Solutions to the Nonlinear Schrödinger Equation in Hyperbolic Space

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Abstract

In this paper we show the existence of radial positive stationary solutions to the energy critical nonlinear Schrödinger equation on $\mathbb{H}^3$ by reducing the problem to an ODE. We also make an observation that Kenig-Merle's variational argument in [2] can work even without the existence of a positive stationary solution, based on this, we sketch a possible strategy which may recover their result in [2] on $\mathbb{H}^3$.

1 Introduction

We are interested in the initial value problem for the energy critical focusing nonlinear Schrödinger equation in three dimensional hyperbolic space

$$i\partial_t u + \Delta_{\mathbb{H}^3} u = -u|u|^4, \quad u(0) = u_0, \tag{1.1}$$

where $\Delta_{\mathbb{H}^3}$ is the Laplacian in $\mathbb{H}^3$. In particular we would like to have conditions on the initial data that guarantee the solution will exist for all time and converge to a solution of the linear equation. The energy quantity on hyperbolic space is defined as

$$E_{\mathbb{H}^3}(u) = \frac{1}{2}\int_{\mathbb{H}^3} |\nabla u(x,t)|^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{H}^3} |u(x,t)|^6 \mathrm{d}x, \tag{1.2}$$

and the analogous one on three dimensional Euclidean space

$$E_{\mathbb{R}}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u(x,t)|^6 \mathrm{d}x. \tag{1.3}$$

Remark 1.1. The hyperbolic energy quantity is conserved for solutions to (1.1). If $u_0 \in C_0^\infty(\mathbb{H}^3)$ this follows from a standard application of the divergence theorem, and the general case holds by a limiting argument.

We recall the definition of $\dot{H}^1(\mathbb{R}^3)$ and $H^1(\mathbb{H}^3)$as function spaces with norms defined as

$$||\nabla u||_{L^2(\mathbb{R}^3)} = ||u||_{\dot{H}^1(\mathbb{R}^3)}$$
$$||\nabla u||_{L^2(\mathbb{H}^3)} = ||u||_{H^1(\mathbb{H}^3)}$$

It is a standard result that

$$||u||_{L^2(\mathbb{H}^3)} \leq ||u||_{H^1(\mathbb{H}^3)}$$

In [2] Kenig and Merle show that

Theorem 1.2. *If $u_0 \in \dot{H}^1(\mathbb{R}^3)$ and u_0 is radial and*

$$E_{\mathbb{R}}(u_0) < E_{\mathbb{R}}(W),$$
$$||\nabla u||_{L^2(\mathbb{R}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)},$$

where W is the unique positive radial solution to

$$-\Delta W = W|W|^4,$$

and can be written explicitly as

$$W(x) = \frac{1}{\sqrt{1 + \frac{|x|^2}{3}}},$$

then the solution $u(x,t)$ of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = -u|u|^4 \\ u(0,x) = u_0(x) \end{cases},$$

is defined for all time and there exists $u_{0,+}, u_{0,-}$ in $\dot{H}^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty} ||u(t) - e^{it\Delta}u_{0,+}||_{\dot{H}^1(\mathbb{R}^3)} = 0, \quad \lim_{t\to-\infty} ||u(t) - e^{it\Delta}u_{0,-}||_{\dot{H}^1(\mathbb{R}^3)} = 0.$$

Ionescu, Pausader and Staffilani show in [1] an analogous result for the defocusing case in $\mathbb{H}^3$.

Theorem 1.3. *If $u_0 \in H^1(\mathbb{H}^3)$. Then the solution u of the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^3} u = u|u|^4 \\ u(0,x) = u_0(x) \end{cases},$$

is defined for all time and there exists $u_{0,+}, u_{0,-}$ in $H^1(\mathbb{H}^3)$ such that

$$\lim_{t\to\infty} ||u(t) - e^{it\Delta}u_{0,+}||_{H^1(\mathbb{H}^3)} = 0, \quad \lim_{t\to-\infty} ||u(t) - e^{it\Delta}u_{0,-}||_{H^1(\mathbb{H}^3)} = 0.$$

We would like to show the analogous result for the focusing case, that is:

Conjecture 1.4. *Consider $u_0 \in H^1(\mathbb{H}^3)$ and assume*

$$E_{\mathbb{H}^3}(u_0) < E_{\mathbb{R}}(W), \tag{1.4}$$
$$||\nabla u_0||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)}, \tag{1.5}$$

where W is as defined in Theorem 1.2. Assume further that u_0 is radial. Then the solution u to (1.1) is defined for all time and there exists $u_{0,+}, u_{0,-}$ in $H^1(\mathbb{H}^3)$ such that

$$\lim_{t\to\infty} ||u(t) - e^{it\Delta}u_{0,+}||_{H^1(\mathbb{H}^3)} = 0, \quad \lim_{t\to-\infty} ||u(t) - e^{it\Delta}u_{0,-}||_{H^1(\mathbb{H}^3)} = 0.$$

In Kenig and Merle [2] there are two key steps to this proof, energy trapping [see Theorem 4.5] and the usage of a concentration compactness method [see Proposition 4.7]. We focus on radial solutions, because part of the second step relies on the result from Kenig and Merle [2].

Because of the prominence of the stationary solution in the Euclidean space one would initially expect that the positive stationary solutions to (1.1) should be well understood. However Mancini and Sandeep [3] showed that the gradient of such solutions do not have a finite L^2 norm. The stationary solutions are still of interest and we can gain a better understanding of those which are radial by transforming the PDE to an ODE by using polar coordinates for hyperbolic space. We obtain the ODE

$$\begin{cases} \frac{d^2}{dr^2}u(r) + 2\coth(r)\frac{d}{dr}u(r) + u(r)^5 = 0 \\ u(0) = A > 0 \\ u'(0) = 0 \end{cases}, \tag{1.6}$$

and we find

Theorem 1.5. *Suppose u solves (1.6) with $A < 2^{1/3}$ then $u(r) > 0$ for $r \geq 0$.*

Remark 1.6. This shows that there exist positive stationary solutions to (1.1)

We are also able to use a numerical solving problem to find approximate solutions to (1.6)

Remark 1.7. The results of our simulation imply that for all $A > 0$ the solution to (1.6) are positive. Furthermore, the simulation results imply that if u solves (1.6) then

$$|(2r)^{-1/4}u(r)| \leq M \tag{1.7}$$

for all r.

2 Preliminaries

2.1 Hyperbolic space

We consider the Minkowski space $\mathbb{R}^4$ with the standard Minkowski metric $-(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2$ and recall the bilinear form on $\mathbb{R}^4 \times \mathbb{R}^4$

$$[x, y] = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3.$$

Hyperbolic space $\mathbb{H}^3$ is defined as

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4; [x, x] = 1 \text{ and } x_0 > 0\}.$$

$\mathbf{0} = (1, 0, 0, 0)$ denotes the origin of $\mathbb{H}^3$. The Minkowski metric of $\mathbb{R}^4$ induces a Riemanian metric $\mathbf{g}$ on $\mathbb{H}^3$ with covariant derivative ∇ and induced measure which we denote by dx.

We let $\mathbb{G}$ denote the connected Lie group of 4×4 matrices that leave the form $[\cdot, \cdot]$ invariant. For any $h \in \mathbb{G}$ the mapping $L_h : \mathbb{H}^3 \to \mathbb{H}^3, L_h(x) = h \cdot x$, defines an isometry of $\mathbb{H}^3$. For any $h \in \mathbb{G}$ there are isometries

$$\pi_h : L^2(\mathbb{H}^3) \to L^2(\mathbb{H}^3), \quad \pi_h(f)(x) = f(h^{-1} \cdot x). \tag{2.1}$$

We would also like be able to pass between functions defined on hyperbolic spaces and functions defined on Euclidean spaces. For any $h \in \mathbb{G}$ there is a diffeomorphism

$$\Psi_h : \mathbb{R}^3 \to \mathbb{H}^3, \quad \Psi_h(v_1, v_2, v_3) = h \cdot (\sqrt{1 + |v|}, v_1, v_2, v_3). \tag{2.2}$$

We will denote the laplacian in three dimensional Hyperbolic space by $\Delta_{\mathbb{H}^3}$

We also recall the sharp Sobolev embedding theorem, as stated in [5] For any $u \in H^1(\mathbb{H}^3)$

$$||\nabla u||_{L^2(\mathbb{H}^3)} \leq K(3, 2)||u||_{L^6(\mathbb{H}^3)}, \tag{2.3}$$

where

$$K(n, 2) = \sqrt{\frac{4}{n(n-2)w_n^{2/n}}},$$

where w_n is the volume of the $n - 1$ sphere in $\mathbb{R}^3$. From [4] we have that in $\mathbb{R}^n$

$$||\nabla u||_{L^2(\mathbb{R}^3)} \leq K(3, 2)||u||_{L^6(\mathbb{R}^3)}. \tag{2.4}$$

So the sharp constants on the Sobolev embedding are the same for $\mathbb{R}^3$ and $\mathbb{H}^3$. We will recall this in our discussion of variational estimates and our energy trapping argument.

2.2 Nonlinear Schrödinger Basics

For any interval $I \subset \mathbb{R}$ and function $u \in C(I, \mathbb{H}^3)$ we utilize the norm

$$||u||_{S(I)} = \int_I \int_{\mathbb{H}^3} |u(x,t)|^{10} \mathrm{d}x\mathrm{d}t. \tag{2.5}$$

Definition 2.1. We say that a solution u to the Nonlinear Schrödinger equation scatters forward if there exists $u_{0,+}$ such that

$$\lim_{t\to\infty} ||u(t) - e^{it\Delta}u_{0,+}||_{H^1}(\mathbb{H}^3).$$

We say that a solution u scatters backwards if there exists $u_{0,-}$ such that

$$\lim_{t\to-\infty} ||u(t) - e^{it\Delta}u_{0,-}||_{H^1}(\mathbb{H}^3).$$

If a solution scatters backwards and scatters forwards then we say that it scatters.

Remark 2.2. From Theorem 1.1 of [1] we have that if a solution is global and has finite $S(I)$ norm then u scatters.

We now cite the classical local well-posedness and small data results

Proposition 2.3. *(Local well-posedness) Assume $\phi \in H^1(\mathbb{H}^3)$. Then there is a unique maximal solution $(I, u) = (I(\phi), u(\phi)), 0 \in I$, of the initial-value problem*

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{H}^3})u = -u|u|^4 \\ u(0) = \phi \end{cases}, \tag{2.6}$$

on $\mathbb{H}^3 \times I$. In addition the energy defined in 1.1 is constant on I and

$$\text{if } I_+ := I \cap [0,\infty) \text{ is bounded then } ||u||_{S(I_+)} = \infty, \tag{2.7}$$

$$\text{if } I_- := I \cap (-\infty,0] \text{ is bounded then } ||u||_{S(I_-)} = \infty. \tag{2.8}$$

Proposition 2.4. *(Small data) Assume $\phi \in H^1(\mathbb{H}^3)$, and $||\phi||_{H^1(\mathbb{H}^3)} < \delta$ some $\delta > 0$ small enough. Then the solution of*

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{H}^3})u = -u|u|^4 \\ u(0) = \phi \end{cases}, \tag{2.9}$$

is defined for all t and scatters.

3 Stationary Solutions

Based on the proof of Kenig and Merle [2] we would expect that the stationary solution to (1.1) to play a role in our proof. However as Mancini and Sandeep [3] showed there are no positive stationary solutions with gradients with a bounded L^2 norm. Although this is the case it can still be interesting to explore stationary solutions and in this section we prove that there exists positive stationary solutions. We proceed by transforming the equation into an ordinary differential equation. We then define a monotone quantity for solutions of the ODE and use a change of variables to write the ODE in a more workable form. We also describe results from a numerical simulation that indicate that for any positive initial data, solutions to the ODE are always positive and that the first order decay rate of solutions at infinite is $(2r)^{-1/4}$

Stationary solutions solve the elliptic PDE

$$-\Delta_{\mathbb{H}^3} u = u|u|^4. \tag{3.1}$$

The Euclidean case is discussed by Tao in [6] and we mimic his technique. We can consider Hyperbolic space under polar coordinates, (r, ξ) where r is the distance to $\mathbf{0}$, and ξ a parameter representing the choice of direction of the geodesic in S^{n-1}, in which case we can rewrite the laplacian as:

$$\Delta_{\mathbb{H}^3} u = \frac{d^2}{dr^2} u + 2\coth(r)\frac{d}{dr} u + \sinh(r)^{-2}\Delta_\xi u,$$

where Δ_ξ is the laplacian on the ordinary unit $n-2$ sphere. We are interested in solutions which are radially symmetric and positive and therefore arrive at the ODE

$$\frac{d^2}{dr^2} u(r) + 2\coth(r)\frac{d}{dr} u(r) + u(r)^5 = 0.$$

Thus we would like to show that there exists a positive solution to (1.6) or

$$\begin{cases} \frac{d^2}{dr^2} u(r) + 2\coth(r)\frac{d}{dr} u(r) + u(r)^5 = 0 \\ u(0) = A > 0 \\ u'(0) = 0 \end{cases},$$

If u is a solution to (1.6) we define an associated energy as

$$\mathcal{E}(r) = \frac{1}{6} u(r)^6 + \frac{1}{2}\left(\frac{d}{dr} u(r)\right)^2. \tag{3.2}$$

Differentiating and using (1.6) we obtain

$$\frac{d}{dr}\mathcal{E}(r) = -2\coth(r)\left(\frac{d}{dr} u(r)\right)^2, \tag{3.3}$$

therefore this energy is monotone decreasing and bounded.

We will also find it useful to make the change of variables $v(r) = u(r)\sinh(r)$, where u solves (1.6). Then v satisfies the following equation

$$\begin{cases} \frac{d^2}{dr^2}v(r) = v(r)\left(1 - \frac{v(r)^4}{\sinh(r)^4}\right) \\ v(0) = 0 \\ \frac{d}{dr}v(0) = u(0) = A > 0 \end{cases}. \tag{3.4}$$

We note that in order to show $u(r) > 0$ for all r it is sufficient to show that $v(r) > 0$ for $r > 0$. We compile some basic results in the following lemmas.

Lemma 3.1. *Suppose u solves* (1.6) *and $u > 0$ (or $u < 0$) for all $r > \tilde{r}$ then there exists r^* such that u is monotone decreasing (increasing) for $r > r^*$.*

Proof. Suppose $u > 0$ and suppose u is not monotone decreasing, that is for some r_0 we have $\frac{d}{dr}u(r_0) = a > 0$ and $\frac{d}{dr}u(r) \leq 0$ for all $r < r_0$. Then for some r_1 we have $\frac{d}{dr}u(r_1) = 0$. We know $u(r_0) > u(r_1)$ and so

$$\mathcal{E}(r_0) = \frac{1}{6}u(r_0)^6 + \frac{1}{2}\left(\frac{d}{dr}u(r_0)\right)^2 > \frac{1}{6}u(r_1)^6 = \mathcal{E}(r_1),$$

but this contradicts the monotonicity of our energy quantity. If we do not have $u > 0$ for all r then it must change sign at least once. So if $u > 0$, (< 0) for all $r > \tilde{r}$ then there exists some r_0 such that $u(r_0) = 0$ and $\frac{d}{dr}u(r_0) > 0$, (< 0), and $u > 0$, (< 0) for all $r > r_0$. Since $u > 0\,(< 0)$ and $\frac{d}{dr}u \geq 0\,(\leq 0)$ we know by (1.6) that $\frac{d^2}{dr^2}u < 0\,(> 0)$ and so for some r_1 we have $\frac{d}{dr}u(r_1) = 0$, and $\frac{d}{dr}u(r) < 0\,(> 0)$ for $r_1 < r < r_1 + \varepsilon$, some $\varepsilon > 0$. We claim that this $r_1 = r^*$. Assume otherwise, so that $\frac{d}{dr}u(r_2) > 0\,(< 0)$ for some $r_2 > r_1$. Then for some $r_3 < r_2$ we must have $u(r_3) < u(r_2)$ $(u(r_3) > u(r_2))$ and $u(r_3) = 0$. But then $E(r_3) < E(r_2)$, contradicting the monotonicity of our energy quantity. ☐

Remark 3.2. In particular if u solves (1.6) and $u > 0$ for all $r \geq 0$ then u is monotone decreasing.

Lemma 3.3. *Suppose u solves* (1.6) *then $\frac{d}{dr}u(r) \to 0$ and $u \to 0$ as $r \to \infty$*

Proof. We will consider three cases separately, u can have a finite number of sign changes, it can have an infinite number of sign changes and tend towards zero, or it can have an infinite number of sign changes and not tend towards zero.

If there are only finitely many sign changes then after some radius r_x we know that $u(r)$ has the same sign. Then by applying lemma 3.1 we know that u must be monotone after some time r_{x*} and so we must have $u \to B$ as $r \to \infty$, for some $B \in \mathbb{R}$. Then we can find a sequence r_n such that $\frac{d}{dr}u(r_n) \to 0$. Therefore $\mathcal{E}(r_n) \to B^6/6$ as $n \to \infty$ and since energy is monotone this means that $\mathcal{E}(r) \to B^6/6$ as $r \to \infty$. Therefore we must have $\frac{d}{dr}u(r) \to 0$ as $r \to \infty$.

If B is not zero then we have that $\frac{d^2}{dr^2}u(r)$ is away from 0 uniformly by the equation, but since $\frac{d}{dr}u \to 0$ we can find a sequence of r_n such that $\frac{d^2}{dr^2}u(r_n) \to 0$ which is a contradiction.

If u has an infinite number of sign changes and converges to zero then we can find a sequence of $\frac{d}{dr}u(r_n) \to 0$ as $n \to \infty$. Then $\mathcal{E}(r_n) \to 0$ as $n \to \infty$ and since energy is monotone $\mathcal{E}(r) \to 0$ as $n \to$ and so $\frac{d}{dr}u(r) \to 0$ as $r \to \infty$.

If u has an infinite number of sign changes and does not tend to zero then we can find a sequence r_n such that $u(r_n) \leq -B$ for some $B > 0$. Then we can find $r_n^* < r_n$ such that $u(r_n^*) = 0$ and u does not attain any zeros between r_n and r_n^*. We can also find r_n^{**} such that $u(r_n^{**}) = -B/2$ and $u(r) > -B/2$ for $r_n^* > r > r_n^{**}$. Then we can obtain an upper bound for the derivative between these points, since for all $s_n \in (r_n^{**}, r_n^*)$ we have $\mathcal{E}(s_n) \leq \mathcal{E}(r_n^{**})$ and so

$$C = \frac{d}{dr}u(s_n) \leq \sqrt{2\left(\frac{1}{2}\left(\frac{d}{dr}u(r_n^{**})\right)^2 - \frac{1}{6}u(s_n)^6\right)} \leq \frac{d}{dr}u(r_n^{**}),$$

and since $\frac{d}{dr}u(r_n^{**})$ is uniformly bounded we have a lower bound for the distance between r_n^{**} and r_n^* which is

$$|r_n^* - r_n^{**}| \geq \frac{B}{2C}.$$

We can also find a lower bound for $\frac{d}{dr}u(r)$ between the two points. We know that $\mathcal{E}(s_n) \geq \mathcal{E}(r_n)$ and so

$$D = \frac{d}{dr}u(s_n) \geq \sqrt{2\left(\frac{1}{2}\left(\frac{d}{dr}u(r_n)\right)^2 + \frac{1}{6}B^6\right)} \geq \frac{\sqrt{3}}{3}B^3.$$

Therefore we can produce a lower bound on the loss of energy on the interval between r_n^{**} and r_n^* that is uniform in n. If we have infinitely many such intervals, then our energy must eventually become negative, a contradiction, and so in fact we cannot have infinite sign changes without u converging to 0. $\square$

Lemma 3.4. *If u solves* (1.1) *then u can only change sign finitely many times.*

Proof. Assume otherwise, so that u changes signs infinitely many times. Since $u \to 0$ as $r \to \infty$ we know that there exists an R such that for all $r > R$ we have $|u(r)| < 1$. Let us take $r_0 > R$ such that $u(r_0) < 0$ and $u'(r_0) < 0$, if no such r_0 exists then u cannot change signs an infinite number of times. If we let $v(r) = u(r)\sinh(r)$ we can see that $v(r_0) < 0, v'(r_0) < 0$ and $v''(r_0) < 0$. Furthermore because $|u(r)| < 1$ we know that $1 - v(r)^4/\sinh(r)^4 > 0$ for all $r > R$ and so the only way for v'' to change signs is if v changes signs. Therefore so long as v is negative its first derivative will become more negative, and if its first derivative is already negative v can never again become positive. But this is exactly the situation at r_0 and so v must remain negative. But since $u(r) = v(r)/\sinh(r)$ this means that u must always be negative, but this contradicts the assumption that u changes signs infinitely many times. $\square$

Now that we have proven some basic and useful results we will proceed to demonstrate the existence of positive solutions to the ODE.

Lemma 3.5. *Suppose v solves* (3.4) *and v has a zero at time r_1 and $v > 0$ for all $0 < r < r_1$, then there exists $r_0 < r_1$ such that $\frac{d^2}{dr^2}v(r_0) = 0$, $v(r_0) \neq 0$ and $\frac{d}{dr}v(r_0) < 0$*

Proof. Since $v(r_1) = 0$ and v is positive for smaller r we know that $\frac{d}{dr}v(r_1) < 0$. We can find a $r_2 < r_1$ such that $1 > v(r_2) > 0, \frac{d}{dr}v(r_2) < 0, \frac{d^2}{dr^2}v(r_2) > 0$. We can also find a $r_3 < r_2$ such that $\frac{d^2}{dr^2}v(r_3) < 0, \frac{d}{dr}v(r_3) < 0$. Thus there exists some $r_0 \in (r_3, r_2)$ such that $\frac{d^2}{dr^2}v(r_0) = 0$ and $\frac{d}{dr}v(r_0) < 0$ and $v(r_0) \neq 0$. $\square$

Proposition 3.6. *Suppose v solves* (3.4) *with $A < 2^{1/3}$ then $v(r) > 0$ for $r > 0$.*

Proof. Assume otherwise, so that for some r_1, $v(r_1) = 0$ and for all $0 < r < r_1$ we have $v(r) > 0$. Then by the above lemma there exists $r_0 < r_1$ such that $\frac{d^2}{dr^2}v(r_0) = 0$ and $\frac{d}{dr}v(r_0) < 0$. Therefore $u(r_0) = 1$ and since $\frac{d}{dr}v(r) = u(r)\cosh(r) + \frac{d}{dr}u(r)\sinh(r)$ we have $\frac{d}{dr}u(r_0) < -\coth(r_0)$. Thus $\mathcal{E}(r_0) = u(r_0)^6/6 + (\frac{d}{dr}u(r_0))^2/2 > 2/3$. However $\mathcal{E}(0) < (2^{1/3})^6/6 = 2/3$, which contradicts the monotonicity of energy. $\square$

Remark 3.7. We can actually improve our bound on A from $2^{1/3}$ to $2^{1/3} + \varepsilon$. A simple method is to find a lower bound for the energy loss that can occur between $r = 0$ and r_0, where $u(r_0) = 1$.

Suppose $A > 1$ and v has its first zero at r_1, then by Lemma 3.4, there exists $r_0 < r_1$ such that $v(r_0) \neq 0, \frac{d^2}{dr^2}v(r_0) = 0, \frac{d}{dr}v(r_0) < 0$. Therefore we have $u(r_0) = 1, \frac{d}{dr}u(r_0) < -1$. Since $A > 1$ and $u'(0) = 0$ there exists s_1 such that $\frac{d}{dr}u(s_1) = -1$ and $\frac{d}{dr}u(r) > -1$ for all $r < s_1$. Furthermore since u is monotone decreasing while positive we have that $u(s_1) > 1$. We can similarly obtain $s_2 < s_1$ such that $\frac{d}{dr}u(s_2) = -1/2$ and $\frac{d}{dr}u(r) < -1/2$ for all $s_2 < t < s_1$. Once again by the monotonicity of u we know $u(s_2) > u(s_1)$.

We know $E(s_2) > E(s_1)$ and so $u(s_2)^6 - u(s_1)^6 > 2.25$ and $u(s_2)^6 - u(s_1)^6 = 6(u(s_2) - u(s_1)) * u(\xi)$ some $\xi \in (s_2, s_1)$ so

$$u(s_2) - u(s_1) > 2.25/(6A).$$

We also know that $|u'(r)| < 1$ on (s_2, s_1) and so

$$s_1 - s_2 > 2.25/(6A).$$

Since $\frac{d}{dr}u(r) < -1/2$ on (s_2, s_1), we know $\frac{d}{dr}\mathcal{E}(r) = -2\coth(r)|\frac{d}{dr}u(r)|^2 > -1/2$. Therefore

$$\mathcal{E}(s_1) - E(0) = \int_0^{s_1} \frac{d}{dr}\mathcal{E}(r)\mathrm{d}r < \int_{s_2}^{s_1} \frac{d}{dr}\mathcal{E}(r)\mathrm{d}r < (s_1 - s_2)(-1/2) < -2.25/(12A).$$

It is clear that for $A < 1.3$ we must have $\mathcal{E}(s_1) < 2/3$, but

$$\mathcal{E}(s_1) = u(s_1)^6/6 + u'(s_1)^2/2 > 1/6 + 1/2 = 2/3.$$

Therefore we can improve our bound from $2^{1/3} = 1.25...$ to 1.3

3.1 Simulation Results

Using a taylor expansion to approximate the solution to the ODE near zero allows us to use numerical methods to obtain approximate solutions to the ODE. This can give us intuition for the asymptotic behavior of solutions to the equation as well as the behavior of solutions for various initial data.

Because our ODE is of second order we can expect to find a relationship between the first three coefficients of the Taylor series. Furthermore because our solution corresponds to a radial equation in higher dimensions we know that it must be even and so do not need coefficients for odd powers of x. After plugging the function $f(r) = B_1 + B_2 r^2 + B_3 r^4$ into (1.6), and ignoring terms of low order we find that

$$B_2 = -B_1^5/6,$$

and

$$B_3 = B_1^9/24.$$

Now using this Taylor series to approximate the value of a solution to the ODE near zero we are able to numerically solve the ODE from a point near zero to large values of r. Now we summarize our simulation results, which can be found in Appendix B.

Simulation Result 3.8. *Suppose u solves* (1.6) *with $A > 0$. Then $u > 0$ for all r*

Simulation Result 3.9. *Suppose u solves* (1.6) *with $A > 0$. Then for $r > 1$ we have* $.5 < u(r)(2r)^{-1/4} \leq 2$

Remark 3.10. If State 3.9 can be shown to be true rigorously then it gives an alternate proof of Mancini and Sandeep's result in [3]. This result is also consistent with the method of dominant balance. That is if we approximate u by $(2r)^{-1/4}$ plus some error term of higher order and plug it into (1.6) we obtain

$$\frac{d^2}{dr^2}u(r) + 2\coth(r)\frac{d}{dr}u(r) + u(r)^5 = \frac{5}{4}(2r)^{-9/4} - \coth(r)(2r)^{-5/4} + (2r)^{-5/4} + o(t^{\alpha}),$$

where $\alpha < -\frac{5}{4}$ Since $\coth(r)$ is approximately 1 for large r, the two terms of order $-\frac{5}{4}$ cancel and we are left with terms of lower order which we can ignore.

4 Nonlinear Schrödinger Equation

In this section we consider radial solutions to the initial value problem

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^3} u = u|u|^4 \\ u(x,0) = u_0(x) \end{cases} . \tag{4.1}$$

We would like to parallel the proof of the scattering result from [2]. That is we would like to show

Claim 4.1. *Suppose Conjecture 1.4 is not true. Then there exists E_{crit} such that for each $E > E_{crit}$ there exists $u_0(x) \in H^1(\mathbb{H}^3)$ with $||\nabla u_0(x)||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)}$ and $E_{\mathbb{H}^3}(u_0) < E$ for which the corresponding solution to* (1.1) *does not scatter. Furthermore we will show there exists a u_{crit} and an interval I^* such that $||\nabla u_{crit}(x)||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)}, E_{\mathbb{H}^3}(u_{crit}) = E_{crit}$ and $||u_{crit}||_{S(I*)} = \infty$, where $||\cdot||_{S(I*)}$ is the norm defined in* (4.14).

At first glance it may be peculiar that the stationary solution in Euclidean space is in our result for hyperbolic space. However after scaling any solution to the nonlinear Schrödinger equation on Euclidean space can be made an approximate solution on hyperbolic space and so we would not expect to be able to get a better constant than those given for Euclidean space by Kenig and Merle in [2]. It is because the sharp Sobolev constants are the same in hyperbolic and Euclidean space that we expect this to work. We know a key component in their result is the variational estimate

Lemma 4.2. *Let W be as in Theorem 1.2. For some $u \in \dot{H}^1(\mathbb{R}^3)$, assume*

$$||\nabla u||^2_{L^2(\mathbb{R}^3)} < ||\nabla W||^2_{L^2(\mathbb{R}^3)}.$$

Assume moreover that $E_{\mathbb{R}}(u) \leq E(W) - \delta$ where $\delta > 0$. Then, there exists $\bar{\delta} = \bar{\delta}(\delta_0)$ such that

$$||\nabla u||^2_{L^2(\mathbb{R})} < ||\nabla W||^2_{L^2(\mathbb{R}^3)} - \bar{\delta},$$

and

$$\int_{\mathbb{R}} |\nabla u|^2 - |u|^6 \geq \bar{\delta} \int |\nabla u|^2.$$

We would like to show an analogue of this in hyperbolic space, but there is no stationary solution to the Schrödinger equation in hyperbolic space with a bounded H^1 norm. So we would not expect tone able to replicate this result in hyperbolic space. However the proof in [2] does not rely on the existence of a ground state element in hyperbolic space and in fact only utilizes the sharp constant from Sobolev embedding and so can be adapted to Hyperbolic space. We can show

Lemma 4.3. *Let W be as in Theorem 1.2 Assume that*

$$||\nabla u||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)}.$$

Assume moreover that

$$E_{\mathbb{H}^3}(u) \leq E_{\mathbb{R}}(W) - \delta,$$

where $\delta > 0$. Then, there exists $\bar{\delta} = \bar{\delta}(\delta_0)$ such that

$$||\nabla u||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)} - \bar{\delta}, \tag{4.2}$$

and

$$\int_{\mathbb{H}^3} |\nabla u|^2 - |u|^6 \geq \bar{\delta} \int_{\mathbb{H}^3} |\nabla u|^2. \tag{4.3}$$

Because the proof of this Lemma is identical to the proof of Lemma 3.4 in [2] we present it in appendix A.

Remark 4.4. We note that a more general statement is true, based on the proof presented. Let M be a noncompact manifold and C be some constant. Assume that

$$||u||_{L^6(M)} \leq C||\nabla u||_{L^2(M)}, \tag{4.4}$$

(the inequality need not be sharp) and

$$||\nabla u||^2_{L^2(M)} < C^{-3}.$$

Assume moreover that

$$\frac{1}{2}\int_M |\nabla u|^2 - \frac{1}{6}\int_M |u|^6 \leq \frac{1}{3}C^{-3} - \delta,$$

where $\delta > 0$. Then, there exists $\bar{\delta} = \bar{\delta}(\delta_0)$ such that

$$||\nabla u||^2_{L^2(M)} < C^{-3} - \bar{\delta}, \tag{4.5}$$

and

$$\int_M |\nabla u|^2 - |u|^6 \geq \bar{\delta} \int_M |\nabla u|^2. \tag{4.6}$$

Once a variational estimate such as the above is shown we can prove an energy trapping result by a standard continuity argument as done in [2]

Theorem 4.5. *(Energy trapping) Let u be a solution of (1.1) with $u(0) = u_0$ such that*

$$||\nabla u_0||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)} \quad \textit{and} \quad E_{\mathbb{H}^3}(u_0) < E_{\mathbb{R}}(W) - \delta \tag{4.7}$$

Let $I \ni 0$ be the maximal interval of existence. Let $\bar{\delta}$ be as in Lemma 4.3. Then for each $t \in I$ we have

$$\int_{\mathbb{H}^3} |\nabla u(t)|^2 \leq ||\nabla W||_{L^2(\mathbb{R}^3)} - \bar{\delta}, \tag{4.8}$$

$$\int_{\mathbb{H}^3} |\nabla u(t)|^2 - |u(t)|^6 \geq \bar{\delta} \int_{\mathbb{H}^3} |\nabla u(t)|^2. \tag{4.9}$$

Proof. By Remark 1.1 we have that $E_{\mathbb{H}^3}(u(t)) = E_{\mathbb{H}^3}(u_0)$ for all $t \in I$ and the theorem follows directly from Lemma 4.3 and a continuity argument. □

Now that we have this energy trapping result we would like to parallel the method of proof and result from [2]

Remark 4.6. Consider $u_0 \in H^1(\mathbb{H}^3)$ and assume $E_{\mathbb{H}^3}(u_0) < E_{\mathbb{R}}(W), ||\nabla u_0||_{L^2(\mathbb{H}^3)} < ||\nabla W||_{L^2(\mathbb{R}^3)}$ and that u_0 is radial. Then the solution u to (1.1) is defined for all time and $||u||_{S(\mathbb{R})} < \infty$.

We consider the set

$$\mathcal{A} = \{E \leq E_{\mathbb{R}}(W); \forall u_0(x) \text{ if } ||\nabla u_0(x)||_{L^2(\mathbb{H}^3)} \leq ||\nabla W||_{L^2(\mathbb{R}^3)} \text{ and } E_{\mathbb{H}^3}(u_0(x)) < E \text{ then } ||u||_{S(I)} < \infty\}. \tag{4.10}$$

If we can show that $\sup \mathcal{A} = E_{\mathbb{R}}(W)$ then our proof is complete. Therefore assume otherwise. That is there exists some $E_{crit} < E_{\mathbb{R}}(W)$ such that for each $E > E_{crit}$ there exists a $u_0(x)$ with $||\nabla u_0(x)||_{L^2(\mathbb{H}^3)} \leq ||\nabla W||_{L^2(\mathbb{R}^3)}$ and $E_{\mathbb{H}^3}(u_0(x)) \leq E$ for which the corresponding solution to (1.1) does not scatter.

We will produce a $u_{crit}(x)$ such that $||\nabla u_{crit}(x)||_{L^2(\mathbb{H}^3)} \leq ||\nabla W||_{L^2(\mathbb{R}^3)}, E(u_{crit}) = E_{crit}$ and the corresponding solution to (1.1) does not scatter. To do so we will find a sequence of functions whose energy converges to E_{crit} and which strongly converges to a limit in $H^1(\mathbb{H}^3)$.

In the below subsection using the profile decomposition described in [1] we will show that for a sequence $\{u_k\}_k$ with $||\nabla u_k||_{L^2(\mathbb{H}^3)} \leq ||\nabla W||_{L^2(\mathbb{R}^3)}$ and $E(u_k) \leq E_{crit} + \frac{1}{k}$ we have that $u_k \to v$ strongly, for some $v \in H^1(\mathbb{H}^3)$.

Based on the proof of Kenig and Merle we expect that the existence of such v will produce the desired contradiction.

4.1 Profile Decomposition

We now utilize the method for profile decomposition in hyperbolic space from Ionescu, Pausader and Staffilani in [1] to show the existence of the desired critical element. The following proposition is the focusing version of the proposition in [1].

Proposition 4.7. *(Profile Decomposition [1]) Assume that $(f_k)_{k\geq 1}$ is a bounded sequence if $H^1(\mathbb{H}^3)$. Then there are sequences of pairs $(\phi^\mu, \mathcal{O}^\mu) \in \dot{H}^1(\mathbb{R}^3) \times \mathcal{F}_e$ and $(\psi^\nu, \tilde{\mathcal{O}}^\nu) \in H^1(\mathbb{H}^3) \times \mathcal{F}_h, \mu, \nu = 1, 2, \dots$ such that, up to a subsequence, for any $J \geq 1$*

$$f_k = \sum_{1\leq\mu\leq J} \tilde{\phi}^\mu_{\mathcal{O}^\mu_k} + \sum_{1\leq\nu\leq J} \tilde{\psi}^\nu_{\tilde{\mathcal{O}}^\nu_k} + r^J_k, \tag{4.11}$$

See Ionescu, Pausader and Staffilani, section 5 in [1] for the defintions of $\tilde{\phi}^\mu_{\mathcal{O}^\mu_k}$ and $\tilde{\psi}^\nu_{\tilde{\mathcal{O}}^\nu_k}$.

$$\lim_{J\to\infty} \limsup_{k\to\infty} \left(\sup_{N\geq 1, t\in\mathbb{R}, x\in\mathbb{H}^3} N^{-1/2} |P_N e^{it\Delta_{\mathbb{H}^3}} r^J_k|(x) \right) = 0. \tag{4.12}$$

Moreover the frames $\{\mathcal{O}^\mu\}_{\mu\geq 1}$ and $\{\tilde{\mathcal{O}}^\nu\}_{\nu\geq 1}$ are pairwise orthogonal. Finally the decomposition is asymptotical orthogonal in the sense that

$$\lim_{J\to\infty} \limsup_{k\to\infty} \left| E(f_k) - \sum_{1\leq\mu\leq J} E(\tilde{\phi}^\mu_{\mathcal{O}^\mu_k}) - \sum_{1\leq\nu\leq J} E(\tilde{\psi}^\nu_{\tilde{\mathcal{O}}^\nu_k}) - E(r^J_k) \right| = 0, \tag{4.13}$$

where E is the energy defined in (1.2).

This is an analogue of Proposition 3.4 in [1], but when dealing with the euclidean profile we utilize the result from [2]. Furthermore because we are interested in radial solutions we do not have space translation elements in the profiles.

Proposition 4.8. *Let $u_k \in C((-T_k, T^k) : H^1(\mathbb{H}^3)), k = 1, 2, \dots$ be a sequence of radial nonlinear solutions of the equation*

defined on open intervals $(-T_k, T^k)$ such that $E_{\mathbb{H}^3}(u_k) \to E_{crit}$. Let $t_k \in (-T_k, T^k)$ be a sequence of times with

$$\lim_{k\to\infty} ||u_k||_{S(-T_k, t_k)} = \lim_{k\to\infty} ||u_k||_{S(t_k, T^k)} = \infty. \tag{4.14}$$

Then there exists $v \in H^1(\mathbb{H}^3)$ such that, after replacing u_k with a suitable subsequence we have, $u_k(t_k, x) \to v(x) \in H^1$ strongly.

Proof. Using the time translation symmetry, we mary assume that $t_k = 0$ for all $k \geq 1$. We apply Proposition 4.7 to the sequence $(u_k(0))_k$ which is bounded in H^1 and we get sequences of pairs $(\phi^\mu, \mathcal{O}^\mu) \in \dot{H}^1(\mathbb{R}^3) \times \mathcal{F}_e$ and $(\psi^\nu, \tilde{\mathcal{O}}^\nu) \in H^1(\mathbb{H}^3) \times \mathcal{F}_h, \mu, \nu = 1, 2, \dots$ such that the conclusion of Proposition 4.7 holds. By Lemma 5.4 (i) of [1] we may assume that for all μ, either $t^\mu_k = 0$ for all k or $(N^\mu_k)^2 |t^\mu_k| \to \infty$ and similarly, for all ν, either t^ν_k or $|t_k|^\nu \to \infty$. There are four possible cases for the distributions of profiles

Case I: all profiles are trivial, $\phi^\mu = 0, \psi^\nu = 0$ for all μ, ν.
Case IIa: There is only one Euclidean profile
Case IIb: There is only one hyperbolic profile
Case III: There are two or more profiles of any variety

We would like to show that all of the cases other than case IIb produce contradictions and that case IIb gives us the desired result. The proof laid out for cases I, IIb, and III in [1] can be repeated for the focusing case because their arguments rely on characteristics of the linear solution and small data results that hold for both the focusing and defocusing cases. In order to show that case IIa leads to a contradiction we can apply the primary result of Kenig and Merle to find that any Euclidean profile must have a bounded $S(I)$ norm contradicting (4.14). This step in particular is why we have restricted ourselves to the radial case. □

Now that we have desired proposition we are able to produce a critical element $v \in H^1(\mathbb{H}^3)$ with some maximal interval of existence I^* such that

$$E_{\mathbb{H}^3}(v) = E_{crit} \quad ||v||_{S(I^*)} = \infty.$$

The next step is to show that $v(t, \cdot), t \in I^*$ forms a pre-compact family, that is to say for any sequence $v(t_k, \cdot)$ there exists a subsequence which is strongly convergent. To show this we use profile decomposition on the sequence $v(t_k, \cdot)$ and repeat Proposition 4.8. This property implies that I^* must be the whole real line. The final step is to show the such a v cannot exist. Namely, given such a v we can define functions $\phi(r) = |r|^2$ and $z_R(t) = \int |v(x,t)|^2 R^2 \phi(\frac{r}{R})\mathrm{d}x$. We expect to be able to find an upper bound for $|z_R'(t)|$ and would like to produce a lower bound for $z_R''(t)$. In computing a lower bound for $z_R''(t)$ it is necessary to make use of Hyperbolic geometry. We expect that this part would be specific to Hyperbolic space as the analogous step in [2] uses a localized version of the Virial identity on $\mathbb{R}^3$ and so we expect a similar level of work specific to the geometry of $\mathbb{H}^3$ would be necessary to complete the proof.

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5 Appendix A(Proof of Lemma 4.3)

Proof. We present the proof of Lemma 3.4 in [2]. Let M be a non-compact manifold and let $C := ||\nabla W||_{L^2(\mathbb{R})}$. From the proof in [2] we know that $\frac{1}{3}||\nabla W||_{L^2(\mathbb{R}^3)} = E_{\mathbb{R}}(W)$ Consider

$$f(y) = \frac{1}{2}y - \frac{C^{-3}}{6}y^3,$$

and let $\bar{y} = ||\nabla u||^2_{L^2(M)}$. Then

$$f(\bar{y}) = \frac{1}{2}||\nabla u||^2_{L^2(M)} - \frac{C^{-3}}{6}||\nabla u||^6_{L^2(M)} \leq \frac{1}{2}||\nabla u||^2_{L^2(M)} - \frac{1}{6}\int_M u^6 \mathrm{d}\mu < \frac{1}{3}C^{-3} - \delta.$$

We note that $f(0) = 0$ and $f'(y) = \frac{1}{2} - \frac{1}{2}C^6 y^2$ so $f'(y) = 0$ if and only if $y = C^{-3}$ and for smaller y is positive. We know $0 < \bar{y} < C^{-3}$ and

$$f(C^{-3}) = \frac{1}{2}C^{-3} - \frac{C^6}{6C^9} = C^{-3}(\frac{1}{2} - \frac{1}{6}) = \frac{1}{3}C^{-3}.$$

Since f is strictly increasing between 0 and C^{-3} we have $\bar{y} \leq C^{-3} - \bar{\delta}$. This shows (4.5). Now to show (4.6) we consider the function

$$g(y) = y - C^6 y^3.$$

Because of (2.3) we have that

$$\int_M |\nabla u|^2 - |u|^6 \geq \int_M |\nabla u|^2 - C^6 \left(\int_M |\nabla u|^2\right)^3 = g(\bar{y}).$$

Note that $g(y) = 0$ if and only if $y = 0$ or $y = C^{-3}$ and that $g'(0) = 1$ and $g'(C^{-3}) = -2$. We then have, for $0 < y < C^{-3}, g(y) \geq D \min\{y, (C^{-3} - y)\}$, and so, since $0 \leq \bar{y} < C^{-3} - \bar{\delta}$ by (4.5) we have (4.6). $\square$

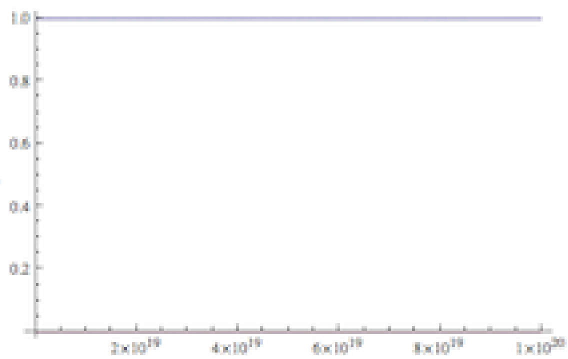
6 Appendix B (Simulation Code and Results)

We compile the Mathematica code used and graphs produced in this section. Each of the three graphs illustrates both the asymptotic behavior of solutions and that solutions are positive up to large values for r.

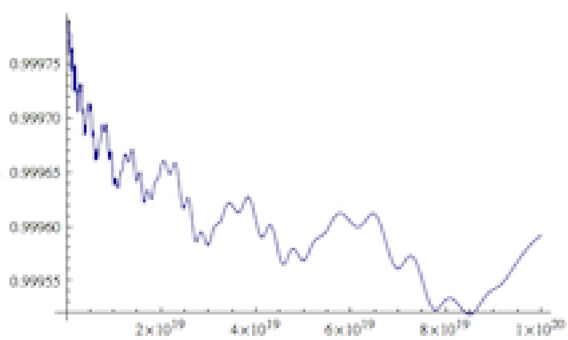
Wolfram *Mathematica* | FOR STUDENTS

```
In[47]:= Bone = 1
         t = .01
         ut = az - Bone^5 / 6 * t^2
         uprimet = -Bone^5 / 3 * t + Bone^9 / 6 * t^3
         s = NDSolve[{u''[x] + 2 Coth[x] u'[x] == -u[x]^5, u[t] == ut, u'[t] == uprimet}, u, {x, t, 1 * 10^20}]
         Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s], 0}, {x, .01, 1 * 10^20}]
         Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s]}, {x, .01, 1 * 10^20}]

Out[47]= 1

Out[48]= 0.01

Out[49]= 0.999983

Out[50]= -0.00333317

Out[51]= {{u -> InterpolatingFunction[{{0.01, 1. × 10^20}}, <>]}}
```

Out[52]=



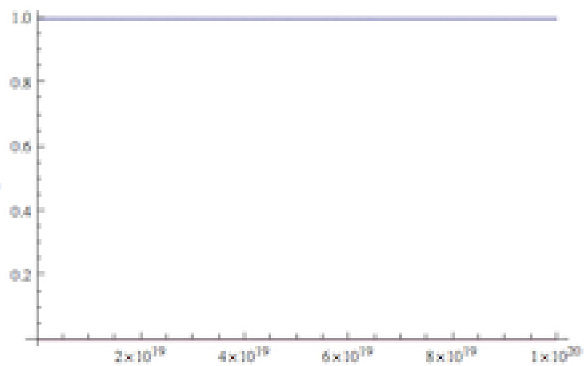
Out[53]=



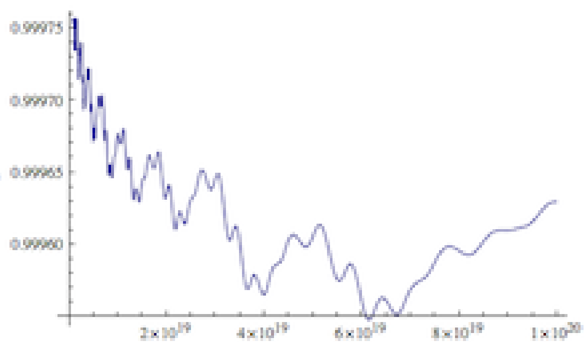
Wolfram *Mathematica* | FOR STUDENTS

```
In[54]:= Bone = 10
         t = .0001
         ut = az - Bone^5 / 6 * t^2
         uprimet = -Bone^5 / 3 * t + Bone^9 / 6 * t^3
         s = NDSolve[{u''[x] + 2 Coth[x] u'[x] == -u[x]^5, u[t] == ut, u'[t] == uprimet}, u, {x, t, 1 * 10^20}]
         Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s], 0}, {x, .01, 1 * 10^20}]
         Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s]}, {x, .01, 1 * 10^20}]

Out[54]= 10

Out[55]= 0.0001

Out[56]= 0.999833

Out[57]= -3.33317

Out[58]= {{u -> InterpolatingFunction[{{0.0001, 1. × 10^20}}, <>]}}
```

Out[59]=



Out[60]=



Wolfram *Mathematica* | FOR STUDENTS

In[68]:=
```
Bone = 1000
t = 1 * 10^-8
ut = az - Bone^5 / 6 * t^2
uprimet = -Bone^5 / 3 * t + Bone^9 / 6 * t^3
s = NDSolve[{u''[x] + 2 Coth[x] u'[x] == -u[x]^5, u[t] == ut, u'[t] == uprimet}, u, {x, t, 1 * 10^20}]
Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s], 0}, {x, .01, 1 * 10^20}]
Plot[{(2 x)^(1 / 4) * Evaluate[u[x] /. s]}, {x, .01, 1 * 10^20}]
```

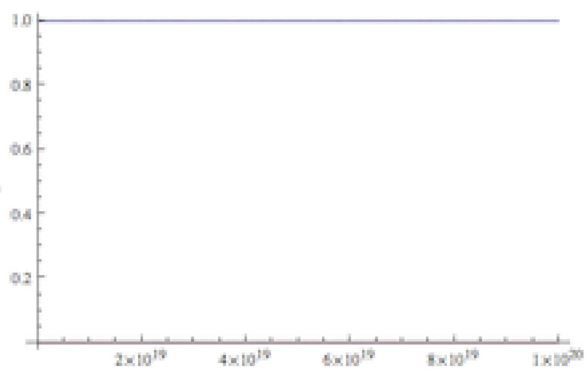
Out[68]= 1000

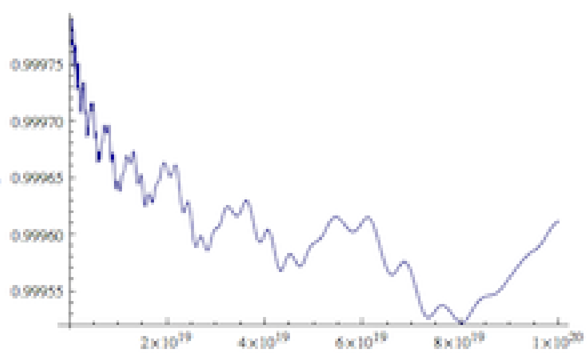
Out[69]= $\frac{1}{100\,000\,000}$

Out[70]= $\frac{59}{60}$

Out[71]= $-\frac{9\,999\,500}{3}$

Out[72]= {{u → InterpolatingFunction[{{1. × 10^{-8}, 1. × 10^{20}}}, <>]}}

Out[73]=


Out[74]=


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