DOMINO-TILING PROBLEM
SPUR FINAL PAPER, SUMMER 2013

SUPANAT KAMTUE
MENTOR: BEN YANG

Abstract. Given a chessborad with a small number of blocks being removed, we want to figure out in which case it is possible to tile the rest of the board with 2×1 dominoes. In this paper, we prove that if the number of removed blocks is small enough, then the tiling is always possible. We give a proof for both an infinite chessboard case and a finite chessboardcase. Moreover, a draft proof for three dimensional case is given as well.

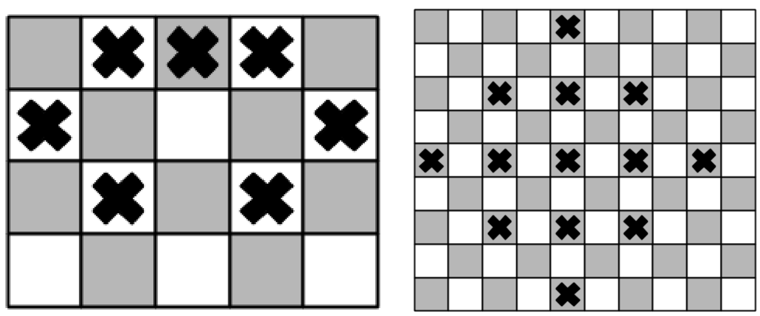
Date: July, 2013.

1. Tileability Problem

Problem. *Given a chessboard with a small number of blocks removed, we want to figure out in which case it is possible to tile the rest of the board with* 2×1 *dominoes.*

The first condition for the board to be tiled with 2×1 dominoes is the numbers of black and whites blocks that are left need to be equal. This first condition is a global condition, and certainly is not enough to guarantee the tileability.

Let consider two examples in Figure 1 where no tiling is possible (though it is not obvious in the second example). These two bad examples suggest that the removed blocks should not be too dense in one area. A further question is how dense the removed blocks should be so that we can guarantee the tileabililty.



(a) (b)

Figure 1. Bad Examples

Let us implement a *density-limit condition*: in any $R \times R$ square, no more than $\lceil cR \rceil$ blocks are removed. The constant c is called a *density constant*.

In this paper we are going to give a proof that we can always tile 2×1 dominoes in a chessboard with some blocks removed in such the way that white and black blocks are remained equally and that the density-limit condition is satisfied.

In section 2, we will discuss about Hall's matching theorem that we will mainly use in this paper. We will prove the tileability for an infinite chessboard (in section 3) and for a finite $n \times n$ chessboard (in section 4). In section 5, we will expand our result from a two-dimensional chess board into a three-dimensional gridboard. Section 6 is a comments section where we will discuss the ideas beyond this paper.

2. Hall's Theorem

Theorem 1 (Phillip Hall's Theorem)**.** [1] *In a finite bipartite graph with the bipartitions* X *and* Y*, there exists a matching that covers* X *if and only if every subset* A *of* X *is connected to at least* $|A|$ *vertices in* Y*.*

Hall's theorem provides a necessary and sufficient condition for the existence of a matching in the graph. It is directly related to our problem here as a chessboard can be viewed as the bipartitions of black blocks and white blocks, and a pair of black and white blocks are connected when they are neighbors (i.e. they share the same edge).

The Hall's theorem is originally for a finite graph, but is later extended for a infinite graph as followed.

Theorem 2 (Extended Hall's Theorem)**.** [1] *In a bipartite graph with the bipartitions X and Y such that the degree of every vertex is finite, there exists a matching that covers X if and only if every finite subset A of X is connected to at least $|A|$ vertices in Y.*

Before investigating more into our problem, as for the convenience we will clarify the denotations that we will be using in this paper. Denoted by W an arbitrary finite set of white blocks that we want to verify the Hall's condition. The set of the neighbor black blocks of W is denoted by $N(W)$. The set of blocks that are removed is called E, which is divided into the whites E_w and the blacks E_b.

The Hall's theorem says that the domino tiling is possible if for any W (supposedly $W \cap E_w = \emptyset$),

$$|N(W) - E_b| \geq |W|$$

or equivalently,

$$|N(W)| - |W| \geq |N(W) \cap E_b| \tag{1}$$

In the other words, the difference $\Delta = |N(W)| - |W|$ needs to be larger than the number of removed black blocks in $N(W)$. Call this number $\mathcal{R}$. Our required Hall's condition for the tileability is in short $\Delta \geq \mathcal{R}$.

Moreover, we define an *expanded* region of W as

$$Exp(W) := W \cup N(W)$$

(the region W and its black neighbors). And we call W to be *semi-connected* if the expanded $Exp(W)$ is a connected region.

3. Infinite Chessboard

What we are going to prove in this section is: for an infinite chessboard, with some blocks removed satisfying a density-limit condition, it is always possible to tile the rest of the board.

We want to verify the Hall's condition for a set W. Without the loss of generality we can assume the W to be *semi-connected*. The reason is that if we partition $W = W_1 \cup W_2$ where $Exp(W_1)$ and $Exp(W_2)$ are disjoint, then the inequality $\Delta \geq \mathcal{R}$ can be obtained by verifying W_1 and W_2 separately.

Consider that a semi-connected set W, so that the connected expanded region $Exp(W)$ occupies row number of rows and col number of columns. Denote $m = \max(row, col)$. Then the region $Exp(W)$ can fit in some square $m \times m$. It follows from the density-limit condition that $\mathcal{R} \leq cm$. On the other hand, in each row of the board, the number of black blocks in $Exp(W)$ will be more than the number of white blocks in $Exp(W)$ by at least 1. This gives $\Delta \geq row$. Similarly, we will also have $\Delta \geq col$, and therefore

$$\Delta \geq m \geq \frac{1}{c}\mathcal{R}.$$

Choosing $c = 1$ results in the desired inequality $\Delta \geq \mathcal{R}$.

4. Finite $n \times n$ Chessboard

In this section, we are going to give a proof of the main result of this paper.

Problem. *For an $n \times n$ chessboard, with some blocks are removed so that the same number of black and white blocks are left, and so that the density-limit condition is satisfied, we can guarantee the domino-tiling on the rest of the board.*

We will consider only the case that n is even. A proof for n being odd will be in a similar manner.

The case of the finite $n \times n$ chessboard is a little more complicated than the infinite one because of the restriction at the boundary of the chessboard. The inequality $\Delta \geq \mathcal{R}$ is more difficult to verify. We also need to use the fact that the numbers of removed black blocks and white ones are equal ($|E_w| = |E_b|$). Consider a set of white blocks W (which does not include any removed white blocks: $W \cap E_W = \emptyset$). By our previous definition, $N(W)$ is the set of black neighbors of W. Let $B = \{\textit{all black blocks}\} - N(W)$ be the set of all black blocks that are not in $N(W)$, and let $N(B)$ be the set of white neighbors of B.

Observe that $Exp(W)$ and $Exp(B)$ are disjoint and the union of both sets contains all but a few white blocks of the entire board. Call the set of those white blocks V.

Consider the following inequality:

$$|N(B)| - |B| \geq |N(B) \cap E_w| \tag{2}$$

This inequality is very similar to the Hall's condition (ineq.(1)):

$$|N(W)| - |W| \geq |N(W) \cap E_b|$$

except that instead of starting with the set W we start with the set B instead.

The inequality (2), in fact, implies our condition (1) because given (2) is true,

$$\begin{aligned} |N(W)| - |W| &= |N(B)| - |B| + |V| \\ &\geq |N(B) \cap E_w| + |V| \\ &\geq |E_w \cap (N(B) \cup V)| \\ &= |E_w| = |E_b| \\ &\geq |N(W) \cap E_b| \end{aligned}$$

The equation $|E_w \cap (N(B) \cup V)| = |E_w|$ comes from the fact that $E_w \cap W = 0$ which means $E_w \in \{\textit{all black blocks}\} - W = N(B) \cup V$.

To verify the Hall's condition, we can do it from either inequality (1) considering the region $Exp(W)$ directly, or inequality (2) which consider the region $Exp(B)$ instead.

Like in the previous section, we consider the $Exp(W)$ to be connected and occupies row number of rows and col number of columns. Let $m = \max(row, col)$, so $Exp(W)$ can be fit in a $m \times m$ rectangle. Before continuing on the proof, let us consider the following lemma.

Lemma 1. *Define a double-stripe as a rectangle of size $2 \times n$ or $n \times 2$. In a double-stripe S which contains some part of $Exp(W)$, the number of black blocks from $Exp(W)$ is more than the number of white from $Exp(W)$ blocks, except the case that the entire S is contained in $Exp(W)$.*

Or algebraically if $Exp(W) \cap S \neq \emptyset$*, then*

$$|N(W) \cap S| - |W \cap S| \begin{cases} = 0, & S \subset Exp(W) \\ \geq 1 & otherwise \end{cases}$$

Proof of Lemma 1 is straightforward and we will not write it down here.

Back to our main problem, we have 3 separate cases to check as follow:

Case 1 There is no double-stripe that is fully occupied by $Exp(W)$. Then in each two consecutive rows which are partially occupied by $Exp(W)$, we will apply the lemma 2. We then have the inequality $\Delta \geq \lfloor \frac{1}{2} row \rfloor$. In fact, one can check that $\Delta \geq \frac{1}{2} row$. Similarly, $\Delta \geq \frac{1}{2} col$. Therefore,

$$\Delta \geq \frac{1}{2} m \geq \mathcal{R}$$

The last inequality is from a density-limit condition with a density constant $c = \frac{1}{2}$.

Case 2 There is either a $2 \times n$ or $n \times 2$ double-stripe (but not both) that is fully occupied by $Exp(W)$. Without lost of generality, assume that there is such a $2 \times n$ double-stripe, and there is no $n \times 2$ one. Then we will have the inequality

$$\Delta \geq \frac{1}{2} col = \frac{1}{2} n \geq \mathcal{R}$$

Case 3 There are double-stripes of both types ($2 \times n$ and $n \times 2$) which are fully occupied by $Exp(W)$. Then it implies that there is no double-stripe that is fully occupied by $Exp(B)$. Then the inequality (2) can be verified, using a similar argument as in Case 1 (also without loss of generality assume $Exp(B)$ to be connected).

In conclusion, with the condition that the number of blacks and whites are equal and with a density-limit condition ($c = \frac{1}{2}$), we verify the Hall's condition. Then it implies the tileability of the rest of the board. Our problem is proved.

5. Three Dimensional Gridboard

The Hall's theorem can be carried out in the case of a three dimensional gridboard as well. The density-limit condition can be adapted for the three dimensional case: in any $R \times R \times R$ cube, no more than $\lceil cR^2 \rceil$ blocks are removed. Our approach to prove the inequality $\Delta \geq \mathcal{R}$ is that . One idea that we have is that Δ (the difference between white blocks and their black neighbors) can be approximated by the surface area of $Exp(W)$, and the $\mathcal{R}$ (the number of removed black blocks) can be approximated by the volume of the region and the density-limit condition. Specifically, we will show that

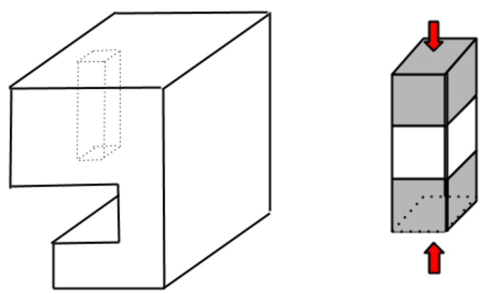
$$\Delta = c_1 \delta Exp(W) \geq \inf \sum_{\mathcal{C}} \lceil c_2 d^2 \rceil \geq \mathcal{R}$$

where $\delta Exp(W)$ denotes the surface area of the region $Exp(W)$. And the summation is summed over a set $\mathcal{C}$ of grid cubes that covers region $Exp(W)$ and d is the side length of each cube.

Next investigate the leftmost inequality. if we slice the region $Exp(W)$ along grid lines that are parallel to z-axis, we will obtain small vertical pieces $1 \times 1 \times k$. For each of this pieces, the top-end and bottom-end are on the surface of $Exp(W)$.

On the other hand, this vertical piece are alternating black and white blocks, and both of the ending block are black (because the region $Exp(W)$ only has black blocks on the boundary. Thus, in this vertical piece, the number of black blocks will be more than the number of white blocks by one. Altogether, we will have $2\Delta =$ surface of $Exp(W)$ that is parallel to the xy-plane. A similar equation will hold for the surface that is parallel to the yz-plane(or zx-plane). As the result,

$$6\Delta = \delta Exp(W)$$



(a) (b)

FIGURE 2. Vertical Slice

And the left equation follows. The rightmost inequality can be obtained immediately from the density-limit condition. What is left to prove is the most important and most difficult part: the inequality

$$\delta Exp(W) \geq \inf \sum_{\mathcal{C}} \lceil kd^2 \rceil$$

We believe that this inequality $\delta S \geq \inf \sum_{\mathcal{C}} \lceil kd^2 \rceil$ is true, and the idea to prove is to find a set of suitable cubes that cover the region. This can be done by staring from a unit grid cube inside S, and we try to enlarge that unit cube into a bigger cube of a suitable size: not too big but contains a good amount of surface area of $Exp(W)$ inside. Then we will delete this cube from our original region $Exp(W)$, and continue investigating the rest region. We do not provide an explicit proof in here.

6. COMMENTS

This problem has started with a matching between black and white blocks in a well-mannered graph, like a chessboard. This can be considered as a partial matching problem: when some blocks are randomly removed and we try to find a matching in the rest of board. One way that this problem can be carried on is to consider a matching in not only a chessboard or a gridboard but a more abstract graph with a similar property. A possible further question is: what is the "similar property", in the way that a matching exists in the original graph, but when some points are removed off a matching becomes questioned.

Acknowledgement

This paper is made possible through supports from everyone involved, and I would like to especially dedicate my acknowledgement of gratitude toward my mentor and supervisors as followed.

First and foremost, I would like to thank my mentor, Ben Yang, for his most support. He worked together with me from the very beginning until the end, and came up with many key ideas in this paper.

Secondly, I would like to thank my and Ben's supervisor, Professor Larry Guth, who passed the original idea of the problem to us, without which this problem would have not been started.

Thirdly, I would like to thank Professor Jacob Fox and Professor Pavel Etingof for adding comments that we can improve on, and for suggesting ways that we can work further.

Finally, I would like to thank my parents for sending the best emotional support oversea. This paper would not be possible if missing any of them.

References

[1] Ron Aharoni, *Infinite matching theory.* Discrete Mathematics 95 (1991), 6-7