CONVERGENCE OF EIGENVALUES TO THE SUPPORT OF THE LIMITING MEASURE IN CRITICAL β MATRIX MODELS

C. FAN [‡], A. GUIONNET [†], Y. SONG [§], AND A. WANG[♯]

Abstract. We consider the convergence of the eigenvalues to the support of the equilibrium measure in the β ensemble model under a critical condition. We show a phase transition phenomenon, namely that all eigenvalues will fall in the support of the limiting spectral measure when $\beta > 1$, whereas this always fails when $\beta < 1$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

1.1. Definitions and Known Results.

Let $\boldsymbol{B}$ be a subset of the real line. $\boldsymbol{B}$ can be chosen as the whole real line, an interval, or the union of finitely many disjoint intervals. For now, let $V : \boldsymbol{B} \to \mathbb{R}$ be an arbitrary function, and let $\beta > 0$ be a positive real number. In this paper, we consider the β ensemble, i.e a sequence of N random variables $(\lambda_1, \ldots, \lambda_N)$ with law $\mu_{N,\beta}^{V;\boldsymbol{B}}$ on $\mathbb{R}^N$.
The β matrix model with potential V on the set $\boldsymbol{B}$ is defined as the probability measure on $\mathbb{R}^N$ given by

$$d\mu_{N,\beta}^{V;\boldsymbol{B}}(\lambda) = \frac{1}{Z_{N,\beta}^{V;\boldsymbol{B}}} \prod_{i=1}^{N} d\lambda_i e^{-\frac{N\beta}{2}V(\lambda_i)} \mathbf{1}_{\boldsymbol{B}}(\lambda_i) \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta , \tag{1.1}$$

where $Z_{N,\beta}^{V;\boldsymbol{B}}$ is the partition function

$$Z_{N,\beta}^{V;\boldsymbol{B}} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i e^{-\frac{N\beta}{2}V(\lambda_i)} \mathbf{1}_{\boldsymbol{B}}(\lambda_i) \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta . \tag{1.2}$$

If β is equal to 1, 2, or 4, $\mu_{N,\beta}^{V;\mathbb{R}}$ is the probability measure induced on the eigenvalues of Ω by the probability measure $d\Omega e^{-\frac{N\beta}{2}\mathrm{Tr}(V(\Omega))}$ on a vector space of real symmetric, Hermitian, and self-dual quaternionic $N \times N$ matrices respectively, see[Meh04].
Therefore, the β ensemble can be viewed as the natural generalization of these matrix models and we will refer to λ_i as "eigenvalues" of a "matrix model". For a

[‡]Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307 USA. email: cjfan@math.mit.edu.
[†] Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307 USA. email: guionnet@math.mit.edu.
[§]Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307 USA. email: yuqisong@.mit.edu.
[♯]Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139-4307 USA. email: wandi@.mit.edu.

quadratic potential, the β ensemble can also be realized as the eigenvalues of tridiagonal matrices [DE02]. Eventhough such a construction is not known for general potentials, β matrix models are natural Coulomb interaction probability measures which appear in many different settings. These laws have been intensively studied, both in physics and in mathematics. In particular, the convergence of the empirical measure of the λ_i's was proved [ST97, Dei99, AGZ10], and its fluctuations analyzed [Joh98, Pas06, Shc13]. Moreover, the partition functions as well as the mean Stieltjes transforms can be expanded as a function of the dimension to all orders [BIPZ78, ACKM93, ACM92, Ake96, CE06, Che06, BG11, BG13]. It turns out that both central limit theorems and all order expansions depend heavily on whether the limiting spectral measure has a connected support. Indeed, when the limiting spectral measure has a disconnected support, it turns out that even though most eigenvalues will stick into one of its connected components, some eigenvalues will randomly switch from one to the other connected components of the support even at the large dimension limit. This phenomenon can invalidate the central limit theorem in [Pas06, Shc13] and result with the presence of a Theta function in the large dimension expansion of the partition function. In the case where the limiting measure has a connected support S, and that the eigenvalues are assumed to belong asymptotically to S, even more refined information could be derived. Indeed, in this case, local fluctuations of the λ_i's could first be established in the case corresponding to Gaussian random matrices, $\beta = 1, 2$ or 4 and $V(x) = x^2$ [Meh04], then to tridiagonal ensembles (all $\beta \geq 0$ but $V(x) = x^2$) [RRV06] and more recently for general potentials and $\beta \geq 0$ [BEY12, BEY, Shc13, FB]. However, local fluctuations have not yet been studied in the case where the limiting measure has a disconnected support nor when it is critical. We shall below define more precisely the later case but let us say already that a non-critical potential prevents the eigenvalues to deviate from the support of the limiting spectral measure as the dimension goes to infinity. In fact, the later property is one of the most fundamental question when one wants to study the local fluctuations of the eigenvalues. We study in this article β models with critical potentials and whether the eigenvalues stay confined in the limiting support. In fact, we exhibit an interesting phase transition; we show that if $\beta > 1$ the eigenvalues stay confined whereas if $\beta < 1$ some deviate towards the critical point with probability one. We postpone the study of the critical case $\beta = 1$ to further research. Let us finally point out that the case where the potential is critical, but with critical parameters tuned with the dimension so that new phenomena occur, was studied in [Cla, Eyn06]. We restrict ourselves to potentials independent of the dimension.

We next describe more precisely some of the results stated above, the definition of criticality and our results.

Consider the *spectral measure* $L_N := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$, where δ_{λ_i} is the Dirac measure centered on λ_i. L_N belongs to the set $M_1(\boldsymbol{B})$ of probability measures on the real line. We endow this space with the weak topology. Then, L_N converges almost surely. This convergence can be derived from the following large deviation result (see [BAG97], and [AGZ10, Theorem 2.6.1]) :

Theorem 1.1. *Assume that V is continuous and goes to infinity faster than $2\log|x|$ (if $\boldsymbol{B}$ is not bounded). The law of L_N under $\mu_{N,\beta}^{V;\boldsymbol{B}}$ satisfies a large deviation principle with speed N^2 and good rate function $\mathcal{E}$, where $\tilde{\mathcal{E}} = \mathcal{E} - \inf\{\mathcal{E}(\mu), \mu \in M_1(\boldsymbol{B})\}$*

with

$$\mathcal{E}[\mu] = \frac{\beta}{2} \iint (V(\xi) + V(\eta) - \log|\xi - \eta|)\, d\mu(\xi) d\mu(\eta)\,. \tag{1.3}$$

In other words,

(1) $\tilde{\mathcal{E}} : M_1(\mathbb{R}) \to [0,\infty]$ *possesses compact level sets* $\{v : \tilde{\mathcal{E}}(v) \le M\}$ *for all* $M \in \mathbb{R}^+$.

(2) *for any open set* $O \subset M_1(\boldsymbol{B})$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mu_{N,\beta}^{V;\boldsymbol{B}} (L_N \in O) \ge -\inf_O \tilde{\mathcal{E}}.$$

(3) *for any closed set* $F \subset M_1(\boldsymbol{B})$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mu_{N,\beta}^{V;\boldsymbol{B}} (L_N \in F) \le -\inf_F \tilde{\mathcal{E}}.$$

The minimizers of $\mathcal{E}$ are described as follows (see [AGZ10, Lemma 2.6.2]):

Theorem 1.2. *$\mathcal{E}$ achieves its minimal value at a unique minimizer $\mu_{\rm eq}$. Moreover, $\mu_{\rm eq}$ has a compact support* $\mathbf{S}$. *In addition, there exists a constant $C_{\beta,V}$ such that:*

$$\begin{cases} \text{for } x \in \mathbf{S} & 2\int_{\mathbb{R}} d\mu_{\rm eq}(\xi) \ln|x-\xi| - V(x) = C_{\beta,V} \\ \text{for } x \text{ almost everywhere in } \mathbf{S}^c & 2\int_{\mathbb{R}} d\mu_{\rm eq}(\xi) \ln|x-\xi| - V(x) < C_{\beta,V}. \end{cases} \tag{1.4}$$

Here the almost everywhere means almost everywhere with respect to Lebesgue measure.

We will refer to $\mu_{\rm eq}$, which is compactly supported, as the equilibrium measure.

Remark 1.3. *Theorem 1.1 and Theorem 1.2 imply that under $\mu_{N,\beta}^{V;\boldsymbol{B}}$, L_N converges to (in weak topology) to the equilibrium measure $\mu_{\rm eq}$ almost surely.*

Once the existence of the equilibrium measure is established, one may explore the convergence of the eigenvalues to the support of the equilibrium measure μ_{eq}. It is shown in [BG11, BG13](also see Theorem 1.4 below) that the probability that eigenvalues escape this limiting support is governed by a large deviation principle with rate function given by

$$\tilde{\mathcal{J}}^{V;\boldsymbol{B}}(x) = \mathcal{J}^{V;\boldsymbol{B}}(x) - C_{\beta,V} \tag{1.5}$$

with

$$\mathcal{J}^{V;\boldsymbol{B}}(x) = \begin{cases} \beta\frac{V(x)}{2} - \beta\int d\mu_{\rm eq}(\xi)\ln|x-\xi| & x \in \boldsymbol{B}\backslash\mathbf{S} \\ C_{\beta,V} & \text{otherwise.} \end{cases} \tag{1.6}$$

The large deviation states as follows:

Theorem 1.4. *Assume V continuous and going to infinity faster than $2\log|x|$ (in the case where B is infinite).*

(1) $\tilde{\mathcal{J}}^{V;\boldsymbol{B}}_{\max}$ *is a good rate function .*

(2) *We have large deviation estimates: for any* $\mathsf{F} \subseteq \overline{\boldsymbol{B}\backslash\mathbf{S}}$ *closed and* $\mathsf{O} \subseteq \boldsymbol{B}\backslash\mathbf{S}$ *open,*

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;\mathsf{B}} [\exists i \quad \lambda_i \in \mathsf{F}] \quad \le \quad -\frac{\beta}{2} \inf_{x\in\mathsf{F}} \tilde{\mathcal{J}}^{V;\boldsymbol{B}}(x),$$

$$\liminf_{N\to\infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;\mathsf{B}} [\exists i \quad \lambda_i \in \mathsf{O}] \quad \ge \quad -\frac{\beta}{2} \inf_{x\in\mathsf{O}} \tilde{\mathcal{J}}^{V;\boldsymbol{B}}(x).$$

The last theorem shows that the support of the spectrum is governed by the minimizers of $\tilde{\mathcal{J}}^{V;\boldsymbol{B}}$.

Definition 1.5. *Assume V is continuous. We say that V is non-critical iff $\mathcal{J}^{V;\boldsymbol{B}}$ is positive everywhere outside of the support of μ_{eq}.*

A consequence of the second part of the aforementioned theorem is the following:

Corollary 1.6. *Let the assumptions in Theorem 1.4 hold. Assume that V is non-critical. Then*

$$\lim_{N\to\infty} \mu_{N,\beta}^{V;\boldsymbol{B}}\left(\exists \lambda_i \notin \boldsymbol{A}\right) = 0\,, \tag{1.7}$$

where $\boldsymbol{A} := \cup_{h=1}^{g}[a_h^-, a_h^+]$ with $\min\{|x-y|, x \in \mathbf{A}^c, y \in \mathbf{S}\} > 0$.

Remark 1.7. *Since the law of the eigenvalues satisfies a large deviation principle with rate N, the eigenvalues actually converges to the support exponentially fast (or be more precisely, $\exists \Gamma > 0$, s.t, $\mu_{N,\beta}^{V;\boldsymbol{B}}\left(\exists \lambda_i \notin \boldsymbol{A}\right) \leq e^{-\Gamma N}$).*

Remark 1.8. *By the definition of the partition function, $1 - \mu_{N,\beta}^{V;\boldsymbol{B}}\left(\exists \lambda_i \notin \boldsymbol{A}\right) = \frac{Z_{N,\beta}^{V;\boldsymbol{A}}}{Z_{N,\beta}^{V;[b-.b+]}}$, thus,*

$$(1.7) \Leftrightarrow \lim_{N\to\infty} \frac{Z_{N,\beta}^{V;\boldsymbol{A}}}{Z_{N,\beta}^{V;\boldsymbol{B}}} = 1. \tag{1.8}$$

In the rest of this article we investigate what happens in the case where V is critical. This investigation will require the uses of precise estimates on β models partitions functions derived in [BG11, BG13] and to apply their results we shall make the following assumption :

Assumption 1.

- $V : \boldsymbol{B} \to \mathbb{R}$ is a continuous function independent of N.
- If $\tau\infty \in \boldsymbol{B}$,

$$\liminf_{x\to\tau\infty} \frac{V(x)}{2\ln|x|} > 1. \tag{1.9}$$

- supp (μ_{eq}) is a finite union of disjoint intervals, i.e. supp (μ_{eq}) of the form $\mathbf{S} = \bigcup_{h=1}^{g} \mathbf{S}_h$, where $\mathbf{S}_h = [\alpha_h^-, \alpha_h^+]$.
- $S(x) > 0$ whenever $x \in \mathbf{S}$, where

$$S(x) = \pi \frac{d\mu_{\mathrm{eq}}}{dx} \sqrt{\left| \frac{\prod_{\tau' \in \mathrm{Hard}}(x - \alpha_{\tau'})}{\prod_{\tau \in \mathrm{Soft}}(x - \alpha_\tau)} \right|}.$$

  and where $\tau \in$ Hard (resp. $\tau \in$ Soft) iff $b_\tau = \alpha_\tau$ (resp. $\tau(b_\tau - \alpha_\tau) > 0$).
- V extends to a holomorphic function in some open neighborhood of $\boldsymbol{S}$.

We want to investigate whether (1.7) still holds when the restriction on $\mathcal{J}^{V;\boldsymbol{B}}$ is weakened so that it vanishes outside the support $\mathbf{S}$. Our working hypothesis will be the following:

Assumption 2. $\tilde{\mathcal{J}}^{V;\boldsymbol{B}}$ vanishes only on the support of the equilibrium measure $\boldsymbol{S}$ and at one point c_0 outside $\boldsymbol{S}$. We also require $\partial^2 \tilde{\mathcal{J}}^{V;\boldsymbol{B}}(c_0) > 0$, and for technical reason we require $\partial^2 V \geq \sigma > 0$ on $\boldsymbol{A}$.

1.2. **Main Results.**

Theorem 1.9. *Given Assumptions 1 and 2, we have the following alternative:*

- *when* $\beta > 1$,

$$\lim_{N\to\infty} \mu^{V;\boldsymbol{B}}_{N,\beta} \left(\exists \lambda_i \notin \boldsymbol{A}\right) = 0, \tag{1.10}$$

- *when* $\beta < 1$,

$$\limsup_{N\to\infty} \mu^{V;\boldsymbol{B}}_{N,\beta} \left(\exists \lambda_i \notin \boldsymbol{A}\right) = 1. \tag{1.11}$$

The behavior below $\beta = 1$ can be illustrated with the case $\beta = 0$ where one would consider a potential V vanishing on a support $\mathbf{S}$ and at a point c_0 (where its second derivative is positive), being strictly positive everywhere else. This corresponds to N independent variables with probability of order $N^{-1/2}$ to belong to a small neighborhood of c_0 (where the latter probability can be estimated by Laplace method). In this case, it is clear that some eigenvalues will lie in the neighborhood of c_0 with positive probability. The existence of a phase transition for this phenomenon at $\beta = 1$ is however new to our knowledge. It suggests that the support of the eigenvalues of matrices with real coefficients corresponding to $\beta = 1$ matrix models might be more sensible to perturbations of the potential than matrices with complex coefficients (corresponding to $\beta = 2$). This is however apparently not supported by finite dimensional perturbations since the BBP transition [BBAP05] does not vary much between these two cases. Let us observe that our arguments could be carried similarly with several critical points similar to c_0 without changing the phase transition. However, if the second derivative of $\mathcal{J}$ at these critical points could vanish so that $\mathcal{J}^{V;\boldsymbol{B}}$ behaves as $|x - c_0|^q$ for some $q > 2$ in the vicinity of c_0, the phase transition would occur at a threshold β_q depending on q (see remark 5.17).

1.3. **Structure of the paper.** In Section 2 we reduce the problem to the analysis of (2.2). (2.2) relies heavily on the estimate for the probability that M eigenvalues are contained in a small neighborhood of c_0 while the rest of the N-M of the eigenvalues are contained in $\boldsymbol{A}$. The entirety of section 5 is devoted to the proof of proposition 2.1, which gives the desired estimate for $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$. Section 3 deals with the case $\beta > 1$ and Section 4 with the case $\beta < 1$. The Appendix contains some useful results for our problem.

1.4. **Notation.** We use the notation $A \lesssim B$ (resp. $A \gtrsim B$) to denote $A \le CB + e^{-\delta N^2}$ (resp. $A \ge CB - e^{-\delta N^2}$) for some universal constant C and some $\delta > 0$ independent of N. $A \approx B$ when both $A \lesssim B$ and $A \gtrsim B$ hold. Moreover, we use $A \lesssim_Q B$ to denote $A \le C_Q B$, i.e, the constant C may depend on Q. c usually denotes a small constant while C denotes a large constant. The values of these constants may change line by line. We sometimes use $a \ll 1$ to denote that a is smaller than any universal constant involved in the proof.

2. Preliminary and Basic analysis

The probability that a specific subset of M eigenvalues are contained in a small neighborhood of c_0 while the other $N - M$ of the eigenvalues are contained in $\boldsymbol{A}$ shall be denoted as $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$:

$$\Phi^{V;\boldsymbol{B}}_{N,M,\beta} := \mu^{V;\boldsymbol{B}}_{N,\beta} \left(\lambda_{N-M+1}, ..., \lambda_N \in [c_0 - \epsilon, c_0 + \epsilon], \lambda_1, ...\lambda_{N-M} \in \mathbf{A}\right). \tag{2.1}$$

where $\epsilon > 0$ is a small fixed constant. The key to prove our main result is calculating the speed at which $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$ approaches 0 as N approaches $+\infty$.

Note that

$$\begin{aligned}
&\mu^{V;\boldsymbol{B}}_{N,\beta}\left(\exists \lambda_i \notin \boldsymbol{A}\right) = \mu^{V;\boldsymbol{B}}_{N,\beta}\left(\exists \lambda_i \notin (\boldsymbol{A} \cup [c_0-\epsilon, c_0+\epsilon])\right) \\
&+ \sum_{\frac{M}{N}>\delta} \mu^{V;\boldsymbol{B}}_{N,\beta}\left(M \text{ eigenvalues are in } [c_0-\epsilon, c_0+\epsilon], N-M \text{ eigenvalues are in } \mathbf{A}\right) \\
&+ \sum_{1\le M\le \delta N} \binom{N}{M} \Phi^{V;\boldsymbol{B}}_{N,M,\beta} \\
&=: P_1 + P_2 + P_3. \qquad (2.2)
\end{aligned}$$

Here $\delta > 0$ is a small fixed constant.

Since $\mathcal{J}$ is a good rate function which is positive outside $\boldsymbol{A} \cup [c_0-\epsilon, c_0+\epsilon]$, Theorem 1.4 implies that for any fixed $\epsilon > 0$, P_1 approaches 0 exponentially fast. In other words, it is controlled by e^{-Nc_ϵ} for some $c_\epsilon > 0$.

By the large deviation principle for the law of the empirical measure L_N described in Theorem 1.1, P_2 is controlled by $e^{-c_\epsilon N^2}$. Therefore we deduce that for any $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\mu^{V;\boldsymbol{B}}_{N,\beta}\left(\exists \lambda_i \notin \boldsymbol{A}\right) = P_3 + O(e^{-c(\delta)N}). \qquad (2.3)$$

Our goal, therefore, is to control the third term P_3.

Since $\mathcal{J}$ goes to infinity at infinity, Theorem 1.4 also shows that the probability to have an eigenvalue above some finite threehold M goes to zero exponentially fast. Therefore, we may assume without loss of generality that $\boldsymbol{B}$ is a bounded set.

Thus, we are left to analyze $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$.

We prove the following upper bound in section 5:

Proposition 2.1. *Let Assumption 1 and 2 hold. Then, there exists a $\delta > 0$ such that when $\frac{M}{N} \le \delta$, we have uniformly in $M \le \delta N$*

$$\Phi^{V;\boldsymbol{B}}_{N,M,\beta} \lesssim \frac{1}{N^{\frac{M(\beta+1)}{2}}}. \qquad (2.4)$$

On the other hand,

$$\frac{1}{N^{\frac{(\beta+1)}{2}}} \frac{Z^{V;\mathbf{A}}_{N,\beta}}{Z^{V;\boldsymbol{B}}_{N,\beta}} \lesssim \Phi^{V;\boldsymbol{B}}_{N,1,\beta}. \qquad (2.5)$$

The calculation is based on the precise estimate derived in [BG13] for the partition function and correlators for fixed filling fraction measure, that is with given number of eigenvalues in each connected part of the support $\mathbf{S}$. The proof of this proposition will be the subject of section 5. We next give the proof of our main result.

3. Convergence of the eigenvalues to the limiting support S when $\beta > 1$

Now, we return to the estimate in (2.3) and use the upper bound provided by Proposition 2.1 to find

$$P_3 \le \sum_{1\le M<\delta N} \binom{N}{M} \Phi^{V;\boldsymbol{B}}_{N,M,\beta} \lesssim \left(1 + N^{-\frac{(\beta+1)}{2}}\right)^N - 1\,. \qquad (3.1)$$

where we finally noted that any error of order e^{-cN^2} in $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}, 1 \leq M \leq \delta N$ is neglectable in the above sum . Hence, when $\beta > 1$, P_3 goes to 0 as N approaches $+\infty$. This finishes the proof of the first half of our main Theorem 1.9.

4. Escaping eigenvalues when $\beta < 1$

We prove that when $\beta < 1$ the probability that no eigenvalues lies in the neighborhood of c_0 goes o zero, that is by Remark 1.8, that we have:

$$\lim_{N\to\infty} \frac{Z^{V;\boldsymbol{A}}_{N,\beta}}{Z^{V,\boldsymbol{B}}_{N,\beta}} = 0. \tag{4.1}$$

This is done by lower bounding the probability $p^V_{N,\beta}$ that one eigenvalue exactly lies in the neighborhood of c_0. Indeed, as $A_i = \{\lambda_i \in [c_0-\epsilon, c_0+\epsilon], \lambda_j \in \mathbf{A}, j \neq i\}$ are disjoint as soon as $[c_0-\epsilon, c_0+\epsilon] \cap \mathbf{A} = \emptyset$, we have

$$p^V_{N,\beta} = N\Phi^{V;\boldsymbol{B}}_{N,1,\beta}.$$

Since $p^V_{N,\beta} \leq 1$, we deduce from (2.5) that there exists a finite constant C such that

$$\frac{N}{N^{\frac{(\beta+1)}{2}}} \frac{Z^{V;\mathbf{A}}_{N,\beta}}{Z^{V;\boldsymbol{B}}_{N,\beta}} \leq C\,,$$

which results with

$$\frac{Z^{V;\mathbf{A}}_{N,\beta}}{Z^{V;\boldsymbol{B}}_{N,\beta}} \leq CN^{\frac{\beta-1}{2}}\,,$$

so that (2.5) follows.

5. Proof of the main Proposition 2.1

Our proof is based on estimates from [BG13] on the fixed filling fraction measure which will allow us to estimate precisely the probability that $N-M$ eigenvalues stay in $\mathbf{A}$, whereas Laplace methods will be used to control the probability that M eigenvalues are close to c_0. Let us introduce some extra notations to describe these estimates. Let $\mathcal{E}_g := \{(\epsilon_1, \cdots, \epsilon_g) | \sum \epsilon_h = 1, \epsilon_1, \ldots, \epsilon_g > 0\}$ denote the interior of the standard $g-1$ simplex. Let

$$\boldsymbol{\epsilon}_\star := (\mu_{\mathrm{eq}}(S_1), \cdots, \mu_{\mathrm{eq}}(S_g)) \tag{5.1}$$

be the g-tuple denoting the value of the equilibrium measure on each of the intervals that comprise the support of the equilibrium measure.

Definition 5.1. *Let the fixed filling fraction probability measure $d\mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}$ be given by:*

$$d\mu^{V;\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta}(\boldsymbol{\lambda}) := \frac{1}{Z^{V;A}_{N,\boldsymbol{\epsilon},\beta}} \prod_{h=1}^{g} \Big[\prod_{i=1}^{N_h} \mathrm{d}\lambda_{h,i}\, \mathbf{1}_{\mathsf{A}_h}(\lambda_{h,i})\, e^{-\frac{\beta N}{2} V(\lambda_{h,i})} \prod_{1\leq i<j\leq N} |\lambda_{h,i}-\lambda_{h,j}|^\beta \Big]$$
$$\times \prod_{1\leq h<h'\leq g} \prod_{\substack{1\leq i\leq N_h \\ 1\leq j\leq N_{h'}}} |\lambda_{h,i}-\lambda_{h',j}|^\beta,$$

where $Z^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}$ is the partition function. The fixed filling fraction measure $d\mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}$ can be viewed as the probability measure induced by $d\mu^{V;\boldsymbol{A}}_{N,\beta}$ conditioned on that the number of the eigenvalues in $[a_h-, a_h+]$ is fixed as $N_h = [N\epsilon_h]$, $1 \leq h \leq g-1$,

$N_g = N - \sum_{h=1}^{g-1} N_h$. *We alternatively denote by* $\mu^{V;\mathsf{A}}_{N,\overrightarrow{\boldsymbol{N}},\beta} = \mu^{V;\mathbf{A}}_{N,\boldsymbol{\epsilon},\beta}$ *for* $N_h = [N\epsilon_h]$, $\overrightarrow{\boldsymbol{N}} = (N_1, \dots, N_g)$.

The following precise estimate from [BG13, Theorem 1.4] will be essential in our proof (up to corrections of order $K = 0$):

Theorem 5.2. *If V satisfies Assumption 1 and 2 on $\mathbf{A}$, there exists $t > 0$ such that, uniformly for $\boldsymbol{\epsilon} \in \mathcal{E}_g$ that satisfies $|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star| < t$, we have:*

$$\frac{N!}{\prod_{h=1}^{g}(N_h)!} Z^{V;\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta} = N^{(\frac{\beta}{2})N+e} \exp\Big(\sum_{k=-2}^{K} N^{-k} F^{\{k\}}_{\boldsymbol{\epsilon},\beta} + o(N^{-K}) \Big), \tag{5.2}$$

Here $N_h := [N\epsilon_h]$. e is a constant depending only whether each edge is soft or hard. $F^{\{k\}}_{\boldsymbol{\epsilon},\beta}$ is a smooth function for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_\star$, and at the value $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_\star$, the derivative of $F^{\{-2\}}_{\boldsymbol{\epsilon},\beta}$ vanishes and its Hessian is negative definite. Moreover, the law of the empirical measure L_N under $\mu^{V;\mathbf{A}}_{N,\boldsymbol{\epsilon},\beta}$ satisfies a large deviation principle with speed N^2 and good rate function $\tilde{\mathcal{E}}_\epsilon$ which is minimized at a unique probability measure $\mu_{\mathrm{eq},\epsilon}$. In particular L_N converges $\mu^{V;\mathbf{A}}_{N,\boldsymbol{\epsilon},\beta}$ almost surely to $\mu_{\mathrm{eq},\epsilon}$.

One can also estimate precisely the correlators of the fixed fraction measure:

Definition 5.3. *The correlators of the fixed filling fraction measure are given for $x \in \mathbb{C}\backslash\mathbf{A}$:*

$$W_{\boldsymbol{\epsilon}}(x) := \mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}\{\sum \frac{1}{x-\lambda_i}\}, \quad W^{\{-1\}}_{\boldsymbol{\epsilon}}(x) := \mu_{\mathrm{eq},\epsilon}(\frac{1}{x-\lambda}). \tag{5.3}$$

A consequence of [BG13, Theorem 1.3] also gives us the following theorem regarding the correlators:

Theorem 5.4. [BG13] *If V satisfies Hypotheses 1 and 2 on $\boldsymbol{A}$, there exists $t > 0$ such that, uniformly for $\boldsymbol{\epsilon} \in \mathcal{E}_g$ and $|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star| < t$, we have an expansion for the correlators:*

$$W_{\boldsymbol{\epsilon}}(x) = N W^{\{-1\}}_{\boldsymbol{\epsilon}}(x) + O(1). \tag{5.4}$$

(5.4) *holds uniformly for x in compact regions outside $\boldsymbol{A}$(in our case in particular near the critical point c_0). $W^{\{-1\}}_{\boldsymbol{\epsilon}}$ is smooth for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_\star$*

Remark 5.5. *Alternatively, we can view $[c_0 - \delta, c_0 + \delta]$ as a degenerate, additional cut of the support of μ_{eq}. However, even if we can achieve an analogue of Theorem 5.2, taking this alternate viewpoint into account, which is likely to be a difficult task, we still cannot exclude the possibility that finite eigenvalues are concentrated near the critical point c_0. Thus, we do not apply this alternate strategy.*

Our goal is to split $\Phi^{V,\boldsymbol{B}}_{N,M,\beta}$ into components and estimate the value of each component. Then, we combine the values of the components to estimate $\Phi^{V,\boldsymbol{B}}_{N,M,\beta}$.

We start with the explicit formula for $\Phi^{V,\boldsymbol{B}}_{N,M,\beta}$:

$$\begin{aligned}\Phi^{V;\boldsymbol{B}}_{N,M,\beta} &= \frac{1}{Z^{V;\boldsymbol{B}}_{N,\beta}}\int_{[c_0-\epsilon,c_0+\epsilon]^M}\prod_{j=N-M+1}^{N} d\lambda_j e^{-\frac{N\beta}{2}V(\lambda_j)}\prod_{N-M+1\le k<l\le N}|\lambda_k-\lambda_l|^\beta\\ &\times\int_{\lambda_i\in\boldsymbol{A}}\left(\prod_{i=1}^{N-M} d\lambda_i e^{-\frac{\beta(N-M)}{2}\left(\frac{N}{N-M}V(\lambda_i)\right)}\prod_{j=N-M+1}^{N}|\lambda_i-\lambda_j|^\beta\right)\\ &\times\left(\prod_{1\le i<j\le N-M}|\lambda_i-\lambda_j|^\beta\right). \end{aligned}\tag{5.5}$$

We want to rewrite $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$ in terms of $\mu^{\frac{N}{N-M}V;\boldsymbol{A}}_{N-M,\beta}$, the law of the $N-M$ eigenvalues in $\boldsymbol{A}$. Note that $\Phi^{V;\boldsymbol{B}}_{N,M,\beta}$ is equal to $\mu^{\frac{N}{N-M}V;\boldsymbol{A}}_{N-M,\beta}$ multiplied by the law of the M eigenvalues in $[c_0-\epsilon,c_0+\epsilon]$ conditioned on the values of the $N-M$ eigenvalues in $\boldsymbol{A}$. Also, note that the $\prod_{N-M+1\le k<l\le N}|\lambda_k-\lambda_l|^\beta$ term is controlled by $(2\epsilon)^{\frac{\beta M(M-1)}{2}}$. Thus, let:

$$\begin{aligned} Y_{N,M} &:= \frac{Z^{\frac{N}{N-M}V;\boldsymbol{A}}_{N-M,\beta}}{Z^{V;\boldsymbol{B}}_{N,\beta}} = \frac{\sum_{N_1+\cdots+N_g=N-M}\frac{(N-M)!}{N_1!\cdots N_g!}\cdot Z^{\frac{N}{N-M}V,\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M},\beta}}{Z^{V;\boldsymbol{B}}_{N,\beta}}\\ \Xi(\eta_1,\cdots,\eta_M) &:= \mu^{\frac{N}{N-M}V;\boldsymbol{A}}_{N-M,\beta}\Big(\prod_{j=1}^{M}e^{\beta\sum_{i=1}^{N-M}\ln|\eta_j-\lambda_i|-\frac{\beta}{2}(N-M)V(\eta_j)}\Big), \qquad (5.6)\\ L_{N,M} &:= \int_{[c_0-\epsilon,c_0+\epsilon]^M}\Xi(\lambda_{N-M+1},\cdots,\lambda_N)\prod_{j=N-M+1}^{N}e^{-\frac{M\beta}{2}V(\lambda_j)}d\lambda_j \end{aligned}$$

where $\overrightarrow{\boldsymbol{N-M}} := (N_1,\cdots,N_g)$ with $\sum N_i = N-M$.
Then, we have:

$$\Phi^{V;\boldsymbol{B}}_{N,M,\beta}\le(2\epsilon)^{\frac{\beta M(M-1)}{2}}Y_{N,M}L_{N,M}. \tag{5.7}$$

When M=1, the inequality (5.7) becomes an equality:

$$\Phi^{V,\boldsymbol{B}}_{N,1,\beta} = Y_{N,1}L_{N,1} \tag{5.8}$$

(5.8) will be applied when we prove (2.5). We wish to split $Y_{N,M}$ into components for further analysis. We make the decomposition;

$$Y_{N,M} = \frac{Z^{V,\mathbf{A}}_{N,\beta}}{Z^{V,\boldsymbol{B}}_{N,\beta}}\tilde{Y}_{N,M}\,,\qquad \tilde{Y}_{N,M} = F_{N,M}G_{N,M}\,, \tag{5.9}$$

where

$$F_{N,M} = \frac{Z^{\frac{N}{N-M}V;\boldsymbol{A}}_{N-M,\beta}}{Z^{V;\boldsymbol{A}}_{N-M,\beta}}\,,\qquad G_{N,M} = \frac{Z^{V;\boldsymbol{A}}_{N-M,\beta}}{Z^{V;\boldsymbol{A}}_{N,\beta}}\,. \tag{5.10}$$

By estimating $F_{N,M}L_{N,M}$ and $G_{N,M}$, we end up with an estimate of $\tilde{Y}_{N,M}$, which is an upper bound on $Y_{N,M}$ as $Z^{V,\mathbf{A}}_{N,\beta}\le Z^{V,\boldsymbol{B}}_{N,\beta}$. It will also provide a lower bound for $\Phi_{1,M}$ to prove (2.5). Thus one rewrites formula (5.7) as:

$$\Phi^{V;\boldsymbol{B}}_{N,M,\beta}\le(2\epsilon)^{\frac{\beta M(M-1)}{2}}G_{N,M}F_{N,M}L_{N,M}\,. \tag{5.11}$$

Remark 5.6. *We always consider* $\frac{M}{N} < \delta \ll 1$. *The* δ *should be chosen to be smaller than the constant t in Theorem 5.2, and in practice we need to make it even smaller s.t we can use Taylor expansions near the* $\boldsymbol{\epsilon}_\star$ *(see Theorem 5.2 and (5.1)).*

We next estimate $G_{N,M}$, $F_{N,M}$ and $L_{N,M}$.

5.1. step 1: Estimating $G_{N,M}$.

Proposition 5.7. *There exists small* $\delta > 0$, *s.t when* $M \leq \delta N$, *we have*

$$G_{N,M} \approx C_M \frac{1}{N^{\frac{M\beta}{2}}} e^{NM(\inf_{\xi\in\boldsymbol{B}} \mathcal{J}^{V;B}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\mathrm{eq}}(\eta))}. \tag{5.12}$$

Furthermore, there exists a finite constant C *such that for all* $1 \leq M \leq \delta N$, C_M *satisfy the control*

$$C_M \leq Ce^{CM^2}. \tag{5.13}$$

Proof. Proposition 5.7 follows directly from the Lemma 5.9 and Corollary 5.12 below. □

Remark 5.8. *The term* e^{CM^2} *does not affect the final estimate due to the presence of* $(2\epsilon)^{\frac{\beta M(M-1)}{2}}$ *in (5.11). Also,* M *is always much smaller than* N, *even though* M *is not fixed for all* N.

To prove Proposition 5.7 we first show the following Lemma.

Lemma 5.9.

$$G_{N,M} \approx e^{MNA_\beta} \frac{1}{N^{\frac{M\beta}{2}}} C_M, \tag{5.14}$$

where A_β *does not depend on* N *or* M *and* $C_M \lesssim e^{CM^2}$

The value of A_β will be given by Corollary 5.12.

Proof. By the partition function estimate from Theorem 5.2, for a sufficiently small δ,

$$\begin{aligned} Z^{V,\boldsymbol{A}}_{N-M,\beta} &= \sum_{N_1+\cdots+N_g=N-M} \frac{(N-M)!}{N_1!\cdots N_g!} \cdot Z^{\frac{N}{N-M}V,\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M},\beta}, \\ &\approx \sum_{\substack{N_1+\cdots+N_g=N-M,\\ |\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}-\boldsymbol{\epsilon}_\star|<\delta}} (N-M)^{(\frac{\beta}{2})(N-M)+e} \exp\left((N-M)^2 F^{\{2\}}_{\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}} + (N-M) F^{\{1\}}_{\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}}\right). \end{aligned} \tag{5.15}$$

In the last step we applied the large deviation principle for the empirical measure L_N; in other words, the sum over all $\overrightarrow{N-M}$ such that $|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M} - \boldsymbol{\epsilon}_\star| \geq \delta$ is negligible. Similarly,

$$\begin{aligned} Z^{V,\boldsymbol{A}}_{N,\beta} &= \sum_{N_1+\cdots+N_g=N} \frac{(N-M)!}{N_1!\cdots N_g!} \cdot Z^{\frac{N}{N}V,\boldsymbol{A}}_{N,\frac{\vec{\boldsymbol{N}}}{N},\beta}, \\ &\approx \sum_{\substack{N_1+\cdots+N_g=N,\\ |\frac{\vec{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|<\delta}} (N)^{(\frac{\beta}{2})(N)+e} \exp\left((N)^2 F^{\{2\}}_{\frac{\vec{\boldsymbol{N}}}{N}} + (N) F^{\{1\}}_{\frac{\vec{\boldsymbol{N}}}{N}}\right). \end{aligned} \tag{5.16}$$

We point out by large deviation of the empirical measure L_N, we need only consider $\frac{\vec{\boldsymbol{N}}}{N}, \frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}$ sufficiently close to $\boldsymbol{\epsilon}_\star$. By Theorem 5.2, for $\frac{\vec{\boldsymbol{N}}}{N}$ sufficiently close to $\boldsymbol{\epsilon}_\star$, $F^{\{1\}}$ and $F^{\{2\}}$ are smooth, and because the Hessian of $F^{\{2\}}$ is negative definite, $F^{\{2\}}_{\boldsymbol{\epsilon}} - F^{\{2\}}_{\boldsymbol{\epsilon}_\star} \approx -|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star|^2$.

All that is left to do is to analyze the limiting behavior of :

$$L_{\boldsymbol{K}} := \sum_{\substack{N_1+\cdots+N_g=K, \\ |\frac{\vec{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|<\delta}} \exp((K)^2 F^{\{2\}}_{\frac{\vec{\boldsymbol{K}}}{K}} + (K) F^{\{1\}}_{\frac{\vec{\boldsymbol{K}}}{K}}). \tag{5.17}$$

Here $\boldsymbol{K} = (N_1, \cdots, N_g)$ with $\sum N_i = K$. The following lemma suffices to complete the proof.

Lemma 5.10.

$$L_{\vec{\boldsymbol{K}}} \approx \exp(K^2 F^{\{2\}}_{\epsilon *} + K F^{\{1\}}_{\boldsymbol{\epsilon}_\star}) \tag{5.18}$$

Clearly, Lemma 5.10 plus (5.15), (5.16) imply (5.14). □

Now we give the proof of the lemma.

Proof of lemma 5.10. According to Theorem 5.2, we can find constant c, C s.t

$$|F^{\{1\}}_{\boldsymbol{\epsilon}} - F^{\{1\}}_{\boldsymbol{\epsilon}_\star}| \le C|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star|, \tag{5.19}$$

$$F^{\{2\}}_{\boldsymbol{\epsilon}} - F^{\{2\}}_{\boldsymbol{\epsilon}_\star} \le -c|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star|^2, \tag{5.20}$$

$$|F^{\{2\}}_{\boldsymbol{\epsilon}} - F^{\{2\}}_{\boldsymbol{\epsilon}_\star}| \le C|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star|^2. \tag{5.21}$$

By the smoothness of $F^{\{1\}}$ and $F^{\{2\}}$ and because there exists at least one $\overrightarrow{\boldsymbol{K_1}}$, s.t $|\frac{\overrightarrow{\boldsymbol{K_1}}}{K} - \boldsymbol{\epsilon}_\star| \lesssim \frac{1}{K}$, we can easily get the lower bound(using (5.19), (5.21):

$$L_{\vec{\boldsymbol{K}}} \gtrsim \exp(K^2 F^{\{2\}}_{\epsilon *} + K F^{\{1\}}_{\boldsymbol{\epsilon}_\star}). \tag{5.22}$$

Next, we calculate the upper bound. Actually, since we only sum over $|\frac{\vec{\boldsymbol{K}}}{K} - \boldsymbol{\epsilon}_\star| \ll 1$ we only need to establish by (5.19) and (5.20) that

$$\sum_{|\frac{\vec{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|\ll 1} \exp(-cK^2|\frac{\vec{\boldsymbol{K}}}{K} - \boldsymbol{\epsilon}_\star|^2 + CK|\frac{\vec{\boldsymbol{K}}}{K} - \boldsymbol{\epsilon}_\star|) \lesssim 1. \tag{5.23}$$

Clearly, (5.23) plus (5.19) and (5.20) will give the upper bound:

$$L_{\vec{\boldsymbol{K}}} \lesssim \exp(K^2 F^{\{2\}}_{\epsilon *} + K F^{\{1\}}_{\boldsymbol{\epsilon}_\star}). \tag{5.24}$$

To prove (5.23), let Q be a large constant such that cQ>100C, then

$$
\begin{aligned}
&\sum_{|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|\ll 1} \exp(-cK^2|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|^2+CK|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|)\\
&\le \sum_{|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|\le\frac{Q}{K}} \exp(-cK^2|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|^2+CK|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|),\\
&+ \sum_{|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|>\frac{Q}{K}} \exp(-cK^2|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|^2+CK|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|),\\
&\lesssim_Q 1+\sum_{|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|>\frac{Q}{K}} \exp(-\frac{1}{2}cK^2|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|^2),\\
&\lesssim 1.
\end{aligned}
\tag{5.25}
$$

which proves the Lemma. □

Having established Lemma 5.9, we complete the proof of Proposition 5.7 by computing A_β. As A_β does not depend on M it is enough to prove the following lemma:

Lemma 5.11. *Given $\eta', \eta'' > 0$ and N large enough,*

$$
\begin{aligned}
G_{N,1} &\ge e^{N(-\eta'+\inf_{\xi\in\boldsymbol{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))},\\
G_{N,1} &\le e^{N(\eta''+\inf_{\xi\in\boldsymbol{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))}.
\end{aligned}
\tag{5.26}
$$

Proof of the lemma almost exactly follows the same computation in the appendix A of [BG11]; For completeness, we include the proof in our Appendix, see lemma A.3.

Thus, we simply take M=1 in Lemma 5.9. By Lemma 5.9, for each $\eta', \eta'' > 0$, and for large N,

$$
\begin{aligned}
N(-\eta'+\inf_{\xi\in\boldsymbol{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta)) &\le NA_\beta + C\ln(N+1),\\
N(\eta''+\inf_{\xi\in\boldsymbol{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta)) &\ge NA_\beta - C\ln(N+1).
\end{aligned}
\tag{5.27}
$$

since we can choose $\eta', \eta'' > 0$ arbitrarily and also note

$$
\inf_{\xi\in\boldsymbol{A}}\mathcal{J} = \inf_{\xi\in\boldsymbol{B}}\mathcal{J}. \tag{5.28}
$$

Thus we get the value of A_β:

Corollary 5.12.

$$
A_\beta = \{\inf_{\xi\in\boldsymbol{B}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta)\}. \tag{5.29}
$$

Thus we get the value of A_β,which completes with Lemma 5.9, the proof of Proposition 5.7.

5.2. **step 2: Estimating the product $F_{N,M} \times L_{N,M}$.** To bound $F_{N,M}$ and $L_{N,M}$ from above uniformly on $|M| \leq \delta N$, we wish to use concentration inequalities under $\mu^{V,\mathbf{A}}_{N,\beta}$ This is the point where we will use the strict convexity of V on each connected components of $\mathbf{A}$, see Assumption 2. However, when $\mathbf{A}$ is not connected, the density of this measure is not strictly log-concave and standard concentration of measure results do not apply. To remedy this point, we will expand the product $F_{N,M} \times L_{N,M}$ in terms of the fixed filing fractions $\mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}$ which have a strictly log-concave density under Assumption 2. Indeed we then have, see [[2.3.2], [4.4.17], [4.4.26]] in [AGZ10] for more details,

Lemma 5.13 (Concentration Inequality). *Let ϵ be given. Let V be a smooth function such that $V''(x) \geq C > 0$ for all $x \in \boldsymbol{A}$. Let f be a function that is class C^1 on $\mathbb{R}^N$. Then*

$$\mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}\left[e^{\left(f-\mu^{V;\boldsymbol{A}}_{N,\boldsymbol{\epsilon},\beta}(f)\right)}\right] \lesssim e^{\frac{1}{NC}\|f\|^2_{\mathcal{L}}}, \tag{5.30}$$

where

$$\|f\|_{\mathcal{L}} := \sqrt{\sum_{i=1}^{N} \|\partial_{\lambda_i} f\|^2_\infty}.$$

Remark 5.14. *In practice, we only need the information of f on $\boldsymbol{A}$, in other words we only require f to be smooth near $\boldsymbol{A}$ and $\|f\|_{\mathcal{L}}$ in the above lemma can be replaced by $\|f\|_{\mathcal{L}(\boldsymbol{A})}$.*

To use this lemma, we write the decomposition over the feeling fractions. By the large deviation principle for the empirical measure Theorem 1.1 it is possible to restrict ourselves to the case where the feeling fractions are closed to that of the equilibrium, up to an error of order e^{-N^2}. Moreover, we first work with fixed $(\lambda_{N-M+1}, \ldots, \lambda_N)$. We shall prove the following upper bound

Proposition 5.15. *There exists $\epsilon > 0$ so that for any $\lambda_{N-M+1}, \cdots, \lambda_N$ belong to $[c_0 - \epsilon, c_0 + \epsilon]$, we have the following uniform estimate:*

$$F_{N,M}\Xi_{N,M}\left(\lambda_{N-M+1}, \cdots, \lambda_N\right) \lesssim Ce^{CM^2} e^{\int -\frac{\beta NM}{2} V(\eta) d\mu_{eq}(\eta)} e^{-N\sum_{j=N-M+1}^{N} \mathcal{J}^{V;\boldsymbol{B}}(\lambda_j)}. \tag{5.31}$$

Proof. From the above considerations, we can write up to this small error

$$F_{N,M}\Xi(\lambda_{N-M+1}, \cdots, \lambda_N) \approx \sum_{\substack{N-M=N_1+\cdots+N_g,\\ |\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}-\boldsymbol{\epsilon}_\star|<\delta}} c_{\overrightarrow{\boldsymbol{N-M}}} d_{\overrightarrow{\boldsymbol{N-M}}}, \tag{5.32}$$

where

- $$c_{\overrightarrow{\boldsymbol{N-M}}} := \frac{1}{Z^{V;\boldsymbol{A}}_{N-M,\beta}} \frac{(N-M)!}{N_1! \cdots N_g!} \cdot Z^{V,\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M},\beta}, \tag{5.33}$$

- $$d_{\overrightarrow{\boldsymbol{N-M}}} := \mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}\beta}\Big(\prod_{j=N-M+1}^{N} e^{\sum_{i=1}^{N-M}(-\frac{\beta}{2}V(\lambda_i)+\ln|\lambda_j-\lambda_i|-\frac{\beta}{2}V(\lambda_j))}\Big). \tag{5.34}$$

We shall use Lemma 5.13 with $f(\lambda_1,\dots,\lambda_{N-M}) = \sum_{i=N-M+1}^{N} g_{\lambda_i}(\lambda_1,\cdots,\lambda_{N-M})$ with

$$g_\lambda(\lambda_1,\cdots,\lambda_{N-M}) := \sum_{j=N-M+1}^{N} (-\frac{\beta}{2}V(\lambda) + \beta\ln|\lambda-\lambda j|)\,.$$

Note that g_λ is smooth for λ close to c_0 and $\lambda_i, i \le N-M$ in $\mathbf{A}$, and such that $\|g_\lambda\|_{\mathcal{L}}^2$ is of order N. As a consequence, $\|f\|_{\mathcal{L}}^2$ is of order NM^2 for $\lambda_{N-M+1},\dots,\lambda_N$ close to c_0.

We first estimate $d_{\overrightarrow{\boldsymbol{N-M}}}$ and then substitute the estimate into (5.32). By the concentration inequality,

$$\begin{aligned} d_{\overrightarrow{\boldsymbol{N-M}}} &= \prod_{j=N-M+1}^{N} e^{-\frac{\beta}{2}(N-M)V(\lambda j)} \mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M};\beta}(e^{\sum_{i=1}^{N-M} f(\lambda_i)}), \\ &= \prod_{j=N-M+1}^{N} e^{-\frac{\beta}{2}(N-M)V(\lambda j)} \mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M};\beta}(e^{\sum_{i=1}^{N-M} f(\lambda_i) - \mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M};\beta}(\sum_{i=1}^{N-M} f(\lambda_i))}), \\ &\quad * e^{\mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M};\beta}(\sum_{i=1}^{N-M} f(\lambda_i))} \\ &\le Ce^{CM^2}(\prod_{j=N-M+1}^{N} e^{-\frac{\beta}{2}NV(\lambda_j)})e^{\mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M};\beta}(\sum_{i=1}^{N-M} f(\lambda_i))}. \end{aligned} \tag{5.35}$$

The last step is a consequence of concentration inequality, Lemma 5.13. We remark here in the following estimate the constant C may change line by line.

Now we use the estimate of Lemma A.1, which is derived from Theorem 5.4. Indeed, since we assumed V analytic in a neighborhood of $\mathbf{A}$, f is also analytic in a neighborhood of $\mathbf{A}$ (for $[c_0-\epsilon, c_0+\epsilon]$ at positive distance of $\mathbf{A}$), so that we can use Lemma A.1 to deduce that

$$\mu^{V;\boldsymbol{A}}_{N-M,\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M},\beta}(\sum_{i=1}^{N-M} f(\lambda_i)) = (N-M)\mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N-M}}}{\mathrm{N-M}}}(f(\lambda)) + O(M). \tag{5.36}$$

Next, we want to substitute $\mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N-M}}}{\mathrm{N-M}}}$ by μ_{eq} By Appendix A.1 in [BG13](i.e the smooth dependence), it is not hard to see(we point out here $\mu_{\mathrm{eq}} = \mu_{\mathrm{eq},\boldsymbol{\epsilon}_\star}$):

$$|\mu_{\mathrm{eq}}\{f\} - \mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N-M}}}{\mathrm{N-M}}}\{f\}| = |\mu_{\mathrm{eq},\boldsymbol{\epsilon}_\star}\{f\} - \mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N-M}}}{\mathrm{N-M}}}\{f\}| \le CM|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M} - \boldsymbol{\epsilon}_\star|. \tag{5.37}$$

Combining (5.35), (5.32), (5.36), (5.37) gives us

$$\begin{aligned} F_{N,M}\Xi(\lambda_{N-M+1},\cdots,\lambda_N) &\lesssim Ce^{CM^2}(\prod_{j=N-M+1}^{N} e^{-\frac{\beta}{2}NV(\lambda_j)}) \\ \times \sum_{|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}-\boldsymbol{\epsilon}_\star|<\delta} & c_{\overrightarrow{\boldsymbol{N-M}}} \exp\left((N-M)(\mu^{V;\boldsymbol{A}}_{\mathrm{eq}}\{g\} + CM|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M} - \boldsymbol{\epsilon}_\star|) + O(M)\right). \end{aligned} \tag{5.38}$$

Now in order to establish our desired estimate:

$$F_{N.M}\Xi(\lambda_{N-M+!},\cdots,\lambda_N)\lesssim Ce^{CM^2}(\prod_{j=N-M+1}^{N}e^{-\frac{\beta}{2}NV(\lambda_j)})\exp\left((N-M)\mu_{\mathrm{eq}}\{f\}\right)$$
$$\leq Ce^{CM^2}e^{\int-\frac{\beta NM}{2}V(\eta)d\mu_{\mathrm{eq}}(\eta)}e^{-N\sum_{j=N-M+1}^{N}\mathcal{J}^{V;\boldsymbol{B}}(\lambda_j)}. \tag{5.39}$$

We need only establish the inequality:

$$\sum_{|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}-\boldsymbol{\epsilon}_\star|<\delta}c_{\overrightarrow{\boldsymbol{N-M}}}e^{CM(N-M)|\frac{\overrightarrow{\boldsymbol{N-M}}}{N-M}-\boldsymbol{\epsilon}_\star|}\leq Ce^{CM^2}, \tag{5.40}$$

which is straightforward since $c_{\overrightarrow{\boldsymbol{N-M}}}$ has a sub-Gaussian tail

$$c_{\overrightarrow{\boldsymbol{K}}}\leq Ce^{-cK^2|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|^2+CK|\frac{\overrightarrow{\boldsymbol{K}}}{K}-\boldsymbol{\epsilon}_\star|}. \tag{5.41}$$

(5.41) follows from Theorem 5.4 and (5.19), (5.20), (5.21).

□

From Proposition 5.15, we can easily obtain an upper bound on $F_{N,M}L_{N,M}$ by using classical Laplace method:

Proposition 5.16. *Under Assumptions 1 and 2,*

$$F_{N,M}L_{N,M}\lesssim Ce^{CM^2}e^{-\frac{\beta}{2}NM\int V(\eta)d\mu_{\mathrm{eq}}(\eta)-NM\inf\mathcal{J}^{V;\boldsymbol{B}}}\frac{1}{N^{\frac{M}{2}}}$$

Proof. The proof is straightforward since by Proposition 5.15

$$F_{N,M}L_{N,M}\lesssim Ce^{CM^2}e^{-\frac{\beta}{2}NM\int V(\eta)d\mu\mathrm{eq}(\eta)-NM\inf\mathcal{J}^{V;\boldsymbol{B}}}\left(\int_{c_0-\epsilon}^{c_0+\epsilon}e^{-N\tilde{\mathcal{J}}^{V;\boldsymbol{B}}(x)}e^{-M\frac{\beta}{2}V(x)}dx\right)^M$$

where we can bound V from below, providing a term e^{CM^2}, and use Laplace method (recall we assume $\tilde{\mathcal{J}}''(c_0)>0$, see [AGZ10, section 3.5.3] for details) to get

$$\int_{c_0-\epsilon}^{c_0+\epsilon}e^{-N\tilde{\mathcal{J}}^{V;\boldsymbol{B}}(\lambda)}d\lambda\approx\frac{1}{\sqrt{N}}.$$

□

Remark 5.17. *Note that if we would have assumed instead of $\tilde{\mathcal{J}}''(c_0)>0$ that for some $q>0$ $\tilde{\mathcal{J}}''(x)\simeq|x-c_0|^q$ in a neighborhood of c_0, we would have obtained*

$$F_{N,M}L_{N,M}\leq Ce^{CM^2}e^{-\frac{\beta}{2}NM\int V(\eta)d\mu_{\mathrm{eq}}(\eta)}\frac{1}{N^{\frac{M}{q}}}$$

and criticality would have occurred at $\beta=2/q$.

Propositions 5.16 and 5.7 give (2.4) since

$$\begin{aligned}\Phi^{V;\boldsymbol{B}}_{N,M,\beta} &\leq (2\epsilon)^{\frac{\beta M(M-1)}{2}}G_{N,M}F_{N,M}L_{N,M}\\ &\leq C'(C'\epsilon)^{\frac{\beta M(M-1)}{2}}\frac{1}{N^{\frac{M}{2}}}\frac{1}{N^{\frac{M\beta}{2}}}\end{aligned} \tag{5.42}$$

for some finite constant C'. Hence, if ϵ is chosen small enough, the result is proved. This concludes the proof.

5.3. **Proof of** (2.5)**.** Simply recall that in this case we have equality:

$$\frac{Z_{N,\beta}^{V;\boldsymbol{B}}}{Z_{N,\beta}^{V;\mathbf{A}}}\Phi_{N,M,\beta}^{V;\boldsymbol{B}} = Y_{N,1}\int_{[c_0-\epsilon,c_0+\epsilon]}\Xi(\lambda_N)e^{-\frac{\beta}{2}V(\lambda_N)}d\lambda_N. \tag{5.43}$$

When $M = 1$, we combine the estimate of (5.14) and apply central limit theorem Lemma A.2. Then we get:

$$F_{N,1}\Xi(\lambda_N) \approx e^{\int -\frac{\beta}{2}NV(\eta)d\mu_{\rm eq}(\eta)}e^{-\mathcal{J}(\lambda_N)}. \tag{5.44}$$

The estimate for $G_{N,1}$ given in Proposition (5.7) holds. Thus,we get

$$\begin{aligned}\frac{Z_{N,\beta}^{V;\boldsymbol{B}}}{Z_{N,\beta}^{V;\mathbf{A}}}\Phi_{N,1,\beta}^{V;\boldsymbol{B}} &\gtrsim \frac{1}{N^{\frac{\beta}{2}}}\int_{[c_0-\epsilon,c_0+\epsilon]}e^{-N\tilde{\mathcal{J}}(\lambda_N)}d\lambda_N,\\ &\gtrsim \frac{1}{N^{\frac{\beta+1}{2}}}.\end{aligned} \tag{5.45}$$

6. Acknowledgments

Part of this work was completed as part of the MIT SPUR program over the summer of 2013. AG was partially supported by the Simons Foundation and by NSF Grant DMS-1307704.

Appendix A. Supplementary Lemmas

A.1. **How to make use of the correlator estimate.** Here we prove the following lemma by using the estimate for correlator.This technique is standard, we include the proof for completeness.It also shows the reason why sometimes we need the assumption that a function is analytic analytic (near the $\boldsymbol{A}$); it guarantees that we can use Stieltjes transform estimates.

Lemma A.1. *Under the assumption in Theorem 5.4, let* $\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\boldsymbol{A}}$*be the fixed filling fractions measure,* $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ *be its limiting measure, let* h *be a function that is holomorphic near* $\boldsymbol{A}$*. Then,*

$$|\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\boldsymbol{A}}(\sum h(\lambda_i)) - N\mu_{\mathrm{eq},\boldsymbol{\epsilon}}(h)| \lesssim C. \tag{A.1}$$

Proof. Using Cauchy's integral and the assumption h is holomorphic near $\boldsymbol{A}$, we find a contour Γ s.t $h(\lambda) = \frac{1}{2\pi i}\oint_\Gamma \frac{h_\xi}{\xi-\lambda}$. Thus

$$\begin{aligned}&|\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\boldsymbol{A}}(\sum h(\lambda_i)) - N\mu_{\mathrm{eq},\boldsymbol{\epsilon}}(h)|\\ &\lesssim |\Gamma|\sup_{\xi\in\Gamma}\{(|h(\xi)|)|\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\boldsymbol{A}}\left(\sum\frac{1}{\xi-\lambda_i}\right) - NW_{\boldsymbol{\epsilon}}^{-1}(\xi)|\},\\ &\lesssim |\Gamma|\sup_{\xi\in\Gamma}\{|h(\xi)||W_\epsilon(\xi) - NW_\epsilon^{-1}(\xi)|\}.\end{aligned} \tag{A.2}$$

Our desired estimate follows from the expansion of the correlators given in Theorem 5.4. □

A.2. **Central limit for analytic functions.** We want to establish estimate (similar to the fixed fractals central limit theorem in section 5.5 of [BG13]). It is essentially contained in[BG13], but we write down the proof to be self-contained. we remark this h must be homomorphic near the neighborhood of $\boldsymbol{A}$(since we want to use the correlator estimate).Then we have:

Lemma A.2. *Let Assumption 1 and 2 hold, and h be holomorphic near the neighborhood of $\boldsymbol{A}$, then:*

$$\mu_{N,\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\} \approx \exp\{N\int h(\eta)d\mu_{\mathrm{eq}}(\eta)\}. \tag{A.3}$$

Proof. By the large deviation principle of the limiting measure:

$$\mu_{N,\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\} \approx \sum_{|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|<\delta} c_{\overrightarrow{\boldsymbol{N}}}\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\}. \tag{A.4}$$

First, we calculate a lower bound. Let $\overrightarrow{\boldsymbol{N}}$ be such that $|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star| < \frac{Q}{N}$ and $c_{\overrightarrow{\boldsymbol{N}}} > c_Q$, by Jensen's Inequality:

$$\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\} \geq \exp(\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}\{\sum h(\lambda_i)\}). \tag{A.5}$$

Next, by Lemma A.1, we can deduce the estimate:

$$|\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}(\sum h(\lambda_i)) - N\mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N}}}{N}}(h)| \lesssim C. \tag{A.6}$$

Using that $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}(h)$ depends smoothly on $\boldsymbol{\epsilon}$ (see appendix A.1 of [BG13]) and $\mu_{\mathrm{eq},\boldsymbol{\epsilon}_\star}$is exactly μ_{eq}, we have:

$$|\mu_{\mathrm{eq},\frac{\overrightarrow{\boldsymbol{N}}}{N}}(h) - \mu_{\mathrm{eq}}(h)| \lesssim C|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|. \tag{A.7}$$

Now , combing (A.4), (A.5), (A.6), (A.7), we get:

$$\mu_{N,\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\} \gtrsim \exp\{N\int h(\eta)d\mu_{\mathrm{eq}}(\eta)\}. \tag{A.8}$$

The upper bound is based on the concentration equality for (fixed fractals measure) and the explicit estimate for the partition function$\frac{(N)!}{N_1!\cdots N_g!}\cdot Z_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}$.Basically, concentration inequality gives the inverse direction of Jensen's inequality.
First from concentration inequality,

$$\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i))\} \lesssim \exp(\mu_{N,\frac{\overrightarrow{\boldsymbol{N}}}{N},\beta}^{V;\boldsymbol{A}}\{\sum h(\lambda_i)\}). \tag{A.9}$$

Now, combine (A.4), (A.6), (A.7), (A.9), and the estimate for the partition function, we get :

$$\begin{aligned}&\mu_{N,\beta}^{V;\boldsymbol{A}}\{\exp(\sum h(\lambda_i)))\} \qquad\qquad \text{(A.10)}\\ &\lesssim \sum_{|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|<\delta} \exp(-cN^2|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|^2 + CN|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|)\cdot\exp(N\mu_{\mathrm{eq}}(h)).\end{aligned}$$

Thus, we finally conclude by

$$\sum_{|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|<\delta} \exp(-cN^2|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|^2 + CN|\frac{\overrightarrow{\boldsymbol{N}}}{N}-\boldsymbol{\epsilon}_\star|) \lesssim 1. \tag{A.11}$$

□

A.3. **lemma for $G_{N,1}$.** We remark the following lemma is follows almost exactly the same argument as the analysis for Y_N in the appendx A.3 in[BG11].
We include the proof for completeness.

Lemma A.3. *Given $\eta > 0$ and N large enough, support set $\mathbf{A} = \cup_{h=1}^{g}\mathbf{A}_h$,*

$$\begin{aligned} G_{N,1} &\geq e^{N(-\eta+\inf_{\xi\in\mathbf{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))}, \\ G_{N,1} &\leq e^{N(\eta+\inf_{\xi\in\mathbf{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))}. \end{aligned} \tag{A.12}$$

Proof. Let $\delta > 0$. By Theorem 1.1, the probability that L_{N-1} is of distance greater or equal to δ from $\mu_{\rm eq}$ is smaller than $e^{-\Gamma_\delta(N-1)^2}$ for some $\Gamma_\delta > 0$. Therefore, when N is large enough,

$$\begin{aligned} \frac{1}{G_{N,1}} &= \frac{1}{Z_{N-1,\beta}^{V;\mathbf{A}}}\int_{\mathbf{A}} d\lambda_N e^{-\frac{N\beta}{2}V(\lambda_N)}\prod_{i=1}^{N-1}|\lambda_N-\lambda_i|^{\beta}e^{-\frac{\beta}{2}V(\lambda_i)} \\ &\times \prod_{i=1}^{N-1} d\lambda_i e^{-\frac{(N-1)\beta}{2}V(\lambda_i)}\prod_{1\leq i<j\leq N-1}|\lambda_i-\lambda_j|^{\beta} \\ &= \mu_{N-1,\beta}^{V;\mathbf{A}}\left(\int_{\mathbf{A}} d\lambda_N e^{-\frac{(N-1)\beta}{2}V(\lambda_N)}e^{-\frac{\beta}{2}V(\lambda_N)}\prod_{i=1}^{N-1}|\lambda_N-\lambda_i|^{\beta}e^{-\frac{\beta}{2}V(\lambda_i)}\right) \\ &\leq e^{-\Gamma_\delta\frac{N^2}{2}} + \int_{\mathbf{A}} d\xi e^{-\frac{\beta}{2}V(\xi)}e^{\beta(N-1)\sup_{d(\mu,\mu_{eq})<\delta}\left(-\frac{V(\xi)}{2}+\int[\ln|\xi-\eta|-\frac{\beta}{2}V(\eta)]d\mu(\eta)\right)}. \end{aligned}$$

As in [[BG11], Appendix 3], note that for all probability measures μ on $\mathcal{A}$:

$$\int_{\mathbf{A}} \ln|\xi-\eta|d\mu(\eta) \leq \int_{\mathbf{A}} \ln[\max(|\xi-\eta|,\zeta)]d\mu(\eta),$$

while the function on the righthand side is continuous in μ and ξ, and thus

$$\lim_{\zeta\to 0}\int_{\mathbf{A}} \ln[\max(|\xi-\eta|,\zeta)]d\mu(\eta) = \int_{\mathbf{A}} \ln|\xi-\eta|d\mu_{\rm eq}(\eta).$$

Therefore,

$$\begin{aligned} \limsup_{\delta\to 0}\sup_{\xi\in\mathbf{A}}\sup_{d(\mu,\mu_{eq})<\delta}\beta\left(\int \ln|\xi-\eta| - \frac{1}{2}V(\eta)d\mu(\eta) - \frac{V(\xi)}{2}\right) &\leq -\inf_{\xi\in\mathbf{A}}\mathcal{J}^{V;\mathbf{A}}(\xi) \\ &\quad -\frac{\beta}{2}\int V(\eta)d\mu_{eq}(\eta) \end{aligned}.$$

Thus, for any $\eta' > 0$ and N large enough,

$$\frac{1}{G_{N,1}} \leq e^{N(\eta'-\inf_{\xi\in\mathcal{A}}\mathcal{J}(\xi)-\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))}. \tag{A.13}$$

As for the upper bound, let $\epsilon > 0$. For any $[a_h^- - \epsilon, a_h^+ - \epsilon]$, there exists $\delta_\epsilon > 0$ such that the following holds by Jensen's inequality:

$$\begin{aligned}\frac{1}{G_{N,1}} &= \mu_{N-1,\beta}^{V;\mathbf{A}}\left(\int_{\mathbf{A}} d\lambda_N e^{-\frac{N\beta}{2}V(\lambda_N)}\prod_{i=1}^{N-1}|\lambda_N-\lambda_i|^\beta e^{-\frac{\beta}{2}V(\lambda_i)}\right),\\ &\geq \mu_{N-1,\beta}^{V;\mathbf{A}}\left(\int_{x-\epsilon}^{x+\epsilon} d\lambda_N e^{-\frac{N\beta}{2}V(\lambda_N)}\prod_{i=1}^{N-1}|\lambda_N-\lambda_i|^\beta e^{-\frac{\beta}{2}V(\lambda_i)}\right),\\ &\geq 2\epsilon e^{-\frac{\beta N}{2}V(x)-N\delta_\epsilon}\mu_{N-1,\beta}^{V;\mathbf{A}}\left(e^{\sum_{i=1}^{N-1}\frac{\beta}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}\ln|\xi-\lambda_i|d\xi-\frac{\beta}{2}V(\lambda_i)}\right).\end{aligned}$$

Since $\lambda\to\frac{1}{2\epsilon}\int_{\mathbf{C}}\ln|\xi-\lambda_i|d\xi$ is bounded continuous on $\mathbf{A}$, by convergence of $L_{N-1}\to\mu_{eq}$, for any given $\eta>0$, when N large enough,

$$\begin{aligned}\frac{1}{G_{N,1}} &\geq 2\epsilon e^{-\frac{\beta N}{2}V(x)-2N\delta_\epsilon}e^{(N-1)\int\frac{\beta}{2\epsilon}\left(\int_{x-\epsilon}^{x+\epsilon}\ln|\xi-\eta|d\xi-\frac{\beta}{2}V(\eta)\right)d\mu_{\rm eq}(\eta)},\\ &\geq e^{-N(\eta+\inf_{\xi\in\mathbf{A}}\mathcal{J}(\xi)+\frac{\beta}{2}\int V(\eta)d\mu_{\rm eq}(\eta))}.\end{aligned}$$

□

References

[ACKM93] J. Ambjørn, L. Chekhov, C. F. Kristjansen, and Yu. Makeenko. Matrix model calculations beyond the spherical limit. *Nuclear Phys. B*, 404(1-2):127–172, 1993.

[ACM92] J. Ambjørn, L. Chekhov, and Yu. Makeenko. Higher genus correlators from the Hermitian one-matrix model. *Phys. Lett. B*, 282(3-4):341–348, 1992.

[AGZ10] Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. An introduction to random matrices. *Cambridge University Press*, 2010.

[Ake96] Gernot Akemann. Higher genus correlators for the Hermitian matrix model with multiple cuts. *Nuclear Phys. B*, 482(1-2):403–430, 1996.

[BAG97] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997.

[BBAP05] Jinho Baik, Gérard Ben Arous, and Sandrine Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5):1643–1697, 2005.

[BEY] P. Bourgade, L. Erdös, and H.-T. Yau. Edge universality of beta ensembles. *http://arxiv.org/abs/1306.5728v1.*

[BEY12] Paul Bourgade, László Erdős, and Horng-Tzer Yau. Bulk universality of general β-ensembles with non-convex potential. *J. Math. Phys.*, 53(9):095221, 19, 2012.

[BG11] Gaëtan Borot and Alice Guionnet. Asymptotic expansion of beta matrix models in the one-cut regime. 2011.

[BG13] Gaëtan Borot and Alice Guionnet. All-order asymptotic expansion of beta matrix models in the multi-cut regime. *arXiv preprint arXiv:1303.1045*, 2013.

[BIPZ78] Edouard Brézin, Claude Itzykson, Giorgio Parisi, and Jean-Bernard Zuber. Planar diagrams. *Communications in Mathematical Physics*, 59(1):35–51, 1978.

[CE06] Leonid Chekhov and Bertrand Eynard. Matrix eigenvalue model: Feynman graph technique for all genera. *arXiv preprint math-ph/0604014*, 2006.

[Che06] L. Chekhov. Matrix models with hard walls: geometry and solutions. *J. Phys. A*, 39(28):8857–8893, 2006.

[Cla] Tom Claeys. The birth of a cut in unitary random matrix ensembles. *arXiv preprint arXiv:0711.2609.*

[DE02] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43:5830, 2002.

[Dei99] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics.* New York University Courant Institute of Mathematical Sciences, New York, 1999.

[Eyn06] B. Eynard. Universal distribution of random matrix eigenvalues near the 'birth of a cut' transition. *J. Stat. Mech. Theory Exp.*, (7):P07005, 33 pp. (electronic), 2006.

[FB] A. Guionnet F. Bekerman, A. Figalli. Transport maps for β-matrix models and universality. *1311.2315*.

[Joh98] Kurt Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.*, 91(1):151–204, 1998.

[Meh04] Madan Lal Mehta. *Random matrices*, volume 142. Access Online via Elsevier, 2004.

[Pas06] L. Pastur. Limiting laws of linear eigenvalue statistics for Hermitian matrix models. *J. Math. Phys.*, 47(10):103303, 22, 2006.

[RRV06] Jose Ramirez, Brian Rider, and Bálint Virág. Beta ensembles, stochastic airy spectrum, and a diffusion. *arXiv preprint math/0607331*, 2006.

[Shc13] M. Shcherbina. Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *J. Stat. Phys.*, 151(6):1004–1034, 2013.

[ST97] Edward B. Saff and Vilmos Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.