A Link Theoretic Perspective on the Isotopy Type of Real Algebraic Curves

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Abstract

We explore the problem of determining the ambient isotopy classes that can be realized by non-singular real algebraic curves of a given degree m on the real affine or projective plane. In particular, we look at a conjecture posed by V. Ragsdale regarding the upper (or lower) bounds on the number of the so-called even connected components of real algebraic curves that assume the maximal number of components in terms of its degree. Following the insights of P. Gilmer, we suggest an approach to the proof of the Ragsdale-Petrovskii hypothesis by the use of link theory. Specifically, given an algebraic curve A , we wish to find an expression, in a form of an inequality, relating the isotopy invariants of $\mathbb{C}A$ in $\mathbb{C}\mathrm{P}^2$ and quantum 3-manifold invariants of a pair of correspondent links in rational homology 3-spheres.

1 Introduction

Given a real algebraic curve, one may ask how its (Hausdorff) connected components are arranged in the ambient real projective plane. To be more precise, given a nonsingular algebraic curve A corresponding to a real polynomial in two variables of degree m, we want to know what are all the ambient isotopy classes its set of real points $\mathbb{R}A$ in $\mathbb{R}\mathrm{P}^2$ can assume. As a real projective algebraic curve is topologically equivalent to a disjoint union of closed loops, an isotopy class can be described in affine space as a nesting structure on a collection of circles and a possible line- representing the *ovals* and *odd component* respectively- on a plane. The isotopy problem of real algebraic curves have been studied in two directions. One is by the method of construction, where one finds real algebraic curves of a given degree that realize isotopy classes. Our project focuses on the second perspective which is the study of prohibitions on isotopy classes of a curve. This refers to determining restrictions on isotopy of a family of curves induced by the topology of related geometric structures. By analyzing the complex variety $\mathbb{C}A$ of A of algebraic degree m in $\mathbb{C}\mathrm{P}^2$, one can deduce restrictions on the number of *even (resp. odd)* ovals- ones that are nested in an even (resp. odd) number of other ovals- in terms of m. The kinds of algebraic curves of special interest are the *maximal curves* (or *M-curves*) that consist of the maximum number of components with respect to its algebraic degree.

V. Ragsdale, after her study of real algebraic plane curves, imposed a conjecture relating an upper bound on the number of even ovals to the algebraic degree of the curves. Her conjecture has later been shown to be false but as of now it has neither been proven nor disproved for maximal curves. The central focus of this project is to address this problem. Because the problem is a statement on isotopy, we believe it is natural to consider a link theoretic approach to the problem. Indeed, there is already a developed formalism for real algebraic curves in terms of link theory by P. Gilmer. Through this formalism, it is suggestive that a proof of Ragsdale's conjecture can be found by a relatively simple two-piece methodology; Given an algebraic curve, create a pair of links in rational homology 3-spheres and a cobordism between them that encodes information about its isotopic and algebraic structure. Select a set of invariants of 3-manifolds and derive an expression on their values for a pair of arbitrary links bounding a (smooth) surface embedded in a 4-manifold. Due to the limited amount of time offered to complete the project, we were not successful in proving Ragsdale's conjecture. However, we have a framework to a possible proof that can be crafted as a later project.

In this paper, we give a more detailed background on the study of real algebraic curves in section 2. An exact statement of the problem is presented in section 3, section 4 provides a discussion on Gilmer's application of link theory to algebraic curves in $\mathbb{R}\mathrm{P}^2$, section 5 gives an abstraction of the main ideas of Gilmer, all the proposed heuristics and ideas for the proof of the conjecture are found in section 6, the results are given in section 7, extensions of the research problem as future project are mentioned in section 8, and section 9 contains final remarks and assessments about our research project. A bibliography is given in the final section.

2 Preliminaries

Many of the fact presented in this section can be found in [De] or [Vi1]. We begin with a series of definitions and conventions. A *real algebraic curve*, or simply *algebraic curve* A is a nonsingular real polynomial in two variable so that its real points $\mathbb{R}A$ can be embedded in $\mathbb{R}\mathrm{P}^2$ and its complex variety $\mathbb{C}A$ in $\mathbb{C}\mathrm{P}^2$. The degree of its polynomial will be denoted m. Each Hausdorff connected component of $\mathbb{R}A$ are homeomorphic to S^1. Its *real projective isotopy* (or *isotopy*) refers to the equivalence class of smooth embeddings $f : \amalg^N S^1 \to \mathbb{R}\mathrm{P}^2$ ambient isotopic to $\mathbb{R}A$, where N is the number of components of $\mathbb{R}A$. Following Gilmer in [Gi1], the *odd component* of A, denoted as C_0, is the component of $\mathbb{R}A$ that has a Möbius neighborhood. All the other components are the *ovals* of A. If A has even degree then it has only ovals while in the case of odd degree it has (at most) one odd component. Given two ovals c_1, c_2, we say that c_1 is nested in c_2, or $c_1 \Subset c_2$, if c_1 is contained in the interior of c_2, the component of $\mathbb{R}\mathrm{P}^2 \setminus c_2$ that is homeomorphic to a disk. We define an oval to be *even* (resp. *odd*) if it is nested in an even (resp. odd) number number of other ovals. Clearly, the isotopy type of A is completely characterized by how its ovals are nested and how they are positioned with respect to C_0 if it exists.

It is often difficult to determine the isotopy of an algebraic curve directly from its polynomial. Rather, we deduce restrictions on the isotopy by analyzing the topology of $\mathbb{C}A$ or related spaces. Perhaps the most important isotopic restriction, which is algebraic geometric in nature, is the Harnack's inequality, giving an upper bounds on the number of components N of $\mathbb{R}A$. Let m be the degree of algebraic curve A. Then

$$N \leq \frac{1}{2}(m-1)(m-2) + 1 \tag{1}$$

We know that the (geometric) genus g of $\mathbb{C}A$ is $(m-1)(m-2)/2$ so (1) can be re-expressed as $N \leq g + 1$. A proof of (1) is given in [Gi1]. We say A is *maximal* if the Harnack inequality is saturated. For each m one can find a maximal curve of degree m. One is also interested in deducing information about the nesting structure of A. Bezout's theorem gives quick restrictions on the nesting structure by considering other algebraic curves of a known degree in an affine subset of $\mathbb{R}\mathrm{P}^2$ that intersect representatives of isotopy classes at a certain number of points. For a fixed affine open set, if B is a representive of an isotopy class $\mathcal{C}$ and if there is an algebraic curve A' that intersects B at more then mm' points in the affine set, then $\mathcal{C}$ is not realizable by an algebraic curve of degree m.

Stronger restrictions on nesting can be derived by considering $\mathbb{C}A$. Many of them are in terms of the number p, n of even and odd ovals respectively. Usually, these restrictions are in the form of a congruence or an inequality. An example of the former is the Gudkov-Rokhlin congruence for A maximal and of degree $m = 2k$,

$$p - n \equiv k^2 \bmod 8 \tag{2}$$

An example of the latter is the Petrovskii inequality for an arbitrary A of degree $2k$,

$$-\frac{3}{2}k(k-1) \leq p - n \leq \frac{3}{2}k(k-1) + 1 \tag{3}$$

This example is actually significant as it is a weaker version of the conjectured inequality for maximal curves that will discuss in the next section.

3 Description of Problem

Ragsdale proposed a conjecture that gives an upper bound on the number of even and odd ovals p, n of an algebraic curve. Because of its resemblance to (3), the inequality is often referred to as the Ragsdale-Petrovskii conjecture. The conjecture says that for any maximal algebraic curve A of degree $2k$ we have,

$$p \leq \frac{3}{2}k(k-1) + 1, \qquad n \leq \frac{3}{2}k(k-1) + 1 \tag{4}$$

There is a similar statement for the odd degree case. The original conjecture says that (4) holds for any curve but O. Viro in [Vi2] was able to construct algebraic curves that violated the inequality. However, these counterexamples are not maximal, and indeed Ragsdale conceived the conjecture based on an analysis of maximal curves.

Statements such as (2) and (3) on the real projective isotopy invariants such as N, p, n were discovered by the first author in [Vi1] as part of a list of known isotopic restrictions in a manner analogous to the case for the study of knots and their classical invariants before a modern understanding from low dimensional topology was developed. The idea of being able to unify many of the classical results found in [Vi1] in the vein of contemporary low dimensional topology was very attractive. Although these results have derived by methods from the study of signatures and Spin structures on branched coverings of spaces as discussed in [De], they can be too technical or rigid to work with. A less complex analytic perspective was more sought after for a problem on isotopy.

A first attempt from both authors to understand the Ragsdale-Petrovskii inequality in terms of link theory was through S. Orevkov's application of link signatures to derive restrictions on the isotopy of low degree algebraic curves, while the links in study were derived from the isotopy of the curves (see [Or]). Our insight was to use link cohomology such as Khovanov homology to find the proper inequalities from the derived links. However, later we discovered that Orevkov's approach is not adequate to prove the conjecture because the only way the algebraic structure of a curve is seen to control the isotopy is through a single inequality by K. Murasugi and A. Tristram, but this obstruction is too weak. What is needed is a way to generate a variety of isotopy restrictions to work with.

Fortunately, P. Gilmer in [Gi1], [Gi2], [Gi3] offers another link theoretic formalism the study of real algebraic curves but it is more fruitful for a unifying theoretical understanding of the classical restrictions in term of the classical real projective isotopy invariants. Not only is this true, but Gilmer was able in [Gi3] to derive many of the classical results that appear in [Vi1] such as (2) and (3). Therefore, we believed it would be helpful to spend a substantial amount of time understanding Gilmer's formalism for the isotopy problem of real algebraic curves to gain a broad perspective on isotopic restrictions (of inequality type) in order to know and find the missing pieces to the puzzle of proving the Ragsdale-Petrovskii conjecture.

4 Gilmer's Link Theoretic Study on Algebraic Curves

Gilmer's formalism describes real projective isotopy invariants and isotopic restrictions of algebraic curves in terms of quantum 3-manifold invariants and cobordisms between links in rational homology 3-spheres. It follows an elegant procedure. Given an algebraic curve A of degree m, one constructs a pair of links $L(A)$, L_m and a cobordism G_A between them, corresponding to its real projective isotopy, algebraic structure, and complex analytic structure respectively as in [Gi1]. One then considers a family of generalized Casson-Gordon invariants and determines a relationship of their values for a pair of concordant colored links in a form of an inequality induced by the topology of an arbitrary cobordism- the generalized Murasugi-Tristam inequality- following [Gi2]. And finally, one finds a specific choice of Casson-Gordon invariants and an appropriate coloring for $L(A)$ and L_m that encodes information on the real projective isotopy invariants in question to specialize the general inequality to derive the corresponding restrictions on isotopy, as seen in [Gi3]. The congruences are derived in the same exact manner but the generalized Arf invariants for colored links are used instead.

The two central rational homology 3-spheres of interest are $\tilde{Q}$ the S^1 tangent bundle of $\mathbb{R}\mathrm{P}^2$ defined as $\{(x,v)|\, x \in \mathbb{R}\mathrm{P}^2,\, v \in \mathrm{T}_p\mathbb{R}\mathrm{P}^2,\, \|v\| = 1\}$, and Q the projective tangent bundle of $\mathbb{R}\mathrm{P}^2$ defined as $\{(x,l)|\, x \in \mathbb{R}\mathrm{P}^2,\, l \in P\mathrm{T}_p\mathbb{R}\mathrm{P}^2\}$. The homology group $H_1(Q)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by a fiber $\mathfrak{f}$ and a line in the base space $\mathfrak{b}$ while $H_1(\tilde{Q}) \simeq \mathbb{Z}_4$. Know, for a given arbitrary (smooth) curve $C \subseteq \mathbb{R}\mathrm{P}^2$, we trace a link $L(C)$ in Q as $\{(x,l)|\, x \in C,\, l \in P\mathrm{T}_p\mathbb{R}\mathrm{P}^2$ is tangent to C at $x\}$. $\tilde{L}(C)$ is a link in $\tilde{Q}$ defined similarly as $\{(x,v)|\, x \in C,\, v \in \mathrm{T}_p\mathbb{R}\mathrm{P}^2$ is unit and tangent to C at $x\}$. If C is assigned an orientation, then we define $L_+(C) \subseteq \tilde{Q}$ as the link traced by the corresponding unit vectors along C. $H_1(\tilde{Q})$

is generated by $L_+(l)$, where l is an oriented line in $\mathrm{R}P^2$. $L(A)$ is defined as $L(\mathbb{R}A)$ and $L_m := \mathfrak{L}_m$ where $\mathcal{L}_m$ denotes a collection of m distinct lines in $\mathrm{R}P^2$. Note that L_m is well defined up to isotopy in Q.

If we embed L_m and $L(A)$ in $Q \times [0,1]$ such that $L_m \subseteq Q \times \{0\}$ and $L(A) \subseteq Q \times \{1\}$, a surface $G_A \subseteq Q \times I$ such that $\partial G_A = L(A) \cup L_m$ can be constructed via the following decomposition of $\mathrm{C}P^2$. We define $h : \mathbb{C}\mathrm{P}^2 \to [0,1]$ as

$$h[z_0, z_1, z_2] := \frac{|z_0^2 + z_1^2 + z_2^2|}{|z_0|^2 + |z_1|^2 + |z_2|^2}$$

Clearly h is well defined. After inspection, one notices that $h^{-1}(1)$ can be identified as $\mathbb{R}\mathrm{P}^2$. $h^{-1}(1-\delta)$ turns out to be orientally diffeomorphic to $\tilde{Q}$ for some δ. If we consider complex complex conjugation conj: $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ ($[z_0, z_1, z_2] \mapsto [\bar{z_0}, \bar{z_1}, \bar{z_2}]$), and for a conj-invariant subset $X \subseteq \mathbb{C}\mathrm{P}^2$ we define $\overline{X}$ as the orbit space of the action then $\overline{h^{-1}(\delta)}, \overline{h^{-1}(1-\delta)}$ are orientally diffeomorphic to Q and $\overline{h^{-1}[\delta, 1-\delta]}$ is orientally diffeomorphic to $Q \times I$ where $I := [0,1]$. Now, $\overline{\mathbb{C}A \cap h^{-1}(1-\delta)}$ and $\overline{\mathbb{C}A \cap h^{-1}(\delta)}$ are identified as $L(A)$ and L_m respectively. Thus we are in good position to define G_A as $\overline{\mathbb{C}A \cap h^{-1}[\delta, 1-\delta]}$. We shall call G_A the Riemann cobordism of A (capturing the fact that it is a quotient of a Riemann surface). All the topological properties of G_A are given in theorem 6.1 in [Gi1]. One of is is that the Euler characteristic $\chi(G_A) = (m - m^2)/2$. More will be discuss briefly.
Note that we follow Gilmer's terminology in that we say a surface G is a cobordism between links L_0, L_1 in a 3-manifold M is $\partial G = L_0 \cup L_1$ and it satisfies the stronger condition that G is smoothly embedded in $M \times I$, and $L_0 \subseteq M \times \{0\}$, $L_1 \subseteq M \times \{1\}$. In other literatures, such G is usually called a (link) concordance between L_i and hence we say that L_i are link concordant.

Often in working with the soon to be defined Casson invariants for a link L in a 3-manifold M, we consider surfaces F in M such that $\partial F = L$ (and no oriented closed connect components). F is called the spanning surface of L. We also talk about a function γ of a surface S defined in the following way. We first deal with the case when S is connected. If S is closed and oriented, then we set $\gamma(S) = 0$. If it is closed and unorientable, $\gamma(S) \in H_1(S, \mathbb{Z}_2)$ is the unique generator of the group. If it has nonempty boundary ∂S with an orientation, then $[\partial S] \in H_1(S, \mathbb{Z})$ is divisible by 2. So it makes sense to define $\gamma(S) := \frac{1}{2}[\partial S] \in H_1(S)$. For the general case that S has many components S_i, we define $\gamma(S) := \oplus_i \gamma(S_i) \in H_1(S)$. With an abuse of notation, we also want to denote $\gamma(S)$ as $j_*\gamma(S) \in H_1(M)$ if S is embedded in 3-manifold M through $j : S \hookrightarrow M$.

It turns out by [Gi2] for a given link, there is a bijection between elements $s \in H_1(M)$ such that $2s = [L]$ and spanning spaces of L up to a simple equivalence extending isotopy via γ. Denote this set as $\Gamma_2(L)$. Lastly, for a fixed integer $d > 1$, we also want to talk about a (d-)characteristic $\psi \in H^1(M - L, \mathbb{Z}_d)$ of an oriented link L. It is convenient to restrict our attention to the ψ that are characterized by a tuple $(\psi(m_i))$ in $\mathbb{Z}_d$ where m_i is the oriented meridian of component L_i linked positively. As discussed in [Gi2], there is an equivalence between these d-restricted characteristics of L and elements $s \in H_1(M)$ such that $ds = [L]$. The two mentioned equivalences are significant in the computation of the Casson invariants and the analysis of the isotopy of algebraic curves. In fact, because of this equivalence, we can talk about finding a spanning surface for a link as assigning a coloring to it.

The Casson-Gordon invariants are defined for a link L in M and a spanning surface class $\gamma \in \Gamma_2(L)$. They are given in terms of the signature and nullity of the symmetric bilinear form $\mathcal{G}_F$- the Göritz form of a surface F and $e(F)$. We wont give an exact definitions of $\mathcal{G}_F$ and $e(F)$ here but we'll say that the former is given for two links a, b in M as a modified linking number $\mathrm{lk}(a, \tau b)$ where τ is a function on links and the latter is given as $-\Sigma_i \mathrm{lk}(L_i, \hat{L}_i)$ where $\hat{L}_i$ is a kind of pertubation of component L_i along F. See [Gi2] for the definitions. The invariants $s(L, \gamma)$ and $\eta(L, \gamma)$ are given as

$$s(L, \gamma) := \mathrm{Sign}(\mathcal{G}_F) + \frac{e(F)}{2}$$

$$\eta(L, \gamma) := \mathrm{Null}(\mathcal{G}_F) + b_0(F) - 1$$

Where F is a spanning surface of L such that $\gamma(F) = \gamma$, $\mathrm{Sign}(\mathcal{G}_F)$, $\mathrm{Null}(\mathcal{G}_F)$ is the signature and nullity of the Göritz form, and b_0 is the zeroth Betti number. There are generalizations σ_d, η_d for any integer $d > 1$ but we will leave it to the reader to refer to [Gi2] for their definitions. The above is for the case $d = 2$.

Gilmer in [Gi2] as theorem 7.3, derives a generalized cobordic inequality as for a pair of links L_1, L_2 in 3-manifold M with nonintersecting cobordism G in terms of the Casson-Gordon invariants and the topology of G. With $\gamma \in \Gamma_2(L_1)$ this is

$$|s(L_2, \gamma') - s(L_1, \gamma) - e(G)/2| + \eta(L_2, \gamma') + \eta(L_1, \gamma) \leq -\chi(G) + 2\Delta \tag{5}$$

Where $\gamma' := \gamma + \gamma(G)$,
$\Delta := \min\{(\eta' + \eta, \eta + b_0(G, L_1), \eta' + b_0(G, L_2), \dim H_1(M \times I, G, \mathbb{Z}_2))\}$, if we set $\eta := \eta(L_1, \gamma)$ and likewise for η'. We will not define $e(G)$ here but we will say that for the case that G is the Riemann cobordism of an algebraic curve A of degree $2k$, $e(G) = 2k^2$. An analogous inequality exists from σ_d, η_d but we won't give them here. See theorem 7.5 in [Gi2]. From this, we can apply (5) to the case where $M = Q$, $L_1 = L_{2k}$, $L_2 = L(A)$, and $G = G_A$. We have then an inequality of Murasugi-Tristram type given in theorem 6.1 in [Gi3] as

$$|s_{\gamma'}(\mathbb{R}A) - s_\gamma(L_{2k}) - k^2| + \eta_{\gamma'}(\mathbb{R}A) + \eta_\gamma(L_{2k}) \leq 2k^2 - k + 2\Delta \tag{6}$$

Where $s_\gamma(C) := s(L(C), \gamma)$,
$\Delta := \min\{\eta_{\gamma'}(\mathbb{R}A), \eta_\gamma(L_{2k}), \dim H_1(Q \times I, G, \mathbb{Z}_2), \dim H_1(X_\gamma, \mathbb{Q})\}$, X_γ is a double covering of $Q \times I$ extending a double covering of $Q \times \{0\}$ branched along

L_{2k} classified by ψ_γ (under the equivalences discussed earlier).

The equation (6) is used by finding the appropriate spanning surface for $L(A)$ and L_{2k} to compute the signatures and the nullities. Ad hoc methods are used to deduce the closed form of Δ in special cases. For a curve C consisting of all ovals, an example of a spanning surface for $L(C)$ can be constructed in the following way. Consider the region B^+, the set of points that lie on C or within an odd number of ovals. We construct a vector field $v : B^+ \to \mathrm{T}\mathbb{RP}^2$ (a section on $\mathrm{T}\mathbb{RP}^2$)that is tangent along C and has a positive zero just inside every even oval and a negative zero just outside every odd oval. This induces a section $s : \breve{B}^+ \to P\mathrm{T}\breve{B}^+$ ($\breve{B}^+$ is B^+ save the zeroes) whose embedded image F_C^+ of $\breve{B}^+$ in Q gives a spanning surface for $L(C)$. F_C^+ is a surface whose first homology group $H_1(F_C^+)$ is spanned by n odd ovals, p positive zeroes, and n negative zeroes. If $n^\pm$ is the number of components of B^+ with positive/negative euler characteristic and similarly for n^0 then we have that [Gi3]

$$\mathrm{Sign}(\mathcal{G}_F) = p - n^0 - 2n^-$$

$$\mathrm{Null}(\mathcal{G}_F) = n^0 + p - 1$$

A spanning surface for L_{2k} is given by first decomposing $\mathcal{L}_{2k}$ into k pairs. For each i-th pair, two lines l_{i1}, l_{i2} intersect at one point p_i. Give the two lines an orientation. Consider a collection $\mathcal{A}_i$ of lines interesting at p_i that sweep an area between them corresponding to the orientations of l_i. Define $P_k := \cup_{i,l\in\mathcal{A}_i} L(l)$. This is clearly a surface bounded by L_{2k}. The signatures and nullities are given as

$$\mathrm{Sign}(\mathcal{G}_P) = 0$$

$$\mathrm{Null}(\mathcal{G}_P) = 2k - 1$$

With these signatures and more, one can deduce stronger forms of the inequality (3) as seen in [Gi3].

5 Further Remarks on Link Theory and Algebraic Curves

Gilmer's contributions of link theory is highly significant in that the isotopy problem of real algebraic curves can be approached from the study of a restricted class of links in a rational homology 3-sphere and link concordances between pairs. For example, one can show that an isotopy class $\mathcal{C}$ is not realized by a real algebraic curve of degree m by proving that $L(B)$, L_m in Q are not concordant. This can be done by considering their link concordant invariants such the Rasmussen invariants, mentioned in [Kho2], and the Milnor invariants. There are severly notions of concordance depending on the conditions of the bounded surface. For example, smooth concordance refers to the surface having a smooth embedding in the corresponding 4-manifold. For maximal curves in particular, we can restrict to concordances that have (total) genus 0 since this is true for their Riemann cobordisms.

To derive formulaic restrictions on isotopy of algebraic curves, we select quantum 3-manifold invariants of our choosing and how their values for a pair of links are controlled by the topological properties of concordances between them. Gilmer only used Casson-type and Arf invariants but one can study other quantum invariants such as the Jones and HOMFLY polynomials and the Witten-Reshetikhin-Turaev invariants. Their corresponding homology theory can be applied to better understand the cobordic expressions that can be derived. For example, we know that the Casson invariants are derivable from Heegaard Floer homology and Khovanov homology is a categorification of the Jones Polynomial (see [Sa] and [Kho1]). Also, link concordant invariants are involved in providing obstructions in the form of inequalities for a link to be *slice*- to be concordant to the unlink. This is seen in Rasmussen's inequality which says that for a link L on the boundary of four-ball B^4 that bounds a surface in B^4 of genus g then,

$$|s(L)| \leq 2g \tag{7}$$

Where $s(L)$ is the Rasmussen invariant of L. It is suggestive that link cobordance invariants are involved in deriving cobordic inqualities such as (5).

6 Heuristics and Methods to Explore

We have a list of ideas and heuristics proposed for proving the Ragsdale-Petrovskii inequality in the direction of Gilmer. However, due to the limited amount of time, only half of them had been carefully analyzed.

1. The first natural idea is to work with the inequality (6) since it was successful in reproducing many of the classical results such as the Petrovskii inequality (3). Because the conjecture (4) is a statement on maximal curves, we thought it would be worthwhile to find some topological properties of the complex variety $\mathbb{C}A$ and the Riemann cobordism G_A whenever A meets the maximality condition. For example, $\chi(\mathbb{C}A_+) = \frac{1}{2}\chi(\mathbb{C}A)$ and G_A is orientable (although the latter is true more generally for dividing curves). The idea is to exploit the topology of G_A to derive stronger inequalities.

2. A similar idea is to find another real function $f : A \to [0, 1]$ for such that $f^{-1}(1) = \mathbb{R}\mathrm{P}^2$ and is more sensitive to maximality of A than the function $h : \mathbb{C}A \to [0, 1]$ defined for the construction of G_A. f would give rise to another cobordism G'_A hoped to give a stronger inequality from (5).

3. An idea that appeared is the consideration of different spanning surfaces for $L(A)$ and L_{2k} to calculate other signatures and nullities to substitute to (6).

4. Another worthy strategy is to consider derived links different but similar to $L(A)$, such as $\tilde{L}(A)$ or $L_+(A)$, that lie in a rational homology 3-sphere, like $\tilde{Q}$ in the given examples. The results derived by Gilmer are only based on Q. In particular, one can use the inequality (5) and perform similar calculations that are given in [Gi3]. The only challenge is that while $h^{-1}(1-\delta) \simeq_{\text{diff}} \tilde{Q}$, $h^{-1}(\delta) \simeq_{\text{diff}} S\text{T}S^2$. However, $\overline{h^{-1}(\delta)} \simeq_{\text{diff}} \overline{h^{-1}(1-\delta)}$ so it is natural to ask if there is an involution $\iota : \mathbb{C}\text{P}^2 \to \mathbb{C}\text{P}^2$ such that $h^{-1}(1-\delta)$ is fixed and ι acts only on the fibers of $\pi : h^{-1}(\delta) \to h^{-1}(0)$ (i.e. $\pi(\iota(\mathbf{z})) = \pi(\mathbf{z})$ for $\mathbf{z} \in h^{-1}(\delta)$). The purpose is to be able to work with $H_1(\tilde{Q}) \simeq \mathbb{Z}_4$ which has elements of 4-torsion, compared to $H_1(Q) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

5. Lastly, what also can be explored are alternative colored link invariants. It may be the case that the 2-signatures and 2-nullities s, η are too weak. This idea is motivated by the simple observation that not even the numbers n^+ nor p^+ could not be isolated by in the signature-nullity formulas. A natural idea is to calculate the 4-signatures and 4-nullities since in this case $\Gamma_4(L(A)) = \mathbb{Z}_4$ since $[L(A)] = 0$ in Q, so all the spanning spaces considered in [Gi3] used to compute the d-signatures and d-nullities for $d = 2$ also applies for $d = 4$. However $\Gamma_4(L_{2k}) = \mathbb{Z}_4$ only when k is even, so one must treat k odd case carefully.

7 Results

For the first heuristic, we deduced that if A is maximal then G_A is a surface such that each of its component has zero genus. This property and the ones mentioned in 1. are the only topology properties of G_A that we could find. In addition, we were unable to derive a nontrivial result that follows from these properties.
For the second heuristic, we were unable to find this described real function.
For the third heuristic, it was realized by the first author that he was not aware primarily that the spanning surfaces constructed in [Gi3] realizes all the elements in the group $H_1(Q)$, and hence the subgroups $\Gamma_2(L(A))$ and $\Gamma_2(L_{2k})$, until it was re-emphasized for him that is isomorphic to the modest group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This means, that creating new spanning surfaces for both $L(A)$ and L_{2k} will not provide any more information.
For the fourth heuristic, we were not able to construct the involution ι described. From this involution, we can deduce about the orbit spaces that $\overline{h^{-1}(1-\delta)} \simeq_{\text{diff}} \tilde{Q}$ and $\overline{h^{-1}(\delta)} \simeq_{\text{diff}} \text{PT}S^2$ (from $h^{-1}(\delta) \simeq_{\text{diff}} S\text{T}S^2$). However, there are two obstacles. One is where or not $\text{PT}S^2$ is diffeomorphic to $S\text{T}\mathbb{R}\text{P}^2$. Also, we were unable to see if $\overline{h^{-1}[\delta, 1-\delta]}$ is diffeomorphic to $\tilde{Q} \times I$, otherwise it is a nontrivial I-bundle over $\tilde{Q}$.
The final heuristic was never attempted.

8 Future Work

Although not much has been accomplished in the duration of the project in terms of written results, there is plenty of unexplored territory to investigate for proving (or disproving) the Ragsdale-Petrovskii. Heuristic 4 can be explored as described in section 7 and the calculations for heuristic 5 should be completed. As mentioned in section 5, we could examine various quantum invariants and homology theories to produce new coboric inequalities (or congruences). These include the Ozsvath-Szabo knot Floer homology, the Khovanov homology, the colored Jones polynomials, the HOMFLY polynomials, the Witten-Reshetikhin-Turaev invariant, the Aarhus integral, and the Froyshov invariant. The many papers for beginning this exploration include [Ba], [Be], [Fr], [Gi4], [Gr], [Kho1], [Kho2], and [Sa]. An important question to address is whether or not there is a quantum invariant that is powerful enough to give bounds on p, n separately.

A curious question is if the Harnack inequality (1) can be understood as an obstruction for $L(A)$ in the 4-manifold $h^{-1}[0, 1-\delta]$ from being slice in a specific sense in a manner similar to the Rasmussen inequality (7) derived from Khovanov homology. If so, then what is the link concordance invariant (or better the 3-manifold invariant or link homology) that gives rise to it? Is the corresponding link homology theory the one most suitable for the study of real algebraic curves?

9 Commentary and Concluding Remarks

The project has been a valuable learning experience for both authors. There were many successes such as the discovery of a new perspective on real projective algebraic curve isotopy involving the rapidly developing field of low dimensional topology. However, there are many things that could have been done differently that would have provided us more time for producing results. Most certainly the reading of literature was done in excess. However it has for a few times saved us from dead-ends such as proceeding by mere computation of the Orevkov braid invariant of algebraic curves. The first author in particular has witnessed the importance of having a strong mastery of the fundamentals (e.g. algebraic topology, algebraic geometry, complex analysis) to be adept in mathematical research but is glad to have experienced the benefits of having a “big picture” mindset.

To conclude the paper, the first author will like to acknowledge the help of the second author for his helpful comments and discussions that allowed for a solid understanding of the read literature. Much thanks is given to P. Etingof for giving him the opportunity to participate in the Summer Program for Undergraduate Research (SPUR) and his feedback on our project. Additional thanks to S. Gerovitch, T. Khovanova, and A. Ferrigno for making the experience enjoyable.

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