Homological Algebra over the Representation Green Functor

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Abstract

In this paper we compute $\mathcal{E}xt_R(F, F)$ for modules over the representation Green functor R with F given to be the fixed point functor. These functors are Mackey functors, defined in terms of a specific group with coefficients in a commutative ring. Previous work by J. Ventura [1] computes these derived functors for cyclic groups of order 2^k with coefficients in $\mathbb{F}_2$. We give computations for general cyclic groups with coefficients in a general field of finite characteristic as well as the symmetric group on three elements with coefficients in $\mathbb{F}_2$.

1 Introduction

Mackey functors are algebraic objects that arise in group representation theory and in equivariant stable homotopy theory, where they arise as stable homotopy groups of equivariant spectra. However, they can be given a purely algebraic definition (see [2]), and there are many interesting problems related to them. In this paper we compute derived functors $\mathcal{E}xt$ of the internal homomorphism functor defined on the category of modules over the representation Green functor R. These Mackey functors arise in Künneth spectral sequences for equivariant K-theory (see [1]). We begin by giving definitions of the Burnside category and of Mackey functors. These objects depend on a specific finite group G. It is possible to define a tensor product of Mackey functors, allowing us to identify 'rings' in the category of Mackey functors, otherwise known as Green functors. We can then define modules over Green functors. The category of R modules has enough projectives, allowing us to do homological algebra. We compute projective resolutions for the R-module F, the fixed point functor, and then proceed to compute $\mathcal{E}xt_R(F, F)$. We perform these computations for cyclic groups with coefficients in a field and for the symmetric group on three elements with coefficients in $\mathbb{F}_2$ using techniques derived from those used by J. Ventura in [1] to compute $\mathcal{E}xt(F, F)$ for the group C_{2^k} with coefficients in $\mathbb{F}_2$.

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3 Background and Notation

3.1 Mackey Functors

We begin with definitions of the Burnside category and Mackey functors. For a finite group G we shall denote the Burnside category as $\mathcal{B}_G$. Before defining $\mathcal{B}_G$ it is convenient to begin with an auxiliary category $\mathcal{B}'_G$. Given a finite group G, $\mathcal{B}'_G$ is the category whose objects are finite G-sets and morphisms from T to S are isomorphism classes of diagrams

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] & \\ T & & S \end{tikzcd}$$

We define composition of morphisms to be the pullback of two diagrams:

$$\begin{tikzcd}[column sep=small] & U_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & & U_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ T & & V \circ & S & & T \end{tikzcd} = \begin{tikzcd}[column sep=small] & & & U_1 \times_T U_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \\ & & U_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & U_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & \\ & S & & T & & V & \end{tikzcd}$$

If we define addition of morphisms as
it follows that $Hom(S, T)$ is a commutative monoid. This construction of $\mathcal{B}'_G$ allows us to define $\mathcal{B}_G$:

Definition 3.1. *Given a finite group G, the Burnside category $\mathcal{B}_G$ is the category whose objects are the same as the objects of $\mathcal{B}'_G$ and for two G-sets S and T, $Hom_{\mathcal{B}_G}(S, T)$ is the Grothendieck group of $Hom_{\mathcal{B}'_G}(S, T)$.*

$$
\begin{tikzcd}[column sep=small]
 & U_1 \arrow[dl] \arrow[dr] & & & U_2 \arrow[dl] \arrow[dr] & & & U_1 \coprod U_2 \arrow[dl] \arrow[dr] & \\
T & + & S & T & & S \arrow[r, phantom, "="] & T & & S
\end{tikzcd}
$$

Given a map of G-sets $f : S \to T$, there are two corresponding morphisms in $\mathcal{B}_G$. The first is depicted on the left in the figure below and is referred to as a 'forward arrow' (denoted by f) and the second is on the right and is referred to as a 'backward arrow' (denoted by $\hat{f}$).

$$
\begin{tikzcd}
 & S \arrow[dl, "="'] \arrow[dr, "f"] & & & S \arrow[dl, "f"'] \arrow[dr, "="] & \\
S & & T & T & & S
\end{tikzcd}
$$

Definition 3.2. *A Mackey functor for a group is an additive contravariant functor from the Burnside Category $\mathcal{B}_G$ to the category of Abelian groups. A morphism of Mackey functors is a natural transformation.*

Recall that as a consequence of additivity and functoriality we have the following:

Proposition 3.3. *It is sufficient to define Mackey functors on G-sets of the form G/H for any subgroup H, quotient maps between these orbits, and conjugation maps from an orbit to an isomorphic orbit.*

Definition 3.4. *Denote by π^H_K, the quotient map from G/K to G/H where $K \subset H$ and by $\gamma_{x,H}$ the conjugation map that maps $H \to Hx^{-1}$. Given a Mackey functor M, we refer to $M(\pi^H_K)$ as a restriction map, $M(\hat{\pi}^H_K)$ as a transfer map, and $M(\gamma_{x,H})$ as a conjugation map. We shall denote restriction maps as r^H_K, transfer maps as t^H_K, and conjugation maps as τ.*

We recall three examples of Mackey functors that we will be concerned with in this paper.

Example 3.5. *We denote by R the representation functor defined by $R(G/H) = RU(H)$ where $RU(H)$ is the complex representation ring of H. The restriction maps are restriction of representations, the transfer maps are induction of representations, and the conjugations $R(\gamma_{x,H})$ send a vector space with an action of H to the same vector space where the action is redefined as $h \cdot v = xhx^{-1}v$.*

Example 3.6. *Let M be an abelian group with G-action. Denote the fixed point functor by FP_M. Then $FP_M(G/H) = M^H$. Restriction maps are inclusions of M^H into M^K and transfer maps send $m \to \sum_{h\in H/K} hm$. The conjugations are maps from $M^H \to M^{^xH}$ which send $m \to xm$. In this paper we only consider the fixed point functor for G-modules M with trivial G-action. Then $FP_M(G/H) = M$, the restrictions are multiplication by 1, the transfers are multiplication by the index $[H : K]$, and the conjugations are identity maps.*

Example 3.7. *The representable functor $[-, X] = Hom_{\mathcal{B}_G}(-, X)$ is a Mackey functor.*

Definition 3.8. *If M is a Mackey functor and X is a G-set, then we can define the Mackey functor M_X where*

$$M_X(Y) \cong M(Y \times X)$$

for any G-set Y.

We can define a tensor product on Mackey functors such that given X, Y, Z there is a one to one correspondence between maps $X \otimes Y \to Z$ and natural transformations $X(S) \otimes Y(T) \to Z(S \times T)$ for G-sets S and T [1]. The unit for this tensor product is $[-, e]$, the burnside ring Mackey functor A_e.

Definition 3.9. *The internal hom functor $\mathcal{H}om(X,Y)$ is the Mackey functor for which*

$$Hom(A, \mathcal{H}om(X,Y)) \cong Hom(A \otimes X, Y).$$

There are several useful identities related to the internal hom functor. We list them here and refer to proofs in [1, 2, 4].

$$\begin{aligned} \mathcal{H}om(M,N)(G/H) \quad \cong \quad & \mathrm{Hom}(M|_H, N|_H) \\ & \mathrm{Hom}(M, N \otimes [-, G/H]) \\ & \mathrm{Hom}(M \otimes [-, G/H], N) \end{aligned}$$

$$\mathcal{H}om_R(R_X, M)(G/H) \cong M_X(G/H) \cong M(G/H \times X).$$

We also have the property of duality which gives us the identity

$$\mathrm{Hom}(M, N \otimes [-, X]) \cong \mathrm{Hom}(M \otimes [-, X], N).$$

3.2 Green Functors

A Green functor should be thought of as a Mackey functor with an additional ring structure where multiplication is given by a tensor product of Mackey functors. There are two equivalent definitions of a Green functor: the first reflects the previous statement and the second simplifies the task of identifying Green functors and modules over Green functors.

Definition 3.10. *A Green functor R for a group G is a Mackey functor for G with maps $R \otimes R \to R$ and $[-, e] \to R$ with associative and unital properties.*

Definition 3.11. *A Mackey functor R is a Green functor if for any G-set S, $R(S)$ is a ring, restriction maps are ring homomorphisms that preserve the unit, and the following Frobenius relations are satisfied for all subgroups $K \subset H$ of G with $a \in R(K)$ and $b \in R(H)$:*

$$b(t_K^H a) = t_K^H((r_K^H b)a)$$

$$(t_K^H a)b = t_K^H(a(r_K^H b))$$

Definition 3.12. *Given a Green functor R, a Mackey functor M is an R-module if for a G-set S, $M(S)$ is an $R(S)$-module, and for a G-map $f : S \to T$ the following are true [1]:*

$$\begin{aligned} M(f)(am) &= R(f)(a)M(f)(m) && \forall\ a \in R(Y), m \in M(Y) \\ aM(\hat{f})(m) &= M(\hat{f})(R(f)(a)m) && \forall\ a \in R(Y), m \in M(X) \\ R(f)(a)m &= M(\hat{f})(aM(\hat{f})(m)) && \forall\ a \in R(X), m \in M(Y) \end{aligned}$$

Definition 3.13. *A morphism f of R-modules is a morphism of Mackey functors that preserves the action of R.*

Example 3.14. *The representation functor R is a Green functor.*

Example 3.15. *The fixed point functor is an R-module where R is the representation functor. The action of R on F is given by the augmentation map $\sigma \mapsto dim(\sigma)$ for a representation σ.*

Example 3.16. *The functor $R_{G/e} \cong R \otimes [-, G/e]$ is an R-module.*

This setup allows us to compute the homology of modules over Green functors.

4 Computation for $G = C_n$

From this point onward we will let R denote $R \otimes K$ for a fixed field K. Simlarly, F will denote FP_K for the trivial G-module K. We intend to compute the derived functors $\mathcal{E}xt_R(F, F)$ for the group $G = C_n$.

There are three steps to computing $\mathcal{E}xt_R(F, F)$. We first compute a projective resolution of F. Next we must apply the functor $\mathcal{H}om_R(-, F)$ to this projective resolution. We then compute the homology groups of the resulting chain complex to get $\mathcal{E}xt_R(F, F)$. One additional computation gives us the ring structure induced by the Yoneda composition products.

4.1 Projective Resolution for F

Proposition 4.1. *The 4-periodic chain complex*

$$0 \longleftarrow F \xleftarrow{aug} R \xleftarrow{(\sigma-1)} R \xleftarrow{1\otimes r} R_{G/e} \xleftarrow{1\otimes(\tau-1)} R_{G/e} \xleftarrow{1\otimes t} R \xleftarrow{\cdots} \tag{4.2}$$

is a four periodic projective resolution for F.

Proof. To show that this is an exact sequence it suffices to show that this sequence is exact when applied to orbits of G-sets. We begin with the Burnside category for $G = C_n$. Since F,R, and $R_{G/e}$ are all Mackey functors, we will apply each functor to orbits in $\mathcal{B}_G$. We will then apply the differentials and show that the resulting sequence is exact.

We start with the diagram of G-maps between groups in the Burnside category:

$$\begin{tikzcd} G/C_l \arrow[rr, bend right, "\pi^H_L"'] & & G/C_m \arrow[ll, bend right, "\hat{\pi}^H_L"'] \end{tikzcd}$$

Let us apply R to $\mathcal{B}_G$. We have that

$$R(G/C_m) = RU(C_m) \cong K + K\sigma + K\sigma^2 + ... + K\sigma^{m-1},$$

where σ is an irreducible representation of C_m. We choose a particular σ as follows: Let g be a fixed generator of C_m, then let σ be the representation that sends $g^{n/m} \mapsto e^{\frac{2\pi i}{m}}$. The restriction and transfer maps are restriction and induction of representations, and the conjugation maps are identity maps. Let $L = C_l$ and $H = C_m$ with $L \subset H$ and $m = kl$. Then if σ_H and σ_L are the above irreducible representations of H and L respectively, we have $r^H_L(\sigma^i_H) = \sigma^i_L$ $t^H_L(\sigma^i_L) = \sigma^i_H \sum_{i=0}^{k-1} \sigma^{il}_H$. Applying R to the diagram of orbits in the Burnside category, we have:

$$\begin{tikzcd} \sum_{i=0}^{l-1} K\sigma^i \arrow[rr, bend right, "\sigma^i_L \mapsto \sigma^i_H \sum_{j=0}^{k-1} \sigma^{jl}_H"'] & & \sum_{i=0}^{m-1} K\sigma^i \arrow[ll, bend right, "\sigma^i_L \leftarrow\!\shortmid \sigma^i_H"'] \end{tikzcd}$$

If we apply the fixed point functor with trivial action F to $\mathcal{B}_G$ each object maps to the coefficient field K. The restrictions are the identity map, the transfers are multiplication by the index $[H : L]$, and the conjugations are the identity. The resulting diagram is

$$\begin{tikzcd} K \arrow[rr, bend right, "{[H:L]=k}"'] & & K \arrow[ll, bend right, "1"'] \end{tikzcd}$$

If we apply $R_{G/e}$ to G/H, then $R_{G/e}(G/H) \cong \bigoplus_{x \in G/H} K$. The transfer map maps $\bigoplus_{x \in G/L} K$ to $\bigoplus_{x \in G/H} K$ by performing the augmentation map that sums the terms in groups of m/l coordinates. The restriction map is a diagonal map that makes m/l copies of each coordinate. There are also n/m conjugations denoted by $\tau_1, ..., \tau_{p^{n-k}}$ (each conjugation corresponds to a power of the generator, so τ_1 corresponds to conjugation by the generator) that operate cyclicly on the coordinates.

$$\begin{tikzcd} \bigoplus\limits_{x \in G/L} K \arrow[rr, bend right, "\text{aug.}"'] & & \bigoplus\limits_{x \in G/H} K \arrow[ll, bend right, "\Delta"'] \end{tikzcd}$$

Now we need a map from $R(G/H) = \sum_{i=0}^{m-1} K\sigma_H^i$ to $F(G/H) = K$. We choose this map to be the augmentation map that sends $\sum_{i=0}^{p^h-1} a_i \sigma_H^i \to \sum_{i=0}^{p^h-1} a_i$. This map is a natural transformation of Mackey functors, so it is a morphism of Mackey functors. Since it preserves the action of R, this map is also a map of R-modules.

The kernel of the augmentation map can be written as $\sum_{i=1}^{m-1} K(\sigma_H^i - 1)$. We choose the map onto the kernel to be multiplication by $(\sigma_H - 1)$. This map is surjective and is clearly a map of R-modules.

The kernel of multiplication by $(\sigma_H - 1)$ is $K \sum_{i=0}^{m-1} \sigma_H^i \cong K$. Now we must map from $R_{G/e}(G/H) \cong \bigoplus_{x \in G/H} R(G/e)$ onto the kernel in $R(G/H)$. We choose this map to be the map $1 \otimes r$ where r is the restriction r_e^G. The map $1 \otimes r$ is induced by the map $1 \times \pi_e^H : G/H \times G/e \to G/H \times G/G$. Note that $R_{G/e}(G/H) \cong R(G/H \times G/e)$, so we actually have a map from $R(G/H \times G/e) \to R(G/H)$. Thus this map is equivalent to R applied to a morphism of G-sets from G/H to $G/H \times G/e$. The morphism can be drawn as

$$\begin{tikzcd} & G/H \times G/e \arrow[dl, "p"'] \arrow[dr, "="] & \\ G/H & & G/H \times G/e \end{tikzcd}$$

Here p is the projection map onto G/H. We can write $G/H \times G/e$ as the disjoint union $\coprod_{x \in G/H} G(x, 1)$. Thus, $G/H \times G/e \cong \coprod_{x \in G/H} G/e$, so the previous morphism can be rewritten as the sum

$$\sum_{x \in G/H} \begin{tikzcd} & G/e \arrow[dl, "\pi_e^H \circ \gamma_{x^{-1},e}"'] \arrow[dr, "i_x"] & \\ G/H & & G/H \times G/e \end{tikzcd}$$

This sum is equivalent to the sum of compositions $\sum_{x \in G/H} i_x \circ \gamma_{x,e} \circ \hat{\pi}_e^H$ where i_x is the inclusion map that sends $1 \mapsto (x, 1)$ (conjugations are trivial in R). Applying R to this sum we get $\sum_{x \in G/H} t_e^H \circ p_x$ where p_x is the projection map. Thus, in mapping from $R_{G/e}(G/H) \to R(G/H)$ we are actually applying the transfer map to each individual component and then adding the results. This gives us a map from $R_{G/e}(G/H)$ onto $K \sum_{i=0}^{m-1} \sigma_H^i$, the kernel of the previous map.

The kernel of this map contains elements of $R_{G/e}(G/H)$ with coordinates that sum to 0, and we must map from $R_{G/e}(G/H)$ onto this kernel. This can be accomplished by applying the map $1 \otimes (\tau_1 - 1)$. Using a computation similar to the one given above, we see that when applied to a vector in $R_{G/e}(G/H)$, this map first cycles the coordinates and then subtracts the original vector. The resulting vector will have coordinates that sum to 0, and any vector with coordinates that sum to zero is hit by this map. Thus, we have a surjection onto the kernel of the previous map. Finally, the kernel of $1 \otimes (\tau_1 - 1)$ contains elements with the same entry in each coordinate, so it is isomorphic to K. We can map from R onto this kernel with the map $1 \otimes t$ where t_e^G is the transfer map. A computation similar to the one given above for $1 \otimes r$ shows that this map simply adds together the coefficients of elements in $R(G/C_m)$. The projective resolution can now repeat, giving us a four periodic projective resolution of F.

$\square$

4.2 Computation of $\mathcal{E}xt_R(F, F)$

We first apply $\mathcal{H}om_R(-, F)$ to the resolution obtained in the previous section. We have

$$\mathcal{H}om_R(R, F) \xrightarrow{(\sigma-1)} \mathcal{H}om_R(R, F) \xrightarrow{\mathcal{H}om(r,1)} \mathcal{H}om_R(R_{G/e}, F) \xrightarrow{\mathcal{H}om(\tau,1)-1} \mathcal{H}om_R(R_{G/e}, F) \xrightarrow{\mathcal{H}om(t,1)} F \to \cdots$$

We can simplify this using the identities $\mathcal{H}om_R(R, F) \cong F$ and $\mathcal{H}om_R(R_{G/e}, F) \cong \mathcal{H}om_R(R \otimes [-, G/e], F) \cong \mathcal{H}om([-G/e], F)$:

$$F \xrightarrow{(\sigma-1)} F \xrightarrow{\mathcal{H}om(r,1)} \mathcal{H}om([-, G/e], F) \xrightarrow{\mathcal{H}om(\tau,1)-1} \mathcal{H}om([-, G/e], F) \xrightarrow{\mathcal{H}om(t,1)} F \xrightarrow{\cdots}$$

Now we apply this chain complex to G/C_m. From the definition of F, $F(G/C_m) = K$. From the identities for $\mathcal{H}om(-, F)$, we see that $\mathcal{H}om_R(R \otimes [-, G/e], F) \cong F(G/C_m \times G/e) \cong \bigtimes_{x \in G/C_m} K$. Since the action of a representation in $R(G/C_m)$ on F is multiplication by the dimension of the representation, $(\sigma - 1)$ is the zero map.

The map $F(G/C_m) \to F(G/C_m \times G/e)$ is F applied to a morphism of G-sets:

$$\begin{tikzcd} & G/H \times G/e \arrow[dl, "p"'] \arrow[dr, "="] & \\ G/H \times G/e & & G/H \end{tikzcd} = \sum_{x \in G/H} \begin{tikzcd} & G/e \arrow[dl, "i_x"'] \arrow[dr, "\pi_e^H \circ \gamma_{x^{-1},e}"] & \\ G/H \times G/e & & G/H \end{tikzcd}$$

The sum of morphisms equals $\sum_{x\in G/H} \pi_e^H \circ \gamma_{x^{-1},e} \circ i_x$ and when we apply F we get $\sum_{x\in G/H} i_x \circ r_e^H$. Since the restrictions of the fixed point functor are all 1, this is equivalent to the diagonal map.

The map $F(G/H \times G/e) \to F(G/H)$ is F applied to the morphism

$$\begin{tikzcd} & G/H \times G/e \arrow[dl, "p"'] \arrow[dr, "="] & \\ G/H & & G/H \times G/e \end{tikzcd}$$

which, as before, can be written as a sum

$$\sum_{x \in G/H} \begin{tikzcd} & G/e \arrow[dl, "\pi_e^H \circ \gamma_{x^{-1},e}"'] \arrow[dr, "i_x"] & \\ G/H & & G/H \times G/e \end{tikzcd}$$

where i_x is the inclusion map. This sum is equal to $\sum_{x\in G/H} i_x \circ \gamma_{x,e} \circ \hat{\pi}_e^H$. Applying F we get $\sum_{x\in G/H} t_e^H \circ p_x$ where p_x is the projection map. However, the transfer map for the fixed point functor is multiplication by the index which in this case is m. Thus given a vector in $F(G/C_m \times G/e)$, this map adds the components and multiplies by m. We denote this operation by $m(\Sigma)$. We obtain the chain

$$K \xrightarrow{0} K \xrightarrow{\Delta} \bigtimes_{x\in G/C_m} K \xrightarrow{\tau - 1} \bigtimes_{x\in G/C_m} K \xrightarrow{m(\Sigma)} K \xrightarrow{\cdots} \tag{4.3}$$

We now compute the homology of this chain. We arrive at the following theorem:

Theorem 4.4. *$\mathcal{E}xt^0_R(F,F)(G/C_m) = K$ and for $j > 0$,*

$$\mathcal{E}xt^j_R(F,F)(G/C_m) = \begin{cases} 0 & j = 1 \\ 0 & j = 2 \\ \frac{Ker(m(\Sigma))}{Im(\tau-1)} & j = 3 \\ K/mK & j = 0 \end{cases} \pmod 4$$

For $\mathcal{E}xt^3_R(F,F)$, a Green functor, we have that $r^H_K = [H : K]$, $t^H_K = 1$, and the conjugations are the identity. For $\mathcal{E}xt^4_R(F,F)$ we have that $r^H_K = 1$, $t^H_K = [H : K]$, and the conjugations are the identity.

It is clear that the results depend on the characteristic of K:

Corollary 4.5. *If K has characteristic 0 or p where $p \nmid m$*

$$\mathcal{E}xt^j_R(F,F)(G/C_m) = \begin{cases} K & j = 0 \\ 0 & j > 0 \end{cases}$$

If K has characteristic p and p divides m, then $\mathcal{E}xt^0_R(F,F) = K$. Since $m = 0$, $\mathcal{E}xt^3_R(F,F) = \frac{Ker(m(\Sigma))}{Im(\tau-1)} = \left(\bigtimes_{x\in G/C_m} K\right)/Im(\tau - 1)$, a one dimensional quotient space of $\bigtimes_{x\in G/C_m} K$. Since the vector $[1, 0, .., 0]$ is not contained in $Im(\tau - 1)$, we denote this subspace as $K[1, 0, ..., 0]$. Also, $\mathcal{E}xt^4_R(F,F) = K/mK = K$. So we have

Corollary 4.6. *If K has characteristic p and p divides m, then $\mathcal{E}xt^0_R(F,F) = K$ and for $j > 0$*

$$\mathcal{E}xt^j_R(F,F)(G/C_m) = \begin{cases} 0 & j = 1 \\ 0 & j = 2 \\ K & j = 3 \\ K & j = 0 \end{cases} \pmod 4$$

4.3 Ring Structure of $\mathcal{E}xt$

We now wish to compute the ring structure of $\mathcal{E}xt_R(F,F)(G/C_m)$ in the case with K a field of characteristic p that divides m. In general, there exists a correspondence between elements of $\mathcal{E}xt^i_R(F,F)(G/H)$ and homotopy classes of chain maps $f : P_{\bullet+i} \otimes [-, G/H] \to P_\bullet$ where $P_\bullet$ is a projective resolution for B and $P_{\bullet+i}$ is $P_\bullet$ shifted by i terms.There exists an associative collection of maps

$$\mathcal{E}xt^i_R(B,C) \otimes \mathcal{E}xt^j_R(A,B) \longrightarrow \mathcal{E}xt^{i+j}_R(A,C)$$

generalizing the composition product. In this case, $A = B = C = F$ so we have a map

$$\mathcal{E}xt^i_R(F,F) \otimes \mathcal{E}xt^j_R(F,F) \longrightarrow \mathcal{E}xt^{i+j}_R(F,F).$$

Thus $\mathcal{E}xt_R(F,F)$ is a Green functor.

Now if we consider the categories of Mackey functors $Mack(G)$ and $Mack(H)$ where $Mack(G)$ is defined on $\mathcal{B}_G$ and likewise for $Mack(H)$, there is an adjoint pair Res^G_H and Ind^G_H where Res^G_H is a functor from $Mack(G)$ to $Mack(H)$ and Ind^G_H is a functor from $Mack(H)$ to $Mack(G)$. In fact, one can prove that

$$A \otimes [-, G/H] \cong Ind^G_H(Res^G_H A),$$

thus there is a one to one correspondence between maps $P_{\bullet+i} \otimes [-, G/H] \to P_\bullet$ and maps $Ind^G_H(Res^G_H P_{\bullet+i}) \to P_\bullet$, which, using the adjoint pair, are in a one to one correspondence with maps $Res^G_H P_{\bullet+i} \to Res^G_H P_\bullet$. However, considering these maps is equivalent to considering the chain maps of projective resolutions for the group H, that is, $P^H_{\bullet+i} \to P^H_\bullet$ where the functors in the projective resolution are functors on $\mathcal{B}_H$.

In our case, $\mathcal{E}xt^i_R(F,F)(G/H) = 0$ when $i = 1+4k, 2+4k$, so we focus on the cases $i = 3+4k$ and $i = 4+4k$. We must compute chain maps $f : P^{C_m}_{\bullet+3} \to P^{C_m}_\bullet$ and $g : P^{C_m}_{\bullet+4} \to P^{C_m}_\bullet$. These chain maps are given in the following proposition.

Proposition 4.7. *The chain map $f : P^{C_m}_{\bullet+3+4k} \to P^{C_m}_\bullet$ given by the vertical maps in the following commutative diagram represent nonzero classes in $\mathcal{E}xt^{3+4k}_R(F,F)(C_n/C_m)$.*

$$\begin{tikzcd}
R_{C_{m/e}} \arrow[d, "1"] & R \arrow[l, "1\otimes t"'] \arrow[d, "\sum_{i=0}^{m-2} -(i+1)\sigma^i"] & R \arrow[l, "(\sigma-1)"'] \arrow[d, "1"] & R_{C_{m/e}} \arrow[l, "1\otimes r"'] \arrow[d, "{[m-1,0,1,...,m-2]}"] & R_{C_{m/e}} \arrow[l, "1\otimes(\tau-1)"'] \arrow[d, "1"] \\
R & R \arrow[l, "(\sigma-1)"] & R_{C_{m/e}} \arrow[l, "1\otimes r"] & R_{C_{m/e}} \arrow[l, "1\otimes(\tau-1)"] & R \arrow[l, "1\otimes t"]
\end{tikzcd}$$

The chain map $g : P^{C_m}_{\bullet+4k} \to P^{C_m}_\bullet$ given by the vertical maps in the following commutative diagram represent nonzero classes in $\mathcal{E}xt^{4k}_R(F,F)(C_n/C_m)$.

$$\begin{tikzcd}
R \arrow[d, "1"] & R \arrow[l, "(\sigma-1)"'] \arrow[d, "1"] & R_{C_{m/e}} \arrow[l, "1\otimes r"'] \arrow[d, "{[1,0,\cdots,0]}"] & R_{C_{m/e}} \arrow[l, "1\otimes(\tau-1)"'] \arrow[d, "{[1,0,\cdots,0]}"] & R \arrow[l, "1\otimes t"'] \arrow[d, "1"] \\
R & R \arrow[l, "(\sigma-1)"] & R_{C_{m/e}} \arrow[l, "1\otimes r"] & R_{C_{m/e}} \arrow[l, "1\otimes(\tau-1)"] & R \arrow[l, "1\otimes t"]
\end{tikzcd}$$

The actual vertical maps are labelled by elements of R evaluated at particular G-sets. For example, a map from $R_{G/e}$ to R is an element of $\mathrm{Hom}(R_{G/e}, R) \cong R(G/e)$, so the map is represented by an element of $R(G/e)$. We define y_3 and x_4 to be the classes represented by these chain maps (for $k = 0$ and $k = 1$ respectively). Note that the chain maps given in the previous proposition are four periodic. This implies that the compositions are also four periodic. Given the chain maps f and g we can now verify that the products on $\mathcal{E}xt_R^{3+4k}(F,F) \otimes \mathcal{E}xt_R^{4l}(F,F)$ and $\mathcal{E}xt_R^{4k}(F,F) \otimes \mathcal{E}xt_R^{4l}(F,F)$ do not correspond to chain maps that are homotopic to zero. This entails computing the compositions $P_{3+4} \xrightarrow{g} P_3 \xrightarrow{f} P_0 \xrightarrow{aug} F$, $P_{4+3} \xrightarrow{f} P_4 \xrightarrow{g} P_0 \xrightarrow{aug} F$, and $P_{4+4} \xrightarrow{g} P_4 \xrightarrow{g} P_0 \xrightarrow{aug} F$. If these compositions represent nonzero $\mathcal{E}xt$ classes, then the chain maps are nonzero and we are finished. Explicitly these maps are $R_{C_{m/e}} \xrightarrow{[1,0,\cdots,0]} R_{C_{m/e}} \xrightarrow{1} R \xrightarrow{aug.} F$, $R_{C_{m/e}} \xrightarrow{1} R \xrightarrow{1} R \xrightarrow{aug.} F$, and $R \xrightarrow{1} R \xrightarrow{1} R \xrightarrow{aug.} F$. All of these maps compose to 1. We have arrived at the following theorem:

Theorem 4.8. *For a field K with characteristic p where $p \mid m$, $\mathcal{E}xt_R(F,F)$ is given by*

$$\mathcal{E}xt_R(F,F)(C_n/C_m) = K[y_3, x_4]/(y_3^2 = 0).$$

where y_3 has degree 3 and x_4 has degree 4.

5 Computation for $G = S_3$

We compute $\mathcal{E}xt_R(F,F)(G/H)$ for $G = S_3$ where the field K is taken to be $\mathbb{F}_2$.

5.1 Projective Resolution for F

Proposition 5.1. *There is a four periodic projective resolution for F of the following form:*

$$0 \longleftarrow F \xleftarrow{aug} R \xleftarrow{V} R \xleftarrow{1\otimes r} R_{G/e} \xleftarrow{1\otimes(\tau_1+\tau_2+\tau_3-1)} R_{G/e} \xleftarrow{1\otimes t} R \leftarrow \cdots \tag{5.2}$$

Proof. The Burnside category for $G = S_3$ takes the form:

$$\begin{tikzcd}
 & G/G \arrow[dl, bend right, "\hat{\pi}^G_{C_2}"] \arrow[dr, bend left, "\hat{\pi}^G_{C_3}"'] & \\
G/C_2 \arrow[ur, bend left, "\pi^G_{C_2}"] \arrow[dr, bend left, "\hat{\pi}^{C_3}_e"] & & G/C_3 \arrow[ul, bend right, "\pi^G_{C_3}"'] \arrow[dl, bend right, "\hat{\pi}^{C_2}_e"'] \\
 & G/e \arrow[ul, bend left, "\pi^{C_2}_e"] \arrow[ur, bend right, "\pi^{C_3}_e"'] &
\end{tikzcd}$$

There are three choices for C_2, we draw our diagrams with $C_2 = \langle (12) \rangle$. Also, there are six conjugations on G/C_2, allowing us to obtain any permutation of the cosets, two conjugations of G/C_3, allowing us to swap the two cosets, and six conjugations of G/e.

If we apply R to the previous diagram, we have

$$\begin{tikzcd}
& \mathbb{F}_2 + \mathbb{F}_2 S + \mathbb{F}_2 V \arrow[dl, bend left, "r^G_{C_2}"] \arrow[dr, bend right, "r^G_{C_3}"'] & \\
\mathbb{F}_2 + \mathbb{F}_2\sigma_2 \arrow[ur, bend left, "t^G_{C_2}"] \arrow[dr, bend left, "r^{C_2}_e"] & & \mathbb{F}_2 + \mathbb{F}_2\sigma_3 + \mathbb{F}_2\sigma_3^2 \arrow[ul, bend right, "t^G_{C_3}"'] \arrow[dl, bend right, "r^{C_3}_e"'] \\
& \mathbb{F}_2 \arrow[ul, bend left, "t^{C_2}_e"] \arrow[ur, bend right, "t^{C_3}_e"'] &
\end{tikzcd}$$

Here S is the sign representation of S_3 and V is the two dimensional representation. The conjugations are trivial since conjugation preserves the character of a representation, except for the orbit G/C_3. The nontrivial conjugation swaps σ_3 and σ_3^2. We avoid using conjugations on this orbit. In this case the restriction and transfer maps are as follows:

$$r^G_{C_2} = \begin{cases} S \mapsto \sigma_2 \\ V \mapsto 1 + \sigma_2 \end{cases} \qquad t^G_{C_2} = \begin{cases} 1 \mapsto 1 + V \\ \sigma_2 \mapsto S + V \end{cases}$$

$$r^G_{C_3} = \begin{cases} S \mapsto 1 \\ V \mapsto \sigma_3 + \sigma_3^2 \end{cases} \qquad t^G_{C_3} = \begin{cases} 1 \mapsto 1 + S \\ \sigma_3 \mapsto V \\ \sigma_3^2 \mapsto V \end{cases}$$

$$r^{C_2}_e = \sigma_2 \mapsto 1 \qquad t^{C_2}_e = 1 \mapsto 1 + \sigma_2$$

$$r^{C_3}_e = \sigma_3 \mapsto 1 \qquad t^{C_3}_e = 1 \mapsto 1 + \sigma_3 + \sigma_3^2$$

If we apply F to the diagram for $\mathcal{B}_G$, every object maps to $\mathbb{F}_2$, the restrictions are identity maps and the transfers are multiplication by the index. The diagram corresponding to F is

$$\begin{tikzcd}
& \mathbb{F}_2 \arrow[dl, bend left, "1"] \arrow[dr, bend right, "1"'] & \\
\mathbb{F}_2 \arrow[ur, bend left, "1"] \arrow[dr, bend left, "1"] & & \mathbb{F}_2 \arrow[ul, bend right, "0"'] \arrow[dl, bend right, "1"'] \\
& \mathbb{F}_2 \arrow[ul, bend left, "0"] \arrow[ur, bend right, "1"'] &
\end{tikzcd}$$

Finally, $R_{G/e}(G/H) \cong R(G/H \times G/e) \cong \bigtimes_{x \in G/H} R(G/e)$. Thus we have a copy of $\mathbb{F}_2$ for each coset of H in G.

The diagram is as follows:

$$\begin{tikzcd}
& \mathbb{F}_2 \arrow[dl, bend left, "\Delta"] \arrow[dr, bend right, "\Delta"'] & \\
\mathbb{F}_2 + \mathbb{F}_2 + \mathbb{F}_2 \arrow[ur, bend left, "+"] \arrow[dr, bend left, "\Delta"] & & \mathbb{F}_2 + \mathbb{F}_2 \arrow[ul, bend right, "+"'] \arrow[dl, bend right, "\Delta"'] \\
& \bigoplus_{i=0}^{5} \mathbb{F}_2 \arrow[ul, bend left, "+"] \arrow[ur, bend right, "+"'] &
\end{tikzcd}$$

In this case the restriction maps are diagonal maps and the transfers add together groups of coordinates. The conjugation maps for each orbit permute the indices of the coordinates. We denote by τ_1, τ_2, τ_3 the conjugations corresponding to $(12), (23), (13)$ respectively.

Now we must verify that the sequence given above is exact. As with the case $G = C_m$, the map from R to F is the augmentation map given in Example 3.15. The kernel of this map applied to each orbit takes the form:

$$\begin{tikzcd}
& \mathbb{F}_2(S-1) + \mathbb{F}_2 V \arrow[dl, bend left] \arrow[dr, bend left] & \\
\mathbb{F}_2(\sigma_2 - 1) \arrow[ur, bend left] \arrow[dr, bend left] & & \mathbb{F}_2(\sigma_3 - 1) + \mathbb{F}_2(\sigma_3 - 1)^2 \arrow[ul, bend left] \arrow[dl, bend left] \\
& 0 \arrow[ul, bend left] \arrow[ur, bend left] &
\end{tikzcd}$$

The transfer and restriction maps for this kernel are induced by the transfer and restriction maps for R. We must show that multiplication by V maps surjectively onto this kernel. Using characters we can calculate the following products of representations: $s^2 = 1$, $V^2 = 1 + S + V$, $SV = V$. If we multiply $\mathbb{F}_2 + \mathbb{F}_2 S + \mathbb{F}_2 V$ by V we get $\mathbb{F}_2 V + \mathbb{F}_2 V + \mathbb{F}_2(1+S+V) \cong \mathbb{F}_2(S-1) + \mathbb{F}_2 V$. Multiplying any element of G/H by V means multiplying that element by the $r^G_H(V)$. Thus $V \cdot (\mathbb{F}_2 + \mathbb{F}_2\sigma_3 + \mathbb{F}_2\sigma_3^2) = (\sigma_3 + \sigma_3^2)(\mathbb{F}_2 + \mathbb{F}_2\sigma_3 + \mathbb{F}_2\sigma_3^2)$. Any choice of coefficients for this expression gives an element of the kernel. For G/C_2, $V \cdot (\mathbb{F}_2 + \mathbb{F}_2\sigma_2) = (1+\sigma_2)(\mathbb{F}_2 + \mathbb{F}_2\sigma_2) = \mathbb{F}_2(1+\sigma_2)$. For G/e, $r^G_e(V) = 0$, so $R(G/e)$ maps onto 0. Thus, the map that multiplies by V surjects onto the kernel of the previous map.

The kernel of multiplication by V is

$$\begin{tikzcd}
& \mathbb{F}_2(S-1) \arrow[dl, bend left] \arrow[dr, bend left] & \\
\mathbb{F}_2(1 + \sigma_2) \arrow[ur, bend left] \arrow[dr, bend left] & & \mathbb{F}_2(1 + \sigma_3 + \sigma_3^2) \arrow[ul, bend left] \arrow[dl, bend left] \\
& \mathbb{F}_2 \arrow[ul, bend left] \arrow[ur, bend left] &
\end{tikzcd}$$

We must map from $R_{G/e}$ onto this kernel. This map is $1 \otimes r$, which, as before, adds together the entries in each vector in $R(G/H)$. The kernel of this map is, for any orbit G/H, the set of vectors V_H so that the sum of the entries in each vector is zero.

$$\begin{tikzcd}
& V_G \arrow[dl, bend left] \arrow[dr, bend left] & \\
V_{C_2} \arrow[ur, bend left] \arrow[dr, bend left] & & V_{C_3} \arrow[ul, bend left] \arrow[dl, bend left] \\
& V_e \arrow[ul, bend left] \arrow[ur, bend left] &
\end{tikzcd}$$

We must now map from $R_{G/e}$ onto the kernel shown above. This is accomplished by the map $1 \otimes (\tau_1 + \tau_2 + \tau_3 - 1)$. If we apply this map to $(a, b, c) \in R_{G/e}(G/C_2)$, $(\tau_1 + \tau_2 + \tau_3)(a, b, c) - (a, b, c) = (a + b + c, a + b + c, a + b + c) - (a, b, c) = (b + c, a + c, a + b)$. This is a map onto the kernel of the previous map in the case of G/C_2. In the case of G/C_2, $(\tau_1 + \tau_2 + \tau_3)(a, b) - (a, b) = (a + b, a + b)$, as desired. For $R_{G/e}(G/e)$, $1 \otimes (\tau_1 + \tau_2 + \tau_3 - 1)$ is a map of rank five. Thus it is a surjection onto the kernel of the previous map which has dimension five. For $R_{G/e}(G/G)$, this map is the zero map. For each of the above orbits, the kernel of the map $1 \otimes (\tau_1 + \tau_2 + \tau_3 - 1)$ is one dimensional. Thus the next kernel looks like

$$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, bend left] \arrow[dr, bend left] & \\ \mathbb{F}_2 \arrow[ur, bend left] \arrow[dr, bend left] & & \mathbb{F}_2 \arrow[ul, bend left] \arrow[dl, bend left] \\ & \mathbb{F}_2 \arrow[ul, bend left] \arrow[ur, bend left] & \end{tikzcd}$$

The map $1 \otimes t$ maps from R into $R_{G/e}$ via the restriction r_e^H and surjects onto the kernel of the previous map. This gives us a four periodic projective resolution.

$\square$

5.2 Computation of $\mathcal{E}xt_R(F, F)$

Applying $\mathcal{H}om_R(-, F)$ we have

$$\mathcal{H}om_R(R, F) \xrightarrow{V} \mathcal{H}om_R(R, F) \xrightarrow{\mathcal{H}om(r,1)} \mathcal{H}om_R(R_{G/e}, F) \xrightarrow{\mathcal{H}om(\tau,1)-1} \mathcal{H}om_R(R_{G/e}, F) \xrightarrow{\mathcal{H}om(t,1)} F \to \cdots$$

Using the identities for $\mathcal{H}om_R(-, F)$ and the fact that the action of V on F is multiplication by 2 which is 0, this simplifies to

$$F \xrightarrow{0} F \xrightarrow{\mathcal{H}om(r,1)} \mathcal{H}om([-, G/e], F) \xrightarrow{\mathcal{H}om(\tau_1+\tau_2+\tau_3,1)-1} \mathcal{H}om([-, G/e], F) \xrightarrow{\mathcal{H}om(t,1)} F \to \cdots$$

We shall apply this sequence to each orbit in $\mathcal{B}_G$ and compute $\mathcal{E}xt_R(F, F)$.

For G/G we have

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{Id.} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to \cdots$$

For G/C_2 we have

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2 + \mathbb{F}_2 + \mathbb{F}_2 \xrightarrow{\tau_1+\tau_2+\tau_3-1} \mathbb{F}_2 + \mathbb{F}_2 + \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to \cdots$$

where Δ denotes the diagonal map. For G/C_3 we have

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2 + \mathbb{F}_2 \xrightarrow{\tau_1+\tau_2+\tau_3-1} \mathbb{F}_2 + \mathbb{F}_2 \xrightarrow{+} \mathbb{F}_2 \to \cdots$$

where Δ denotes the diagonal map, and $+$ denotes the augmentation map that adds coordinates. For G/e we have

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{\Delta} \bigoplus_{i=0}^{5} \mathbb{F}_2 \xrightarrow{\tau_1+\tau_2+\tau_3-1} \bigoplus_{i=0}^{5} \mathbb{F}_2 \xrightarrow{+} \mathbb{F}_2 \to \cdots$$

where Δ denotes the diagonal map, and $+$ denotes the augmentation map that adds coordinates.

We summarize the results of the $\mathcal{E}xt_R(F, F)$ computations for $G = S_3$ in the following theorem:

Theorem 5.3. *For the orbit G/G, $\mathcal{E}xt_R^0(F, F)(G/G) = \mathbb{F}_2$, and for $j > 0$,*

$$\mathcal{E}xt_R^j(F, F)(G/G) = \begin{cases} 0 & j = 1 \\ 0 & j = 2 \\ \mathbb{F}_2 & j = 3 \\ \mathbb{F}_2 & j = 0 \end{cases} \pmod 4$$

For the orbit G/C_2, $\mathcal{E}xt_R^0(F, F)(G/G) = \mathbb{F}_2$, and for $j > 0$,

$$\mathcal{E}xt^j_R(F,F)(G/C_2) = \begin{cases} 0 & j=1 \\ 0 & j=2 \\ \mathbb{F}_2 & j=3 \\ \mathbb{F}_2 & j=0 \end{cases} \pmod 4$$

For the orbit G/C_3,

$$\mathcal{E}xt^j_R(F,F)(G/C_3) = \begin{cases} \mathbb{F}_2 & j=0 \\ 0 & j>0 \end{cases}$$

For the orbit G/e,

$$\mathcal{E}xt^j_R(F,F)(G/C_3) = \begin{cases} \mathbb{F}_2 & j=0 \\ 0 & j>0 \end{cases}$$

For both $\mathcal{E}xt^3_R(F,F)$ and $\mathcal{E}xt^4_R(F,F)$,Green functors, we have that $r^H_K = 1$, $t^H_K = 1$, and the conjugations are the identity.

5.3 Ring Structure of $\mathcal{E}xt$

We now wish to identify the ring structure of $\mathcal{E}xt^i_R(F,F)(G/C_2)$ and $\mathcal{E}xt^i_R(F,F)(G/G)$. For the first case, $\mathcal{E}xt^i_R(F,F)(G/C_2)$ corresponds to chain maps $P^{C_2}_{\bullet+i} \to P^{C_2}_\bullet$. However, we already computed the chain maps and the products for $G = C_2$ in the previous section, and this gives us the ring structure:

Theorem 5.4. *Over the field $\mathbb{F}_2$, $\mathcal{E}xt_R(F,F)(S_3/C_2)$ is given by*

$$\mathcal{E}xt_R(F,F)(G/C_2) = \mathbb{F}_2[y_3, x_4]/(y_3^2 = 0).$$

where y_3 has degree 3 and x_4 has degree 4.

For G/G, elements of $\mathcal{E}xt^3_R(F,F)(G/G)$ correspond to the chain maps $P_{\bullet+3} \to P_\bullet$. This follows since $H = G$, so $Res^G_H = Res^G_G$ is trivial. Similarly elements of $\mathcal{E}xt^4_R(F,F)(G/G)$ correspond to the chain maps $P_{\bullet+4} \to P_\bullet$. We compute these chain maps in the following proposition.

Proposition 5.5. *The chain map $f : P^{C_m}_{\bullet+3+4k} \to P^{C_m}_\bullet$ given by the vertical maps in the following commutative diagram represent nonzero classes in $\mathcal{E}xt^{3+4k}_R(F,F)(C_n/C_m)$.*

$$\begin{tikzcd}
R_{G/e} \arrow[d, "1"] & R \arrow[l, "1\otimes t"'] \arrow[d, "1+V"] & R \arrow[l, "V"'] \arrow[d, "1"] & R_{G/e} \arrow[l, "1\otimes r"'] \arrow[d, "{[0,1,1,1,0,0]}"] & R_{G/e} \arrow[l, "1\otimes(\tau_1+\tau_2+\tau_3-1)"'] \arrow[d, "1"] \\
R & R \arrow[l, "V"] & R_{G/e} \arrow[l, "1\otimes r"] & R_{G/e} \arrow[l, "1\otimes(\tau_1+\tau_2+\tau_3-1)"] & R \arrow[l, "1\otimes t"]
\end{tikzcd}$$

The chain map $g : P^{C_m}_{\bullet+4k} \to P^{C_m}_\bullet$ given by the vertical maps in the following commutative diagram represent nonzero classes in $\mathcal{E}xt^{4k}_R(F,F)(C_n/C_m)$.

$$\begin{tikzcd}
R \arrow[d, "1"] & R \arrow[l, "V"'] \arrow[d, "1"] & R_{G/e} \arrow[l, "1\otimes r"'] \arrow[d, "{[1,0,0,0,0,0]}"] & R_{G/e} \arrow[l, "1\otimes(\tau_1+\tau_2+\tau_3-1)"'] \arrow[d, "{[1,0,0,0,0,0]}"] & R \arrow[l, "1\otimes t"'] \arrow[d, "1"] \\
R & R \arrow[l, "V"] & R_{G/e} \arrow[l, "1\otimes r"] & R_{G/e} \arrow[l, "1\otimes(\tau_1+\tau_2+\tau_3-1)"] & R \arrow[l, "1\otimes t"]
\end{tikzcd}$$

As before, these chain maps represent the classes x_3 and x_4. Now we must compute the compositions $P_{3+4} \xrightarrow{g} P_3 \xrightarrow{f} P_0 \xrightarrow{aug} F$, $P_{4+3} \xrightarrow{f} P_4 \xrightarrow{g} P_0 \xrightarrow{aug} F$, and $P_{4+4} \xrightarrow{g} P_4 \xrightarrow{g} P_0 \xrightarrow{aug} F$. Explicitly these maps are $R_{G/e} \xrightarrow{[1,0,\cdots,0]} R_{G/e} \xrightarrow{1} R \xrightarrow{aug} F$, $R_{G/e} \xrightarrow{1} R \xrightarrow{1} R \xrightarrow{aug} F$, and $R \xrightarrow{1} R \xrightarrow{1} R \xrightarrow{aug} F$. These maps all compose to 1. We have arrived at the following theorem:

Theorem 5.6. *Over the field $\mathbb{F}_2$, $\mathcal{E}xt_R(F,F)(S_3/S_3)$ is given by*

$$\mathcal{E}xt_R(F,F)(G/G) = \mathbb{F}_2[y_3, x_4]/(y_3^2 = 0).$$

where y_3 has degree 3 and x_4 has degree 4.

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