'_i, and
– ρ'_{i+1} is obtained from ρ'_i by reversing the order of the pair t'_i.

From this it follows immediately that there exists some $\rho^* = \rho'_{s+1}$ for which $x \notin S$, and such that $\overline{S}(\rho^*) \subset \overline{S}(\rho)$.

Let $\rho_1 = \rho$ and ρ_{i+1} be obtained from ρ_i by inverting the pair t_i. Firstly, note that each clearly each pair in $\mathbf{t}'$ must commute, as it belongs to $\mathbf{t}$. Secondly, note that reversing the order of some pair t_i which is in $\mathbf{t}$ but not in $\mathbf{t}'$ does not change the relative positions of elements of T with respect to the order ρ_i. Finally, reversing the order of some t'_i, if it can be done, preserves the property that T forms a chain.

As such, consider any pair $t'_i = t_p$. If we assume that the sequence is valid up to t'_{i-1}, then ρ'_i agrees with ρ_p on the elements of T. Since t'_i is adjacent in ρ_p and the elements of T form a chain with respect to ρ'_i, we can conclude that t'_i is adjacent in ρ'_i, and the sequence is valid through t'_i. The lemma is proved by induction on the length of $\mathbf{t}'$. □

Proof of Lemma 17. Fix $S = P_B(K)$. If some x blocks S in ρ, then certainly $K \notin N([\rho])$. For the converse, suppose that no element blocks K in ρ. If $\overline{S}(\rho) = S$ we are already finished. If not, we can select some $y \in \overline{S}(\rho) \setminus S$. By Lemma 1, we can produce some ρ' such that $y \notin \overline{S}(\rho')$, and $\overline{S}(\rho') \subset \overline{S}(\rho)$. Then, we have $\overline{S}(\rho) \supsetneq \overline{S}(\rho') \supset S$. Since ρ was arbitrary, we have by induction that $\overline{S}(\rho^*) = S$ for some $\rho^* \in [\rho]$, so $K \in N([\rho])$ as desired. □

Proof of Lemma 18. Let $\rho \in A_B(J_n, k)$ be given, such that there exists $K \in C_B(J_n, 3)$, $K \notin N(\rho)$. Fix $K = [i,\, j,\, k]$, where $i < j < 0$ by convention. Once again, fix $S = P_B(K)$, and $\overline{S}(\rho)$ is the minimal chain containing S in ρ.

First, we show that there exists some element which blocks S in ρ, which does not commute with every element of S. Suppose to the contrary that every element which blocks S commutes with every element of S. Then, by applying Lemma 1, we can produce some $\rho' \in [\rho]$ for which the only elements in $\overline{S}(\rho') \setminus S$ are those which block ρ in S. But then there exists an equivalent ordering ρ^* for which $\overline{S}(\rho^*) = S$, which contradicts.

Suppose $K \in C^1_B(J_n, 3)$. Then, we can conclude from the above that there exists some b which blocks S in ρ, and b has the form $[i,\, x]$, $[j,\, x]$, or $[k,\, x]$. Either b falls into one of the stated cases, or one of the following:

(1) $[i,\, j] < [k,\, x] < [i,\, k]$
(2) $[i,\, k] < [i,\, x] < [j,\, k]$

In case (1), consider the set $D = \{d : [i,\, j] < d < [k,\, x]\}$. If every element of D commutes with $[i,\, j]$, then there exists an equivalent ordering which does not contain b, a contradiction. We may thus conclude that there is an element $b' \in D$ of the form $[i,\, x]$ or $[j,\, x]$ such that $[i,\, j] < b' < [i,\, k] < [j,\, k]$.

In case (2), consider the set $D = \{d : [i,\, k] < d < [j,\, k]\}$. If every element of D commutes with $[j,\, k]$, then there exists an equivalent ordering which does not contain $[i,\, x]$, a contradiction. We may thus conclude that there is an element $b' \in D$ of the form $[k,\, x]$ or $[j,\, x]$ such that $[i,\, j] < [i,\, k] < b' < [j,\, k]$.

If $K \in C^2_B(J_n, 3)$, then we can conclude from the above that there exists some $b \in \overline{S}(\rho)$ of the form $[i,\, x]$ or $[j,\, x]$. Either b belongs to one of the stated cases, or we have $[i,\, -j] < [i,\, x] < [j,\, \star]$. In this case, consider the set $D = \{d : [i,\, -j] < d < [j,\, \star]\}$. If every element of D commutes with $[j,\, \star]$, then there exists an equivalent ordering which does not contain $[i,\, x]$, a contradiction. We may thus conclude that there is an element b' of the form $[j,\, x]$ such that $[i,\, -j] < b' < [j,\, \star]$.

We have shown that one of the stated cases must hold if $K \notin N([\rho])$, proving the lemma. □

Proof of Lemma 19. Lemma 4 is proved by case work. For the complete case analysis, refer to the Appendix [Sec. 5.3]. □

Theorem 20. For $k = 1, 2$, there is a bijection

$$\text{maximal chains in } B_B(J_n, k) \xrightarrow{\sim} A_B(J_n, k+1),$$

which sends

$$[\rho_{min}] = [\rho_1] \leq [\rho_2] \leq \cdots [\rho_i] \cdots \leq [\rho_{max}] \mapsto K_1 \cdots K_m,$$

where $\rho_i \in p_{K_{i-1}}(\ldots p_{K_1}(r_{min}))$, $m = |C(J_n, 2)|$ and $K_i \in N(\rho_i) \setminus \mathrm{Inv}\,(\rho_i)$ for all i.

Proof. Our map is clearly injective. Assuming the previous theorem, it also produces a total ordering of $C_B(J_n, k+1)$. We must show that its into $A_B(J_n, k+1)$, and that it is surjective.

First, let a $(k+1)$-packet $T \subset C_B(J_n, k+1)$ be given, and set $S = \cup_{X \in T} P(X)$. For any s, $t \in S$, the unique k-packet which contains s and t, if it exists, is given by $P(X)$ for some $X \in T$.

In other words, given some $\rho \in A_B(J_n, k)$, the only packet-flips which may affect $\rho\big|_S$ are those corresponding to elements of T. Furthermore, if some $P(X)$, $X \in T$ doesn't form a chain with respect to $\rho\big|_S$, then it cannot form a chain in ρ. With these considerations, one can check that for any such T, the only possible sequences of packet-flips carrying ρ_{min} to ρ_{max} correspond to an admissible ordering of T. This shows that our map is into $A_B(J_n, k+1)$.

To show that our map is onto, we will prove that given any admissible ordering $K_1 \ldots K_m$ as in the statement of the proposition, $K_i \in N(\rho_i)$ for all i. The proof is by contradiction. Suppose, for arbitrary $\rho \in A(J_n, 2)$ that there exists some $K_i \notin N(\rho)$. We will show that the corresponding ordering is inadmissible.

Both statements are proven by casework; for the reader's convenience, the full case analysis is removed to the Appendix [Sec. 5.1].

□

3. The group B_n

Once again, let J_n be the set of indices $\{-n \ldots n\} \setminus \{0\}$. Recall that the group B_n acts faithfully on J_n by permutations. More specifically, B_n consists of exactly those permutations π for which $\pi(-i) = -\pi(i)$. In this way, we have a natural inclusion $B_n \hookrightarrow \mathcal{S}_{2n}$.

Likewise, the set of all admissible orderings of J (in our sense), namely $A_B(J_n, 1) = B_B(J_n, 1)$, includes into the set of *all* total orderings of J, $A(J_n, 1) = B(J_n, 1)$, which are trivially "admissible" in the sense of Manin and Schechtman.

Recall from the Preliminaries section that $\mathcal{S}_{2n}$ is related to $A(J_n, 1)$ by the bijection φ, which is also an isomorphism of posets. Our first task in this section will be showing that in the following diagram, the images of B_n and $B_B(J_n, 1)$ coincide, giving the dotted map:

$$\begin{array}{ccc} B(J_n, 1) & \xrightarrow{\varphi} & \mathcal{S}_{2n} \\ \uparrow & & \uparrow \\ B_B(J_n, 1) & \xrightarrow{\varphi} & B_n \end{array}$$

We will see that the induced byjection $B_B(J_n, 1) \to B_n$ is an isomorphism of posets. The construction we present is, in this way, a 'natural' one.

Definition 21. For an arbitrary total ordering ρ of J_n, there exists a unique, strictly increasing bijection $\pi_\rho : (J_n, \rho) \to (J_n, <)$, where '$<$' denotes the usual ordering of J_n. Let φ be the map sending $\rho \mapsto \pi_\rho$.

Proposition 22. The images $B_B(J_n, 1) \hookrightarrow B(J, 1) \xrightarrow{\varphi} S_{2n}$ and $B_n \hookrightarrow S_{2n}$ coincide.

Proof. From the way B_n acts on J_n and the construction of φ, it is evident that the image of B_n in $\mathcal{S}_{2n}$ coincides with the image of exactly those orderings ρ for which reversing the order of ρ is the same as negating each element. Such an ordering must belong to $A_B(J_n, 1)$, so it suffices to show the converse.

To this end, let $\rho \in A_B(J_n, 1)$ be given, and let x be the maximal element with respect to ρ. Then, for every other element $y \in J \setminus \{x\}$, we have $y < x$ and hence

$-x < -y$. From this we conclude that $-x$ is the minimal element. Our result follows by induction on n. □

Remark. In order to prove that φ defines a poset isomorphism, we utilize the theory of reflection groups.

Recall that B_n consists of those permutations $\pi : J \to J$ for which $\pi(-k) = -\pi(k)$. Being a reflection group, B_n also has a standard reflection representation in $O(\mathbb{R}^n)$. Having chosen a basis $\{e_i\}_{i=1}^n$ for $\mathbb{R}^n$, B_n acts on $\mathbb{R}^n$ by the rule:

$$e_i \mapsto \operatorname{sign}(k) \cdot e_{|k|}, \text{ where } k = \pi(i).$$

This structure provides an easy formula to compute the length of some $\pi \in B_n$.

Definition 23 (Root System)**.** Let $\Phi_+ \subset \mathbb{R}^n$ be the system of positive roots in type B given by the choice of roots $\{e_i\}_{i=1}^n \cup \{e_i - e_j\}_{i>j}$. The corresponding simple roots are $\{e_n\} \cup \{e_i - e_{i-1}\}_{i=2}^n$.

Remark. This choice of simple roots corresponds to the choice of generators $\{(n \;\; n{+}1)\} \cup \{(n{+}1{-}i \;\; n{+}2{-}i)(n{+}i \;\; n{-}2{+}i)\}_{i=2}^n$ for B_n as a subset of $\mathcal{S}_{2n}$, written in cycle notation. For the remainder of the discussion, we will assume that B_n is presented with these generators. Then $\ell(x) = \operatorname{card}(\{\alpha \in \Phi_+ : x(\alpha) \notin \Phi_+\})$.

Definition 24. Let f be the map $C_B(J_n, 2) \to \Phi_+$ given by

$$f(K) = \begin{cases} e_i - e_j & \text{if } K = [i, j] \text{ for } i > j > 0 \\ e_i + e_j & \text{if } K = [i, -j] \text{ for } i > j > 0 \\ e_k & \text{if } K = [k, \star] \text{ for } k > 0 \end{cases}$$

Clearly f defines a bijection. We will refer to the root $f(P)$ as α_P.

Proposition 25. Let $x \in B_n$ be given, and ρ_x be the corresponding total ordering of J_n. Then, for each $P \in C_B(J_n, 2)$, we have

$$P \in \operatorname{Inv}(\rho_x) \iff x(\alpha_P) \notin \Phi_+.$$

Proof. As the proof is simply casework, it is removed to the appendix [Sec. 5.2]. □

In other words, the following diagram commutes.

$$\begin{array}{ccc} B_B(J_n, 1) & \xrightarrow{\quad\varphi\quad} & B_n \\ \big\downarrow{\scriptstyle \mathrm{Inv}} & & \big\downarrow \\ \mathcal{P}(C_B(J_n, 2)) & \xrightarrow{\mathcal{P}(f)} & \mathcal{P}(\Phi_+) \end{array}$$

Theorem 26. φ defines a poset isomorphism $B_B(J_n, 1) \to B_n$, where B_n is ordered by the weak left Bruhat order.

Proof. The proof follows directly from the following observation, which is simply a consequence of φ's construction. For some $x \in B_n$ and s, a simple generator, we have that $\varphi : sx \rightsquigarrow p_K(\rho_x)$ for some $K \in C_B(J_n, 2)$. Likewise, $\varphi^{-1} : p_K(\rho) \rightsquigarrow s'\varphi^{-1}(\rho)$ for some simple generator s'.

Let us then examine the covering relations. In B_n, under the weak left Bruhat order, x' covers x if there exists a simple generator s such that $x' = sx$ and $\ell(x') > \ell(x)$. Under φ^{-1}, this corresponds to $\rho_{x'} = p_K(\rho_x)$ for some $K \in C_B(J_n, 2)$. But by Proposition 3, $\ell(x') > \ell(x) \implies \operatorname{Inv}(\rho_{x'}) \supsetneq \operatorname{Inv}(\rho_x) \implies K \notin \operatorname{Inv}(\rho_x)$. So $\rho_{x'}$ covers ρ_x.

Conversely, let $\rho_{x'} = p_K(\rho_x)$ for some $K \notin \operatorname{Inv}(\rho_x)$. Under φ, this corresponds to $x' = sx$ for a simple generator s. Once again, Proposition 3 tells us that $\operatorname{Inv}(\rho_{x'}) \supsetneq \operatorname{Inv}(\rho_x) \implies \ell(x') > \ell(x)$, so x' covers x. As desired, we have an isomorphism of posets. □

Corollary 27. φ induces a bijection $A_B(J_n, 2) \to \mathrm{R}(w_0)$, where w_0 is the longest element of B_n.

Proof. From Theorem 20, we have a bijection

$$\{\text{max chians in } B_B(J_n, 1)\} \to A_B(J_n, 2).$$

Similarly, each maximal chain in the weak left Bruhat order gives a unique reduced expression for w_0. As φ is an isomorphism of posets, it also gives a bijection between the set of maximal chains. We get our bijection by composing these maps. □

Proposition 28. In the induced bijection $A_B(J_n, 2) \to \mathrm{R}\,(w_0)$, two words differ by a commutation if and only if the corresponding total orderings possess elementary equivalence.

Proof. In $C_B(J_n, 2)$, two elements commute if their packets are disjoint. In $B_n \subset \mathcal{S}_{2n}$, two cycle products commute if they operate on disjoint sets of indices. Under the correspondence given by φ, these two statements are exactly the same. □

Remark. The higher Bruhat orders were defined by Manin and Schechtman for $\mathcal{S}_n$, and B_n can be realized as the wreath product $\mathcal{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. It is therefore natural to ask whether our construction can be generalized to the complex reflection group $\mathcal{S}_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$.

4. Conjectures and current work

4.1. **Direction of current work.** The central goal of our current work is to prove the following conjecture.

Conjecture 1. Theorems 14 and 20 hold for general k.

By assuming the existence of such a result for $k = 3$, one can classify possible packet-flip sequences and so determine the orderings of 4-packets. Then it will be possible investigate admissible orderings of $C_B(J_n, 4)$, and whether the packet orderings admit a coherent description. In this manner, we have been able to compute the posets $B_B(J_4,\, 3)$ and $B_B(J_5,\, 4)$.

The main challenge faced by this program is choosing the standard ordering from the set of admissible orderings. Short of observing and proving a coherent pattern, our computations have motivated a number of conjectures about ρ_{min} for $k > 3$. Once ρ_{min} is systematically defined, we will have the all of the necessary structure to state, and hopefully prove, Conjecture 1.

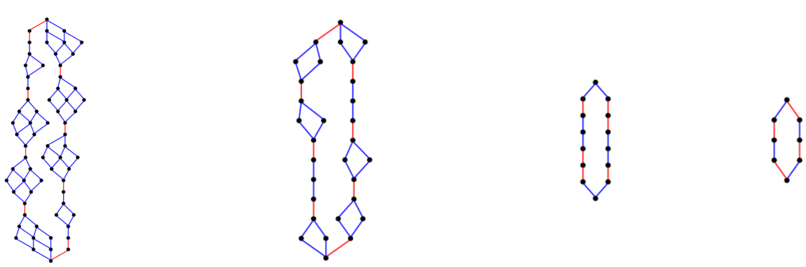
Conjecture 2. Note that we have a map $f_i : A_B(J_{n+1},\, k) \to A_B(J_n, k)$ by removing all elements containing the index i from some ordering $\rho \in A_B(J_{n+1}, k)$. We expect that every f_i sends $\rho_{min} \in A_B(J_{n+1},\, k)$ to $\rho_{min} \in A_B(J_n,\, k)$.

Conjecture 3. The elements of $C_B^2(J_n,\, 2)$ occur in lexicographic order in ρ_{min}, if one ignores $\star$.

Assuming Conjecture 1, we should be able to understand the structure of the posets $B_B(J_n,\, k)$ directly in terms of the posets $B_B(J_{k+1},\, k)$ and $B(I_n,\, n{-}2)$. With this in mind, we have the following conjecture about the structure of $B_B(J_n,\, n-1)$

Conjecture 4. $B_B(J_n,\, n{-}1)$ is composed of squares, and $(2^n{+}2n)$-gons. $B_B(J_n, k)$ is composed of these cycles, as well as $2n$-gons similar to $B(I_n,\, n-2)$.

The computed posets $B_B(J_n,\, n-1)$ are presented below. Note that red edges correspond to flipping the packet of some $K \in C_B^2(J_n,\, n-1)$, while blue edges correspond to flipping the packet of some $K' \in C_B^1(J_n,\, n-1)$.



5. Appendix

5.1. **Theorem 20.** Firstly, given some $K \in C_B(J_n, k+1)$, we examine the restriction of ρ_{min} to $S = \cup_{Z \in P(K)} P(Z)$, and consider which sequences of packet flips may occur. (An ideal representative is assumed in the left-hand column only.)

Case k = 1:

K	$\rho_{min}\big\vert_S$	Possible flip sequence (up to reverse)
$[i,\, j,\, 0]$	$-i < -j < j < i$	$[i,\, j] \prec [i,\, \star] \prec [i,\, -j] \prec [j,\, \star]$
$[i,\, j,\, k]$	$i < j < k$	$[i\, j] \prec [i,\, k] \prec [j,\, k]$

Case k = 2:

K	$\rho_{min}\big\vert_S$	Possible flip sequence (up to reverse)
$[i,\, j,\, k,\, 0]$	$[-i,\, -j] < [-i,\, -k] < [-j,\, -k] < [i,\, \star] < [-i,\, j] < [-i,\, k] < [j,\, \star] < [-j,\, k] < [k,\, \star]$	$[-i,\, -j,\, -k] \prec [i,\, j,\, \star] \prec [-i,\, -j,\, k] \prec [-i,\, -k,\, j] \prec [i,\, k,\, \star] \prec [j,\, k,\, \star] \prec [-i,\, j,\, k]$
$[i,\, j,\, k,\, l]$, where $k < 0$	$[i,\, j] < [i,\, k] < [j,\, k] < [i,\, l] < [j,\, l] < [k,\, l]$	$[i\, j,\, k] \prec [i,\, j,\, l] \prec [i,\, k,\, l] \prec [j,\, k,\, l]$
$[i,\, j,\, l,\, k]$, where $k > 0$	$[i,\, j] < [-k,\, -l] < [i,\, l] < [i,\, k] < [j,\, l] < [j,\, k]$	$[i,\, j,\, k] \prec [i,\, j,\, l] \prec [j,\, k,\, l] \prec [i,\, k,\, l]$

From this it's clear that any maximal chain in $B_B(J_n, k)$ is sent to an ordering of $C_B(J_n,\, k+1)$ that respects the standard or reverse-standard ordering of $(k+1)$-packets. This shows our map is into $A_B(J, k+1)$.

Suppose $K_N \dots K_1$ is an admissible order of $C_B(J_n, k+1)$. Let r_0 denote the class of the standard ordering of $C_B(J_n, k)$. We want to show that $K_N \dots K_1$ gives a valid sequence of packet flips $p_{K_N} \dots p_{K_1}$ on r_0. With the empty sequence of packet flips as base case, assume inductively that $p_{K_i} \dots p_{K_1}$ is a valid sequence of packet flips on r_0 for some $i \geq 0$. Then writing $r_i = p_{K_i} \dots p_{K_1}(r_0)$, we need to check that $K_{i+1} \in N(r_i)$. Noting that K_{i+1} is the minimal element of $C_B(J_n, k+1) \setminus \operatorname{Inv}(r_i)$ with respect to the admissible order $K_N \dots K_1$, it suffices to prove the following statement: If ρ is an admissible ordering of $C_B(J_n, k)$ and $K \in C_B(J_n, k+1) \setminus (\operatorname{Inv}(\rho) \cup N(\rho))$, then K is not minimal in the restriction of any admissible ordering to $C_B(J_n, k+1) \setminus Inv(\rho)$. This is what we check by casework below.

The cases we consider are equivalent to the statement that $K \notin N([\rho]) \cup \operatorname{Inv}(\rho)$ by Lemma 17 and Lemma 18. To clear up ambiguity: the first column gives the order of elements in some ρ for which the packet K cannot be flipped. The fourth column gives lists some features of an ordering corresponding to a chain in which K would be flipped next, and the fifth column gives the necessary packet order, showing that the ordering in column 3 is in fact inadmissible.

Case k = 1: We consider cases where the 1-It is only packet containing $k, l \in J_n$ cannot be inverted. (Without loss of generality, $k < l$.) necessary that we consider the upper half of a given ordering. Using the fact that elements whose packet is inverted precede $[k,\, l]$, and that $[k,\, l]$ must precede other uninverted packets, we conclude that the corresponding order is inadmissible. (Note that in the table below, the assignment $l = -k$ is possible, yielding the packet $[k,\, \star] \approx [k,\, l]$.)

Case	Condition	Implied order	Admissible order (up to reverse)
$k \prec x \prec l$	$x < k < l$	$[x,\, k] \prec [k,\, l] \prec [x,\, l]$	$[x,\, k] < [x,\, l] < [k,\, l]$
	$k < x < l$	$[k,\, l] \prec [k,\, x]$ and $[k,\, l] \prec [x,\, l]$	$[k,\, x] < [k,\, l] < [x,\, l]$
	$k < l < x$	$[l,\, x] \prec [k,\, l] \prec [k,\, x]$	$[k,\, l] < [k,\, x] < [x,\, l]$

Case k = 2: We do the same for $k = 2$. Here, we examine the packet of $[k,\, l,\, m]$ (ideal representative), and the packet of $[k,\, l,\, \star]$.

Case	m condition	x condition	Implied order	Admissible order (up to reverse)
$[k, l] \prec [k, x] \prec [k, m] \prec [l, m]$	$k < l < m < 0$	$x > m$	$[k, m, x] \prec [k, l, m] \prec [k, l, x]$	$[k, l, m] < [k, l, x] < [k, m, x]$
		$l < x < m$	$[l, x, m] \prec [k, l, m] \prec [k, x, m]$	$[k, l, m] < [k, x, m] < [l, x, m]$
		$k < x < l$	$[k, l, m] \prec [k, x, m]$ and $[k, l, m] \prec [x, l, m]$	$[k, x, m] < [k, l, m] < [x, l, m]$
		$x < k$	$[x, k, m] \prec [k, l, m] \prec [x, l, m]$	$[x, k, m] < [x, l, m] < [k, l, m]$
	$k < l < 0 < m$	$0 < x < m$	$[k, m, x] \prec [k, l, m] \prec [k, l, x]$	$[k, l, m] < [k, l, x] < [k, m, x]$
		$x > m$	$[k, l, m] \prec [k, l, x]$ and $[k, l, m] \prec [k, x, m]$	$[k, l, x] < [k, l, m] < [k, x, m]$
		$l < x < 0$	same as ↑	same as ↑
		$k < x < l$	$[k, x, l] \prec [k, l, m] \prec [k, x, m]$	$[k, x,] < [k, x, m] < [k, l, m]$
		$x < k$	$[x, k, l] \prec [k, l, m] \prec [x, k, m]$	$[x, k, l] < [x, k, m] < [k, l, m]$.
$[k, l] \prec [k, m] \prec [x, m] \prec [l, m]$	$k < l < m < 0$	$x > m$	$[l, m, x] \prec [k, l, m] \prec [k, m, x]$	$[k, l, m] < [k, m, x] < [l, m, x]$
		$l < x < m$	$[l, x, m] \prec [k, l, m] \prec [k, x, m]$	$[k, l, m] < [k, x, m] < [l, x, m]$
		$k < x < l$	$[k, l, m] \prec [k, x, m]$ and $[k, l, m] \prec [x, l, m]$	$[k, x, m] < [k, l, m] < [x, l, m]$
		$x < k$	$[x, k, m] \prec [k, l, m] \prec [x, l, m]$	$[x, k, m] < [x, l, m] < [k, l, m]$
	$k < l < 0 < m$	$x > 0$	$[l, m, x] \prec [k, l, m] \prec [k, m, x]$	$[k, l, m] < [k, m, x] < [l, m, x]$
		$l < x < 0$	$[l, x, m] \prec [k, l, m] \prec [k, x, m]$	$[k, l, m] < [k, x, m] < [l, x, m]$
		$k < x < l$	$[k, l, m] \prec [k, x, m]$ and $[k, l, m] \prec [j, x, m]$	$[k, x, m] < [k, l, m] < [j, x, m]$
		$x < k$	$[x, k, m] \prec [k, l, m] \prec [x, l, m]$	$[x, k, m] < [x, l, m] < [k, l, m]$
$[k, l] \prec [l, x] \prec [l, m]$	$k < m < 0$	$x > m$	$[l, m, x] \prec [k, l, m] \prec [k, l, x]$	$[k, ,l, m] < [k, l, x] < [l, m, x]$
		$k < x < m$	$[k, l, m] \prec [k, l, x]$ and $[k, l, m] \prec [l, x, m]$	$[k, l, x] < [k, l, m] < [l, x, m]$
		$x < k$	$[x, k, m] \prec [k, l, m] \prec [x, l, m]$	$[x, k, m] < [x, l, m] < [k, l, m]$.
	$k < 0 < m$	$x > m$	$[k, l, m] \prec [k, l, x]$ and $[k, l, m] \prec [l, m, x]$	$[k, l, x] < [k, l, m] < [l, m, x]$
		$k < x < 0$	same as ↑	same as ↑
		$0 < x < m$	$[l, m, x] \prec [k, l, m] \prec [k, l, x]$	$[k, l, m] < [k, l, x] < [l, m, x]$
		$x < k$	$[x, k, l] \prec [k, l, m] \prec [x, l, m]$	$[x, k, l] < [x, l, m] < [k, l, m]$
$[i, j] \prec [i, x] \prec [i, \star] \prec [i, -j] \prec [j, \star]$	$i < j < 0$	$x < i$	$[x, i, j] \prec [i, j, \star] \prec [x, i, \star]$	$[x, i, j] < [i, x, \star]$
		$i < x < j$	$[i, x, j] \prec [i, j, \star] \prec [i, x, \star]$	$[i, x, j] < [i, x, \star] < [i, j, \star]$
		$0 < x < j$	$[i, j, \star] \prec [i, j, x]$ and $[i, j, \star] \prec [i, x, -j]$	$[i, j, x] < [i, j, \star] < [i, x, -j]$
		$0 < x < -i$	$[i, x, 0] \prec [i, j, \star] \prec [i, j, x]$	$[i, j, \star] < [i, j, x] < [i, x, \star]$
		$x > -i$	$[x, i, \star] \prec [i, j, \star] \prec [x, -i, j]$	$[x, i, \star] < [x, -i, j] < [i, j, \star]$

5.2. **Proposition 25.** Recall that the function f defines a bijection $C_B(J_n, 2) \to \Phi_+$, and is defined by

$$f(K) = \begin{cases} e_i - e_j & \text{if } K = [i,\, j] \text{ for } i > j > 0 \\ e_i + e_j & \text{if } K = [i,\, -j] \text{ for } i > j > 0 \\ e_k & \text{if } K = [k,\, \star] \text{ for } k > 0 \end{cases}.$$

From now on we will write α_P to denote the root $f(P)$.

Proof of Proposition 25. Let i, j such that $i > j > 0$ be given. Fix $k = x(i)$ and $l = x(j)$. We will consider each case separately. Suppose that $[-j,\, -i] \notin \mathrm{Inv}\,(\rho_x)$. Then the image of the positive root $\alpha_P = e_i - e_j$ under x is positive. In particular, either:

(1) $k > l > 0$, and $x(\alpha_P) = e_k - e_l$.
(2) $k > 0 > l$, and $x(\alpha_P) = e_k + e_{-l}$, or
(3) $0 > k > l$, and $x(\alpha_P) = e_{-l} - e_{-k}$.

Conversely, if $[-i,\, -j] \in \mathrm{Inv}\,(\rho_x)$, we have one of the following:

(1) $l > k > 0$, and $x(\alpha_P) = e_k - e_l$.
(2) $l > 0 > k$, and $x(\alpha_P) = -e_{-k} - e_l$, or
(3) $0 > l > k$, and $x(\alpha_P) = e_{-l} - e_{-k}$.

Therefore $x(\alpha_P)$ is not positive.

Next, we show that if $[-j,\, i] \notin \mathrm{Inv}\,(\rho_x)$, then the image of $\alpha_P = e_i + e_j$ under x is positive. If this were the case, then we would have one of the following:

(1) $-k < l < 0$, and $x(\alpha_P) = e_k - e_{-l}$,
(2) $-k < 0 < l$, and $x(\alpha_P) = e_k + e_l$, or
(3) $0 < -k < l$, and $x(\alpha_P) = e_l - e_{-k}$.

Analagous to the previous case, if $[-j,\, i] \in \mathrm{Inv}\,(\rho_x)$ then the image of α is not positive. The cases to consider here are:

(1) $l < -k < 0$, and $x(\alpha_P) = e_k - e{-}l$,
(2) $l < 0 < -k$, and $x(\alpha_P) = -e_k - e_l$, or
(3) $0 < l < -k$, and $x(\alpha_P) = e_l - e_{-k}$.

Lastly, if $[i,\, \star] \in \mathrm{Inv}\,(\rho_x)$, then then the image of $\alpha_P = e_i$ is clearly positive. Otherwise, it is not; this follows directly from $\pi(-i) = -\pi(i)$. □

5.3. **Lemma 19.** As the lemma is merely casework, it was checked by a computer algorithm. We will describe this algorithm, and prove its correctness.

Definition 29. For some $\rho \in A_B(J_n, k)$, and two elements a, $b \in C_B(J_n, k)$, *a crosses b in ρ* if $a < b$ in ρ, and $b < a$ in some $\rho' \in [\rho]$. If this is the case, then it is also true that b crosses a in ρ.

We now describe an algorithm $\mathcal{A}$ which, on inputs $\rho \in A_B(J_n, k)$ and a, $b \in C_B(J_n, k)$, outputs the predicate [a crosses b in ρ].

Algorithm 1. Let S denote the chain of elements in ρ greater than a and less than or equal to b. Without loss of generality, suppose $a < b$ in ρ. We initialize a list, called `right`, containing only the element a. For each element q in S, in ascending order, we compute whether q commutes with every element in `right`. If so, we continue. If not, we add q to `right`. Finally, we output the value of the predicate [b commutes with every q contained in `right`].

Proof of correctness. Suppose that there exists some $\rho' \in [\rho]$ such that of the first k elements of S, each element is either less than than a or contained in `right`. Furthermore, suppose that the elements of $S \cup \{a\}$ form a chain in ρ', and those elements of S which are not among the first k conserve their positions from ρ. If the $(k+1)^{\text{th}}$ element commutes with all of `right` than clearly we can find a ρ'' in which the same is true for the first $k+1$ elements, by commuting it below a. If not then we add this element to `right` and proceed with ρ'. Since ρ suffices as a base case, it follows by induction on the size of S that $\mathcal{A}$ outputs $1 \implies a$ crosses b.

Conversely, suppose that $x > a$ in ρ and x does not commute with a. Then clearly $x > a$ in ρ', for all $\rho' \in [\rho]$. Now suppose that $\mathcal{A}$ outputs 0. Then b did not commute with some element $r < b$ of `right`, hence $r < b$ for all $\rho' \in [\rho]$. But for each element r in `right` which is not a, we can find some smaller element r' in `right` such that $r' < r$ for all $\rho' \in [\rho]$. By induction on the size of S, which bounds the size of `right`, we are guaranteed to arrive in this manner at some element which is greater than a for all $\rho' \in [\rho]$. By transitivity, the same is true of b. □

Remark. In the following algorithm, posets are represented as directed acylclic graphs, where elements are nodes and covering relations are directed edges. Clearly, two elements are comparable in a poset if and only if they are connected in the corresponding graph. The *transitive union* of two posets is computed by taking the transitive binary relation given by the union of two directed graphs. This relation is reflexive and transitive by definition, and it is antisymmetric so long as the union graph contains no cycles. Linear extensions are computed using the topological sorting algorithm.

Notation. A 2-packet P is understood to be an ordered set with the ordering inherited from the standard ordering ρ_{min}. $\operatorname{Rev} P$ is understood to be the same set, with the ordering relation inherited from ρ_{max}.

Algorithm 2. Now we describe the general algorithm.

Recall that each case from Lemma 18 consists of a sequence of 4 or 5 elements of $C_B(J_n, 2)$; all but one belong to some packet $P_B(K)$ for $K \in C_B(J_n, 3)$, and the remaining element shares exactly one index with K. As such, there is a unique element $R \in C_B(J_n, 4)$ such that the set $T = \cup_{S \in P_B(R)} P_B(S)$ contains our sequence. Furthermore, the unique 2-packet containing any pair of elements in our sequence is contained in T.

Our algorithm receives as input the sequence mentioned above, as well as a total ordering of the indices in the sequence.

The algorithm initializes an empty list `L`, and iterates over all pairs of elements in our sequence. If any pair is contained in a common 2-packet, P, it does the following:

- If the pair of elements is in standard order, it adds the poset given by the standard order on P to `L`.
- Otherwise, it adds the poset given by the reverse-standard order to `L`.

Let U be the set of packets $P_B(S)$ for $S \in P_B(R)$ whose order is not recorded in this manner. Our algorithm iterates over $\mathcal{P}(U)$ and does the following.

For some $B \in \mathcal{P}(U)$, the algorithm creates a new list `L'` containing the elments of `L`. For each element $P \in U \cap B$, it adds the poset P to `L'`. For each element $P \in U \setminus B$, it adds the poset $\operatorname{Rev} P$ to `L'`. The algorithm computes the transitive union over the relations in `L'`. If there are no cycles, then the algorithm records a linear extension of the corresponding poset.

For each recorded linear extension, the algorithm iterates over the packets $P_B(S)$ for $S \in P_B(R)$ until it finds a 2-packet P^* which is in lexicographic order such that either

- $\min_\rho P^* > \min_\rho P_B(K)$ and $\min_\rho P_B(K)$ does not cross $\min_\rho P^*$, or
- $\min_\rho P^* = \min_\rho P_B(K)$, $\max_\rho P^* < \max_\rho P_B(K)$, and $\max_\rho P^*$ does not cross $\max_\rho P_B(K)$.

If this is the case, the algorithm continues. Otherwise, it outputs 0. If every linear extension recorded has been checked in this way, the algorithm outputs 1.

Proof of correctness. Firstly, we note that every $\rho \in A_B(J_n, k)$ which respects the ordering of our sequence must have the property that $\operatorname{Inv}(\rho)$ contains all of those packets in `L` which are in reverse-standard order, and none of those which are in standard order. Now, for some such ρ, consider the poset obtained by taking the transitive union over $\operatorname{Rev} P_B(S)$ for each $S \in \operatorname{Inv}(\rho)$ and $P_B(S')$ for each $S' \in C_B(J_n, 3) \setminus \operatorname{Inv}(\rho)$. This poset, which we will denote Q, is clearly extended by ρ. As such, we can conclude that the relation obtained in this way is indeed a partial ordering. Now consider the relation obtained by taking the transitive union of $P_B(S)$ or $\operatorname{Rev} P_B(S)$ for just those $S \in P_B(R)$. This relation includes in P, and it is transitive. Therefore $\rho\Big|_T$ was considered by our algorithm. Moreover, if two elements cannot cross in $\rho\Big|_T$ then they can't cross in ρ, for obvious reasons. This completes the proof. □

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