Spectrum Asymptotics of Airy's Operator under Perturbation

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Abstract

We study the eigenvalue problem of perturbed Airy operator $-\triangle + (x + V(x))$ on positive real axis, with Dirichlet boundary condition. Given the asymptotic expansion of $V(x)$ at $+\infty$, we provide asymptotic expansion of the eigenvalues λ_n to certain order. Then we will provide an example to show that in general, merely from the asymptotic expansion of $V(x)$, the asymptotic expansion of λ_n cannot be completely determined.

1 INTRODUCTION

In this paper, we study the perturbed Airy operator $-\triangle + (x + V(x))$ on positive real axis, such that $V(x)$ has an asymptotic expansion up to certain order,

$$V(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_k x^{\alpha_k} + o(x^{\alpha_k}),$$

where $\{a_i\}_1^k$, $\{\alpha_i\}_1^k$ are constants and $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_k$. From the general theory of second-order differential operators, it is known that under these assumptions the spectrum of this purturbed operator consists of discrete eigenvalues of multiplicity one,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \longrightarrow +\infty.$$

The main result of the present paper is that given the asymptotic expansion above, λ_n also has a similar asymptotic expansion,

$$\lambda_n = \begin{cases} (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{\kappa_m} + o(n^{\frac{2}{3}\alpha_k}), & \alpha_k > -1 \\ (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n + o(n^{-\frac{2}{3}} \ln n), & \alpha_k = -1 \\ (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n + O(n^{-\frac{2}{3}}), & \alpha_k < -1 \end{cases}$$

where $\{d_i\}$ and $\{\kappa_i\}$ are constants, depend on the asymptotic expansion of $V(x)$. One may notice that the asymptotic expansion of λ_n is determined up to at most $O(n^{-\frac{2}{3}})$, this is because the information of $V(x)$ at infinity can only determine λ_n up to $O(n^{-\frac{2}{3}})$. At the end of this paper, I will provide two perturbations $V_1(x)$, $V_2(x)$ such that they have the same asymptotic expansion, but the coefficients of $n^{-\frac{2}{3}}$ in their eigenvalue expansions are different.

The main method is based on the minimax principle, namely, if two potentials $q_1(x) \le q_2(x)$, then their corresponding eigenvalues satisfy $\lambda_n^{(1)} \le \lambda_n^{(2)}$. Given the asymptotic expansion of $V(x)$, we can construct two well-behaved (in the aspect of monotonicity,differentiability, etc.) potentials $q_1(x)$, $q_2(x)$ such that $q_1(x) \le x + V(x) \le q_2(x)$, then the minimax principle implies that $\lambda_n^{(1)} \le \lambda_n \le \lambda_n^{(2)}$. Thus if the two potentials q_1 and q_2 are sufficiently close such that the asymptotic expansion of eigenvalues $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ agree with each other up to $o(n^\kappa)$, then we get the asymptotic expansion of λ_n up to $o(n^\kappa)$ by comparison.

The computation of eigenvalue asymptotics follows from a detailed investigation of the generalized Weyl law, which concerns about the asymptotic eigenvalue counting problem for a family of differential operators. A brief introduction to Weyl law, we refer to [1]. For Schrödinger operator

$$H = -h^2 \triangle + q(x),$$

the distribution of eigenvalues satisfies the following asymptotics

$$N(\lambda) = \#\{\lambda_i \le \lambda\} \sim \frac{1}{2\pi h} \int_{\{\xi^2 + q(x) \le \lambda\}} dx d\xi, \tag{1.1}$$

as λ tends to $+\infty$ and h tends to 0.

In this paper, we obtain a more precise approximation of $N(\lambda_n)$

$$\frac{1}{\pi} \int_0^{X_n} (\lambda_n - q(x))^{\frac{1}{2}} dx = n - \frac{1}{4} + O(\frac{1}{n}), \tag{1.2}$$

under the condition that the potential $q(x)$ is monotonically increasing and has an expansion with the form of

$$q(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_k x^{\alpha_k}, \qquad x \geq M$$

where X_n is the unique solution of $q(X_n) = \lambda_n$. The proof of (1.2) is based on the proof given by Titchmarsh [6] by Langer's method, where he proves (1.2) under the condition that $q(x)$ is increasing and convex downward, and as $x \to +\infty$

$$\frac{q'(x)}{q(x)} = O(\frac{1}{x}), \quad \frac{q''(x)}{q'(x)} = O(\frac{1}{x}), \quad \frac{q'''(x)}{q''(x)} = O(\frac{1}{x}).$$

We also believe that the approximation (1.2) holds for a much more general class of potentials.

Finally we apply these to the inverse spectral problem, reconstructing the asymptotic expansion of $V(x)$ from the asymptotic expansions of the eigenvalues. We will show the asymptotic expansion of $V(x)$ can be recovered but only up to $o(x^{-1})$. It can be explained by the fact that, terms like $x^{-1-\epsilon}$ in asymptotics of $V(x)$ have a very small influence on eigenvalues, which can be offset by the change of $V(x)$ on a finite interval. One way to think about this result is to view the right hand side of (1.1), which shows that the asymptotics of eigenvalue distribution is determine by $\text{Area}(\xi^2 + V(x) \leq \lambda)$. For terms like $x^{-1-\epsilon}$, the area $\int x^{-1-\epsilon}$ is finite. Therefore one can not distinguish between the perturbation on eigenvalue asympototics caused by $x^{-1-\epsilon}$ or merely by the behavior of $V(x)$ on finite interval.

2 PRELIMINARY

2.1 Airy operator and its spectrum

Airy's operator with Dirichlet boundary condition

$$-\frac{d^2}{dx^2} + x$$

is an unbounded self-adjoint operator with domain

$$\{u \in H^2([0,\infty)); xu \in L^2([0,\infty)), u(0) = 0\} \subset L^2([0,\infty)).$$

Since $\lim\limits_{x\to\infty} x = +\infty$, Airy operator has discrete spectrum, which consists of distinct eigenvalues. Say λ is an eigenvalue of Airy operator with eigenfunction u, then we have

$$-\frac{d^2 u(x)}{dx^2} + (x - \lambda)u(x) = 0,$$

After a translation by λ, we get

$$-\frac{d^2 u(x+\lambda)}{dx^2} + xu(x+\lambda) = 0.$$

Therefore $y(x) = u(x+\lambda)$ is a solution of equation

$$-\frac{d^2 y}{dx^2} + xy = 0, \tag{2.1}$$

on interval $[-\lambda, \infty)$, which lies in $L^2([0, +\infty))$, with initial condition $v(-\lambda) = 0$. Hence, such solution exists if and only if λ is the negative of zeros of Airy function, which is the L^2 solution of the above equation.

In order to know the asymptotic behavior of eigenvalues of Airy operator, we solve the Airy equation. And we are only interested in the L^2 solution of (2.1). Take a Fourier transform, and let $Y(t) = \hat{y}(x)$,

$$-y''(x) + xy(x) = 0 \Longleftrightarrow t^2 Y(t) - iY(t) = 0.$$

It is easy to give the solution to this first order differential equation, which is

$$Y(t) = e^{-\frac{t^3}{3}i}.$$

By Fourier inverse formula,

$$y(x) = \int_{-\infty}^{+\infty} e^{-\frac{t^3}{3}i} e^{-ixt} dt = 2\int_0^{+\infty} \cos(\frac{t^3}{3} + xt) dt$$

Normalize this solution, we get

$$Ai(x) = \int_0^{+\infty} \cos(\frac{t^3}{3} + xt) dt$$

By stationary phase methods, which is available in literatures, for example Chapter 3 of [5], one can get the asymptotic expansion of Airy function

$$Ai(x) \sim \begin{cases} \frac{1}{2}\pi^{\frac{1}{2}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} & (x \geq 0) \\ \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}} \sin(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}) & (x \leq 0) \end{cases}$$

As a result of the asymptotic expansion above, the $n-$th eigenvalue of Airy operator λ_n is around the size of the n-th zero of $\sin(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4})$, which implies $\lambda_n \sim \{\frac{3}{2}(n - \frac{1}{4})\pi\}^{\frac{2}{3}}$.

2.2 Singular Sturm-Liouville operators

In general the perturbed Airy operator $-\triangle + x + V(x)$ on positive real axis is a special case of a singular Sturm-Liouville operator. In this section, we give some results on singular Sturm-Liouville operator $-\triangle + q(x)$, especially for the case $\lim_{x \to +\infty} q(x) = +\infty$.

Theorem 2.1. *Consider a Sturm-Liouville operator $L = -\triangle + q(x)$ on positive real axis, with the potential $q(x)$ satisfying*

$$q(x) \in C([0, +\infty)), \qquad \lim_{x \to +\infty} q(x) = +\infty.$$

and with the boundary condition

$$\{u, u' \text{ are absolutely continuous}, u(0) = 0, Lu \in L^2([0, \infty))\} \in L^2[0, \infty).$$

The spectrum of L is discrete, and we can write it as

$$spec(L) = \{\lambda_j\}_{j=1}^{\infty}, \qquad \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$$

We denote by $z(\lambda, x)$ the unique solution of the following differential equation,

$$Lz(\lambda, x) = \lambda z(\lambda, x), \qquad z(\lambda, 0) = 0 \text{ and } \frac{dz}{dx}(\lambda, 0) = 1. \tag{2.2}$$

Then for $\lambda \in (\lambda_{k-1}, \lambda_k]$, the solution $z(\lambda, \cdot)$ has exactly k zeros on $[0, +\infty)$. Here we define $\lambda_0 = -\infty$.

One thing to mention here, for any fixed λ, the zeros of the solution $z(\lambda, x)$ are discrete. Otherwise, it has a limit point, denoted as x_0, such that $\lim_{n\to\infty} x_n = x_0$, where $\{x_n\}_1^\infty$ are zeros of $z(\lambda, \cdot)$. Since $z(\lambda, \cdot) \in C^2([0, \infty))$, we have

$$z(\lambda, x_0) = 0, \qquad \partial_x z(\lambda, x_0) = 0.$$

But by the uniqueness of 2nd-order ODE solution, $z(\lambda, \cdot) \equiv 0$, which contradicts with the initial condition (2.5)! Again by the same argument, all the zeros of $z(\lambda, x)$ are simple ones, which means $z(\lambda, x)$ will change sign when passing through each zeros.

Lemma 2.1. *For any fixed λ, $z(\lambda, x)$ has finitely many zeros.*

Proof. Say y satisfies

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0,$$

$y(0)$ and $y'(0)$ are not both equal to 0. Since $\lim_{x\to+\infty} q(x) = +\infty$, there exists an x_1 such that for $x \geq x_1$, $q(x) > \lambda$. Without loss of generality, we can assume $y(x_1) > 0$, next we show there is at most one zero on interval $[x_1, \infty)$ by listing all the possible cases.

1) $y'(x_1) \geq 0$, then for $x \geq x_1$, $y''(x) = \{q(x) - \lambda\}y(x) \geq 0$. So $y(x)$ will keep going upward to $+\infty$ on interval $[x_1, +\infty)$.

2) $y'(x_1) < 0$,

i) $y(x)$ remains positive on (x_1, ∞), then $y''(x) > 0$. If at some point $y'(x)$ changes sign, this turns to 1). Otherwise $y'(x)$ remains negative on interval (x_1, ∞), the only possibility is $\lim_{x\to+\infty} y'(x) = 0$, $\lim_{x\to+\infty} y(x) = 0$, and $\lim_{x\to+\infty} y''(x) = 0$.

ii) $y(x)$ equals zero at some point $x = x_2$, then it must be $y'(x_2) < 0$. $y(x)$ changes sign at $x = x_2$, so for any $x > x_2$, $y(x) < 0$, $y''(x) < 0$, and $y'(x) < 0$. This leads to $\lim_{x\to+\infty} y(x) = -\infty$.

Hence, there is at most one zero on the interval $[x_1, \infty)$, since the zeros of $y(x)$ are discrete, the total number of zeros is finite. □

Remark 2.1. *The only interesting case above is where $\lim_{x\to+\infty} y'(x) = 0$, $\lim_{x\to+\infty} y(x) = 0$, and $\lim_{x\to+\infty} y''(x) = 0$, in such circumstances*

$$\int_{x_1}^{\tau} y''(x)dx = y'(\tau) - y'(x_1) = \int_{x_1}^{\tau} (q(x) - \lambda)y(x)dx \leq -y'(x_1),$$

Hence, $y \in L^1([0, +\infty))$. Combined with $\lim_{x\to+\infty} y(x) = 0$, then $y \in L^2([0, +\infty))$. Moreover, from the discussion above there are only two cases, either $y \in L^2([0, +\infty))$ and $\lim_{x\to+\infty} y(x) = 0$, or $\lim_{x\to+\infty} y(x) = \infty$. Thus u is an eigenfunction of L, if and only if $u(+\infty) = 0$.

The whole theory of Sturm-Liouville operator depends on the following fundamental theorem due to Sturm.

Theorem 2.2. *Let u be a solution of*

$$\frac{d^2u}{dx^2} + g(x)u = 0$$

and v a solution of

$$\frac{d^2v}{dx^2} + h(x)v = 0$$

where $g(x)$, $h(x)$ are continuous function and $g(x) \leq h(x)$. Then between any two consecutive zeros of u there is at least one zero of v.

Proof. Say x_1 and x_2 are two consecutive zeros of u, and there are no zero of v between them. Without lose of generality, we can assume that u and v are positive on the interval (x_1, x_2). Multiplying by v, u respectively, and subtracting,

$$u''v - v''u = [h(x) - g(x)]uv. \tag{2.3}$$

Integrating (2.3) from x_1 to x_2,

$$\int_{x_1}^{x_2} (u''v - v''u)dx = [u'v - v'u]|_{x_1}^{x_2} = u'(x_2)v(x_2) - u'(x_1)v(x_1) = \int_{x_1}^{x_2} [h(x) - g(x)]uv \tag{2.4}$$

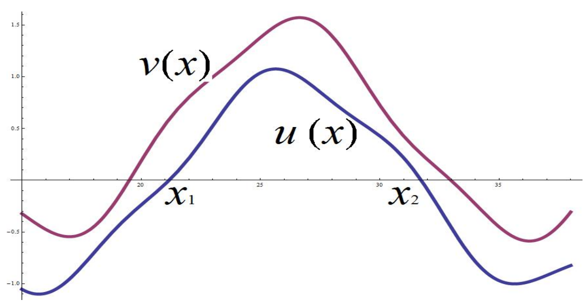

Figure 1: Comparison Theorem

From the right-hand of (2.4), we know that $u'(x_2)v(x_2) - u'(x_1)v(x_1) \geq 0$. However, as shown in Figure 1

$$u'(x_2) < 0, \quad v(x_2) \geq 0, \quad u'(x_1) > 0, \quad v(x_1) \geq 0,$$

sum them up, we get that

$$u'(x_2)v(x_2) - u'(x_1)v(x_1) \leq 0 \leq \int_{x_1}^{x_2} [h(x) - g(x)]uv,$$

equality holds if and only if $h(x) = g(x)$. $\square$

Corollary 2.1. *If we have $\lim\limits_{x \to +\infty} u(x) = 0$, then $+\infty$ can be regarded as a zero of u and the above theorem also holds. Say x_1 is the maximum zero of $u(x)$, then there is at least one zero of v on interval $[x_0, +\infty)$.*

Corollary 2.2. *For $\lambda_1 \geq \lambda_2$, $\#\{zeros\ of\ z(\lambda_1, x)\ on\ [0,\eta)\} \geq \#\{zeros\ of\ z(\lambda_2, x)\ on\ [0,\eta)\}$.*

Equipped with what has been discussed, we can come back to Theorem 2.1 now. The following proof is based on Titchmarsh[7].

Proof. Define

$$n(\lambda) = \#\{\text{zeros of } z(\lambda, x)\}.$$

From Theorem 2.2, for any $\lambda_1 > \lambda_2$, consider the differential equations,

$$\partial_{xx} z(\lambda_1, x) + (\lambda_1 - q(x))z(\lambda_1, x) = 0,$$
$$\partial_{xx} z(\lambda_2, x) + (\lambda_2 - q(x))z(\lambda_2, x) = 0.$$

Then between any two consecutive zeros of $z(\lambda_2, x)$, there is at least one zero of $z(\lambda_1, x)$. Moreover, since 0 is a zero point for both of them. The total number of zeros of $z(\lambda_1, x)$ is more than that of $z(\lambda_2, x)$. And $n(\lambda)$ is a non-decreasing function of λ. Next we show that $n(\lambda)$ is right continuous, and at each jump point $n(\lambda+) = n(\lambda) + 1$.

For any fixed λ_0, denote $n(\lambda_0-) = \lim_{\lambda \nearrow \lambda_0} n(\lambda) = k$. So there exists some λ_1, such that $n(\lambda) = k$ on the interval $[\lambda_1, \lambda_0)$. Say the k zeros of $z(\lambda, x)$ are

$$0 = a_1(\lambda) < a_2(\lambda) < a_3(\lambda) < \cdots < a_k(\lambda).$$

By Theorem 2.2, $a_i(\lambda)$ are all decreasing functions. Given $\lambda_2 \in [\lambda_1, \lambda_0)$, denote $a_i(\lambda_2-) = \lim_{\lambda \nearrow \lambda_2} a_i(\lambda)$. By continuity of $z(\lambda, x)$, $a_i(\lambda_2-)$ are all zeros of $z(\lambda_2, \cdot)$. Since the zeros of $z(\lambda_2, \cdot)$ are simple, $a_i(\lambda_2-)$ $i = 1, 2, \cdots, k$ are different zeros of $z(\lambda_2, \cdot)$. Hence, $a_i(\lambda_2-) = a_i(\lambda_2)$, and $a_i(\lambda)$ are all continuous function on $[\lambda_1, \lambda_0)$.

Similarly, $a_i(\lambda_0-)$ are zeros of $z(\lambda_0, \cdot)$. In order to show $n(\lambda_0-) = n(\lambda_0)$, we have to prove $a_i(\lambda_0-)$ are all the zeros of $z(\lambda_0, \cdot)$. For any i, without loss of generality, assume that $\partial_x z(\lambda_1, a_i(\lambda_1)) > 0$, then by continuity of $\partial_x z(\lambda, x)$, $\partial_x z(\lambda, a_i(\lambda)) > 0$ for all $\lambda \in [\lambda_1, \lambda_0)$ (notice here we use that $\partial_x z(\lambda, a_i(\lambda)) \neq 0$). Thus $z(\lambda, x) \geq 0$ for any $\lambda \in [\lambda_1, \lambda_0)$ and $x \in [a_i(\lambda), a_{i+1}(\lambda)]$. Moreover, by continuity of $z(\lambda, x)$, $z(\lambda_0, x) = \lim_{\lambda \nearrow \lambda_0} z(\lambda, x) \geq 0$ for $x \in [a_i(\lambda_0-), a_{i+1}(\lambda_0-)]$, which means there is no zero on interval $(a_i(\lambda_0-), a_{i+1}(\lambda_0-))$ (here we use the fact that $z(\lambda, \cdot)$ will change sign on each zero). Therefore $a_i(\lambda_0, -)$ are all the zeros of $z(\lambda_0, \cdot)$, and $n(\lambda_0-) = n(\lambda_0) = k$.

Next, we show that either $n(\lambda+) = n(\lambda)$ or $n(\lambda+) = n(\lambda) + 1$, and λ is an eigenvalue of L iff $n(\lambda+) = n(\lambda) + 1$. For any fixed λ_0, denote $n(\lambda_0+) = \lim_{\lambda \searrow \lambda_0} n(\lambda) = k$. So there exists some λ_1, such that $n(\lambda) = k$ on the interval $(\lambda_0, \lambda_1]$. Use the same notation, the k zeros of $z(\lambda, x)$ are

$$0 = a_1(\lambda) < a_2(\lambda) < a_3(\lambda) < \cdots < a_k(\lambda).$$

Similarly we have that $a_i(\lambda)$ are continuous decreasing functions on $(\lambda_0, \lambda_1]$.

Since $\lim_{x \to +\infty} q(x) = +\infty$, there exists some $C > 0$, such that $q(x) > \lambda_1$ for any $x \geq C$. From Lemma 2.1, $z(\lambda, \cdot)$ has at most one zeros on the interval $[C, \infty)$ for any $\lambda \leq \lambda_1$. Thus, $a_i(\lambda) \leq C$ on interval $(\lambda_0, \lambda_1]$ for $i = 1, 2, \cdots k - 1$. $a_i(\lambda_0+) = \lim_{\lambda \searrow \lambda_0} a_i(\lambda)$ exists for $i = 1, 2, \cdots, k - 1$, and are all zeros of $z(\lambda_0, x)$. For $i = k$ there are two possibilities,

1) $\lim_{\lambda \searrow \lambda_0} a_k(\lambda) = a_k(\lambda_0+)$, then $z(\lambda_0, x)$ has k zeros.

2) $\lim\limits_{\lambda\searrow\lambda_0} a_k(\lambda) = +\infty$, then $z(\lambda_0, x)$ has at most $k-1$ zeros on $[0, a_k(\lambda))$ for any $\lambda \in (\lambda_0, \lambda_1]$. But $a_k(\lambda) \to \infty$ as $\lambda \searrow \lambda_0$, $z(\lambda_0, x)$ has $k-1$ zeros in all.

If 2) is the case, assume that $\partial_x z(\lambda, a_k(\lambda)) > 0$ for $\lambda \in (\lambda_0, \lambda_1]$. Then for $\lambda_2 \in [C, a_k(\lambda))$,

$$z(\lambda_2, x) < 0, \quad \partial_{xx} z(\lambda_2, x) < 0, \quad \partial_x z(\lambda_2, x) > \partial_x z(\lambda_2, a_k(\lambda_2)) > 0.$$

As a result of continuity

$$z(\lambda,x) \leq 0, \quad \partial_{xx} z(\lambda_0, x) \leq 0, \quad \partial_x z(\lambda_0, x) \geq 0.$$

for $x \in [C, +\infty)$. From the remark of Lemma 2.1, $z(\lambda_0, x) \in L^2([0, +\infty))$, which means λ_0 is an eigenvalue of L.

If λ is an eigenvalue of L and $z(\lambda, x)$ is the corresponding eigenfunction, then $\lim\limits_{x\to\infty} z(\lambda, x) = 0$. Say the zeros of $z(\lambda, x)$ are $0 = a_1 < a_2 < a_3 < \cdots < a_k$. Then for any $\lambda_1 > \lambda$, there exists at least one zero on interval (a_i, a_{i+1}) for $i = 1, 2, 3, \cdots, k$ (here $a_{k+1} = +\infty$). Also $a_1 = 0$ is a zero of $z(\lambda_1, x)$, the total number of zero is at least $k+1$, thus $n(\lambda+) = k+1$.

Finally we show that

$$n(-\infty) = \lim_{\lambda\to-\infty} n(\lambda) = 1, \qquad n(+\infty) = \lim_{\lambda\to+\infty} n(\lambda) = +\infty,$$

For λ small enough, such that $\lambda - q(x) \leq -M$, consider the differential equation

$$u'' - Mu = 0, \qquad u = \frac{e^{\sqrt{M}x} - e^{-\sqrt{M}x}}{2\sqrt{M}}$$

by Theorem 2.2, $1 \leq n(\lambda) \leq \#\{\text{zeros of } u\} = 1$. Therefore $n(-\infty) = \lim\limits_{\lambda\to-\infty} n(\lambda) = 1$.

For λ big enough, such that $\lambda - q(x) \geq M$ on interval $[0, 1]$, consider the differential equation

$$u'' + Mu = 0, \qquad u = \frac{\sin(\sqrt{M}x)}{\sqrt{M}}$$

by Theorem 2.2, $n(\lambda) \geq \#\{\text{zeros of } u \text{ on } [0, 1)\} = \lceil \frac{\sqrt{M}}{\pi} \rceil$. Therefore $n(+\infty) = \lim\limits_{\lambda\to+\infty} n(\lambda) = +\infty$.

To sum up, $n(\lambda)$ is a non-decreasing right continuous function and $n(\lambda+) - n(\lambda) \in \{0, 1\}$, with range $\{1, 2, 3, \cdots, \}$. Those discontinuity points of $n(\lambda)$ are exactly the eigenvalues of L. Moreover, since $n(\lambda)$ is non-decreasing, L has discrete spectrum,

$$\lambda_1 < \lambda_2 < \lambda_3, \cdots \to +\infty.$$

And $n(\lambda) = k$ on interval $(\lambda_{k-1}, \lambda_k]$. □

To conclude this section, we prove the Minimax Principle which will be the main argument we use to derive the eigenvalue asymptotics later.

Theorem 2.3. ***(Minimax Principle)*** *Given two Sturm-Liouville operators $L_i = -\triangle + q_i(x)$, such that $\lim\limits_{x\to+\infty} q_i(x) = +\infty (i = 1, 2)$. If $q_1(x) \leq q_2(x)$, then their corresponding eigenvalues satisfy $\lambda_n^{(1)} \leq \lambda_n^{(2)}$ $n = 1, 2, 3, \cdots$.*

Proof. Assume $q_1(x) \neq q_2(x)$. Say $u_n(x)$ is the eigenfunction of L_2 with eigenvalue $\lambda_n^{(2)}$. Then

$$u_n'' + (\lambda_n^{(2)} - q_2(x))u_n = 0.$$

Consider the differential equation

$$v'' + (\lambda_n^{(2)} - q_1(x))v = 0,$$

Since $\lambda_n^{(2)} - q_1(x) \geq \lambda_n^{(2)} - q_2(x)$, and $u_n \in L^2([0, +\infty))$, by Theorem 2.2, we have

$$\#\{\text{zeros of } v(x) \text{ on } [0, +\infty)\} \geq \#\{\text{zeros of } u_n(x) \text{ on } [0, +\infty)\} + 1.$$

This implies $\lambda_n^{(1)} \leq \lambda_n^{(2)}$. □

As an application of Minimax principle we prove the following theorem,

Theorem 2.4. *Let $\lambda_n(q(x))$ is the $n-$th eigenvalue of Sturm-Liouville operator $-\triangle + q(x)$ on positive real axis with Dirichlet boundary condition. Suppose $\lim_{x\to+\infty} \frac{V(x)}{x} = 0$, then $\lambda_n(x + V(x)) = \lambda_n(x) + o(\lambda_n(x))$ as $n \to +\infty$.*

Proof. For simplicity we denote $\lambda_n = \lambda_n(x)$. Since $V(x) = o(x)$, for any $\epsilon > 0$, there exists constant C such that

$$(1-\epsilon)x - C < x + V(x) < (1+\epsilon)x + C.$$

By Minimax principle,

$$\lambda_n((1-\epsilon)x - C) \leq \lambda_n(x + V(x)) \leq \lambda_n((1+\epsilon)x + C). \tag{2.5}$$

In order to obtain an approximation of $\lambda_n(x + V(x))$, consider the perturbed Airy operator $-\triangle + (x \pm (\epsilon x + C))$. For the equation,

$$-u'' + [(1-\epsilon)x - C]u = \lambda u, \qquad u(0) = 0 \text{ and } u \in L^2([0, +\infty)),$$

which is only a translation of Airy equation, so $u(x) = Ai((1-\epsilon)^{\frac{1}{3}}(x - \frac{\lambda+C}{(1-\epsilon)^{\frac{2}{3}}}))$ and $u(0) = Ai(-\frac{\lambda+C}{(1-\epsilon)^{\frac{1}{3}}}) = 0$. This implies $-\frac{\lambda+C}{(1-\epsilon)^{\frac{1}{3}}} = -\lambda_n$ for some $n \in N^+$, thus $\lambda_n((1-\epsilon)x - C) = (1-\epsilon)^{\frac{1}{3}}\lambda_n - C$. Similarly $\lambda_n((1+\epsilon)x + C) = (1+\epsilon)^{\frac{1}{3}}\lambda_n + C$. Since $\lambda_n = (\frac{3}{2}n\pi)^{\frac{2}{3}} + O(1) \to +\infty$ as $n \to +\infty$,

$$\liminf_{n\to+\infty} \frac{\lambda_n(x+V(x))}{\lambda_n} \geq \liminf_{n\to+\infty} \frac{\lambda_n((1-\epsilon)x - C)}{\lambda_n} = \liminf_{n\to+\infty} \frac{(1-\epsilon)^{\frac{1}{3}}\lambda_n - C}{\lambda_n} = (1-\epsilon)^{\frac{1}{3}},$$
$$\limsup_{n\to+\infty} \frac{\lambda_n(x+V(x))}{\lambda_n} \leq \limsup_{n\to+\infty} \frac{\lambda_n((1+\epsilon)x + C)}{\lambda_n} = \limsup_{n\to+\infty} \frac{(1+\epsilon)^{\frac{1}{3}}\lambda_n + C}{\lambda_n} = (1+\epsilon)^{\frac{1}{3}}.$$

Let $\epsilon \searrow 0$, we get

$$1 = \lim_{\epsilon\searrow 0}(1-\epsilon)^{\frac{1}{3}} \leq \liminf_{n\to+\infty} \frac{\lambda_n(x+V(x))}{\lambda_n} \leq \limsup_{n\to+\infty} \frac{\lambda_n(x+V(x))}{\lambda_n} \leq \lim_{\epsilon\searrow 0}(1+\epsilon)^{\frac{1}{3}} = 1$$

Therefore,

$$\lim_{n\to+\infty} \frac{\lambda_n(x+V(x))}{\lambda_n} = 1, \text{ and } \lambda_n(x + V(x)) = \lambda_n + o(\lambda_n).$$

□

Remark 2.2. *The above theorem shows if the perturbation is relatively small compared with x, $V(x) = o(x)$, the first term in the asymptotic expansion of eigenvalues will remain the same. On the other hand, if we take $V(x)$ so large, say $V(x) = \pm x$, the distribution of eigenvalues may change a lot, or even fail to remain discrete.*

3 A FORMULA OF EIGENVALUE DISTRIBUTION

In order to obtain more information about the distribution of perturbed eigenvalues, we have now to consider the relation between the potential and eigenvalue distribution in more detail . However by Theorem 2.1, the eigenvalue distribution coincides with the distribution of the zeros of eigenfunctions. Actually what we need to do is to count zeros of eigenfunctions.

Theorem 3.1. *Suppose the potential $q(x)$ is monotonically increasing and in the form*

$$q(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_k x^{\alpha_k}, \qquad x \geq M$$

where $\{a_i\}_1^k$, $\{\alpha_i\}_1^k$ and M are constants. Then the $n-$th eigenvalue λ_n of operator $-\triangle + q(x)$ satisfies

$$\frac{1}{\pi}\int_0^{X_n} (\lambda_n - q(x))^{\frac{1}{2}} dx = n - \frac{1}{4} + O(\frac{1}{n}), \tag{3.1}$$

where X_n is the unique solution of $q(x) = \lambda_n$.

Proof. The formula (3.1) has been much discussed in books on quantum mechanics, which all require the potential $q(x)$ to satisfy certain favorable properties. Here the proof is based on the method given by Langer and developed by Titchmarsh[6], where they prove formula (3.1) under the assumption that $q(x)$ has derivatives up to the third order, $q(x)$ is increasing and convex downward, and as $x \to +\infty$

$$\frac{q'(x)}{q(x)} = O(\frac{1}{x}), \quad \frac{q''(x)}{q'(x)} = O(\frac{1}{x}), \quad \frac{q'''(x)}{q''(x)} = O(\frac{1}{x}).$$

We consider the following Langer transformation. For any real number λ, there is a unique X such that $q(X) = \lambda$, then let

$$\zeta(x) = \int_X^x \{\lambda - q(t)\}^{\frac{1}{2}} dt, \qquad \eta(x) = \{\lambda - q(x)\}^{\frac{1}{4}} \psi(x)$$

where $\psi(x)$ is the solution of the basic equation,

$$\psi''(x) + \{\lambda - q(x)\}\psi = 0, \tag{3.2}$$

with boundary condition $\psi(0) = 0$. The power functions are defined on $\mathbb{C} - \{(-\infty, 0]\}$, such that $0 \leq \arg\{\{\lambda - q(x)\}^p\} \leq p\pi$ $(p = \frac{1}{2}, \frac{1}{4})$. Thus the argument for ζ satisfies $\arg \zeta(x) = \frac{1}{2}\pi (x > X)$, $-\pi (x < X)$. Then $\zeta(x)$ is a C^1 function, range from $-\int_0^X \{\lambda - q(t)\}^{\frac{1}{2}} dt$ to 0, then along the $y-$axis to $+\infty i$.

$$\frac{d\eta}{d\zeta} = \frac{d\eta}{dx}\frac{dx}{d\zeta} = \frac{\psi'(x)}{\{\lambda - q(x)\}^{\frac{1}{4}}} - \frac{q'(x)\psi(x)}{4\{\lambda - q(x)\}^{\frac{5}{4}}}$$

$$\begin{aligned}\frac{d^2\eta}{d\zeta^2} &= \frac{d\frac{d\eta}{d\zeta}}{d\zeta}\frac{dx}{d\zeta} = -\{\lambda - q(x)\}^{\frac{1}{4}}\psi(x)\{1 + \frac{q''(x)}{4\{\lambda - q(x)\}^2} + \frac{5q'^2(x)}{16\{\lambda - q(x)\}^3}\} \\ &= -\{1 + \frac{q''(x)}{4\{\lambda - q(x)\}^2} + \frac{5q'^2(x)}{16\{\lambda - q(x)\}^3}\}\eta.\end{aligned}$$

Plug back into the original equation (3.2),

$$\frac{d^2\eta}{d\zeta^2} + (1 + \frac{5}{36\zeta^2})\eta = f(x)\eta, \tag{3.3}$$

where

$$f(x) = \frac{5}{36\zeta^2} - \frac{q''(x)}{4\{\lambda - q(x)\}^2} - \frac{5q'^2(x)}{16\{\lambda - q(x)\}^3}.$$

The main purpose here it to count the zeros of eigenfunctions. Observe that, if $\psi(x)$ is an eigenfunction, from Lemma 2.1, $\lim_{x \to +\infty} \psi(x) = 0$, and in this range $\psi(x)$ is either convex downwards where it is positive, or concave upwards where it is negative, so it has no zeros for $x \geq X$. As a result we only need to count zeros on the interval $[0, X)$. Since $\lambda - q(x) > 0$ on the interval $[0, X)$, the zeros of $\psi(x)$ and $\eta(x)$ coincides with each other, which means we just need to investigate the distribution of the zeros of solution of equation (3.3) on interval $[0, X)$.

It will be shown that the right-hand side of (3.3) is relatively small when λ tends to $+\infty$, and what determines the behavior of solution $\eta(x)$ is actually the following equation,

$$\frac{d^2\eta}{d\zeta^2} + (1 + \frac{5}{36\zeta^2})\eta = 0. \tag{3.4}$$

Note that $(\frac{1}{2}\pi\zeta)^{\frac{1}{2}}J_{\frac{1}{3}}(\zeta)$ and $(\frac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta)$ are basic solutions of (3.4), where $J_{\frac{1}{3}}$ is the first-type Bessel function and $H^{(1)}_{\frac{1}{3}}$ is the first Hankel H-function. By variation of constants, $\eta(x)$ satisfies the following formal integral equation

$$\eta(x,\lambda) = (\frac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta) + \frac{1}{2}\pi i\int_x^{+\infty}[H^{(1)}_{\frac{1}{3}}(\zeta)J_{\frac{1}{3}}(\theta) - J_{\frac{1}{3}}(\zeta)H^{(1)}_{\frac{1}{3}}(\theta)]\zeta^{\frac{1}{2}}\theta^{\frac{1}{2}}f(t)\eta(t,\lambda)\{\lambda - q(t)\}^{\frac{1}{2}}dt, \tag{3.5}$$

where $\theta = \zeta(t)$. Next we will show that the above integral equation can be solved by iteration methods and that part of integral is relatively small as $\lambda \to +\infty$.

Lemma 3.1. *Consider the integral equation,*

$$\phi(x) = f(x) + \int_x^{+\infty} K(x,t)\phi(t)dt, \tag{3.6}$$

where $f(x)$ is bounded and $|k(x,t)| \leq U(t) \in L^1([0,+\infty))$. Then there is a unique solution of phi(x) given by iteration method, and the solution is uniformly bounded on $[0,+\infty)$.

Proof. Denote $\phi_0(x) = f(x)$,

$$\phi_{n+1} = f(x) + \int_x^{+\infty} K(x,t)\phi_n(t)dt.$$

By subtraction,

$$\phi_{n+1}(x) - \phi_n(x) = \int_x^{+\infty} K(x,t)(\phi_n(t) - \phi_{n-1}(t))dt.$$

Then

$$|\phi_1(x) - \phi_0(x)| \leq \|f\| \int_x^{+\infty} U(t)dt = \|f\| j(x),$$

where $j(x) = \int_x^{+\infty} U(t)dt$. Next we prove by induction that $|\phi_{n+1}(x) - \phi_n(x)| = \|f\| \frac{j^{n+1}(x)}{(n+1)!}$.

$$\begin{aligned}
|\phi_{n+1}(x) - \phi_n(x)| &\leq \int_x^{\infty} |K(x,t)||(\phi_n(t) - \phi_{n-1}(t))|dt \\
&\leq \int_x^{\infty} \|f\| U(t) \frac{j^n(t)}{n!} dt \\
&= \|f\| \int_x^{\infty} d\frac{j^{n+1}(t)}{(n+1)!} \\
&= \|f\| \frac{j^{n+1}(x)}{(n+1)!}.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} |\phi_{n+1}(x) - \phi_n(x)| \leq \|f\| \sum_{n=0}^{\infty} \frac{j^n(t)}{n!} \leq \|f\| e^{j(0)}.$$

Since $\phi_0(x) + \sum \phi_n(x)$ converge uniformly, we can define

$$\phi_(x) = \phi_0(x) + \sum_{n=0}^{\infty} \phi_n(x),$$

it is easy to verify that $\phi(x)$ is the unique solution of (3.6), which coincides with $f(x)$ at $+\infty$. $\square$

Come back to (3.5), in order to solve it by iteration, we need to show,

Lemma 3.2. *As λ tends to $+\infty$,*

$$\int_0^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt = O(\lambda^{-\frac{3}{2}}) \tag{3.7}$$

Proof. For any fixed λ, denote X the unique solution of $q(x) = \lambda$. Since $q(x) = x + o(x)$ as $x \to +\infty$, we have that $X \sim \lambda$. Also for sufficiently large x,

$$\begin{aligned}
q'(x) &= 1 + a_1\alpha_1 x^{\alpha_1 - 1} + \cdots, & q'(x) &\sim 1 \\
q''(x) &= a_1\alpha_1(\alpha_1 - 1)x^{\alpha_1 - 2} + \cdots, & q'(x) &= O(x^{\alpha_1 - 2}) \\
q'''(x) &= a_1\alpha_1(\alpha_1 - 1)(\alpha_1 - 2)x^{\alpha_1 - 3} + \cdots, & q'(x) &= O(x^{\alpha_1 - 3}).
\end{aligned}$$

Thus there exists some constant M, such that $q'(x) \in [1-\epsilon, 1+\epsilon]$ and $(1-\epsilon)x \le q(x) \le (1+\epsilon)x$ for $x \ge M$. Notice here X is a singular point of $f(t)$, in order to approximate (3.7), we split the integral into four parts,

$$\begin{aligned}\int_0^\infty |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt &= \int_0^{\frac{1}{2}X} + \int_{\frac{1}{2}X}^{X} + \int_X^{2X} + \int_{2X}^{+\infty} \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Now we compute I_i one by one. On the interval $[0, \frac{1}{2}X)$,

$$\zeta(x) = -\int_x^X \{\lambda - q(t)\}^{\frac{1}{2}} dt$$

For λ sufficiently large, $(1-\epsilon)X \le q(X) = \lambda \le (1+\epsilon)X$

$$\begin{aligned}|\zeta(x)| &\ge \int_{\frac{1}{2}X}^{X} \{\lambda - q(t)\}^{\frac{1}{2}} dt \\ &\ge \int_{\frac{\lambda}{2(1-\epsilon)}}^{\frac{\lambda}{1+\epsilon}} (\lambda - (1+\epsilon)t))^{\frac{1}{2}} dt \\ &= A\lambda^{\frac{3}{2}},\end{aligned}$$

therefore $\frac{1}{\zeta(x)} = O(\lambda^{-\frac{3}{2}})$ on $[0, \frac{1}{2}X)$, so

$$I_1 \le \frac{5}{36}\int_0^{\frac{1}{2}X} \frac{\{\lambda - q(t)\}^{\frac{1}{2}}}{\zeta^2(t)} dt + \frac{1}{4}\int_0^{\frac{1}{2}X} \frac{q'^2(t)dt}{\{\lambda - q(t)\}^{\frac{5}{2}}} dt + \frac{5}{16}\int_0^M \frac{q''(t)dt}{\{\lambda - q(t)\}^{\frac{3}{2}}} dt$$

And we estimate the three terms on the right hand side one by one

$$\begin{aligned}\int_0^{\frac{1}{2}X} \frac{\{\lambda - q(t)\}^{\frac{1}{2}}}{\zeta^2(t)} &= \int_0^{\frac{1}{2}X} \frac{\zeta'(t)}{\zeta^2(t)} dt = \frac{1}{\zeta(0)} - \frac{1}{\zeta(\frac{1}{2}X)} = O(\lambda^{-\frac{3}{2}}) \\ \int_0^{\frac{1}{2}X} \frac{q'^2(t)dt}{\{\lambda - q(t)\}^{\frac{5}{2}}} dt &\le \int_0^M \frac{q'^2(t)dt}{\{\lambda - q(t)\}^{\frac{5}{2}}} dt + (1+\epsilon)\int_M^{\frac{1}{2}X} \frac{q'(t)}{\{\lambda - q(t)\}^{\frac{5}{2}}} dt \\ &= O(\lambda^{-\frac{5}{2}}) + (1+\epsilon)[-\frac{2}{3}\{\lambda - q(t)\}^{-\frac{3}{2}}]|_M^{\frac{1}{2}X} = O(\lambda^{-\frac{3}{2}}) \\ \int_0^{\frac{1}{2}X} \frac{q''(t)dt}{\{\lambda - q(t)\}^{\frac{3}{2}}} dt &= \frac{q'(t)}{\{\lambda - q(t)\}^{\frac{3}{2}}}\Big|_0^{\frac{1}{2}X} - \frac{3}{2}\int_0^{\frac{1}{2}X} \frac{q'^2(t)dt}{\{\lambda - q(t)\}^{\frac{5}{2}}} dt = O(\lambda^{-\frac{3}{2}})\end{aligned}$$

Sum them up we get that $I_1 = O(\lambda^{-\frac{3}{2}})$, similarly, one can show that $I_4 = O(\lambda^{-\frac{3}{2}})$. Now consider

I_3, on the interval $(X, 2X]$

$$
\begin{aligned}
-i\zeta(x) &= \int_X^x \{q(t)-\lambda\}^{\frac{1}{2}} dt \\
&= \int_X^x \frac{2\{q(t)-\lambda\}^{\frac{1}{2}}}{q'(t)} dt \\
&= \frac{2}{3}\frac{\{q(x)-\lambda\}^{\frac{3}{2}}}{q'(x)} + \frac{2}{3}\int_X^x \frac{\{q(t)-\lambda\}^{\frac{3}{2}} q''(t)}{q'^2(t)} dt \\
&= \frac{2}{3}\frac{\{q(x)-\lambda\}^{\frac{3}{2}}}{q'(x)} + \frac{4}{15}\frac{q''(x)\{q(x)-\lambda\}^{\frac{5}{2}}}{q'^3(x)} - \frac{4}{15}\int_X^x \{q(t)-\lambda\}^{\frac{5}{2}} d(\frac{q''(t)}{q'^3(t)}) \\
&= \frac{2}{3}\frac{\{q(x)-\lambda\}^{\frac{3}{2}}}{q'(x)} \{1 + \frac{2}{5}\frac{q''(x)\{q(x)-\lambda\}}{q'^2(x)} + S\},
\end{aligned}
$$

where,

$$
S = \frac{2}{5}\frac{q'(x)}{\{q(x)-\lambda\}^{\frac{3}{2}}} \int_X^x \{q(t)-\lambda\}^{\frac{5}{2}} \frac{q'''(t)q'(t) - 3q''^2(t)}{q'^4(t)} dt.
$$

Since for $X \le x \le 2X$, $q''(x) = O(X^{\alpha_1 - 2})$ and $q'''(x) = O(X^{\alpha_1 - 3})$

$$
\begin{aligned}
&\frac{2}{5}\frac{q''(x)\{q(x)-\lambda\}}{q'^2(x)} = O(X^{\alpha_1-2}\lambda) = O(\frac{1}{X^{1-\alpha_1}}) \longrightarrow 0 \\
&\frac{q'''(x)q'(x) - 3q''^2(x)}{q'^4(x)} = O(\frac{1}{X^{3-\alpha_1}}) + O(\frac{1}{X^{4-\alpha_2}}) = O(\frac{1}{X^{3-\alpha_1}}) \\
&S = O\{\frac{q'(x)}{X^{3-\alpha_1}\{q(x)-\lambda\}^{\frac{3}{2}}} \int_X^x \{q(t)-\lambda\}^{\frac{5}{2}} q'(t) dt\} \\
&\quad = O\{\frac{q'(x)\{q(x)-\lambda\}^2}{X^{3-\alpha_1}}\} = O(\frac{1}{X^{1-\alpha_1}}) \longrightarrow 0.
\end{aligned}
$$

For λ sufficiently large,

$$
-\frac{1}{\zeta^2(x)} = \frac{9q'^2(x)}{4\{q(x)-\lambda\}^3}\{1 - \frac{4}{5}\frac{\{q(x)-\lambda\}q''(x)}{q'^2(x)} + O(\frac{\{q(x)-\lambda\}^2 q''^2(x)}{q'^4(x)}) + O(|S|)\}.
$$

Plug back into $f(x)$,

$$
\begin{aligned}
f(x) &= \frac{5}{36\zeta^2} - \frac{q''(x)}{4\{\lambda - q(x)\}^2} - \frac{5q'^2(x)}{16\{\lambda - q(x)\}^3} \\
&= O(\frac{q''^2(x)}{q'^2(x)\{q(x)-\lambda\}}) + \frac{9q'^2(x)}{4\{q(x)-\lambda\}^3} O(|S|) \\
&= O(\frac{1}{X^{4-2\alpha_1}\{q(x)-\lambda\}}) + O(\frac{1}{X^{3-\alpha_1}\{q(x)-\lambda\}}) \\
&= O(\frac{1}{X^{3-\alpha_1}\{q(x)-\lambda\}}),
\end{aligned}
$$

thus,

$$
\begin{aligned}
I_3 &= O(\int_X^{2X} \frac{1}{X^{3-\alpha_1}\{q(t)-\lambda\}^{\frac{1}{2}}} dt) \\
&= O(\int_X^{2X} \frac{q'(t)}{X^{3-\alpha_1}\{q(t)-\lambda\}^{\frac{1}{2}}} dt) \\
&= O(\frac{1}{X^{3-\alpha_1}}(q(2X)-\lambda)) \\
&= O(\lambda^{-\frac{3}{2}}).
\end{aligned}
$$

Similarly it can be proved that $I_2 = O(\lambda^{-\frac{3}{2}})$, sum them up we get (3.7). □

Denote

$$
\chi(x) = e^{-i\zeta}\eta(x), \qquad \alpha(x) = e^{-i\zeta}(\frac{1}{2}\pi\zeta)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta), \qquad \beta(x) = e^{i\zeta}(\frac{1}{2}\pi\zeta)^{\frac{1}{2}} J_{\frac{1}{3}}(\zeta),
$$

Integral Equation (3.5) can be written as,

$$
\chi(x) = \alpha(x) + i\int_x^{+\infty} [\alpha(x)\beta(t) - e^{2i(\theta-\zeta)}\beta(x)\alpha(t)] f(t)\{\lambda - q(t)\}^{\frac{1}{2}}\chi(t)dt \tag{3.8}
$$

If $x \geq X$, then $\arg\zeta = \frac{1}{2}\pi$, say $\zeta = i\omega$ where ω is real and positive,

$$
\begin{aligned}
J_{\frac{1}{3}}(i\omega) &= e^{\frac{1}{6}\pi i} I_{\frac{1}{3}}(\omega), \qquad I_{\frac{1}{3}}(\omega) \sim \frac{e^{\omega}}{(2\pi\omega)^{\frac{1}{2}}} \\
H_{\frac{1}{3}}(i\omega) &= \frac{K_{\frac{1}{3}}(\omega)}{\frac{1}{2}\pi i e^{\frac{1}{6}\pi i}}, \qquad K_{\frac{1}{3}}(\omega) \sim (\frac{\pi}{2\omega})^{\frac{1}{2}} e^{-\omega}.
\end{aligned}
$$

If $x < X$, then $\arg\zeta = -\pi$, say $\zeta = -\zeta'$, where ζ' is real and positive,

$$
\begin{aligned}
H^{(1)}_{\frac{1}{3}}(\zeta) &= \frac{2e^{\frac{1}{3}i\pi}}{i\sqrt{3}}(J_{\frac{1}{3}}(\zeta') + J_{-\frac{1}{3}}(\zeta')) \\
J_{\frac{1}{3}}(\zeta) &= e^{-\frac{1}{3}i\pi} J_{\frac{1}{3}}(\zeta').
\end{aligned}
$$

Moreover,

$$
im(\theta - \zeta) = im\{\int_x^t (\lambda - q(u))^{\frac{1}{2}} du\} \geq 0.
$$

Therefore $\alpha(x)\beta(t) - e^{2i(\theta-\zeta)}\beta(x)\alpha(t)$ is bounded. By Lemma 3.1, (3.8) can be solved by iteration methods and the solution $\chi(x)$ is uniformly bounded. Thus

$$
|\int_x^{+\infty} [\alpha(x)\beta(t) - e^{2i(\theta-\zeta)}\beta(x)\alpha(t)] f(t)\{\lambda - q(t)\}^{\frac{1}{2}}\chi(t)dt| \leq A\int_0^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt = O(\lambda^{-\frac{3}{2}}),
$$

and

$$
\chi(x) = \alpha(x) + O(\lambda^{-\frac{3}{2}}).
$$

Come back to integral equation (3.5), if $\eta(x,\lambda)$ is the solution, then $\phi(x,\lambda) = \{\lambda - q(x)\}^{-\frac{1}{4}}\eta(x,\lambda)$ is a solution for the basic differential equation (3.2),

$$\phi(x,\lambda) = \{\lambda - q(x)\}^{-\frac{1}{4}}\{(\frac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta) + O(e^{i\zeta}\lambda^{-\frac{3}{2}})\} \tag{3.9}$$

As $x \to +\infty$, $e^{i\zeta} = e^{-\omega}$, so $\phi(x,\lambda)$ is exponentially decaying. If we take $\lambda = \lambda_n$ as an eigenvalue, and ϕ_n the corresponding eigenfunction, then the Wronskian of $\phi(x,\lambda_n)$ and $\phi_n(x)$ tends to zero at infinity, so there exists a constant C_n

$$C_n\phi(x,\lambda_n) = \phi_n(x).$$

As $x \to +\infty$, $\arg\phi(x,\lambda) \to (-\frac{1}{4}\pi - \frac{1}{4}\pi - \frac{1}{6}\pi) = -\frac{2}{3}\pi$, we can take $C_n = e^{-\frac{2}{3}\pi i}$, then $e^{\frac{2}{3}\pi i}\phi(x,\lambda_n)$ is a real-valued function. Thus

$$\phi_n(x) = e^{\frac{2}{3}\pi i}\phi(x,\lambda_n) = e^{\frac{2}{3}\pi i}\{\lambda - q(x)\}^{-\frac{1}{4}}\{(\frac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta) + O(e^{i\zeta}\lambda^{-\frac{3}{2}})\}$$

Combined with the asymptotic expansion of Bessel function,

$$J_{\frac{1}{3}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}}\{\cos(z - \frac{5}{12}) + O(\frac{1}{z})\} \tag{3.10}$$
$$J_{-\frac{1}{3}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}}\{\cos(z - \frac{1}{12}) + O(\frac{1}{z})\} \tag{3.11}$$

Take $x = 0$ in the formula (3.9), we get

$$\phi(0,\lambda) = 2e^{-\frac{2}{3}\pi i}\{\lambda - q(0)\}^{-\frac{1}{4}}\{\cos(Z - \frac{1}{4}\pi) + O(\frac{1}{Z})\},$$

where

$$Z = \int_0^X \{\lambda - q(u)\}^{\frac{1}{2}}du, \qquad Z \sim \lambda^{\frac{3}{2}}.$$

Then the boundary condition $\phi(0) = 0$ gives that

$$Z_n - \frac{3}{4}\pi = m\pi + \rho_n,$$

here $Z_n = \int_0^{X_n}\{\lambda_n - q(t)\}^{\frac{1}{2}}dt$, ρ_n satisfies $\sin(\rho_n) = O(\frac{1}{Z_n})$, so $\rho_n = O(\frac{1}{Z_n})$. Define $z(x) = \int_x^X(\lambda - q(t)^{\frac{1}{2}}dt$, then on the interval $[0, X)$

$$\begin{aligned}\eta(x,\lambda) &= (\frac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta) + O(\lambda^{-\frac{3}{2}})\\ &= (\frac{1}{2}\pi(-z))^{\frac{1}{2}}\frac{2e^{\frac{1}{3}\pi i}}{\sqrt{3}i}(J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)) + O(\lambda^{-\frac{3}{2}})\\ &= e^{-\frac{2}{3}\pi i}(\frac{2\pi}{3})^{\frac{1}{2}}v(z) + O(\lambda^{-\frac{3}{2}})\end{aligned}$$

Write $\varphi(x) = (\frac{3}{2\pi})^{\frac{1}{2}}e^{\frac{2}{3}\pi i}\eta(x,\lambda)$, then

$$\varphi(x) = v(z) + w(x), \qquad w(x) = O(\lambda^{-\frac{3}{2}}).$$

There exists some constant A such that $|w(x)| \leq A\lambda^{-\frac{3}{2}}$. Also if we differentiate (3.5) on both sides with respect to x, the integral in the formula for $\frac{d\eta}{dx}$ remains the same except that the differentiation produces an extra factor $\{\lambda - q(x)\}^{\frac{1}{2}}$. Hence, by the same method one can show that there exists a constant A such that

$$|w'(x)| \leq A\lambda^{-\frac{3}{2}}\{\lambda - q(x)\}^{\frac{1}{2}}.$$

Sum up formula (3.10) and (3.11), we get the asymptotic expansion for $v(z)$,

$$v(z) \sim (\frac{6}{\pi})^{\frac{1}{2}} \cos(z - \frac{1}{4}\pi)$$

Similarly, $v'(z)$ has the following asymptotics

$$v'(z) \sim -(\frac{6}{\pi})^{\frac{1}{2}} \sin(z - \frac{1}{4}\pi)$$

Since we need to count the zeros of $\phi(x)$ on the interval $[0, X)$, here $\phi(x)$ is the sum of $v(z)$ and $w(x)$. Compared with $v(z)$, $w(x)$ is relatively small, it is taken granted that the behavior of $\phi(x)$ is mainly determined by $v(z)$. So in order to understand the distribution of zeros of $\phi(x)$, we onl need to get a more detailed zero distribution of $v(z)$.

Lemma 3.3. *If n is a sufficiently large large integer, the function*

$$v(x) = x^{\frac{1}{2}}\{J_{\frac{1}{3}}(x) + J_{-\frac{1}{3}}(x)\}$$

has exactly n zeros in the interval $0 < x < (n + \frac{1}{4})\pi$.

Proof. There has been an extensive dicussion about Bessel function and its zeros. For the proof of this lemma, we refer to [8]. ☐

Since $Z_n = (m + \frac{3}{4})\pi + \rho_n$, by the above lemma there are exactly m zeros of $v(z)$ on the interval $(0, (m + \frac{1}{4})\pi)$ and there is a zero around Z_n. Take $\delta > 0$, such that δ is smaller than any maximum of $v(z)$ and $-\delta$ is bigger than any minimum of $v(z)$. The line $y = \delta$ intersects the graph of $v(z)$ at $z'_0 < z'_1 < z'_2 < \cdots$, the corresponding value of x are $x'_0 > x'_1 > x'_2 > \cdots$. Similarly line $y = -\delta$ intersects $v(z)$ at $z''_1 < z''_2 < z''_3 < \cdots$, and corresponding value of x are $x''_1 > x''_2 > x''_3 > \cdots$. As depicted in Figure 2. Then there is exactly one zero of v(z) in each interval $[z'_i, z''_i]$ $(i = 1, 2, \cdots, m)$. Since $v'(z) \neq 0$ on each interval $[z'_i, z''_i]$ $(i = 1, 2, \cdots, m)$, there exists a constant $B \geq 0$ such that $|v'(z)| \geq B$ on those intervals, thus

$$|\frac{dv(z)}{dx}| = |\frac{dv(z)}{dz}\frac{dz}{dx}| \geq B\{\lambda - q(x)\}^{\frac{1}{2}}.$$

If we take λ large enough, such that $|w(x)| < \delta$ then $\varphi(x)$ has no zeros on interval $[x'_{2i+1}, x'_{2i}]$ and $[x''_{2i}, x''_{2i-1}]$. Further, if λ is chosen so large that $|w'(x)| < B\{\lambda - q(x)\}^{\frac{1}{2}}$, then $\varphi(x)$ is monotonic on each interval $[x'_i, x''_i]$. Since $\varphi(x'_i) > 0$ and $\varphi(x''_i) < 0$, there is unique zero of $\phi(x)$ on interval $[x'_i, x''_i]$. By the above lemma, there are m such zeros in all.

Now consider the ends of interval $[0, X_n)$. On the interval $[x'_0, X_n)$, $\varphi(x)$ is monotonic, so there is at most one zero on the interval $[x'_0, X_n]$. Since $\phi(x) = C\{\lambda - q(x)\}^{\frac{1}{4}}\phi_n(x)$, it must be the end point X_n, but this is not a zero of $\psi_n(x)$. On the lower end of $[0, X_n)$, suppose the greatest of z'_i,

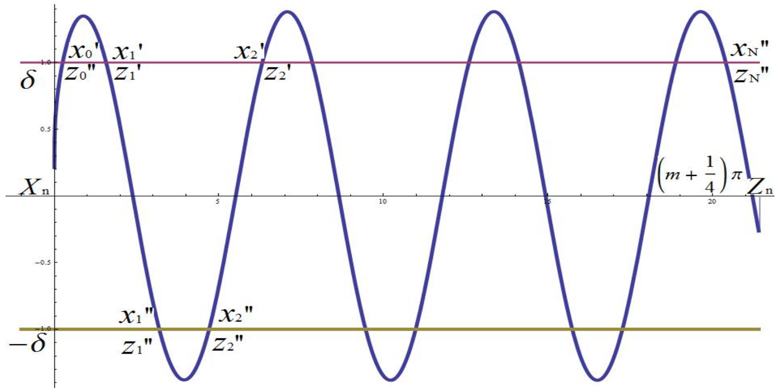

Figure 2: $v(z)$

z_i'' is z_N', and the corresponding value of x is x_N'. On the interval $[0, x_N']$, $\varphi(x)$ is monotonic. So $\psi(x)$ and $\psi_n(x)$ has at most one zero on this interval, actually it is the zero of $\psi_n(x)$ at $x = 0$.

In all, there are $m + 1$ zeros on the interval $[0, X_n]$, thus from Theorem 2.1, λ_n is the $m + 1$-th eigenvalue and $\phi(x, \lambda_n)$ is its eigenfunction. So we have $m + 1 = n$, and

$$Z_n = \int_0^X \{\lambda - q(x)\}^{\frac{1}{2}} = (n-1)\pi + \frac{3}{4}\pi + O(\frac{1}{Z_n}) = (n - \frac{1}{4})\pi + O(\frac{1}{n})$$

which completes the proof og Theorem 3.1. □

4 ASYMPTOTIC EXPANSION OF PERTURBED EIGENVALUES

4.1 Asymptotic behavior of eigenvalues w.r.t. potential perturbation

Suppose the perturbation $V(x)$ can be described in terms of its asymptotic expansion (complete expansion or to certain order) at $+\infty$

$$V(x) \sim a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + \cdots ,$$

where $\{a_k\}_1^\infty$, $\{\alpha_k\}_1^\infty$ are constants, and $1 > \alpha_1 > \alpha_2 > \alpha_3 > \cdots$. And for $\forall k \geq 1$,

$$V(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + o(x^{\alpha_k}).$$

Given any $\epsilon > 0$, denote

$$\widetilde{q_1}(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + (a_k - \epsilon) x^{\alpha_k}$$
$$\widetilde{q_2}(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + (a_k + \epsilon) x^{\alpha_k}$$

Then there exists some constant M, such that for $x \geq M$, both $\widetilde{q_1}(x)$ and $\widetilde{q_2}(x)$ are monotonically increasing, and

$$\widetilde{q_1}(x) \leq x + V(x) \leq \widetilde{q_2}(x), \qquad x \geq M$$

Take M sufficiently large, such that

$$\widetilde{q_2}(M) > \sup_{0\le x\le M}\{x+V(x)\}.$$

Then we can replace $\widetilde{q_i}(x)$ by strictly increasing smooth functions $q_i(x)$, such that $q_i(x)=\widetilde{q_i}(x)$ for $x\ge M$ $(i=1,2)$ and

$$q_1(x)\le x+V(x)\le q_2(x), \qquad x\ge 0.$$

The monotonicity of $q_i(x)$ implies the existence of inverse function $Q_i(s)$ on $[q_i(0),+\infty)$. With the

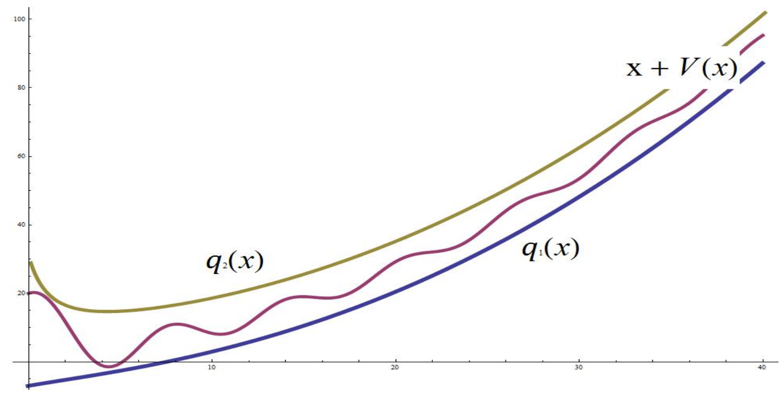

Figure 3: $q_1(x)\le x+V(x)\le q_2(x)$

change of variable $x=Q_i(s)$,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{q_i^{-1}(\lambda)}(\lambda-q_i(x))^{\frac12}dx=\frac{1}{\pi}\int_{q_i(0)}^{\lambda}(\lambda-s)^{\frac12}dQ_i(s)\\
=&\frac{1}{\pi}(\lambda-s)^{\frac12}Q_i(s)\big|_{q_i(0)}^{\lambda}+\frac{1}{2\pi}\int_{q_i(0)}^{\lambda}(\lambda-s)^{-\frac12}Q(s)ds\\
=&\frac{1}{2\pi}\int_{q_i(0)}^{\lambda}\frac{Q_i(s)}{(\lambda-s)^{\frac12}}ds, \qquad\qquad i=1,2. \qquad (4.1)
\end{aligned}$$

The following lemma enables us to determine the behavior of $Q_i(s)$ at infinity from the asymptotic expansion of $q_i(x)$.

Lemma 4.1. *Suppose $y(x)$ has an asymptotic expansion*

$$y(x)=x+a_1x^{\alpha_1}+a_2x^{\alpha_2}+a_3x^{\alpha_3}+\cdots+a_kx^{\alpha_k}+o(x^{\alpha_k}) \qquad (4.2)$$

where $\{a_i\}_1^k$ and $\{\alpha_i\}_1^k$ are constants, and $1>\alpha_1>\alpha_2>\alpha_3>\cdots>\alpha_k$. Then its inverse function $x(y)$ (not necessarily continuous) also has asymptotic expansion at $+\infty$ up to $o(y^{\alpha_k})$

$$x(y)=y+c_1y^{\gamma_1}+c_2y^{\gamma_2}+c_3y^{\gamma_3}+\cdots+c_ly^{\gamma_l}+o(y^{\alpha_k}), \qquad (4.3)$$

where $\{c_i\}_1^l$ and $\{\gamma_i\}_1^l$ are constants, $1>\gamma_1>\gamma_2>\gamma_3>\cdots>\gamma_{l-1}>\gamma_l=\alpha_k$, and are uniquely determined by $\{a_i\}_1^k$ and $\{\alpha_i\}_1^k$. Actually $\{c_i\}_1^l$ and $\{\gamma_i\}_1^l$ are all polynomials of $\{a_i\}_1^k$ and $\{\alpha_i\}_1^k$.

Proof. From expression (4.2), $y(x) = x + o(x)$, so $x(y) = y + o(y)$. Assume we already have the asymptotic expansion of $x(y)$

$$x(y) = y + c_1 y^{\gamma_1} + c_2 y^{\gamma_2} + c_3 y^{\gamma_3} + \cdots + c_j y^{\gamma_j} + o(y^{\gamma_j}).$$

If $\gamma_j > \alpha_k$, we show $x(y)$ has higher order asymptotic expansion up to $o(y^{\max\{\gamma_j+\alpha_1-1,\alpha_k\}})$

$$x(y) = y + c_1 y^{\gamma_1} + c_2 y^{\gamma_2} + c_3 y^{\gamma_3} + \cdots + o(y^{\max\{\gamma_j+\alpha_1-1,\alpha_k\}}). \tag{4.4}$$

Denote

$$\Delta = x(y) - (y + c_1 y^{\gamma_1} + c_2 y^{\gamma_2} + c_3 y^{\gamma_3} + \cdots + c_j y^{\gamma_j}), \qquad \Delta = o(y^{\gamma_j}). \tag{4.5}$$

Notice that

$$\begin{aligned} a_i x^{\alpha_i} &= a_i (y + c_1 y^{\gamma_1} + c_2 y^{\gamma_2} + c_3 y^{\gamma_3} + \cdots + c_j y^{\gamma_j} + o(y^{\gamma_j}))^{\alpha_i} \\ &= a_i y^{\alpha_i} (1 + c_1 y^{\gamma_1 - 1} + c_2 y^{\gamma_2 - 1} + c_3 y^{\gamma_3 - 1} + \cdots + c_j y^{\gamma_j - 1} + o(y^{\gamma_j - 1}))^{\alpha_i} \\ &= a_i y^{\alpha_i} + c_1 a_i y^{\alpha_i + \gamma_1 - 1} + \cdots + o(y^{\alpha_i + \gamma_j - 1}). \end{aligned}$$

Plug (4.5) into (4.2), we get

$$\begin{aligned} y =& (y + c_1 y^{\gamma_1} + c_2 y^{\gamma_2} + c_3 y^{\gamma_3} + \cdots + c_j y^{\gamma_j} + \Delta) \\ &+ (a_1 y^{\alpha_1} + \cdots + o(y^{\alpha_1 + \gamma_j - 1})) + \cdots + (a_k y^{\alpha_k} + \cdots + o(y^{\alpha_k + \gamma_j - 1})) + o(y^{\alpha_k}) \end{aligned}$$

Compare the coefficients on both sides, we can get an asymptotic expansion of Δ up to

$$o(y^{\max\{\gamma_j+\alpha_1-1,\alpha_k\}}).$$

Substitute the expression of Δ into (4.5), we get formula (4.4). Since $\alpha_1 - 1 < 0$, repeat the above process, we can get the asymptotic expansion (4.3). □

Suppose $q(x)$ is in the form

$$q(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + \epsilon x^{\alpha_k} \tag{4.6}$$

where $\{a_i\}_1^k$ and $\{\alpha_i\}_1^k$ are constants, and $1 > \alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_k$. Since $q(x)$ is monotonically increasing for large x, there exists $Q(s)$, such that $Q(q(x)) = x$ for large x.

Lemma 4.2. *Under the assumption above $Q(s)$ has an asymptotic expansion at $+\infty$ up to $o(s^{\alpha_k})$*

$$Q(s) = s + c_1 s^{\gamma_1} + c_2 s^{\gamma_2} + \cdots c_l s^{\gamma_l} - \epsilon s^{\alpha_k} + o(s^{\alpha_k}), \tag{4.7}$$

where $\{c_i\}_1^l$ and $\{\gamma_i\}_1^l$ are constants (independent of ϵ), $1 > \gamma_1 > \gamma_2 > \gamma_3 > \cdots > \gamma_{l-1} > \gamma_l = \alpha_k$.

Proof. From Lemma 4.1, $Q(s)$ has the form (4.7), and $\{c_i\}_1^{l-1}$ are uniquely determined by $\{a_1\}_0^{k-1}$ and $\{\alpha_i\}_0^{k-1}$. Plug the asymptotic expansion of $Q(s)$ into (4.6), the coefficient of s^{α_k} can be obtained by comparing the coefficients of s^{α_k} on both sides. □

With the two lemmas above, we can prove the asympototic expansion of eigenvalues, which is the main results of this paper.

Theorem 4.1. *Suppose $V(x)$ (not necessarily increasing or smooth) has asymptotic expansion up to $o(x^{\alpha_k})$,*

$$V(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + o(x^{\alpha_k}).$$

Then $\lambda_n(x + V(x))$ has asymptotic expansion

$$\lambda_n(x+V(x)) = \begin{cases} (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{\kappa_m} + o(n^{\frac{2}{3}\alpha_k}), & \alpha_k > -1 \\ (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n + o(n^{-\frac{2}{3}} \ln n), & \alpha_k = -1 \\ (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1} n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n + O(n^{-\frac{2}{3}}), & \alpha_k < -1 \end{cases}$$

where $\{d_i\}$ and $\{\kappa_i\}$ are constants, depend on the asymptotic expansion of $V(x)$.

Proof. Construct $q_1(x)$ and $q_2(x)$ as above, such that $q_1 \le x + V(x) \le q_2$. By Lemma 4.2 their inverse functions have the form

$$\begin{aligned} Q_1(x) &= s + c_1 s^{\gamma_1} + c_2 s^{\gamma_2} + \cdots c_l s^{\gamma_l} + \epsilon s^{\alpha_k} + \theta_1(s), \qquad \theta_1(s) = o(s^{\alpha_k}) \\ Q_2(x) &= s + c_1 s^{\gamma_1} + c_2 s^{\gamma_2} + \cdots c_l s^{\gamma_l} - \epsilon s^{\alpha_k} + \theta_2(s), \qquad \theta_2(s) = o(s^{\alpha_k}), \end{aligned}$$

on the interval $[T, \infty)$ ($T \ge \max\{1, q_1(0), q_2(0)\}$), where $\gamma_1 > \gamma_2 > \cdots > \gamma_l = \alpha_k$. Plug them into formula (4.1),

$$\begin{aligned} &\frac{1}{2\pi}\int_{q_i(0)}^{\lambda} \frac{Q_i(s)}{(\lambda - s)^{\frac{1}{2}}} ds = \frac{1}{2\pi}\int_{T}^{\lambda} \frac{Q_i(s)}{(\lambda - s)^{\frac{1}{2}}} ds + \frac{1}{\pi}\int_{q_i(0)}^{T} \frac{Q_i(s)}{(\lambda - s)^{\frac{1}{2}}} ds \\ =&\frac{1}{2\pi}\int_{T}^{\lambda} \frac{s + c_1 s^{\gamma_1} + c_2 s^{\gamma_2} + \cdots c_l s^{\gamma_l} + (\pm \epsilon s^{\alpha_k}) + \theta_i(s)}{(\lambda - s)^{\frac{1}{2}}} ds + O(\lambda^{-\frac{1}{2}}) \\ =&\frac{1}{2\pi}\int_{T}^{\lambda} \frac{s}{(\lambda - s)^{\frac{1}{2}}} ds + \sum_{i=1}^{l} \frac{1}{2\pi}\int_{T}^{\lambda} \frac{c_i s^{\gamma_i}}{(\lambda - s)^{\frac{1}{2}}} ds + \frac{1}{2\pi}\int_{T}^{\lambda} \frac{\pm \epsilon s^{\alpha_k}}{(\lambda - s)^{\frac{1}{2}}} ds + \qquad (4.8) \\ &+ \frac{1}{2\pi}\int_{T}^{\lambda} \frac{\theta_i(s)}{(\lambda - s)^{\frac{1}{2}}} ds + O(\lambda^{-\frac{1}{2}}) \qquad (4.9) \end{aligned}$$

where $i = 1, 2$. In the above formula there are three types of integrals,

$$\begin{aligned} &\int_T^{\lambda} \frac{s^{\alpha}}{(\lambda - s)^{\frac{1}{2}}} = \int_0^{\lambda} \frac{s^{\alpha}}{(\lambda - s)^{\frac{1}{2}}} - \int_0^{T} \frac{s^{\alpha}}{(\lambda - s)^{\frac{1}{2}}} = \frac{\sqrt{\pi}\Gamma(1+\alpha)}{\Gamma(\frac{3}{2}+\alpha)} \lambda^{\frac{1}{2}+\alpha} + O(\lambda^{-\frac{1}{2}}), \qquad \alpha > -1 \\ &\int_T^{\lambda} \frac{s^{\alpha}}{(\lambda - s)^{\frac{1}{2}}} = \int_T^{\lambda} \frac{1}{s\lambda^{\frac{1}{2}}} ds + \int_T^{\lambda} \left(\frac{1}{s(\lambda - s)^{\frac{1}{2}}} - \frac{1}{s\lambda^{\frac{1}{2}}}\right) ds \\ =&(\ln \lambda - \ln T)\lambda^{-\frac{1}{2}} + \int_T^{\lambda} \frac{1}{\lambda^{\frac{1}{2}}(\lambda - s)^{\frac{1}{2}}[\lambda^{\frac{1}{2}} + (\lambda - s)^{\frac{1}{2}}]} = \lambda^{-\frac{1}{2}} \ln \lambda + O(\lambda^{-\frac{1}{2}}), \qquad \alpha = -1 \\ &\int_T^{\lambda} \frac{\theta_i(s)}{(\lambda - s)^{\frac{1}{2}}} = \int_T^{\lambda} \frac{o(s^{\alpha_k})}{(\lambda - s)^{\frac{1}{2}}} = o(\lambda^{\frac{1}{2}+\alpha_k}), \qquad \theta_i(s) = o(s^{\alpha_k}). \end{aligned}$$

Consider the case with $\alpha_k = -1$, as the other two cases can be proved in the same way. Substitute the above integration results into (4.8),

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{q_i(0)}^{\lambda}\frac{Q_i(s)}{(\lambda-s)^{\frac{1}{2}}}ds=\frac{1}{2\pi}\int_{T}^{\lambda}\frac{s}{(\lambda-s)^{\frac{1}{2}}}ds+\sum_{i=1}^{l}\frac{1}{2\pi}\int_{T}^{\lambda}\frac{c_i s^{\gamma_i}}{(\lambda-s)^{\frac{1}{2}}}ds+\\
&+\frac{1}{2\pi}\int_{T}^{\lambda}\frac{\pm\epsilon s^{\alpha_k}}{(\lambda-s)^{\frac{1}{2}}}ds+\frac{1}{2\pi}\int_{T}^{\lambda}\frac{\theta_i(s)}{(\lambda-s)^{\frac{1}{2}}}ds+O(\lambda^{-\frac{1}{2}})\\
&=\frac{2}{3\pi}\lambda^{\frac{3}{2}}+\sum_{i=1}^{l-1}\frac{c_i\Gamma(1+\gamma_i)}{2\sqrt{\pi}\Gamma(\frac{3}{2}+\gamma_i)}\lambda^{\frac{1}{2}+\gamma_i}+\frac{c_l\pm\epsilon}{2\pi}\lambda^{-\frac{1}{2}}\ln\lambda+O(\lambda^{-\frac{1}{2}}).
\end{aligned}
$$

Say λ'_n is the n-th eigenvalue of operator $-\triangle+q_1(x)$ and λ''_n is the n-th eigenvalue of operator $-\triangle+q_2(x)$.

$$
\begin{aligned}
n-\frac{1}{4}+O(\frac{1}{n})&=\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}}+\sum_{i=1}^{l-1}\frac{c_i\Gamma(1+\gamma_i)}{2\sqrt{\pi}\Gamma(\frac{3}{2}+\gamma_i)}\lambda^{\frac{1}{2}}+\frac{c_l}{2\pi}\lambda_n'^{-\frac{1}{2}}\ln\lambda'_n+\frac{\epsilon}{2\pi}\lambda_n'^{-\frac{1}{2}}\ln\lambda'_n+O(\lambda_n'^{-\frac{1}{2}})\\
n-\frac{1}{4}+O(\frac{1}{n})&=\frac{2}{3\pi}\lambda_n''^{\frac{3}{2}}+\sum_{i=1}^{l-1}\frac{c_i\Gamma(1+\gamma_i)}{2\sqrt{\pi}\Gamma(\frac{3}{2}+\gamma_i)}\lambda^{\frac{1}{2}}+\frac{c_l}{2\pi}\lambda_n''^{-\frac{1}{2}}\ln\lambda''_n+\frac{-\epsilon}{2\pi}\lambda_n''^{-\frac{1}{2}}\ln\lambda''_n+O(\lambda_n''^{-\frac{1}{2}}).
\end{aligned}
$$

Since $O(\frac{1}{n})=O(\lambda_n'^{-\frac{3}{2}})=O(\lambda_n'^{-\frac{1}{2}})$, also $O(\frac{1}{n})=o(\lambda_n''^{-\frac{1}{2}})$. Rearrange above expressions,

$$
\begin{aligned}
n&=\{\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}}+\sum_{i=1}^{l-1}\frac{c_i\Gamma(1+\gamma_i)}{2\sqrt{\pi}\Gamma(\frac{3}{2}+\gamma_i)}\lambda_n'^{\frac{1}{2}}+\frac{c_l}{2\pi}\lambda_n'^{-\frac{1}{2}}\ln\lambda'_n+\frac{1}{4}\}+\frac{\epsilon}{2\pi}\lambda_n'^{-\frac{1}{2}}\ln\lambda'_n+O(\lambda_n'^{-\frac{1}{2}})\\
n&=\{\frac{2}{3\pi}\lambda_n''^{\frac{3}{2}}+\sum_{i=1}^{l-1}\frac{c_i\Gamma(1+\gamma_i)}{2\sqrt{\pi}\Gamma(\frac{3}{2}+\gamma_i)}\lambda_n''^{\frac{1}{2}}+\frac{c_l}{2\pi}\lambda_n''^{-\frac{1}{2}}\ln\lambda''_n+\frac{1}{4}\}+\frac{-\epsilon}{2\pi}\lambda_n''^{-\frac{1}{2}}\ln\lambda''_n+O(\lambda_n''^{-\frac{1}{2}}).
\end{aligned}
$$

Lemma 4.3. *Suppose the following equation holds asymptotically for λ_n*

$$
n=\frac{2}{3\pi}\lambda_n^{\frac{3}{2}}+c_1\lambda_n^{\gamma_1}+\cdots+c_{l-1}\lambda_n^{\gamma_{l-1}}+c_l\lambda_n^{-\frac{1}{2}}\ln\lambda_n+O(\lambda_n^{-\frac{1}{2}}),\tag{4.10}
$$

then λ_n has the following asymptotic expansion,

$$
\lambda_n=(\frac{3\pi}{2}n)^{\frac{2}{3}}+d_1n^{\kappa_1}+\cdots+d_{m-1}n^{\kappa_{m-1}}+d_mn^{-\frac{2}{3}}\ln n+O(n^{-\frac{2}{3}}),\tag{4.11}
$$

where $\{d_i\}_1^m$ and $\{\kappa_i\}_1^m$ are constants, depend only on $\{c_i\}_1^l$ and $\{\gamma_i\}_1^{l-1}$.

Proof. With a change of variable $\mu_n=\frac{2}{3\pi}\lambda_n^{\frac{3}{2}}$, and substitute into (4.10),

$$
n=\mu_n+c'_1\mu_n^{\frac{2}{3}\gamma_1}+\cdots+c'_{l-1}\mu_n^{\frac{2}{3}\gamma_{l-1}}+c'_l\mu_n^{-\frac{1}{3}}\ln\mu_n+O(\mu_n^{-\frac{1}{3}}).
$$

In the same way as Lemma 4.1, we can get the asymptotic expansion μ_n in terms of n up to $O(n^{-\frac{1}{3}})$. Then put the expansion into $\lambda_n=(\frac{3\pi}{2}\mu_n)^{\frac{2}{3}}$, we get the asymptotic expansion of λ_n up to $O(n^{-\frac{2}{3}})$. Moreover plug (4.11) into (4.10) and compare the coefficients of $n^{-\frac{1}{3}}\ln n$ on both sides, we get

$$
d_m=-(\frac{2}{3})^{\frac{5}{3}}\pi^{-\frac{1}{3}}c_l
$$

□

By the above lemma, there exists constants $\{d_i\}_1^m$ and $\{\kappa_i\}_1^m$, $\frac{2}{3} > \kappa_1 > \kappa_2 > \cdots > \kappa_m = -\frac{2}{3}$ such that

$$\lambda_n' = (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1}n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n - \epsilon R n^{-\frac{2}{3}} \ln n + O(n^{-\frac{2}{3}})$$
$$\lambda_n'' = (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_{m-1}n^{\kappa_{m-1}} + d_m n^{-\frac{2}{3}} \ln n + \epsilon R n^{-\frac{2}{3}} \ln n + O(n^{-\frac{2}{3}}),$$

where R is a constant. Since $\lambda_n' \le \lambda_n \le \lambda_n''$, let $\epsilon \searrow 0$, we get

$$\lambda_n = (\frac{3\pi}{2}n)^{\frac{2}{3}} + d_1 n^{\kappa_1} + d_2 n^{\kappa_2} + \cdots + d_m n^{-\frac{2}{3}} \ln n + o(n^{-\frac{2}{3}} \ln n)$$

□

Remark 4.1. *We are only interested in perturbations $V(x)$ such that $V(x) = o(x)$. Otherwise the first term of asymptotic expansion of $\lambda_n(x+V(x))$ may be changed. Such potential has unique asymptotic expansion to some order $o(x^{\alpha_k})$ or to $o(x^{-\infty})$, corresponds to one of the following forms:*

$$q(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + o(x^{\alpha_k}), \quad \textit{can not be expanded further.}$$
$$q(x) = x + a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots + a_k x^{\alpha_k} + \cdots.$$

From the existence of asymptotic expansion of $x + V(x)$, eigenvalues can always be asymptotically expanded to certain order.

4.2 Example: asymptotic expansion of potential does not completely determine eigenvalues

On the other hand, even if $V(x)$ is a smooth function with complete asymptotic expansion, this does not necessarily guarantee the existence of complete asymptotic expansion of λ_n as $n \to +\infty$. Moreover only the information of $V(x)$ at infinite cannot completely determine the asymptotic expansion of λ_n, which means the behavior of $V(x)$ on finite interval, say $[0,1]$, can affect the higher order terms in the asymptotic expansion of λ_n. The following theorem gives such an example.

Take

$$V(x) = \begin{cases} e^{-\frac{2}{1-x}} & (x \in [0,1)) \\ 0 & (x \in [1,+\infty)), \end{cases}$$

the two smooth functions x and $x + V(x)$ have the same asymptotic expansion at $+\infty$.

Theorem 4.2. *With V(x) defined above, the eigenvalues of $-\triangle + x$ (denoted as $\lambda_n(x)$) and $-\triangle + x + V(x)$ ($\lambda_n(x+V(x))$) satisfy the following relation*

$$\lambda_n(x+V(x)) = \lambda_n(x) + \frac{C}{2}\lambda_n(x)^{-1} + O(\lambda_n(x)^{-2}),$$

here constants $C = \int_0^1 V(t)dt$.

Proof. It is easy to verify that both x and $x + V(x)$ satisfies the conditions in Theorem 3.1. For simplicity denote $\lambda_n = \lambda_n(x)$ and $\lambda_n' = \lambda_n(x+V(x))$. By theorem 3.4, for n large enough

$$\frac{1}{\pi}\int_0^{\lambda_n} (\lambda_n - x)^{\frac{1}{2}} dx = n - \frac{1}{4} + O(\frac{1}{n}) \tag{4.12}$$
$$\frac{1}{\pi}\int_0^{\lambda_n'} (\lambda_n' - x - V(x))^{\frac{1}{2}} dx = n - \frac{1}{4} + O(\frac{1}{n}), \tag{4.13}$$

since here $X_n = \lambda_n$ and $X'_n = \lambda'_n$. Rearrange expression (4.13)

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\lambda'_n}(\lambda'_n - x - V(x))^{\frac{1}{2}}dx \\
=&\frac{1}{\pi}\int_0^{\lambda'_n}(\lambda'_n - x - V(x))^{\frac{1}{2}}dx - \frac{1}{\pi}\int_0^1 (\lambda'_n - x)^{\frac{1}{2}} - (\lambda'_n - x - V(x))^{\frac{1}{2}}dx \\
=&\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}} - \frac{1}{\pi}\int_0^1 \frac{V(x)}{(\lambda'_n - x)^{\frac{1}{2}} + (\lambda'_n - x - V(x))^{\frac{1}{2}}}dx \\
=&\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}} - \frac{1}{\pi}\int_0^1 \frac{V(x)}{2\lambda_n'^{\frac{1}{2}}}dx - \frac{1}{\pi}\int_0^1 \frac{V(x)}{(\lambda'_n - x)^{\frac{1}{2}} + (\lambda'_n - x - V(x))^{\frac{1}{2}}} - \frac{V(x)}{2\lambda_n'^{\frac{1}{2}}}dx \\
=&\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}} - \frac{C}{2\pi}\lambda_n'^{-\frac{1}{2}} + O(\lambda_n^{-\frac{3}{2}}). \qquad (4.14)
\end{aligned}
$$

From expression (4.13), we have $\lambda_n^{\frac{3}{2}} \sim \frac{3\pi}{2}n$, so $O(\lambda_n^{-\frac{3}{2}}) = O(\frac{1}{n})$. Combine (4.12) (4.13) and (4.14)

$$
\frac{2}{3\pi}\lambda_n'^{\frac{3}{2}} - \frac{2}{3\pi}\lambda_n^{\frac{3}{2}} - \frac{C}{2\pi}\lambda_n'^{-\frac{1}{2}} = O(\lambda_n^{-\frac{3}{2}}) \qquad (4.15)
$$

Denote $\Delta = \lambda_n'^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}}$. From expression (4.15), there exists constant $A > 0$,

$$
\begin{aligned}
A\lambda_n^{-\frac{3}{2}} \geq& \frac{2}{3\pi}\lambda_n'^{\frac{3}{2}} - \frac{2}{3\pi}\lambda_n^{\frac{3}{2}} - \frac{C}{2\pi}\lambda_n'^{-\frac{1}{2}} = O(\lambda_n^{-\frac{3}{2}}) \\
=& \frac{2}{3\pi}(\lambda_n'^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}})(\lambda'_n + \lambda_n'^{\frac{1}{2}}\lambda_n^{\frac{1}{2}} + \lambda_n) - \frac{C}{2\pi}\lambda_n'^{-\frac{1}{2}} \\
\geq& \frac{2}{\pi}\lambda_n\Delta - \frac{C}{2\pi}\lambda_n^{-\frac{1}{2}}.
\end{aligned}
$$

Therefore $\Delta = O(\lambda_n^{-\frac{3}{2}})$. Plug this back into formula (4.15), we get

$$
3\lambda_n\Delta + 3\lambda_n^{\frac{1}{2}}\Delta^2 + \Delta^3 = \frac{3C}{4}\lambda_n^{-\frac{1}{2}} + O(\lambda_n^{-\frac{3}{2}}).
$$

So $\Delta = \frac{C}{4}\lambda_n^{-\frac{3}{2}} + O(\lambda_n^{-\frac{5}{2}})$ and

$$
\lambda'_n = (\lambda_n^{\frac{1}{2}} + \Delta)^2 = (\lambda_n^{\frac{1}{2}} + \frac{C}{4}\lambda_n^{-\frac{3}{2}} + O(\lambda_n^{-\frac{5}{2}}))^2 = \lambda_n + \frac{C}{2}\lambda_n^{-1} + O(\lambda_n^{-2}). \qquad (4.16)
$$

□

With the above formula, it is easy to write down the asymptotic expansion of λ_n and λ'_n,

$$
\begin{aligned}
\lambda_n &= (\frac{3\pi}{2}n)^{\frac{2}{3}} + \frac{1}{6}(\frac{3\pi}{2})^{\frac{2}{3}}n^{-\frac{1}{3}} + O(n^{-\frac{4}{3}}) \\
\lambda'_n &= (\frac{3\pi}{2}n)^{\frac{2}{3}} + \frac{1}{6}(\frac{3\pi}{2})^{\frac{2}{3}}n^{-\frac{1}{3}} + \frac{C}{2}(\frac{3\pi}{2})^{-\frac{2}{3}}n^{-\frac{2}{3}} + O(n^{-\frac{4}{3}})
\end{aligned}
$$

Remark 4.2. *From Theorem 4.1, if $V(x)$ has complete asymptotic expansion, from the asymptotics of $V(x)$, one can determine the asymptotic expansion of λ_n up to $O(n^{-\frac{2}{3}})$. However, the example above shows merely from the asymptotic expansion of $V(x)$, it is impossible to get the complete asymptotic expansion of λ_n. More precisely, behavior of $V(x)$ on a finite interval will affect the asymptotic expansion of λ_n up to $O(n^{-\frac{2}{3}})$.*

Since the asymptotic expansion of λ_n is determined by the asymptotic expansion of $V(x)$ up to $O(n^{-\frac{2}{3}})$. It is taken for granted one may believe that the asymptotic expansion of $V(x)$ can be recovered by the asymptotic expansion of λ_n. Here we introduce the results on inverse spectral problems.

Theorem 4.3. *If we restrict us into a special class of perturbations, which has complete asymptotic expansion at $+\infty$, then those coefficients $\{a_i\}_1^k$ and indexes $\{\alpha_i\}_1^k$ with $\alpha_i \geq -1$ are spectral invariant (uniquely determined by its spectral).*

Proof. From Theorem 4.1, if $V(x)$ has complete asymptotic expansion at $+\infty$, then λ_n has asymptotic expansion up to $O(n^{-\frac{2}{3}})$. From the asymptotic expansion of λ_n, we can first recover the asymptotic expansion of n in terms of λ_n, and then recover the asymptotic expansion of $Q(s)$ up to $o(s^{-1})$. Finally get the asymptotic expansion of $V(x)$ at infinity up to $o(x^{-1})$. ☐

Remark 4.3. *The terms like $x^{-1-\epsilon}$ in expansions of V(x) has a very small influence on eigenvalues, which can be offset by the change of $V(x)$ on finite interval. By Weyl law the asymptotics of eigenvalue distribution is determine by Area$(\xi^2 + V(x) \leq \lambda)$. For terms like $x^{-1-\epsilon}$, the area $\int x^{-1-\epsilon}$ is finite. Therefore one can not determine from the asymptotic expansion whether its affect on eigenvalues is caused by $x^{-1-\epsilon}$ or merely by the behavior of $V(x)$ on finite interval. Furthermore if we do not restrict the asymptotics of $V(x)$ to be sum of power functions, any terms $\psi(x)$ in the asymptotic expansion, as long as the integral $\int \psi(x)$ diverges, can be recovered from the eigenvalues.*

A Continuity of the Solution of a System of Ordinary Differential Equations

Here we consider the eigenfunctions of Strum-Liouville operator $-y'' + q(x)y = \lambda y$ on positive real axis, with boundary condition $y(0) = 0, y'(0) = 1$. Let $y_1(x) = y(x)$, $y_2(x) = y'(x)$ and $z(\lambda, x) = [y_1(x), y_2(x)]^T$, consider the first-order differential system,:

$$\frac{dz(\lambda,x)}{dx} = \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} y_2 \\ (q(x)-\lambda)y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{bmatrix} z(\lambda,x) = A(\lambda,x)z(\lambda,x), \tag{A.1}$$

where $A(\lambda, x) = \begin{bmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{bmatrix}$.

Theorem A.1. *$z(\lambda, x)$, $\partial_x z(\lambda, x)$ are continuous functions, and $z(\lambda, x)$ is an analytic function of λ.*

Proof. For simplicity here we only consider the case when $x \geq 0$. First we change (A.1) into the Volterra type integral equation,

$$z(\lambda,x) = z(\lambda,0) + \int_0^x A(\lambda,t)z(\lambda,t)dt. \tag{A.2}$$

Solve this by Picard's process of successive approximation. We start by setting $\phi_0(x) = z(\lambda, 0) = [0, 1]^T$, and the recursive relations,

$$\phi_{n+1}(\lambda,\tau) = z(\lambda,0) + \int_0^\tau A(\lambda,t)\phi_n(t)dt \qquad (n = 0, 1, 2\cdots)$$

Setting $\psi_n(\lambda, x) = \phi_n(\lambda, x) - \phi_{n-1}(\lambda, x) (n \geq 1)$ and $\psi_0(\lambda, x) = \phi_0(\lambda, x)$, we observe that

$$\psi_n(\lambda,\tau) = \int_0^\tau A(\lambda,t)\psi_{n-1}(\lambda,t)dt \qquad (n = 1, 2, 3\cdots), \tag{A.3}$$

hence by interchanging the order of integration

$$\begin{aligned}
\psi_1(\lambda,\tau) &= \int_0^\tau A(\lambda,t)\psi_0(\lambda,t)dt \\
\psi_2(\lambda,\tau) &= \int_0^\tau A(\lambda,x_1)\psi_1(\lambda,x_1)dx_1 \\
&= \int_0^\tau \int_t^\tau A(\lambda,x_1)dx_1 A(\lambda,t)\psi_0(\lambda,t)dt \\
&= \int_0^\tau A_1(\lambda,\tau,t)A(\lambda,t)\psi_0(\lambda,t)dt \\
&\cdots\cdots \\
\psi_n(\lambda,\tau) &= \int_0^\tau A(\lambda,x_{n-1})\psi_{n-1}(\lambda,x_{n-1})dx_{n-1} \\
&= \int_0^\tau A(\lambda,x_{n-1}) \int_0^{x_{n-1}} A_{n-2}(\lambda,x_{n-1},t)A(\lambda,t)\psi_0(\lambda,t)dtdx_{n-1} \\
&= \int_0^\tau \int_t^\tau A(\lambda,x_{n-1})A_{n-2}(\lambda,x_{n-1},t)dx_{n-1}A(\lambda,t)\psi_0(\lambda,t)dt \\
&= \int_0^\tau A_{n-1}(\lambda,\tau,t)A(\lambda,t)\psi_0(\lambda,t)dt, \qquad \text{(A.4)}
\end{aligned}$$

here $A_n(\lambda,\tau,t)$ is defined recursively as

$$\begin{aligned}
A_0(\lambda,\tau,t) &= I_2 \\
A_n(\lambda,\tau,t) &= \int_t^\tau A(\lambda,x_n)A_{n-1}(\lambda,x_n,t)dx_n \qquad (n=1,2,3\cdots)
\end{aligned}$$

If we can somehow show that the infinite series $\sum_{n=0}^\infty \psi_n(\lambda,x)$ converges uniformly to $\psi(\lambda,x)$ for $(\lambda,x)\in K$, where K is any compact set of $C\times R$. Then let $z(\lambda,x)=\sum_0^\infty \psi_n(\lambda,x)$, we have

$$\begin{aligned}
&\psi_0(\lambda,x) + \int_0^x A(\lambda,t)z(\lambda,t)dt \\
=&\psi_0(\lambda,x) + \int_0^x A(\lambda,t)\sum_0^\infty \psi_n(\lambda,t)dt \\
=&\psi_0(\lambda,x) + \sum_{n=0}^\infty \int_0^x A(\lambda,t)\psi_n(\lambda,t)dt \\
=&\psi_0(\lambda,x) + \sum_{n=0}^\infty \psi_{n+1}(\lambda,x) \\
=&z(\lambda,x).
\end{aligned}$$

Thus solution of (A.2) is given by $\sum_{n=0}^\infty \psi_n(\lambda,x)$.

Now we begin to show that the infinite sum $\sum_{n=0}^\infty \psi_n(\lambda,x)$ converges absolutely on each compact set. For any matrix $A=[a_{ij}]$ define the norm to be $\|A\|=\max\{|a_{ij}|\}$. Say for fixed $x\in[0,\tau]$ and $|\lambda|\le R$, we have $\|A(\lambda,x)\|\le M$. Now we prove by induction for $\tau\ge t\ge 0$, we have

$$\|A_n(\lambda,\tau,t)\| \le \frac{[2(\tau-t)M]^n}{n!}. \qquad \text{(A.5)}$$

It is easy to check (A.5) for $n = 1$. Assume it hold for $n - 1$,

$$\begin{aligned}
\|A_{n+1}(\lambda,\tau,t)\| &= \|\int_t^\tau A(\lambda,x_n)A_{n-1}(\lambda,x_n,t)dx_n\| \\
&\le \int_t^\tau 2\|A(\lambda,x_n)\|\|A_{n-1}(\lambda,x_n,t)\|dx_n \\
&\le \int_t^\tau 2M\frac{[2(x_n-t)M]^n}{n!}dx_n \\
&= \frac{[2(\tau-t)M]^{n+1}}{(n+1)!}
\end{aligned}$$

Thus (A.5) holds for all integers.

Plug (A.5) back into (A.4), we get

$$\begin{aligned}
\|\psi_n(\lambda,\tau)\| &= \|\int_0^\tau A_{n-1}(\lambda,\tau,t)A(\lambda,t)\psi_0(\lambda,t)dt\| \\
&\le \int_0^\tau 2\|A_{n-1}(\lambda,\tau,t)\|\|A(\lambda,t)\|dt \\
&\le -\frac{[2(\tau-t)M]^n}{n!}\Big|_0^\tau \\
&= \frac{(2\tau M)^n}{n!}
\end{aligned} \tag{A.6}$$

Therefore

$$\sum_{n=0}^{\infty}\|\psi_n(\lambda,x)\| \le \sum_0^\infty \frac{(2xM)^n}{n!} = e^{2xM} \le e^{2\tau M} \tag{A.7}$$

Since for any fixed x, $\psi_n(\lambda,x)$ are analytic functions of λ, moreover from (A.7), $\sum_{n=0}^\infty \psi_n(\lambda,x)$ converges uniformly on any compacts. So $z(\lambda,x)$ is analytic function of λ for each fixed x. Hence, we have proved that $z(\lambda,x) = \sum_{n=0}^\infty \psi_n(\lambda,x)$ is the only solution of (A.2), which is a continuous function and for any fixed x, each term of $z(\cdot,x)$ is an analytic function. □

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