QUANTITATIVE BOUNDS FOR HURWITZ STABLE POLYNOMIALS UNDER MULTIPLIER TRANSFORMATIONS

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Abstract. We extend results of Borcea and Brändén to quantitatively bound the movement of the zeros of Hurwitz stable polynomials under linear multiplier operators. For a multiplier operator T on polynomials of degree up to n, we show that the ratio between the maximal real part of the zeros of a Hurwitz stable polynomial f and that of its transform $T[f]$ is minimized for the polynomial $(z+1)^n$. For T operating on polynomials of all degrees, we use the classical Pólya-Schur theorem to derive an asymptotic formula for the maximal real part of the zeros of $T[(z+1)^n]$, and consider the action of T on classes of entire functions. We develop interlacing controls on the transformations of real-rooted polynomials. On the basis of numerical evidence, we conjecture several further quantitative controls on the zeros of Hurwitz stable polynomials under multiplier operators.

1. Introduction

Polynomials which do not vanish in some prescribed region $\Omega \subset \mathbb{C}$ are called Ω-stable. These polynomials play a fundamental role in a wide range of mathematical disciplines, including operator theory, algebraic geometry, and combinatorics [8]. A linear operator is said to *preserve Ω-stability* if the image of any Ω-stable polynomial under T is either Ω-stable or identically zero. These linear operators are naturally of great interest, and have recently been characterized by Borcea and Brändén when Ω is a circular domain or a strip [2]. Their work encompasses important special cases, including standard stability, Hurwitz stability, and Schur stability. We focus on Hurwitz stability, for which Ω is the right half-plane:

Definition 1.1. *A polynomial $f \in \mathbb{C}[z]$ is* weakly Hurwitz stable *if $f(z) \neq 0$ whenever* $\mathrm{Re}(z) > 0$. *A polynomial f is* strictly Hurwitz stable *if $f(z) \neq 0$ whenever* $\mathrm{Re}(z) \geq 0$. *We denote the class of weakly Hurwitz stable polynomials by $\mathcal{HS}$.*

Hurwitz stable polynomials are of particular significance in control theory, where they represent stable linear dynamic systems [8]. Note that in other literature the term "Hurwitz stability" often refers to strict Hurwitz stability. Borcea and Brändén have precisely characterized the linear operators which preserve weak Hurwitz stability. Their results concern two types of operators: those that act on polynomials of all degrees, and those that act only on polynomials of degree up to some $n \in \mathbb{N}$. We therefore define restricted classes of polynomials which are bounded in degree:

Definition 1.2. *For $n \in \mathbb{N}$, let $\mathbb{C}_n[z]$ and $\mathcal{HS}_n$ denote the subsets of $\mathbb{C}[z]$ and $\mathcal{HS}$ respectively consisting of polynomials f with* $\deg(f) \leq n$.

Problems dealing with operators on polynomials of all degrees are referred to as "transcendental" while those concerning polynomials of bounded degree are "algebraic" [1].

The recent results of Borcea-Brändén are qualitative by nature: they characterize operators which preserve the class of polynomials which do not vanish in the open right half-plane. Our quantitative results measure "how stable" polynomials are, and how this measure of stability behaves under certain linear transformations on the polynomials. For a weakly Hurwitz stable polynomial f, we measure the distance between the roots of f and the boundary of the left half-plane, namely the imaginary axis:

Definition 1.3. *For $f \in \mathcal{HS}$, we define*

$$\mathscr{R}\{f\} := \min\{-\mathrm{Re}(\zeta);\ f(\zeta) = 0\}. \tag{1.1}$$

If f is a nonzero constant, we set $\mathscr{R}\{f\} = +\infty$.

Since we have defined $\mathscr{R}$ only for weakly Hurwitz stable polynomials, it is always nonnegative. For a linear operator T which preserves weak Hurwitz stability, the simplest form of quantitative control is a bound for $\mathscr{R}\{T[f]\}$ in terms of $\mathscr{R}\{f\}$. To obtain such a bound, we restrict attention to the class of "multiplicative transformations."

Definition 1.4. *A linear* multiplicative *operator $T\colon \mathbb{C}[z] \to \mathbb{C}[z]$ is determined by $T[z^k] = c_k z^k$ with $c_k \in \mathbb{C}$ for all $k \geq 0$.*

An analogous definition holds for operators on $\mathbb{C}_n[z]$. Multiplicative operators are simply the diagonal operators in the standard basis of $\mathbb{C}[z]$. This class includes numerous significant combinatorial transformations, as well as the theta operator $\theta\colon f \mapsto (zf)'$. We may now state our main "algebraic" result:

Theorem 1.5. *Let $n \in \mathbb{N}$ and $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ be a multiplicative operator. Then $T\colon \mathcal{HS}_n \to \mathcal{HS}_n \cup \{0\}$ if and only if $T[(z+1)^n]$ has all real, nonpositive zeros or is identically zero. Furthermore, if $T\colon \mathcal{HS}_n \to \mathcal{HS}_n \cup \{0\}$ then for all $f \in \mathcal{HS}_n$ we have either $T[f] = 0$ or*

$$\mathscr{R}\{T[f]\} \geq \mathscr{R}\{T[(z+1)^n]\} \cdot \mathscr{R}\{f\}. \tag{1.2}$$

Essentially, this theorem establishes $(z+1)^n$ as a "worst case" polynomial: out of all strictly Hurwitz stable polynomials, its zeros are proportionally moved closest to the imaginary axis. Furthermore, if $\mathscr{R}\{T[(z+1)^n]\}$ is strictly positive, (1.2) shows that T preserves *strict* Hurwitz stability. For multiplicative transformations, Theorem 1.5 therefore resolves the open problem in [2] concerning the preservation of strict stability.

Since a "transcendental" operator $T\colon \mathbb{C}[z] \to \mathbb{C}[z]$ may be restricted to $\mathbb{C}_n[z]$, Theorem 1.5 controls the behavior of T on polynomials of degree up to n. However, to apply the theorem to any given polynomial in $\mathbb{C}[z]$, we must control $\mathscr{R}\{T[(z+1)^n]\}$ for all $n \in \mathbb{N}$. To obtain this control we consider the formal power series

$$\Phi(z) := \sum_{k=0}^{\infty} \frac{c_k}{k!} z^k. \tag{1.3}$$

Due to the classical Pólya-Schur theorem, if T preserves weak Hurwitz stability then Φ defines an entire function of order 1 with all real, nonpositive zeros. We may extend the definition of the stability-measure to the entire function Φ:

$$\mathscr{R}\{\Phi\} := \inf\{-\operatorname{Re}(\zeta);\ \Phi(\zeta) = 0\}. \tag{1.4}$$

The largest zero of Φ asymptotically controls the largest zero of $T[(z+1)^n]$ as n becomes large:

Theorem 1.6. *Let $T\colon \mathbb{C}[z] \to \mathbb{C}[z]$ be a multiplicative operator such that T preserves weak Hurwitz stability, and let Φ be as above. If Φ has zeros,*

$$\mathscr{R}\{T[(z+1)^n]\} \sim \frac{\mathscr{R}\{\Phi\}}{n} \tag{1.5}$$

as $n \to \infty$. Otherwise $\Phi(z) = Ce^{az}$ for some $a \geq 0$ and $C \in \mathbb{C}$, and

$$\mathscr{R}\{T[(z+1)^n]\} = \frac{1}{a}$$

for all $n \in \mathbb{N}$.

In Section 2 we adapt several results of Borcea-Brändén to prove Theorem 1.5. The proof also relies on a composition theorem of Szegő, which illuminates the singular role of the polynomials

$(z+1)^n$. In Section 3 we discuss the connections between the algebraic and transcendental problems, derive Theorem 1.6 from the proof of Pólya-Schur, and extend transcendental operators to classes of entire functions. Section 4 offers stronger results for polynomials with all real roots. Finally, in Section 5 we offer numerical evidence for several conjectured refinements of Theorem 1.5.

2. Algebraic Operators

We first consider algebraic multiplicative transformations, which operate on some space $\mathbb{C}_n[z]$ of bounded degree. Multiplicative operators enjoy an elementary scaling property which greatly simplifies their theory. If $\lambda \in \mathbb{C}$, we may define $S_\lambda\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ by substituting $z \mapsto \lambda z$, so that $S[z^k] = \lambda^k z^k$. S_λ is itself multiplicative, so it clearly commutes with all multiplicative operators. In particular, when $\lambda > 0$ the operator S_λ maps $\mathcal{HS}$ to itself bijectively. On the other hand, we may let $\lambda = e^{i\theta}$ for $\theta \in [0, 2\pi)$ to see that Hurwitz stability is not distinguished for multiplicative operators. Following the notation of Borcea-Brändén, let $\mathbb{H}_\theta$ denote the open half-plane given by $\mathrm{Im}(e^{i\theta} z) > 0$ for $\theta \in [0, 2\pi)$.

Lemma 2.1. *Let $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ be a multiplicative operator. Then T preserves weak Hurwitz stability if and only if T preserves $\mathbb{H}_\theta$-stability for all $\theta \in [0, 2\pi)$.*

Proof. Weak Hurwitz stability is simply $\mathbb{H}_{\frac{\pi}{2}}$-stability, so the sufficiency of the condition is immediate. Therefore suppose $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ preserves weak Hurwitz stability. Let $f \in \mathbb{C}_n[z]$ be $\mathbb{H}_\theta$-stable for some $\theta \in [0, 2\pi)$. The lemma is trivial if $T[f] = 0$, so assume $T[f]$ is not identically zero. Let $\lambda = e^{i\left(\frac{\pi}{2} - \theta\right)}$. Then $S_\lambda[f]$ is weakly Hurwitz stable, and therefore so is $(T \circ S_\lambda)[f]$. Finally, this implies that $(S_\lambda^{-1} \circ T \circ S_\lambda)[f]$ is $\mathbb{H}_\theta$-stable. But $S_\lambda^{-1} \circ T \circ S_\lambda = T$. $\square$

For concreteness, we focus our results on weak Hurwitz stability. This choice is particularly convenient, since the preservation of weak Hurwitz stability is intimately connected with the preservation of a well-studied type of real stability.

Definition 2.2. *$\mathcal{P}^{\leq}$ is the class of real polynomials with all real, nonpositive roots. $\mathcal{P}_n^{\leq}$ is the subset of $\mathcal{P}^{\leq}$ consisting of polynomials of degree up to $n \in \mathbb{N}$.*

The study of transcendental linear operators which preserve $\mathcal{P}^{\leq}$ has a distinguished history (see for instance [5, 9]), but the algebraic problem for operators on bounded degrees has only been solved recently. In [1, Corollary 4.6], Borcea-Brändén show that the preservation of weak Hurwitz stability is equivalent to the preservation of $\mathcal{P}_n^{\leq}$ for multiplicative operators:

Theorem 2.3 (Borcea-Brändén)**.** *Let $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ be a multiplicative linear operator. Then $T\colon \mathcal{HS}_n \to \mathcal{HS}_n \cup \{0\}$ if and only if $e^{i\theta} T\colon \mathcal{P}_n^{\leq} \to \mathcal{P}_n^{\leq} \cup \{0\}$ for some phase $\theta \in [0, 2\pi)$.*

Remark 2.4. *We consider transformations which map $\mathcal{HS}_n$ to $\mathcal{HS}_n \cup \{0\}$ to admit simple cases like $c_0 = 0$, in which case $T[1] = 0$.*

Note that if the condition in Theorem 2.3 holds, $T = e^{-i\theta} T_0$ for some phase $\theta \in [0, 2\pi)$ and some real operator T_0. If $T[z^k] = c_k z^k$, the finite sequence $\{e^{i\theta} c_0, \ldots, e^{i\theta} c_n\}$ must therefore be a real sequence. Since constant multiples of T behave identically to T with respect to polynomial roots, we may assume without loss of generality that $c_k \in \mathbb{R}$ for all $1 \le k \le n$. Furthermore, since the coefficients of polynomials in $\mathcal{P}^{\le}$ necessarily have the same sign, $\{c_0, \dots, c_n\}$ must also have the same sign if $T\colon \mathcal{P}_n^{\le} \to \mathcal{P}_n^{\le} \cup \{0\}$. Hence we may assume $c_k \ge 0$ for all k. Such a sequence which preserves $\mathcal{P}_n^{\le}$ is termed a (nonnegative) n-multiplier sequence [1].

The proof of Theorem 1.5 also relies on a classical composition result of Szegő, which involves the following modified form of the traditional Hadamard product [10].

Definition 2.5. *For $f, g \in \mathbb{C}_n[z]$, let $f = \sum_{k=0}^n a_k z^n$, $g = \sum_{k=0}^n b_k z^n$. Define the modified Hadamard product $*'$ by*

$$f *' g := \sum_{k=0}^{n} \binom{n}{k}^{-1} a_k b_k z^k.$$

Theorem 2.6 (Szegő)**.** *Take $f, g \in \mathbb{C}_n[z]$ such that the zeros of f lie in a closed circular domain A. Then if γ is a zero of $f *' g$, there exists a point α in A and a zero β of g such that $\gamma = -\alpha\beta$.*

Remark 2.7. *"Circular domain" denotes a disk, the complement of a disk, or a half-plane. If $\deg f < n$, we consider f to have a root of multiplicity $n - \deg f$ at ∞ in the extended complex plane. In this case the circular region A must be either the complement of a disc or a half-plane.*

We may now prove Theorem 1.5, which characterizes the multiplicative algebraic operators which preserve weak Hurwitz stability, and establishes a quantitative bound in (1.2). The necessity that $T[(z+1)^n] \in \mathcal{P}_n^{\le}$ follows directly from Theorem 2.3. We derive sufficiency and the attendant quantitative control as a corollary of Theorem 2.6.

Proof of Theorem 1.5. Let $n \in \mathbb{N}$ and $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ be a multiplicative operator. First suppose $T\colon \mathcal{HS}_n \to \mathcal{HS}_n \cup \{0\}$. By Theorem 2.3, $T[(z+1)^n]$ has only real nonpositive roots or is identically zero. This establishes the necessary condition in Theorem 1.5.

To prove sufficiency, first note that the claims of Theorem 1.5 follow trivially if either $T[(z+1)^n]$ or f are constant. Therefore suppose $T[(z+1)^n]$ is non-constant with all real nonpositive roots and $f \in \mathcal{HS}_n$ is non-constant. For concision, let $\lambda := \mathscr{R}\{T[(z+1)^n]\}$ and $\mu := \mathscr{R}\{f\}$, so that $\lambda, \mu \in [0, +\infty)$. Write $f = \sum_{k=0}^n a_k z^k$. By the definition of the modified Hadamard product,

$$T[f] = \sum_{k=0}^{n} a_k c_k z^k = \sum_{k=0}^{n} \binom{n}{k}^{-1} a_k \binom{n}{k} c_k z^k = f *' T[(z+1)^n]. \tag{2.1}$$

Hence we may apply Theorem 2.6 to control the roots of $T[f]$. All roots of f are contained in the half-plane $A := \{z \in \mathbb{C};\ \mathrm{Re}(z) \le -\mu\}$, so every root of $T[f]$ must have the form $-\alpha\beta$ where $\alpha \in A$ and β is a root of $T[(z+1)^n]$. But all roots of $T[(z+1)^n]$ are real numbers, so $\mathrm{Re}(-\alpha\beta) = -\beta\,\mathrm{Re}(\alpha) \le -\lambda\mu$. Therefore each root of $T[f]$ has real part at most $-\lambda\mu$, i.e. $\mathscr{R}\{T[f]\} \ge \lambda\mu$ as claimed. □

Now consider a nonzero multiplicative operator $T\colon \mathbb{C}_n[z] \to \mathbb{C}_n[z]$ which preserves weak Hurwitz stability. For the remainder of the section, let $g_T := T[(z+1)^n]$. Theorem 1.5 implies g_T has all real nonpositive roots. If $\deg g_T < n$, we may consider g_T to have $n - \deg g_T$ roots at $-\infty$. Hence g_T has n roots in the extended ray $[-\infty, 0]$, the one-point compactification of the half-line $(-\infty, 0]$.

Conversely, a multiset $Z = \{\zeta_1, \ldots, \zeta_n\} \subset [-\infty, 0]$ uniquely determines a polynomial p_Z with those zeros (up to a constant factor) by

$$p_Z(z) = \prod_{j=1}^{n} \left(\frac{z - \zeta_j}{1 - \zeta_j} \right).$$

We identify this polynomial p_Z with g_T for some T. We may recover the multiplier sequence $\{c_0, \ldots, c_n\}$ from the coefficients of g_T. Therefore the set $\mathscr{T}_n$ of nonzero multiplicative operators (up to constant factors) on $\mathbb{C}_n[z]$ which preserve weak Hurwitz stability is in bijection with $[-\infty, 0]^n/\mathcal{S}_n$, where we have modded out by the symmetric group $\mathcal{S}_n$ to account for permutations of the roots. If we assign $\mathscr{T}_n$ the weak topology induced by action on the topological vector space $\mathbb{C}_n[z]$, the bijection is continuous in both directions. Hence $\mathscr{T}_n$ is homeomorphic to $[-\infty, 0]^n/\mathcal{S}_n$. Of course $[-\infty, 0]$ is homeomorphic to $[0, 1]$, so we have shown:

Corollary 2.8. *Let $\mathscr{T}_n$ be the set of nonzero multiplicative operators on $\mathbb{C}_n[z]$ which preserve weak Hurwitz stability, in which operators differing by a constant factor are equivalent. Then $\mathscr{T}_n$ is homeomorphic to $[0, 1]^n/\mathcal{S}_n$, and is therefore compact and path-connected.*

The proof of Corollary 2.8 demonstrates that we may effectively identify an operator $T \in \mathscr{T}_n$ with the polynomial $g_T \in \mathcal{P}_n^{\leq}$. As shown in (2.1), $T[f] = f *' g_T$, so we may consider the action of a multiplicative T on a polynomial $f \in \mathcal{HS}_n$ to be the combination of f with the polynomial g_T via the modified Hadamard product. In this perspective there is a natural duality between the input f and the multiplicative operator T. Theorems 1.5 and 2.3 together imply:

Corollary 2.9. *The modified Hadamard product $*'$ defines bilinear maps*

$$\mathcal{P}_n^{\leq} \times \mathcal{P}_n^{\leq} \to \mathcal{P}_n^{\leq} \cup \{0\} \quad \text{and} \quad \mathcal{HS}_n \times \mathcal{P}_n^{\leq} \to \mathcal{HS}_n \cup \{0\}.$$

Furthermore, in this dual perspective the bound (1.2) in Theorem 1.5 assumes a particularly simple form:

Corollary 2.10. *Let $f \in \mathcal{HS}_n$ and $g \in \mathcal{P}_n^{\leq}$. Then either $f *' g = 0$ identically or*

$$\mathscr{R}\left\{f *' g\right\} \geq \mathscr{R}\{f\} \cdot \mathscr{R}\{g\}.$$

3. Transcendental Operators

We turn now to the connection between algebraic transformations on bounded degree polynomials and transcendental transformations on all degrees. Certainly any transcendental operator on $\mathbb{C}[z]$ may be restricted to $\mathbb{C}_n[z]$ for all $n \in \mathbb{N}$. The converse does not hold:

Proposition 3.1. *There exists an operator $T\colon \mathbb{C}_3[z] \to \mathbb{C}_3[z]$ which preserves weak Hurwitz stability and which cannot be extended to a transcendental operator that preserves weak Hurwitz stability.*

Proof. We prove the proposition by constructing a specific example. Define $T\colon \mathbb{C}_3[z] \to \mathbb{C}_3[z]$ by the sequence $\left\{\frac{4}{5}, 1, 1, \frac{1}{2}\right\}$. Then $T[(z+1)^3]$ has all real negative roots. By Theorem 1.5, T preserves weak Hurwitz stability. However, suppose $\left\{\frac{4}{5}, 1, 1, \frac{1}{2}, c_4\right\}$ is a 4-multiplier sequence for some $c_4 \geq 0$.

Let $\tilde{T}$ be the corresponding operator on $\mathbb{C}_4[z]$. As shown in [8], the sequence $\left\{\frac{4}{5}, 1, 1, \frac{1}{2}, c_4\right\}$ must be log-convex if $\tilde{T}[(z+1)^4]$ has all real zeros. So $\left(\frac{1}{2}\right)^2 \geq 1 \cdot c_4$. Hence $c_4 \in \left[0, \frac{1}{4}\right]$. Then for all $x \in \mathbb{R}$:

$$2x^3 + 6x^2 + 4x + \frac{4}{5} \leq \tilde{T}[(z+1)^4](x) \leq \frac{1}{4}x^4 + 2x^3 + 6x^2 + 4x + \frac{4}{5}.$$

A simple computation shows that $\tilde{T}[(z+1)^4]$ must have a local minimum at some $x \in (0,1)$, but that $\tilde{T}[(z+1)^4](x)$ is positive for all $x \in (0,1)$. This implies that $\tilde{T}[(z+1)^4]$ cannot have 4 real roots. So $\tilde{T}$ cannot preserve weak Hurwitz stability for any choice of c_4. □

To avoid this issue, the rest of this section will consider transcendental operators on $\mathbb{C}[z]$ or the restrictions of such operators to $\mathbb{C}_n[z]$. By applying Theorem 2.3 to the restrictions of a transcendental multiplicative operator T to $\mathbb{C}_n[z]$, we see that T preserves weak Hurwitz stability if and only if $e^{i\theta}T \colon \mathcal{P}^{\leq} \to \mathcal{P}^{\leq} \cup \{0\}$ for some $\theta \in [0, 2\pi)$. Let $\mathscr{T}$ be the set of such operators which are also nonzero, where operators differing by a constant factor are equivalent. The nonnegative infinite sequence (c_k) associated with T is a (nonnegative) "multiplier sequence." These sequences have been studied for over a century, and are completely characterized by a classical theorem due to Pólya and Schur [2, 9, 7]:

Theorem 3.2 (Pólya-Schur). *Let $T \colon \mathbb{R}[z] \to \mathbb{R}[z]$ be a linear operator determined by $T[z^k] = c_k z^k$ for some sequence (c_k) of nonnegative real numbers. Define the formal power series*

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!} z^k.$$

The following are equivalent:

(i) $T \colon \mathcal{P}^{\leq} \to \mathcal{P}^{\leq} \cup \{0\}$, *i.e.* (c_k) *is a nonnegative multiplier sequence.*

(ii) Φ *defines an entire function on* $\mathbb{C}$ *which is the limit, uniform on compact subsets of* $\mathbb{C}$, *of a sequence of polynomials in* $\mathcal{P}^{\leq}$.

(iii) Φ *is an entire function with the following Hadamard product decomposition:*

$$\Phi(z) = C z^m e^{az} \prod_{j=1}^{\infty} (1 + \alpha_j z), \tag{3.1}$$

where $C, a, \alpha_k \geq 0$ *for all* k, $m \in \mathbb{Z}_{\geq 0}$, *and* $\sum_{j=1}^{\infty} \alpha_j < \infty$.

(iv) $T[(z+1)^n] \in \mathcal{P}^{\leq} \cup \{0\}$ *for all* $n \in \mathbb{N}$.

The proof of Theorem 1.6 closely follows the proof in [7] of the equivalence between (i), (ii), and (iv) in Theorem 3.2. The proof also relies on a well-known continuity result due to Hurwitz, which we state for completeness [11].

Theorem 3.3 (Hurwitz). *Let (f_n) be a sequence of analytic functions on an open domain $A \subset \mathbb{C}$, such that (f_n) converges, uniformly on compact subsets of A, to a function f which is not identically zero. Then for all $z_0 \in A$, f is analytic at z_0. Furthermore, for any sufficiently small neighborhood U of z_0, f_n and f have the same number of zeros (with multiplicity) in U when n is sufficiently large. Within U, the set of zeros of f_n converges to the set of zeros of f.*

Proof of Theorem 1.6. Let $T \in \mathscr{T}$ be determined by $T[z^k] = c_k z^k$. Define Φ by (1.3). By Theorem 2.3, without loss of generality we may assume T satisfies statement (i) in Theorem 3.2.

First suppose Φ has no zeros, so by part (iii) in Theorem 3.2, $\Phi(z) = Ce^{az}$ for some $C > 0$, $a \geq 0$. Then $c_k = Ca^k$, so $T[z^k] = C(az)^k$. Hence $T[(z+1)^n] = C(az+1)^n$ for all n. $C(az+1)^n$ vanishes precisely at $z = -\frac{1}{a}$. So

$$\mathscr{R}\{T[(z+1)^n]\} = \mathscr{R}\{C(az+1)^n\} = \frac{1}{a}.$$

Note that this formula holds even in the degenerate case $a = 0$ where we take $\frac{1}{0} = +\infty$, for then $T[(z+1)^n] = 1$, which does not vanish.

Now suppose Φ has zeros. Define $\lambda := \mathscr{R}\{\Phi\}$ by (1.4). Let

$$q_n(z) := \left(\frac{z}{n} + 1\right)^n$$

for $n \in \mathbb{N}$. Then

$$T[q_n](z) = \sum_{k=0}^{n} \frac{1}{n^k} \binom{n}{k} c_k z^k = \sum_{k=0}^{n} \frac{c_k}{k!} z^k \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right).$$

As shown in Chapter VIII of [7], the collection $\{T[q_n]\}_{n\in\mathbb{N}}$ constitutes a normal family of analytic functions. Because the coefficients of $T[q_n]$ converge to those of Φ as $n \to \infty$, $T[q_n] \to \Phi$ uniformly on compact sets. Because $T \in \mathscr{T}$ is nonzero, Φ is not identically zero. We may therefore apply Hurwitz' theorem to conclude that a sequence of zeros of $T[q_n]$ converges to $-\lambda$. Also, $T[q_n](z) = T[(z+1)^n](\frac{z}{n})$, so $\mathscr{R}\{T[q_n]\} = n\mathscr{R}\{T[(z+1)^n]\}$. If $\lambda = 0$, $c_0 = 0$ and we are done. If $\lambda > 0$, take $\varepsilon \in (0, \lambda)$. The interval $I_\varepsilon := [-\lambda + \varepsilon, 0]$ is compact, so if $T[q_n]$ has a zero in I_ε for infinitely many values of n, these zeros have a limit point in I_ε. This sequence of zeros corresponds to a subsequence of $(T[q_n])$, which also converges uniformly on compact sets to Φ. Hence by Hurwitz's theorem, Φ would vanish at the limit point in I_ε, contradicting the definition of λ. So I_ε contains only finitely many zeros of the sequence $(T[q_n])$. That is, $T[q_n]$ does not vanish on T_ε for sufficiently large n. It follows that $\mathscr{R}\{T[q_n]\} \geq \lambda - \varepsilon$ for sufficiently large n. Since $T[q_n]$ has a sequence of zeros converging to $-\lambda$ and $\varepsilon > 0$ was arbitrary, we must have

$$\lim_{n\to\infty} \mathscr{R}\{T[q_n]\} = \lim_{n\to\infty} n\mathscr{R}\{T[(z+1)^n]\} = \lambda. \tag{3.2}$$

□

Because transcendental operators are defined by infinite sequences, they may be extended to act on formal power series. We are particularly interested in power series describing entire functions, which are the uniform limits on compact sets of sequences of polynomials. Let $\mathscr{E}$ denote the space of entire functions with the topology induced by uniform convergence on compact sets. Considering the form of Φ in Theorem 3.2, we define:

Definition 3.4. *For formal powers series $f = \sum_{k=0}^{\infty} a_k z^k$ and $g = \sum_{k=0}^{\infty} b_k z^k$, define the Hadamard composition by*

$$f \star g := \sum_{k=0}^{\infty} k! a_k b_k z^k. \tag{3.3}$$

Remark 3.5. *We must initially define the Hadamard composition on formal power series because the composition does not generally map entire functions to entire functions.*

Let $\overline{\mathcal{P}^{\leq}}$ and $\overline{\mathcal{HS}}$ denote the closures of $\mathcal{P}^{\leq}$ and $\mathcal{HS}$ in $\mathscr{E}$. Theorem 3.2 shows that a transcendental operator which preserves weak Hurwitz stability may be represented by Hadamard composition with an element of $\overline{\mathcal{P}^{\leq}}$. That is, we may identify $\mathscr{T}$ with $\overline{\mathcal{P}^{\leq}} - \{0\}$. To extend the action of $\mathscr{T}$ from polynomials to entire functions, we require the following regularity result:

Proposition 3.6. *If $f \in \mathscr{E}$ and $g \in \overline{\mathcal{P}^{\leq}}$, the formal power series $f \star g$ describes an entire function. Furthermore, the map $T_g \colon \mathscr{E} \to \mathscr{E}$ given by $T_g[f] := f \star g$ is continuous.*

Proof. Write $f = \sum_{k=0}^{\infty} a_k z^k$ and $g = \sum_{k=0}^{\infty} b_k z^k$. By (3.1) in Theorem 3.2 and Theorem 3 in Lecture 5 of Levin's *Lectures*, g is an entire function of order at most 1 with finite type [6]. Hence by a well-known formula for the type of an entire function [6],

$$\limsup_{k\to\infty} k \sqrt[k]{|b_k|} := L < \infty. \tag{3.4}$$

Because f is entire, the root test shows that $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 0$. By (3.3), (3.4), and Stirling's formula,

$$\lim_{k\to\infty} \sqrt[k]{|k! a_k b_k|} = \lim_{k\to\infty} k \sqrt[k]{|a_k b_k|} \leq L \lim_{k\to\infty} \sqrt[k]{|a_k|} = 0.$$

Hence $f \star g$ is entire. To demonstrate the continuity of T_g, we require the following lemma:

Lemma 3.7. *A sequence (f_n) of entire functions given by $f_n = \sum_{k=0}^{\infty} a_{n,k} z^k$ converges to zero uniformly on compact subsets of $\mathbb{C}$ if and only if*

$$\lim_{n\to\infty} \sup_{k\geq 0} \sqrt[k+1]{|a_{n,k}|} = 0.$$

Proof. First suppose $\lim_{n\to\infty} \sup_{k\geq 0} \sqrt[k+1]{|a_{n,k}|} = 0$. To prove uniform convergence on any compact set, it is sufficient to prove uniform convergence on the disc $D(R) := \{z \in \mathbb{C};\ |z| \leq R\}$ for all $R > 0$. Fix $R > 0$ and $\varepsilon > 0$, and set $\delta := \frac{1}{2} \min\{\varepsilon, R^{-1}\}$. There exists $N \in \mathbb{N}$ such that $\sup_k \sqrt[k+1]{|a_{n,k}|} < \delta$ for all $n \geq N$. For such n and for $z \in D(R)$,

$$|f_n(z)| \leq \sum_{k=0}^{\infty} |a_{n,k}|\, R^k < \sum_{k=0}^{\infty} \delta^{k+1} R^k \leq \delta \sum_{k=0}^{\infty} 2^{-k} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $f_n \to 0$ uniformly on $D(R)$, as desired. Now suppose instead that $f_n \to 0$ uniformly on every compact subset of $\mathbb{C}$. Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $|f_n(z)| < \varepsilon$ for all $z \in D(\varepsilon^{-1})$ and all $n \geq N$. For such n, the Cauchy integral formula shows that

$$|a_{n,k}| \leq \frac{1}{2\pi} \int_{\partial D(\varepsilon^{-1})} \frac{|f(z)|}{|z|^{k+1}} dz < \varepsilon^{k+1}$$

for all $k \geq 0$. Hence $\sup_k \sqrt[k+1]{|a_{n,k}|} < \varepsilon$ for all $n \geq N$. $\square$

(Proof of Proposition 3.6) Lemma 3.7 shows that $\mathscr{E}$ is metrizable as a topological space, for the metric $d(f_1, f_2) := \sup_k \sqrt[k+1]{|a_{1,k} - a_{2,k}|}$ induces the topology of uniform convergence on compact

sets. Therefore to establish the continuity of the map T_g it is sufficient to verify sequential continuity at 0. Suppose a sequence (f_n) of entire functions satisfies $f_n \to 0$ in $\mathscr{E}$. By Lemma 3.7, $\lim_{n\to\infty} \sup_k \sqrt[k+1]{|a_{n,k}|} = 0$. Hence by (3.4) and Stirling's formula,

$$\lim_{n\to\infty} \sup_k \sqrt[k+1]{|k!a_{n,k}b_k|} \leq \left(\sup_k \sqrt[k+1]{k!b_k}\right)\left(\lim_{n\to\infty} \sup_k \sqrt[k+1]{|a_{n,k}|}\right) = 0.$$

Therefore by the lemma we have $T_g[f_n] = f_n \star g \to 0$ in $\mathscr{E}$. $\square$

This continuity result allows us to extend Corollary 2.9 to entire functions:

Corollary 3.8. *The Hadamard composition $\star$ defines bilinear maps*

$$\overline{\mathcal{P}^{\leq}} \times \overline{\mathcal{P}^{\leq}} \to \overline{\mathcal{P}^{\leq}} \quad \textit{and} \quad \overline{\mathcal{HS}} \times \overline{\mathcal{P}^{\leq}} \to \overline{\mathcal{HS}}.$$

Furthermore, the first map is continuous in both arguments, while the second map is continuous only in the first argument.

Remark 3.9. *Because* 0 *is contained in the closure of $\mathcal{P}^{\leq}$ in $\mathscr{E}$, we need not explicitly include it in the codomains of these maps.*

Remark 3.10. *Corollary 3.8 may be deduced from a sequence of more general results due to Borcea and Brändén, but we felt a more direct proof for multiplicative operators was desirable* [3, 1, 4].

Proof. Fix $g \in \overline{\mathcal{P}^{\leq}}$. For $f \in \overline{\mathcal{P}^{\leq}}$ there exists a sequence (p_n) of polynomials in $\mathcal{P}^{\leq}$ such that $p_n \to f$ in $\mathscr{E}$. By Theorem 3.2, $p_n \star g \in \mathcal{P}^{\leq} \cup \{0\}$, and Proposition 3.6 ensures that $p_n \star g \to f \star g$. Hence $f \star g \in \overline{\mathcal{P}^{\leq}}$. The proof for $f \in \overline{\mathcal{HS}}$ is analogous. Proposition 3.6 proves the claimed continuity for both maps. To see that the map $\overline{\mathcal{HS}} \times \overline{\mathcal{P}^{\leq}} \to \overline{\mathcal{HS}}$ is discontinuous in its second argument, consider the function $f = e^{z^2}$ and the sequence $g_n = e^{n(z-n)}$. Theorem 3 in Ch. 8 of Levin shows that $f \in \overline{\mathcal{HS}}$, while Theorem 3.2 shows that $g_n \in \overline{\mathcal{P}^{\leq}}$ [7]. It is easy to check that $g_n \to 0$ in $\overline{\mathcal{P}^{\leq}}$ but that $f \star g_n \not\to 0$ in $\overline{\mathcal{HS}}$. $\square$

Having established the essential qualitative behavior of transcendental operators on entire functions, we consider now quantitative control of zeros. Unfortunately, Theorem 1.6 shows that no inequality of the form found in Theorem 1.5 can hold for transformations in $\mathscr{T}$. Controlling only the minimum distance between the zeros of a function in $\overline{\mathcal{HS}}$ and the imaginary axis is not sufficient to bound the locus of zero of the transformed entire function, because there is no bound on the *number* of zeros which may accumulate a fixed distance from the axis. This fact enables the zeros of $T[(z+1)^n]$ to approach the imaginary axis as n grows large. Therefore more strict quantitative control is necessary to bound the zeros of transformed entire functions. We have formulated a result for $\overline{\mathcal{P}^{\leq}}$, but the problem for the larger class $\overline{\mathcal{HS}}$ remains open. To concisely present the proposition for $\overline{\mathcal{P}^{\leq}}$, we use the following notation:

Definition 3.11. *Suppose $f = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathcal{P}^{\leq}}$ has zero set $Z = \{z \in \mathbb{C};\ f(z) = 0\}$ and that $0 \notin Z$. Then define*

$$\sigma(f) := \frac{1}{e} \limsup_{k\to\infty} k \sqrt[k]{|a_k|} \tag{3.5}$$

and

$$\ell(f) := -\sum_{\zeta\in Z}\zeta^{-1},$$

where the zeros in Z are counted with multiplicity in the above sum. Take $\ell(f)=0$ if Z is empty.

These values are related to the product expansion in (3.1): $\sigma(f)$ is the constant factor in the exponential and $\ell(f)$ is the (negative) ℓ^1 sum of the reciprocals of the zeros of f counted with multiplicity [6]. This sum is guaranteed to converge by Theorem 3.2. Note also that the zeros of f are all negative real numbers, so $\ell(f)$ is always a nonnegative real number.

Proposition 3.12. *Suppose $f, g \in \overline{\mathcal{P}^{\leq}}$ do not vanish at the origin. Then*

$$\ell(f\star g) = \sigma(f)\ell(g) + \sigma(g)\ell(f) + \ell(f)\ell(g).$$

Remark 3.13. *To extend this result to the entirety of $\overline{\mathcal{P}^{\leq}}$, we note that if f or g vanishes at 0, so does $f\star g$, and if either are identically zero, so is $f\star g$.*

Proof. Write $f=\sum_{k=0}^{\infty}a_kz^k$ and $g=\sum_{k=0}^{\infty}b_kz^k$. Because f and g do not vanish at the origin, we may assume without loss of generality that $f(0)=g(0)=1$. By simple absolute convergence results,

$$f(z) = e^{\sigma(f)z}\prod_{\zeta\in Z}\left(1-\frac{z}{\zeta}\right) = (1+\sigma(f)z+\ldots)\left(1-z\sum_{\zeta\in Z}\frac{1}{\zeta}+\ldots\right) = 1+[\sigma(f)+\ell(f)]z+\ldots$$

where the ellipses denote terms of order higher than one and the sum and product account for root multiplicity. In particular, we see that $a_1=\sigma(f)+\ell(f)$. Analogously, $b_1=\sigma(g)+\ell(g)$. Now the first Taylor coefficient of $f\star g$ is a_1b_1, so we have

$$\sigma(f\star g)+\ell(f\star g) = a_1b_1 = [\sigma(f)+\ell(f)][\sigma(g)+\ell(g)]. \tag{3.6}$$

It only remains to determine $\sigma(f\star g)$. Using (3.5) and Stirling's formula, we have

$$\sigma(f\star g) = \frac{1}{e}\limsup_{k\to\infty} k\sqrt[k]{|k!a_kb_k|} = \frac{1}{e^2}\limsup_{k\to\infty} k^2\sqrt[k]{|a_kb_k|} = \sigma(f)\sigma(g). \tag{3.7}$$

Combining (3.6) and (3.7), we see that

$$\ell(f\star g) = \sigma(f)\ell(g) + \sigma(g)\ell(f) + \ell(f)\ell(g).$$

□

Because the roots of nonzero elements of $\overline{\mathcal{P}^{\leq}}$ are nonpositive real numbers, the value of $\ell(f\star g)$ in Proposition 3.12 may be used to quantitatively bound the roots of $f\star g$ away from zero.

4. Real Operators on $\mathcal{P}^{\leq}$

As Theorem 2.3 shows, multiplicative operators which preserve weak Hurwitz stability are (up to a constant factor) precisely those which preserve nonpositive real-rootedness. In this section we take advantage of the well-developed theory of real-rooted polynomials by restricting attention to operators on $\mathcal{P}^{\leq}$. Many of the stronger results available for $\mathcal{P}^{\leq}$ are due to the properties of interlacing polynomials:

Definition 4.1. *Let f and g be degree n real-rooted polynomials. Then f and g* interlace *if the ordered roots $\alpha_1 \leq \ldots \leq a_n$ and $\beta_1 \leq \ldots \leq \beta_n$ of f and g respectively satisfy*

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_n \leq \beta_n$$

or

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \beta_n \leq \alpha_n.$$

In the former case we write $f \lessdot g$, in the latter $g \lessdot f$. If we take $f, g \in \mathcal{P}^{\leq}$, we may extend the definition of interlacing when $\deg(f) \neq \deg(g)$, by considering the polynomial of lower degree to have roots at $-\infty$.

Interlacing polynomials have a simple characterization found in [5] in terms of linear combination:

Proposition 4.2. *Let f and g be polynomials of the same degree with all real roots. Then f and g interlace if and only if $\lambda f + \mu g$ has all real roots for all $\lambda, \mu \in \mathbb{R}$.*

This characterization leads to the following important result in [5]. We have modified the hypotheses of the theorem, so we provide proof.

Theorem 4.3. *Let $T\colon \mathbb{R}_n[z] \to \mathbb{R}_n[z]$ be a multiplicative operator such that $T\colon \mathcal{P}_n^{\leq} \to \mathcal{P}_n^{\leq} \cup \{0\}$. Then for $f, g \in \mathcal{P}_n^{\leq}$ of equal degree, $f \lessdot g$ implies $T[f] \lessdot T[g]$ or either $T[f]$ or $T[g]$ is identically zero.*

Proof. Let T, f, and g satisfy the hypotheses of the theorem. Suppose neither $T[f]$ nor $T[g]$ are identically zero. By Theorem 4.2, $\lambda f + \mu g$ has all real roots for all $\lambda, \mu \in \mathbb{R}$. By Theorem 1.5 and Lemma 2.1, T preserves $\mathbb{H}_0$-stability. Since $T[\lambda f + \mu g]$ is real and $\mathbb{H}_0$-stable, it has all real roots. So $\lambda T[f] + \mu T[g]$ has all real roots for all $\lambda, \mu \in \mathbb{R}$. Since T is multiplicative, $T[f]$ and $T[g]$ have the same degree. By Theorem 4.2 again, $T[f]$ and $T[g]$ interlace. Since the coefficients c_k are nonnegative, the proof that $T[f] \lessdot T[g]$ (rather than $T[g] \lessdot T[f]$) is identical to that in [5]. $\square$

The preservation of interlacing is a specific case of the monotonicity of T on a certain partial order of the sets of roots.

Definition 4.4. *Let f and g be degree n real-rooted polynomials. Then we write $f \blacktriangleleft g$ if the ordered roots $\alpha_1 \leq \ldots \leq a_n$ and $\beta_1 \leq \ldots \leq \beta_n$ of f and g respectively satisfy $\alpha_j \leq \beta_j$ for all j. If we take $f, g \in \mathcal{P}^{\leq}$, we may extend the definition of $\blacktriangleleft$ when $\deg(f) \neq \deg(g)$, by considering the polynomial of lower degree to have roots at $-\infty$.*

Theorem 4.5. *Let $T\colon \mathbb{R}_n[z] \to \mathbb{R}_n[z]$ be a multiplicative operator such that $T\colon \mathcal{P}_n^{\leq} \to \mathcal{P}_n^{\leq} \cup \{0\}$. Then for any $f, g \in \mathcal{P}_n^{\leq}$, $f \blacktriangleleft g$ implies $T[f] \blacktriangleleft T[g]$, or either $T[f]$ or $T[g]$ is identically zero.*

Proof. Assume $T[f]$ and $T[g]$ are nonzero. Because $f \blacktriangleleft g$, $\deg(f) \leq \deg(g)$. We may assume $\deg(f) = \deg(g) = n$. The result for polynomials of lower degree will then follow by taking limits as some roots move to $-\infty$, and using the continuity of T on the roots in the compactified space

$[-\infty, 0]^n/\mathcal{S}_n$ as in Corollary 2.8. Let f and g have roots $\alpha_1 \le \ldots \le a_n$ and $\beta_1 \le \ldots \le \beta_n$ respectively. For $1 \le k \le n-1$, let

$$h_k = (x-\alpha_1)\ldots(x-\alpha_{n-k})(x-\beta_{n-k+1})\ldots(x-\beta_n)$$

and $h_0 = f$, $h_n = g$. By the ordering on the roots, $h_k \leqslant h_{k+1}$ for all $0 \le k \le n-1$. By Theorem 4.3, $T[h_k] \leqslant T[h_{k+1}]$ and hence $T[h_k] \blacktriangleleft T[h_{k+1}]$. It is clear that $\blacktriangleleft$ is transitive, so $f = h_0 \blacktriangleleft h_n = g$. $\square$

Remark 4.6. *If $f \in \mathcal{P}_n^{\le}$, the bound (1.2) in Theorem 1.5 follows easily from the monotonicity in Theorem 4.5 and the scaling property described in Section 2.*

Theorem 4.5 controls the behavior of polynomials in $\mathcal{P}_n^{\le}$ under a fixed transformation T. However, the symmetry in Corollary 2.9 between the polynomial and the transformation allows us to apply the ordering $\blacktriangleleft$ to operators in $\mathscr{T}_n$. As in Section 2, we identify an operator $T \in \mathscr{T}_n$ with $g_T = T[(z+1)^n] \in \mathcal{P}_n^{\le}$ by $T[f] = f *' g_T$. We obtain the following corollary:

Corollary 4.7. *Let $T, S \in \mathscr{T}_n$ satisfy $g_T \blacktriangleleft g_S$. Then for all $f \in \mathcal{P}_n^{\le}$,*

$$T[f] \blacktriangleleft S[f]$$

or either $T[f]$ or $S[f]$ is identically zero.

This corollary lets us compare the action of different operators on some fixed polynomial. In particular, Corollary 4.7 can be applied to study $T[(z+1)^n]$.

Proposition 4.8. *Let $T \in \mathscr{T}_n$. Then either $T[(z+1)^{m_0}] = 0$ for some $1 \le m_0 \le n$ and $\mathscr{R}\{T[(z+1)^m]\} = 0$ whenever it is defined, or the sequence (λ_m) for $1 \le m \le n$ given by*

$$\lambda_m = m \cdot \mathscr{R}\{T[(z+1)^m]\}$$

is nondecreasing.

Proof. If $T[(z+1)^{m_0}] = 0$ for any m_0, $c_k = 0$ for all $k \le m_0$, and $\mathscr{R}\{T[(z+1)^m]\} = 0$ whenever it is defined. Suppose $T[(z+1)^m] \ne 0$ for all m. As above let $g_T = T[(z+1)^n]$. Define $S \in \mathscr{T}_n$ by $g_S(z) = z + \mathscr{R}\{g_T\}$, so $S[1] = \mathscr{R}\{g_T\}$, $S[z] = \frac{z}{n}$, and $S[z^k] = 0$ for all $k \ge 2$. Then $g_S \blacktriangleleft g_T$. By Corollary 4.7, $S[(z+1)^{n-1}] \blacktriangleleft T[(z+1)^{n-1}]$. Hence

$$\mathscr{R}\left\{S[(z+1)^{n-1}]\right\} = \frac{n}{n-1}\mathscr{R}\{g_T\} \ge \mathscr{R}\left\{T[(z+1)^{n-1}]\right\}.$$

Rearranging, we have $n\mathscr{R}\{T[(z+1)^n]\} \ge (n-1)\mathscr{R}\left\{T[(z+1)^{n-1}]\right\}$. By restricting the domain of T to $\mathbb{C}_m[z]$ for $2 \le m \le n$, we see that

$$m\mathscr{R}\{T[(z+1)^m]\} \ge (m-1)\mathscr{R}\left\{T[(z+1)^{m-1}]\right\}$$

for all $2 \le m \le n$. $\square$

Remark 4.9. *A transcendental operator $T \in \mathscr{T}$ may be restricted to $\mathbb{C}_n[z]$ to apply Proposition 4.8. Hence the sequence in (3.2) given by*

$$\lambda_m = \mathscr{R}\{T[q_m]\} = m \cdot \mathscr{R}\{T[(z+1)^m]\}$$

for all $m \in \mathbb{N}$ is nondecreasing. So $m \cdot \mathscr{R}\{T[(z+1)^m]\} \nearrow \mathscr{R}\{\Phi_T\}$ as $m \to \infty$.

5. Conjectured Bounds for $\mathcal{HS}$ and $\mathcal{P}^{\leq}$

Theorem 1.5 is the first quantitative result concerning weakly Hurwitz stable polynomials under multiplicative transformations. There are several directions in which the theorem could conceivably be strengthened. In particular, Theorem 1.5 considers only the effect of the rightmost zero of f on $\mathscr{R}\{T[f]\}$. In doing so, it essentially bounds f using the polynomial with an n-repeated root at $-\mathscr{R}\{f\}$. Theorem 1.5 would be strongly improved if it incorporated information about all the roots of f, rather than just the rightmost root. There are well-developed techniques to yield such bounds in the case that f is real-rooted. Therefore it is critical to connect the complex theory for $\mathcal{HS}$ with the real theory for $\mathcal{P}^{\leq}$. In this vein, we suggest the following:

Conjecture 1. *Let $f \in \mathcal{HS}_n$ have zeros $\zeta_1, \ldots, \zeta_n$. Let $f_{\mathrm{Re}} \in \mathcal{P}_n^{\leq}$ be the polynomial with zeros $\mathrm{Re}(\zeta_1), \ldots, \mathrm{Re}(\zeta_n)$. Then if $T \in \mathscr{T}_n$,*

$$\mathscr{R}\{T[f]\} \geq \mathscr{R}\{T[f_{\mathrm{Re}}]\}.$$

This conjecture is supported by numerical evidence. We simulated $\sim 10^6$ random polynomials in $\mathcal{HS}_n$ under varying transformations in $\mathscr{T}_n$ for several values of n, and observed no violation of Conjecture 1. The polynomial zeros were randomly generated using a gamma distribution for the real part and a normal distribution for the imaginary part. The parameters of these distributions were varied in different trials. One sample trial is shown in Figure 1.

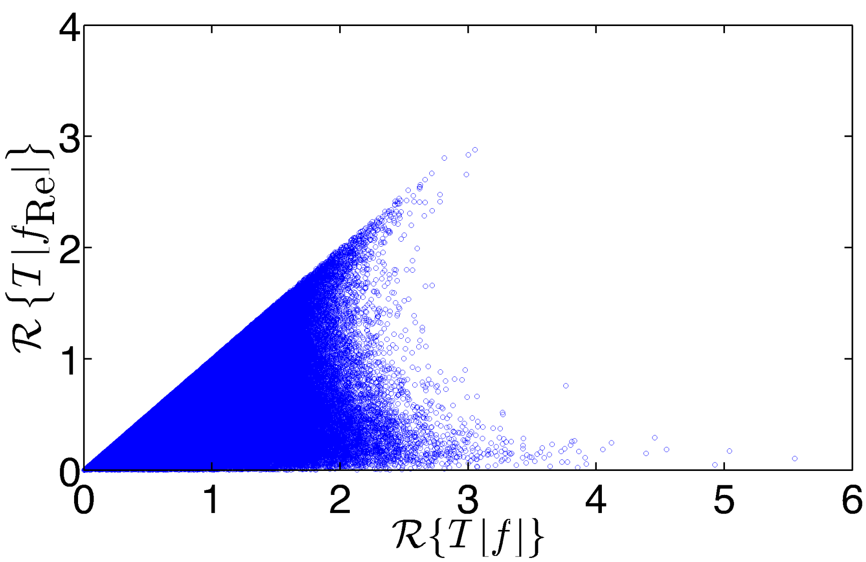

Figure 1. This trial used 400,000 degree 5 polynomials with the multiplier sequence $c_k = \frac{1}{(2k)!}$ from [5]. The maximal ratio of $\mathscr{R}\{T[f_{\mathrm{Re}}]\}$ to $\mathscr{R}\{T[f]\}$ was 0.9999.

If Conjecture 1 holds, we may control $\mathscr{R}\{T[f]\}$ in terms of bounds on $\mathscr{R}\{T[f_{\mathrm{Re}}]\}$ yielded by the theory of real-rooted polynomials. We therefore turn to the real-rooted case, and consider the action of multiplier sequences on $\mathcal{P}^{\leq}$.

We seek a measure on the roots of $f \in \mathcal{P}^{\leq}$ which will determine in some sense the "distance" between the set of roots of f and an n-repeated root at $-\mathscr{R}\{f\}$. Based on numerical evidence, a good candidate seems to be a mean-distance between the inverses of the roots and $-\frac{1}{\mathscr{R}\{f\}}$. Note that if $\mathscr{R}\{f\} = 0$ we must have $\mathscr{R}\{T[f]\} = 0$, so any further bound would be trivial. Hence we consider only f such that $\mathscr{R}\{f\} > 0$.

Definition 5.1. *Let $f \in \mathcal{P}_n^{\leq}$ with zeros $\zeta_j < 0$ for all j. Let $T \in \mathscr{T}_n$. We define*

$$\mathcal{M}\{f\} \coloneqq \frac{1}{n-1} \sum_{j=1}^{n} \left(1 + \frac{\mathscr{R}\{f\}}{\zeta_j}\right),$$

which measures the deviation of the roots of f from an n-multiple root at $-\mathscr{R}\{f\}$. We define

$$\mathcal{N}_T\{f\} \coloneqq \frac{n}{n-1} \left(1 - \frac{\mathscr{R}\{T[(z+1)^n]\}\, \mathscr{R}\{f\}}{\mathscr{R}\{T[f]\}}\right),$$

which measures the deviation of the rightmost root of $T[f]$ from the upper bound given in (1.2).

The definition of $\mathscr{R}$ implies that $\mathcal{M}\{f\} \in [0,1]$. Theorem 1.5 shows that $\mathcal{N}_T\{f\} \geq 0$. Theorem 4.5 and Corollary 4.7 may be combined to show that in fact $\mathcal{N}_T\{f\} \in [0,1]$. Based on numerical evidence, we suggest:

Conjecture 2. *Let $f \in \mathcal{P}_n^{\leq}$ with $\mathscr{R}\{f\} > 0$, and let $T \in \mathscr{T}_n$. Then*

$$\frac{\mathcal{N}_T\{(z+1)^{n-1}\}}{\mathcal{M}\{(z+1)^{n-1}\}} \leq \frac{\mathcal{N}_T\{f\}}{\mathcal{M}\{f\}} \leq 1.$$

This too was tested using $\sim 10^6$ of random polynomials with various degrees, distributions, and transformations. Sample trials are shown in Figures 2 and 3. Each figure shows a clearly defined admissible region. The upper boundary appears to correspond to the polynomials $f_\xi \coloneqq (z+1)(\xi z+1)^{n-1}$ for $\xi \in [0,1]$. The lower boundary has a distinctive cusped shape, with n evenly spaced cusps (including the endpoints) corresponding to the polynomials $f_k = (z+1)^k$ for $1 \leq k \leq n$. In between the cusps at f_{k+1} and f_k, the lower boundary seems to correspond to the polynomials $f_k^\xi \coloneqq (z+1)^k(\xi z+1)$ for $\xi \in [0,1]$. In summary, we believe the upper boundary is formed by moving all but one of the zeros of $(z+1)^n$ together away to $-\infty$, while the lower boundary is formed by moving one zero of $(z+1)^n$ at a time away to $-\infty$. The lower bound in Conjecture 2 corresponds to the ratio of $\mathcal{N}_T$ to $\mathcal{M}$ at the first cusp, i.e. at $(z+1)^{n-1}$. The upper bound corresponds to the slope of the upper boundary curve at $\mathcal{M} = 0$, i.e. near $(z+1)^n$.

The two bounds conjectured in this section would combine to yield far more powerful control on $\mathscr{R}\{T[f]\}$ in terms of the zeros of $f \in \mathcal{HS}$. Given upper bounds on the real parts of the roots of $f \in \mathcal{HS}$, Conjecture 1 would allow us to convert the complex roots of f into the real roots of f_{Re}. Conjecture 2 would then bound $\mathscr{R}\{T[f]\}$ in terms of the bounds on the original zeros of f. Control of this form on the entire class of Hurwitz stable polynomials would be immediately valuable in the other branches of mathematics entwined with the theory of stable polynomials.

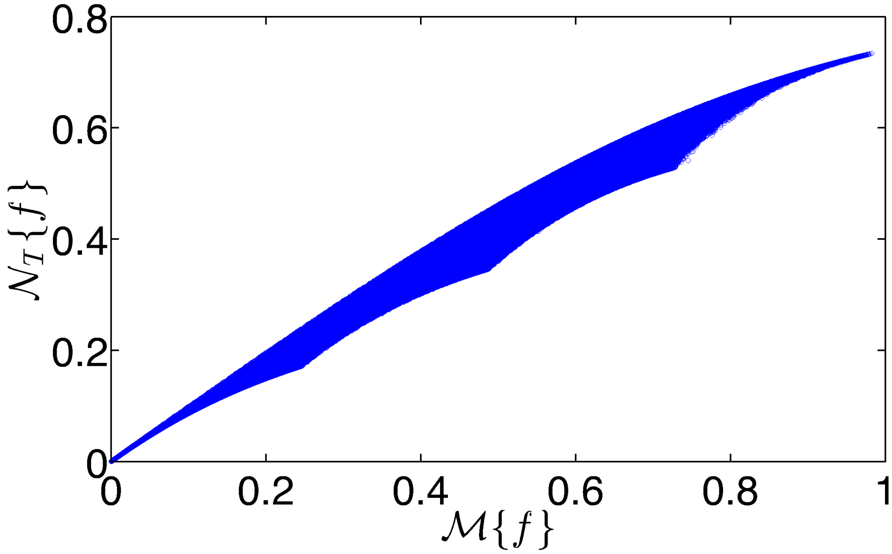

FIGURE 2. This trial used 400,000 degree 5 polynomials with the multiplier sequence $\{12, 9.8, 6.75, 4, 2.1, 1\}$. The maximal ratio of $\mathcal{N}_T\{f\}$ to $\mathcal{M}\{f\}$ was 1 to within algorithmic error. The minimal ratio was 0.6% above the conjectured minimum limit.

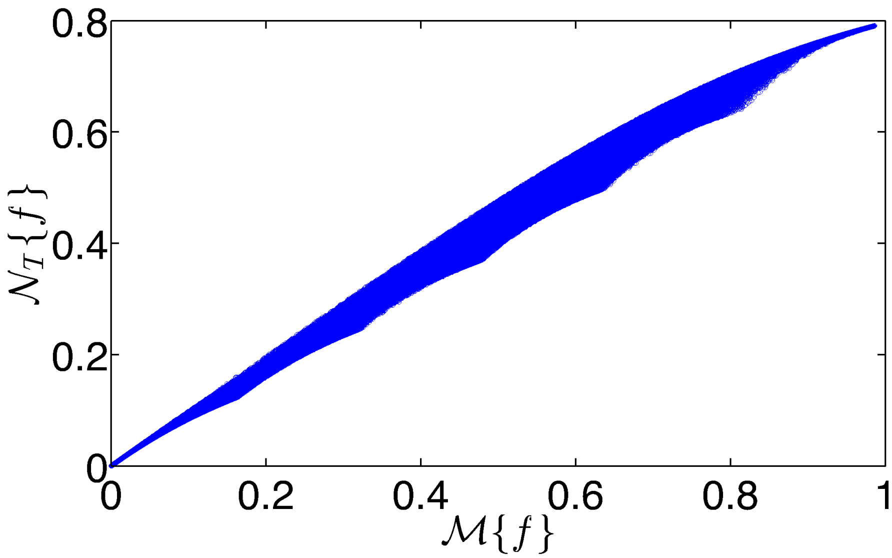

FIGURE 3. This trial used 400,000 degree 7 polynomials with the multiplier sequence $\{360, 228.86, 122.67, 56.9, 23.5, 8.83, 3.07, 1\}$. The maximal ratio of $\mathcal{N}_T\{f\}$ to $\mathcal{M}\{f\}$ was 1 to within algorithmic error. The minimal ratio was 0.4% above the conjectured minimum limit.

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