Discrete-time anisotropic KPZ growth of random surface in 2+1 dimensions

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Abstract

We study the discrete-time analog of the stochastic growth models on a system of interacting particles in 2+1 dimensions, which belong to the anisotropic KPZ class introduced in [1]. We consider the discrete time dynamics described in [1], derive the formulas for the limit shape and discover the Gaussian free field. Then we do some simulations to observe the limiting behavior of this system, and study the phenomena concerning the convergence of such moments of demeaned height functions which describe the behavior. We also do some simulations which shows some evidence of Conjecture 1.4 in [1].

1 Introduction

The system discussed here consists of $\frac{n(n+1)}{2}$ particles, denoted as $\{x_k^m\}$ (where m ranges from 1 through n, and k ranges from 1 through m), each with a position in the integer lattice $\mathbb{Z}^2$, evolving in discrete time (in our model, we simply consider $\mathbb{N}_0-$ valued time) with a determinantal structure described by the determinantal kernel.

The principal object we study here is a randomly growing surface, embedded in the four-dimensional space-time. There are two kinds of interesting projections of this model. One is to reduce the spatial dimension by one, the other is by fixing time, each yielding some random surface. For the first kind, we should note that the projections of our growth models described in Section 2 and 3 to $\{x_1^m\}_{m\geq 1}$ and $\{x_m^m\}_{m\geq 1}$ give the "Bernoulli jumps with blocking" and "Bernoulli jumps with pushing" discussed in [15]. On the opther hand, $\{x_1^m\}$ in the continuous time context is a totally asymmetric simple exclusion process (TASEP). The second kind (fixed-time) projection will be mentioned in §3.2.

The other object discussed in this paper is the Gaussian free field, which is commonly assumed to describe the fluctuations of random surfaces appearing in a wide class of models in statistical physics, in particular in the case with dimension equal to two (for more general discussion, see [11], for its connection to SLE_4 see [12], to dimer model see [7], [8], and to other growth models on a system of interlacing particles see [1], [13]). And in our case we expect the two dimensional GFF to describe the fluctuation of the random surface induced by our discrete time growth model.

Our main result is the behavior of the height functions that integrate the particle configuration and reflect the limit shape of the growing surface consisting of facets interpolated by a curved piece. We see that the curved region has a Gaussian fluctuation by exploiting the determinantal structure of the process. In the same flavor, we observe the presence of a Gaussian free field as the pushforward of the random surface we consider. As a result, we can express the limit of the moment of the height fluctuations at multiple points in terms of the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary condition.

Section 2 gives the background of our model and some analytic tools to describe the limit shape. Section 3 gives the theoretical results. §3.1 cites the transition kernel for four types of discrete-time interacting particle systems on the plane, computes critical points for each system. And using the critical points, we obtain the growth velocity of the surface and show the processes are in AKPZ class. §3.2 introduces some important notions of Gaussian free fields, and presents the main theorems about the growing surface and GFF. Following Duits' constructions in [13], §3.3 proves one of the main results. Section 4 discusses the continuous-time AKPZ growth model in [1] and its left-jumping counterpart as a limit of the discrete case, and demonstrates how to derive the theorems for the height fluctuations and their moments in the continuous time scheme from the discrete time scheme. Section 5 presents the simulation result testing conjecture 1.4 in [1]. Some interesting phenomena in the simulations, and suggestions about topics for further research are provided, which ends with a remark about the conditions in which

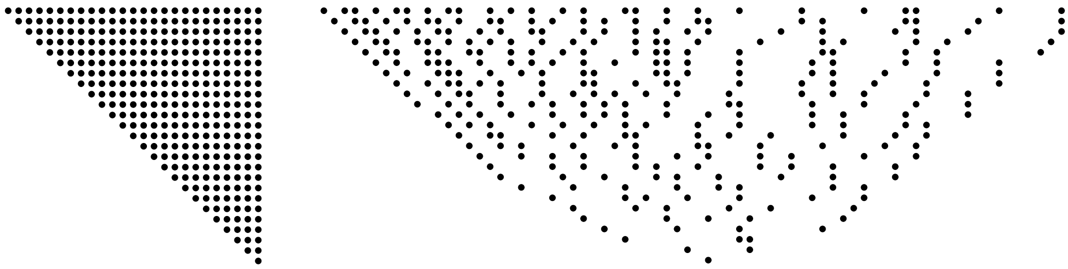

Figure 1: The initial configuration and random configuration for the case $n = 25$, and F of the form (B) or (D), which will be introduced after (2.8).

the main results in this paper holds.

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2 The model

We consider the discrete-time anisotropic 2-dimensional growth model as follows: consider a Markov chain on the (locally compact, separable) state space $\mathcal{X}$ of interlacing variables

$$\mathcal{S}^{(n)} = \big\{\{x_k^m\}_{\substack{m=1,\cdots,n \\ k=1,\cdots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} | x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1}\big\}, \qquad n \in \mathbb{Z}^+,$$

where the double-indexed x_k^m (which also refers to the particle itself) can be interpreted as the position of particle with label (k, m). The initial condition we consider is a fully-packed one, which means $x_k^m(0) = k - m - 1$ for all k, m.

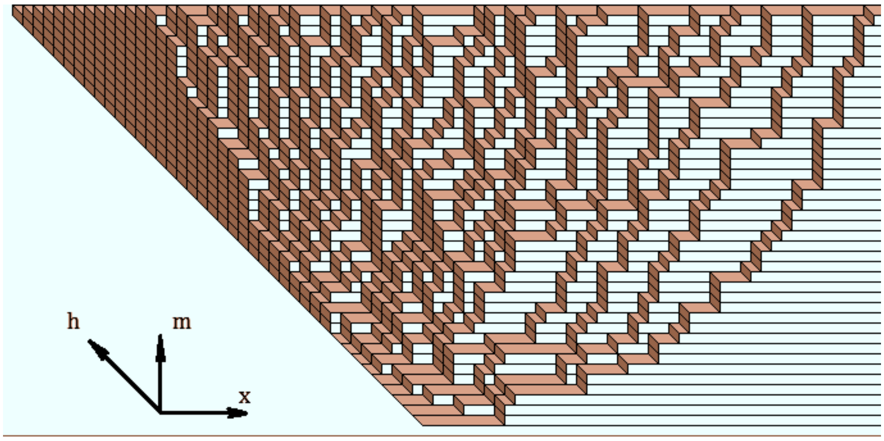

Figure 2: Demonstration for the height function, with a random configuration of the model with $n = 40$ rows.

The particles evolve according to the following dynamics. At each time $t \in \mathbb{Z}_+$, each particle x_k^m attempts to jump to either left or right by one unit according to some function F_t, and each particle x_k^m can be pushed by lower ones or be blocked by lower ones. Define the **height function** as $h(x, m, t) = \text{card}\{k | x_k^m(t) > x\}$. Intuitively, we can describe the anisotropy as follows: particles with smaller upper indices are heavier than those with larger upper indices, so that the heavier particles block and push the lighter ones in order to preserve the interlacing conditions.

To analyze the properties of such random point processes, let us start with introducing some definitions given in [4] and [14].

Definition 2.1. *We call a subset of $\mathcal{X}$ as a **point configuration**, and define Conf($\mathcal{X}$) to be the space of point configurations in $\mathcal{X}$; we call a relatively compact subset A of $\mathcal{X}$ a **window**, and for a window A, define N_A: Conf($\mathcal{X}$) $\to \mathbb{N}_0$ by $N_A(X) =$ card($X \cap A$). Put the smallest Borel measure on Conf($\mathcal{X}$) such that these functions are measurable. A **random point process** is a probability measure on Conf($\mathcal{X}$).*

Definition 2.2. *For a finite subset $A \subset \mathcal{X}$, the **correlation function** of A is*

$$\rho(A) = Prob\{X \in Conf(\mathcal{X}) | A \subset X\}.$$

For $A = \{x_1, \cdots, x_n\}$, we write $\rho(A) = \rho_n(x_1, \cdots, x_n)$, which is a symmetric function on $\mathcal{X}^n$.

Definition 2.3. *A point process on $\mathcal{X}$ is **determinantal** if there is a kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that*

$$\rho_n(x_1, \cdots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n, \tag{2.1}$$

*where $K(x, y)$ is called the **correlation kernel**.*

For $m = 1, \cdots, n$, let $\mathcal{X}_m$ denote the set of particles $\{x_1^m, \cdots, x_m^m\}$, and P_m be a stochastic matrix defining a Markov chain on $\mathcal{X}_m$, and let $\Lambda_1^2, \cdots, \Lambda_{n-1}^n$ be Markov links between these sets:

$$P_k : \mathcal{X}_k \times \mathcal{X}_k \to [0,1], \quad \sum_{y \in \mathcal{X}_k} P_k(x,y) = 1, \quad x \in \mathcal{X}_k, \quad k = 1, \cdots, n;$$

$$\Lambda_{k-1}^k : \mathcal{X}_k \times \mathcal{X}_{k-1} \to [0,1], \quad \sum_{y \in \mathcal{X}_{k-1}} \Lambda_{k-1}^k(x,y) = 1, \quad x \in \mathcal{X}_k, \quad k = 2, \cdots, n.$$

We expect these matrices to satisfy the commutation relations:

$$\Delta_{k-1}^k := \Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k, \quad , k = 2, \cdots, n, \tag{2.2}$$

The commutation relations can be provided using Toeplitz-like transition probabilities.

Proposition 2.1. *Choose n nonzero complex numbers $\alpha_1, \cdots, \alpha_n$, and let $F(z)$ be an analytic function in an annulus center at 0 that contains each α_i^{-1}, and that $F(\alpha_i^{-1}) \neq 0$. Then*

$$\frac{1}{\prod_{i=1}^n F(\alpha_i^{-1})} \sum_{\{y_1 < \cdots < y_n\} \subset \mathbb{Z}} \det(\alpha_i^{y_j})_{i,j=1}^n \det(f(x_i - y_j))_{i,j=1}^n = \det(\alpha_i^{x_j})_{i,j=1}^n,$$

where $f(m) = \frac{1}{2\pi i} \int dz\ z^{-(m+1)} F(z)$, or equivalently, $F(z) = \sum_{m \in \mathbb{Z}} f(m) z^m$.

Assume y_n is virtual with $f(x_k - virt.) = \alpha_n^{x_k}$. Then

$$\frac{1}{\prod_{i=1}^{n-1} F(\alpha_i^{-1})} \sum_{\{y_1 < \cdots < y_n\} \subset \mathbb{Z}} \det(\alpha_i^{y_j})_{i,j=1}^{n-1} \det(f(x_i - y_j))_{i,j=1}^n = \det(\alpha_i^{x_j})_{i,j=1}^n.$$

See proof in [1].

Definition 2.4. *Set $X = (x_1 < \cdots < x_n)$ and $Y = (y_1 < \cdots < y_n)$, $Y' = (y_1, \cdots, y_{n-1}, y_n = virt.)$. Define the* ***Toeplitz matrix*** *of F as $(f(i-j))_{i,j \in \mathbb{Z}}$, and* ***Toeplitz-like transition probability***

$$T_n(\alpha_1, \cdots, \alpha_n; F)(X,Y) := \frac{1}{\prod_{i=1}^n F(\alpha_i^{-1})} \det(f(x_i - y_j))_{i,j=1}^n \frac{\det(\alpha_i^{y_j})_{i,j=1}^n}{\det(\alpha_i^{x_j})_{i,j=1}^n}, \tag{2.3}$$

$$T_{n-1}^n(\alpha_1, \cdots, \alpha_n; F)(X,Y') := \frac{1}{\prod_{i=1}^{n-1} F(\alpha_i^{-1})} \det(f(x_i - y_j))_{i,j=1}^n \frac{\det(\alpha_i^{y_j})_{i,j=1}^{n-1}}{\det(\alpha_i^{x_j})_{i,j=1}^n}. \tag{2.4}$$

The last fraction of equation (2.3) is called ***Doob's h-transform***.

We have some nice properties for Toeplitz-like transition probability: let F_1, F_2 be two holomorphic functions in an annulus with $F_i(\alpha_j^{-1}) \neq 0$, then

$$T_n(F_1)T_n(F_2) = T_n(F_2)T_n(F_1) = T_n(F_1F_2), \tag{2.5}$$

$$T_n(F_1)T_{n-1}^n(F_2) = T_{n-1}^n(F_1)T_{n-1}(F_2) = T_{n-1}^n(F_1F_2). \tag{2.6}$$

From [1], we have the following two lemmas (see [1] for their proofs):

Lemma 2.1. *Consider $F(z) = 1 + pz$, that is,*

$$f(m) = \begin{cases} p, & m = 1, \\ 1, & m = 0, \\ 0, & \textit{otherwise} \end{cases} .$$

For integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, we have

$$\det[f(x_i - y_j)]_{i,j=1}^n = p^{\sum_{i=1}^n (x_i - y_i)} \prod_{i=1}^n \mathbb{1}_{\{0,1\}}(x_i - y_i).$$

Lemma 2.2. *Consider $F(z) = (1 - qz)^{-1}$, that is*

$$f(m) = \begin{cases} q^m, & m \geq 0, \\ 0, & \textit{otherwise} \end{cases} .$$

For integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$,

$$\det[f(x_i - y_j)]_{i,j=1}^n = \begin{cases} q^{\sum_{i=1}^n (x_i - y_i)}, & x_{i-1} < y_i \leq x_i, 1 \leq i \leq n, \\ 0, & \textit{otherwise}; \end{cases}$$

for integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_{n-1}$, and $y_n = virt.$ such that $f(x - virt.) = q^x$,

$$\det[f(x_i - y_j)]_{i,j=1}^n = \begin{cases} (-1)^{n-1} q^{\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i}, & x_i < y_i \leq x_{i+1}, 1 \leq i \leq n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Now we continue to discuss the multivariate Markov chains with the Toeplitz matrices which are tools to give the commutation relations (2.2) via relations (2.5), (2.6). Take

$$\begin{aligned} \Lambda_{k-1}^k &= T_{k-1}^k(\alpha_1, \cdots, \alpha_k; (1 - \alpha_k z)^{-1}), \quad k = 2, \cdots, n, \\ P_m(t) &= T_m(\alpha_1, \cdots, \alpha_m; F_t(z)), \qquad\quad m = 1, \cdots, n, \end{aligned}$$

where $F_t(z) = (1 + \beta_t^+ z)$ or $(1 + \beta_t^- / z)$ or $(1 - \gamma_t^+ z)^{-1}$ or $(1 - \gamma_t^- / z)^{-1}$, on the sequential update state space

$$\begin{aligned} \mathcal{S}_\Lambda^{(n)} &= \Big\{ (x^1, \cdots, x^n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n | \prod_{m=2}^n \Lambda_{m-1}^m (x^m, x^{m-1}) > 0 \Big\} \\ &= \Big\{ \{x_k^m\}_{\substack{m=1,\cdots,n \\ k=1,\cdots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} | x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1} \Big\}, \end{aligned}$$

with transition probabilities as (using notation $X_n = (x^1, \cdots, x^n), Y_n = (y^1, \cdots, y^n)$)

$$P_\Lambda^{(n)}(X_n, Y_n) = \begin{cases} P_1(x^1, y^1) \prod_{k=2}^n \frac{P_k(x^k, y^k) \Lambda_{k-1}^k (y^k, y^{k-1})}{\Delta_{k-1}^k (x^k, y^{k-1})}, & \prod_{k=2}^n \Delta_{k-1}^k (x^k, y^{k-1}) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

We choose $\alpha_1 = \cdots = \alpha_n = 1$, and assume

$$\beta_t^{\pm},\ \gamma_t^{\pm} > 0,\ \gamma_t^{+} < \min\{\alpha_1, \cdots, \alpha_n\} = 1,\ \gamma_t^{-} < \min\{\alpha_1^{-1}, \cdots, \alpha_n^{-1}\} = 1. \tag{2.8}$$

The dynamics on $\mathcal{S}_\Lambda^{(n)}$ can be described as follows. Given $\{x_k^m(t)\} \in \mathcal{S}_\Lambda^n$, to obtain $\{x_k^m(t+1)\}$, we perform the sequential update from $\mathcal{X}_1$ to $\mathcal{X}_n$. When we are at $\mathcal{X}_m$, $1 \le m \le n$, the new positions of the particles $x_1^m < \cdots < x_m^m$ are decided independently. Here we summarize the results directly following from [1]:

(A) For $F_t(z) = (1 + \beta_t^+ z)$, x_k^m either is forced to stay if $x_{k-1}^{m-1}(t+1) = x_k^m(t)$, or is forced to jump to the left by 1 if $x_k^{m-1}(t+1) = x_k^m(t)$, or chooses between staying or jumping to the left by 1 with probability of staying as $1/(1+\beta_t^+)$.

(B) For $F_t(z) = (1 + \beta_t^-/z)$, x_k^m either is forced to stay if $x_k^{m-1}(t+1) = x_k^m(t) + 1$, or is forced to jump to the right by 1 if $x_{k-1}^{m-1}(t+1) = x_k^m(t) + 1$, or chooses between staying or jumping to the right by 1 with probability of staying as $1/(1+\beta_t^-)$

(C) For $F_t(z) = (1 - \gamma_t^+ z)^{-1}$, x_k^m chooses its new position according to a geometric random variable with parameter $1/\gamma_t^+$ conditioned to stay in the segment

$$[\max(x_{k-1}^m(t) + 1, x_{k-1}^{m-1}(t+1)), \min(x_k^m(t), x_k^{m-1}(t+1) - 1)].$$

(D) For $F_t(z) = (1 - \gamma_t^-/z)^{-1}$, x_k^m chooses its new position according to a geometric random variable with parameter γ_t^- conditioned to stay in the segment

$$[\max(x_k^m(t), x_{k-1}^{m-1}(t+1)), \min(x_{k+1}^m(t) - 1, x_k^{m-1}(t+1) - 1)].$$

Remark 2.1. *Note that there is another type of update scheme, parallel update, which has the same correlation functions as sequential update Markov chains. For more details about parallel update, see* [*1*].

3 Main results

3.1 Limit shape of the growing surface

One main result of [1] is that, in the continuous-time model, the growing surface in the $(\nu, \eta, \mathbf{h})-$space has a limit shape consisting of facets interpolated by a curved piece. Here we otain a similar result for discrete-time model, which is the expression (3.21).

First we would like to quantitatively describe our model in languages introduced in Section 2. Taking the element x_j of $\mathcal{X}$ in definition 3 to be of form $\varkappa_j = (y_j, m_j.t_j) \in \mathbb{Z} \times \{1, \cdots, n\} \times \mathbb{N}_0$, or $\mathbb{Z} \times \{1, \cdots, n\} \times \mathbb{R}_{\ge 0}$, $j = 1, \cdots, M$, the correlation function

$$\rho_M(\varkappa_1, \cdots, \varkappa_M) = \\ \text{Prob}\{\text{For } j = 1, \cdots, M \text{ there exists a } 1 \le k_j \le m_j \text{ such that } x_{k_j}^{m_j}(t_j) = y_j\}.$$

As in [1] and [3], we introduce a partial order on pairs $(m, t) \in \{1, \cdots, n\} \times \mathbb{N}_0$ or $\{1, \cdots, n\} \times \mathbb{R}_{\ge 0}$:

$$(m_1, t_1) \prec (m_2, t_2) \ \text{ iff } \ m_1 \le m_2, t_1 \ge t_2 \text{ and } (m_1, t_1) \ne (m_2, t_2). \tag{3.1}$$

From [1], we have the following theorem

Theorem 3.1. *Consider the Markov chain $P_\Lambda^{(n)}$ with the densely packed initial condition and $F_t(z)$ be one of the four kinds we discuss above. Assume that triplets $\varkappa_j = (y_j, m_j, t_j)$, $j = 1, \cdots, M$, are such that any two distinct pairs (m_i, t_i), (m_j, t_j) are comparable with respect to $\prec$. Then*

$$\rho_M(\varkappa_1, \cdots, \varkappa_M) = \det[\mathcal{K}(\varkappa_i, \varkappa_j)]_{i,j=1}^M, \tag{3.2}$$

where

$$\begin{aligned}\mathcal{K}(y_1, m_1, t_1;\, y_2, m_2, t_2) &= -\frac{1}{2\pi i}\oint_{\Gamma_0} \frac{dw}{w^{y_2-y_1+1}} \frac{\prod_{t=t_2}^{t_1-1} F_t(w)}{\prod_{l=m_1+1}^{m_2}(1-\alpha_l w)} \mathbb{1}_{[(m_1,t_1)\prec(m_2,t_2)]} \\ &+ \frac{1}{(2\pi i)^2}\oint_{\Gamma_0} dw \oint_{\Gamma_{\alpha^{-1}}} dz \frac{\prod_{t=0}^{t_1-1} F_t(w)}{\prod_{t=0}^{t_2-1} F_t(z)} \frac{\prod_{l=1}^{m_1}(1-\alpha_l w)}{\prod_{l=1}^{m_2}(1-\alpha_l z)} \frac{w^{y_1}}{z^{y_2+1}} \frac{1}{w-z},\end{aligned} \tag{3.3}$$

where the contours $\Gamma_0, \Gamma_{\alpha^{-1}}$ are closed and positively oriented, and include poles 0 and $\{\alpha_i^{-1}\}, i = 1, \cdots, n$, respectively, and no other poles.

In particular, in the cases we consider $\alpha_i = 1$. Define a shifted and conjugate kernel K by

$$K(x_1, n_1, t_1; x_2, n_2, t_2) = (-1)^{n_1-n_2}\mathcal{K}(x_1 - n_1, n_1, t_1; x_2 - n_2, n_2, t_2),$$

then with change of variables $w \mapsto \frac{1}{1-z}, z \mapsto \frac{1}{1-w}$, and using residue formula for the first integral in (3.3), we get

$$\begin{aligned}&K(x_1, n_1, t_1; x_2, n_2, t_2) \\ &= \begin{cases} \frac{1}{(2\pi i)^2}\oint_{\Gamma_1} dz \oint_{\Gamma_0} dw \frac{z^{n_1}}{w^{n_2}} \frac{(1-w)^{x_2}}{(1-z)^{x_1+1}} \frac{1}{w-z} \frac{\prod_{t=0}^{t_1-1} F_t(\frac{1}{1-z})}{\prod_{t=0}^{t_2-1} F_t(\frac{1}{1-w})}, & (n_1,t_1) \not\prec (n_2,t_2) \\ \frac{1}{(2\pi i)^2}\oint_{\Gamma_1} dz \oint_{\Gamma_{0,z}} dw \frac{z^{n_1}}{w^{n_2}} \frac{(1-w)^{x_2}}{(1-z)^{x_1+1}} \frac{1}{w-z} \frac{\prod_{t=0}^{t_1-1} F_t(\frac{1}{1-z})}{\prod_{t=0}^{t_2-1} F_t(\frac{1}{1-w})}, & (n_1,t_1) \prec (n_2,t_2). \end{cases}\end{aligned} \tag{3.4}$$

Next step, we want to calculate the limit shape via $\mathbf{h}(\nu, \eta) := \lim_{L\to\infty} L^{-1}\mathbb{E}h(\nu, \eta)$ when $(n_1, t_1) \prec (n_2, t_2)$. Taking the hydrodynamic limit, and writing $x_i \simeq \nu_i L$, $n_i \simeq \eta_i L$, $t_i \simeq \tau_i L$, $L \to \infty$, we have the bulk scaling limit (see section 3.2 of [10])

$$\begin{aligned}K^{bulk}(x_1, n_1, t_1; x_2, n_2, t_2) &\simeq \frac{1}{(2\pi i)^2}\oint\oint \exp(L(S_1(w) - S_2(z)))\frac{dzdw}{(1-z)(w-z)} \quad (3.5) \\ &= \frac{1}{2\pi i}\int_{\overline{\Omega}}^{\Omega} \exp(L(S_1(w) - S_2(w)))\frac{dw}{(1-w)} \quad (3.6)\end{aligned}$$

where

$$S_i(u) = -\eta_i \ln u + \nu_i \ln(1-u) - \tau_i \ln\Big(F_t\Big(\frac{1}{1-u}\Big)\Big), \tag{3.7}$$

and the integration contour of 3.6 crosses $\mathbb{R}_+$ if $(n_1, t_1) \not\prec (n_2, t_2)$, and crosses $\mathbb{R}_-$ if $(n_1, t_1) \prec (n_2, t_2)$.

We want to deform the contours as in [10] so that

$$\mathrm{Re}(S(w)) < S_0,\ \mathrm{Re}(S(z)) > S_0,\quad \text{for some constant } S_0. \tag{3.8}$$

We want to find a domain $\mathcal{D}$ of $\{(\nu, \eta, \tau)\}$ in $\mathbb{R}^3_+$, such that the $x-$density, which is the local average number of particles on unit length in the $x-$direction, $\simeq -L^{-1}\partial h/\partial\nu$, is asymptotically strictly between 0 and 1. Notice that given $(\nu_1, \eta_1, \tau_1) = (\nu_2, \eta_2, \tau_2) \in \mathcal{D}$, they correspond to same critical points, and we will show later that $L^{-1}\mathbb{E}h(\nu L, \eta L, \tau L)$ has a limit when L tends to infinity for $(\nu, \eta, \tau) \in \mathcal{D}$, hence we can also view $\mathcal{D}$ in $P\mathbb{R}^3 \cap \mathbb{R}^3_+$, and identify $\mathcal{D}$ in $\mathbb{R}^2_+$ via $(\nu, \eta, \tau) \sim (\nu', \eta') = (\nu/\tau, \eta/\tau)$.

Then we consider the critical points of $S(w)$, that is, roots of $S'(w) = 0$. As mentioned in [14], if both roots $r_0 \leq r_1$ are real, then the triplet $\varkappa = (\nu, \eta, \tau)$ is not in the domain. Here for the four types of F_t mentioned in Section 2 (we first assume β and γ constant over time and remove the assumption later), we first calculate the critical points for those $\varkappa \in \mathcal{D}_F$, and then compute the domain $\mathcal{D}_F$. Here we denote the critical points by $\Omega, \overline{\Omega}$ since they are solution of the quadratic equation $S'_F(w) = 0$, and notice that we can choose Ω to be in $\mathbb{H}$ and identify with $(|\Omega|, |\Omega - 1|)$. It is not hard to check that Ω is a homeomorphism from the domain $\mathcal{D}$ to $\mathbb{H}$, hence it maps $\partial\mathcal{D}$ to $\mathbb{R}$.

(A) $F_t(z) = 1 + \beta^+ z$: $(\nu - \eta)w^2 + (2\eta + \beta\eta - \nu - \beta\nu - \beta\tau)w - (1 + \beta)\eta = 0$.

$$|\Omega| = \sqrt{\frac{(1 + \beta^+)\eta}{\eta - \nu}},\ (\eta > \nu),\ |\Omega - 1| = \sqrt{\frac{\beta^+(\tau + \nu)}{\eta - \nu}}. \tag{3.9}$$

$$\Delta < 0 \Leftrightarrow (\beta\eta - (1 + \beta)\nu - \beta\tau)^2 < 4\beta\eta\tau,$$

$$\mathcal{D} : |\sqrt{\beta^+\eta} - \sqrt{\tau}| < \sqrt{(1 + \beta^+)(\nu + \tau)} < \sqrt{\beta^+\eta} + \sqrt{\tau};\ \eta > \tau, \nu. \tag{3.10}$$

(B) $F_t(z) = 1 + \beta^-/z$: $\beta(\nu - \eta - \tau)w^2 + ((2\beta + 1)\eta - (1 + \beta)\nu + \beta\tau)w - (1 + \beta)\eta = 0$.

$$|\Omega| = \sqrt{\frac{(1 + \beta^-)\eta}{\beta^-(\tau + \eta - \nu)}},\ (\tau + \eta > \nu),\ |\Omega - 1| = \sqrt{\frac{\nu}{\beta^-(\tau + \eta - \nu)}}. \tag{3.11}$$

$$\Delta < 0 \Leftrightarrow (\eta - (1 + \beta)\nu + \beta\tau)^2 < 4\beta\tau\eta,$$

$$\mathcal{D} : |\sqrt{\eta} - \sqrt{\beta^-\tau}| < \sqrt{(1 + \beta^-)\nu} < \sqrt{\eta} + \sqrt{\beta^-\tau};\ \tau + \eta > \nu. \tag{3.12}$$

(C) $F_t(z) = (1 - \gamma^+ z)^{-1}$: $(\nu - \eta)w^2 + (2\eta - \gamma\eta - \nu + \gamma\nu - \gamma\tau)w - (1 - \gamma)\eta = 0$.

$$|\Omega| = \sqrt{\frac{(1 - \gamma^+)\eta}{\eta - \nu}},\ (\eta > \nu),\ |\Omega - 1| = \sqrt{\frac{\gamma^+(\tau - \nu)}{\eta - \nu}},\ (\tau > \nu). \tag{3.13}$$

$$\Delta < 0 \Leftrightarrow (\gamma\eta + (1 - \gamma)\nu + \gamma\tau)^2 < 4\gamma\eta\tau,$$

$$\mathcal{D} : |\sqrt{\gamma^+\eta} - \sqrt{\tau}| < \sqrt{(1 - \gamma^+)(\tau - \nu)} < \sqrt{\gamma^+\eta} + \sqrt{\tau};\ \eta, \tau > \nu. \tag{3.14}$$

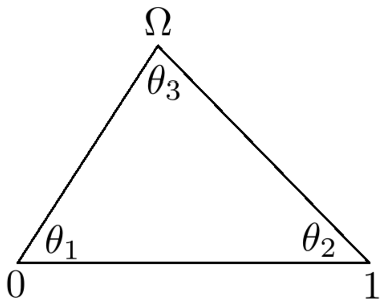

Figure 3: the triangle for the critical point in the $\mathbb{H}$ defined using $|\Omega|$ and $|1-\Omega|$

(D) $F_t(z) = (1-\gamma^-/z)^{-1}$: $\gamma(\eta-\nu-\tau)w^2+((1-2\gamma)\eta-(1-\gamma)\nu+\gamma\tau)w-(1-\gamma)\eta = 0$.

$$|\Omega| = \sqrt{\frac{(1-\gamma^-)\eta}{\gamma^-(\tau-\eta+\nu)}},\ (\tau+\nu>\eta),\ |\Omega-1| = \sqrt{\frac{\nu}{\gamma^-(\tau-\eta+\nu)}}. \tag{3.15}$$

$$\Delta < 0 \Leftrightarrow (\eta-(1-\gamma)\nu+\gamma\tau)^2 < 4\gamma\tau\eta,$$

$$\mathcal{D}: |\sqrt{\eta}-\sqrt{\gamma^-\tau}| < \sqrt{(1-\gamma^-)\nu} < \sqrt{\eta}+\sqrt{\gamma^-\tau};\ \tau+\nu>\eta. \tag{3.16}$$

Let us denote the angles of the triangle of vertices 0,1,Ω in $\mathbb{H}$ as in figure 3. Then we take S_0 in (3.8) to be Re$S(\Omega)$, and by (3.6), write

$$K(x_1,n_1,t_1;x_1,n_1,t_1) = \frac{1}{2\pi i}\int_{\overline{\Omega}}^{\Omega}\frac{1}{1-w}dw = \frac{1}{2\pi i}\int_{1-\Omega}^{1-\overline{\Omega}}\frac{1}{u}\,du. \tag{3.17}$$

Then by theorem 3.1, we have

$$\begin{aligned}\rho(x,n,t) &= \det[\mathcal{K}(x,n,t;x,n,t)] = K(x,n,t;x,n,t)\\ &= \frac{1}{2\pi i}\int_{1-\Omega}^{1-\overline{\Omega}}\frac{du}{u} = \frac{-\arg(1-\Omega)}{\pi} = \frac{\theta_2}{\pi},\end{aligned} \tag{3.18}$$

hence

$$\begin{aligned}\mathbf{h}(\nu,\eta,\tau) &= \lim_{L\to\infty}\frac{1}{L}\mathbb{E}h([\nu L],[\eta L],[\tau L]) = \lim_{L\to\infty}\frac{1}{L}\sum_{[\nu L]+\frac{1}{2}}K(x+n,n,t;x+n,n,t)\\ &= \frac{-1}{\pi}\int_{\nu}^{\infty}\arg(1-\Omega(\nu,\eta,\tau))d\nu_1 = \frac{-1}{\pi}\int_{\nu}^{u}\arg(1-\Omega(\nu,\eta,\tau))d\nu_1,\end{aligned} \tag{3.19}$$

where u is an upper bound for ν so that Ω is in the domain.

Remark 3.1. *By its homogeneity, we observe that*

$$(\nu\frac{\partial}{\partial\nu}+\eta\frac{\partial}{\partial\eta}+\tau\frac{\partial}{\partial\tau})\mathbf{h} = \mathbf{h}. \tag{3.20}$$

Using the method mentioned in [14], we can obtain

Theorem 3.2.

$$\mathbf{h}(\nu, \eta, \tau) = \frac{1}{\pi}\big(-\nu\theta_2 + \eta(\pi - \theta_1) + \tau \mathbf{v}_F\big), \tag{3.21}$$

where $\mathbf{v}_F$ *denotes the growing velocity* $\mathbf{v}$ *corresponding to different functions* F.

Proof. Using the kernel in (3.4) and taking partial derivatives, by (3.7) we have

$$S'(\Omega) = -\frac{\eta}{\Omega} - \frac{\nu}{1-\Omega} - \tau \frac{F_t'\left(\frac{1}{1-\Omega}\right)}{F_t\left(\frac{1}{1-\Omega}\right)},$$

$$S''(\Omega)\frac{\partial \Omega}{\partial \nu} + \frac{1}{1-\Omega} = 0,$$

$$S''(\Omega)\frac{\partial \Omega}{\partial \eta} + \frac{1}{\Omega} = 0,$$

$$S''(\Omega)\frac{\partial \Omega}{\partial \tau} + \frac{F_t'\left(\frac{1}{1-\Omega}\right)}{F_t\left(\frac{1}{1-\Omega}\right)} = 0,$$

$$(1-\Omega)\Omega_\nu = \Omega\Omega_\eta = \frac{F_t\left(\frac{1}{1-\Omega}\right)}{F_t'\left(\frac{1}{1-\Omega}\right)}\Omega_\tau, \tag{3.22}$$

and the first equality of (3.22) gives the complex Burgers equation, cf. [1], [9].

Taking partial derivatives with respect to (3.19), we get

$$\mathbf{h}_\nu = \frac{\arg(1-\Omega)}{\pi} = -\frac{\theta_2}{\pi} \tag{3.23}$$

$$\begin{aligned}\mathbf{h}_\eta &= \mathrm{Im}\ \frac{1}{\pi}\int_\nu^u \frac{1}{1-\Omega}\Omega_\eta d\nu = \mathrm{Im}\ \frac{1}{\pi}\int_\nu^u \frac{\Omega_\nu}{\Omega} d\nu \\ &= \frac{1}{\pi}\mathrm{Im}\ \ln(\Omega)|_\nu^u = \frac{1}{\pi}(\pi - \theta_1).\end{aligned} \tag{3.24}$$

$$\mathbf{v} = \mathrm{Im}\ \frac{1}{\pi}\int_\nu^u \frac{1}{1-\Omega}\Omega_\tau d\nu = \mathrm{Im}\ \frac{1}{\pi}\int_\nu^u \frac{F_t'\left(\frac{1}{1-\Omega}\right)}{F_t\left(\frac{1}{1-\Omega}\right)}\Omega_\nu d\nu. \tag{3.25}$$

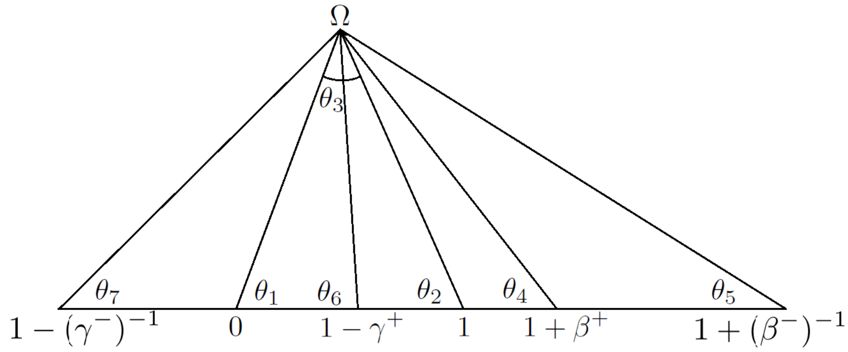

Figure 4: triangle to describe the discrete-time growth velocity $\mathbf{v}$

We compute $\mathbf{v}$ for the four cases. (For the angles, see figure 4).

$$
\begin{aligned}
\text{(A)}: \mathbf{v} &= \operatorname{Im} \frac{1}{\pi} \int_{\nu}^{u} \frac{\beta^+}{(1-\Omega)(\beta^+ + 1 - \Omega)} \Omega_\nu d\nu \\
&= \operatorname{Im} \frac{1}{\pi} \int_{\nu}^{u} \left(\frac{1}{1-\Omega} - \frac{1}{\beta^+ + 1 - \Omega} \right) \Omega_\nu d\nu \\
&= \frac{1}{\pi} \arg(1-\Omega) - \frac{1}{\pi} \arg(\beta^+ + 1 - \Omega) = \frac{\theta_4 - \theta_2}{\pi}; \qquad (3.26) \\
\text{(B)}: \mathbf{v} &= \operatorname{Im} \frac{1}{\pi} \int_{\nu}^{u} \frac{-1}{\frac{1}{\beta^-} + 1 - \Omega} \Omega_\nu d\nu \\
&= \frac{-1}{\pi} \arg \left(\frac{1}{\beta^-} + 1 - \Omega \right) = \frac{\theta_5}{\pi}; \qquad (3.27) \\
\text{(C)}: \mathbf{v} &= \operatorname{Im} \frac{1}{\pi} \int_{\nu}^{u} \left(\frac{1}{1-\gamma^+ - \Omega} - \frac{1}{1-\Omega} \right) \Omega_\nu d\nu \\
&= \frac{1}{\pi} \arg(1-\gamma^+ - \Omega) - \frac{1}{\pi} \arg(1-\Omega) = \frac{\theta_2 - \theta_6}{\pi}; \qquad (3.28) \\
\text{(D)}: \mathbf{v} &= \operatorname{Im} \frac{1}{\pi} \int_{\nu}^{u} \frac{1}{1 - \frac{1}{\gamma^-} - \Omega} \Omega_\nu d\nu \\
&= \frac{1}{\pi} \arg \left(1 - \frac{1}{\gamma^-} - \Omega \right) = \frac{\theta_7 - \pi}{\pi}. \qquad (3.29)
\end{aligned}
$$

Notice that the Ω's under different functions F are different. As a result of the above and remark 3.1, we prove (3.21). □

Remark 3.2. *By calculating the Hessians of $\mathbf{v}(\mathbf{h}_\nu, \mathbf{h}_\eta)$ with respect to $text\mathbf{h}_\nu, text\mathbf{h}_\eta$, we conclude that the discrete-time growth models we consider are in AKPZ class. See [16].*

By theorem 3.2, we know the limit shape of the growing surface. Our next goal is to study the fluctuation of the surface, that is, the fluctuation of the height function.

To do this, we need to introduce some notions of Gaussian free field, cf. [11]. In the previous theorem, we show that the height function grows with L in order 1; in contrast, our later result will show that the variance of the height function at a given point grows logarithmically with L.

Remark 3.3. *Now we relax the assumption that β and γ fixed over time, and reduce the new case to the case with time-fixed β, γ by setting*

$$\tilde{F}(z) = \left(\prod_{j=1}^{t} F_j(z) \right)^{1/t}, \tag{3.30}$$

for example for $F_t(z) = 1 + \beta_t^+(z)$, we let $\tilde{F}(z) = 1 + \tilde{\beta^+}(z) := \left(\prod_{j=1}^{t} (1 + \beta_j^+(z)) \right)^{1/t}$. (The expression (3.30) *seems more natural when we are considering the continuous time, which can be viewed as the derivative of a strictly increasing function.)*

Next, inspired by [5], we consider a mixture of the four growth models above.

Remark 3.4. *At each time $t \in \mathbb{N}$, we randomly choose F to be one of the form $1+\beta^+ z$, $1 + \beta^-/z$, $(1 - \gamma^+ z)^{-1}$, $(1 - \gamma^-/z)^{-1}$ with equal probability (of course we can choose different probability; the idea is the same as the argument below, all to do is to put different weight on F's). We have our newly defined F and* (3.7) *as*

$$\begin{aligned} F(z) &= \left((1 + \beta^+ z)(1 + \beta^-/z)((1 - \gamma^+ z)^{-1})((1 - \gamma^-/z)^{-1}) \right)^{1/4} \\ S(u) &= -\eta \ln u + \nu \ln(1 - u) - \tau \ln\Big(F\Big(\frac{1}{1 - u} \Big) \Big) \end{aligned}$$

Using the method we used for calculating Ω and $\mathcal{D}$ for cases $(A), (B), (C), (D)$, we can obtain the corresponding critical points and domain for our newly defined mixture model. We can also compute the limit shape. $\mathbf{h}_\nu$ and $\mathbf{h}_\eta$ are defined in terms of Ω as in (3.23), (3.24)*, and for integral* (3.25)*, note that*

$$\frac{F'(\frac{1}{1-\Omega})}{F(\frac{1}{1-\Omega})} = \left(\ln F \left(\frac{1}{1 - \Omega} \right) \right)' = \frac{1}{4} \sum_{A,B,C,D} \frac{F'(\frac{1}{1-\Omega})}{F(\frac{1}{1-\Omega})}$$

hence with the figure 3 for the new critical point Ω, we have

$$\mathbf{v} = \frac{\theta_4 + \theta_5 - \theta_6 + \theta_7 - \pi}{4\pi}$$

Combine with Remark 3.3, we obtain the limit shape for a general class of growth model.

Dealing with more general $F_t(z)$ in the discrete-time model by letting β and γ vary over time and alternatively choosing one of the four F_t's at time $t \in \mathbb{N}$, is technically harder- it is harder to establish the convergence of the Markov chains. we need We will briefly discuss the continuous analog of alternating step Markov chain in Section 4.

3.2 Gaussian fluctuations and Gaussian free field

Denote the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions by Δ and take the Sobolev space $W_0(\mathbb{H})$ (later simply denoted W_0) as the completion of $C_c^\infty(\mathbb{H})$ equipped with Dirichlet inner product [11]

$$(\phi_1, \phi_2)_\nabla = \int_{\mathbb{H}} \nabla\phi_1 \cdot \nabla\phi_2 \sim -\int_{\mathbb{H}} \phi_1 \Delta\phi_2 = (\phi_1, -\Delta\phi_2). \tag{3.31}$$

with " $\sim$ " to be " $=$ " if ϕ_2 is smooth.

Definition 3.1. [*11*](*definition* 12) *The* ***Gaussian free field*** *on* $\mathbb{H}$*, denoted* $GFF(\mathbb{H})$*, is any Gaussian Hilbert space* $\mathcal{G}(\mathbb{H})$ *of random variables denoted by* $\{\langle F, \phi\rangle_\nabla\}-$ *one variable for each* $\phi \in W_0-$ *that inherits the Dirichlet inner product structure of* ϕ_{W_0}*, also called the covariance structure, namely,*

$$\mathbb{E}[\langle F, \phi_1\rangle_\nabla \langle F, \phi_2\rangle_\nabla] = (\phi_1, \phi_2)_\nabla. \tag{3.32}$$

Consider $\mathbf{h}$ as a function on (ν', η'), so $h - \mathbb{E}h$ becomes $h_t - \mathbb{E}h_t = h(x, m, t) - L\mathbf{h}(\nu', \eta', 1)$, where $(x, m, t) = ([\nu L, [\eta L], [\tau L])$, hence $h_t - \mathbb{E}h_t$, as a function of $(x, m) \in \mathbb{Z} \times \mathbb{N}$, has a discrete domain. We restrict the test functions to those in $C_c^2(\mathbb{H})$, and define a **discretization** of the inner product (3.31):

$$\langle F, \phi\rangle := \langle F, \phi\rangle_\nabla = -\frac{\sqrt{\pi}}{L^2} \sum_{(x,m)\in L\mathcal{D}} F(x, m)\Delta\phi(\Omega(x/L, m/L))J(x/L, m/L), \tag{3.33}$$

where $J = J_\Omega$ is the Jacobian of the map $\Omega : \mathcal{D} \to \mathbb{R}_+^2$, $F : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$ and $\phi \in C_c^2(\mathbb{H})$.

Then using the techniques in [13] dealing with linear statistics, we can show the following result

Theorem 3.3. *Let* $\phi \in C_c^2(\mathbb{H})$*, and* h *be defined as above. Then with the discretization* $\langle\cdot, \cdot\rangle$ *given in* (3.33)*,*

$$\lim_{L\to\infty} \mathbb{E}[\exp i\xi\langle h_t - \mathbb{E}h_t, \phi\rangle] = e^{-\frac{\|\phi\|_\nabla^2 \xi^2}{2}}. \tag{3.34}$$

Hence, as $L \to \infty$*, the pushforward of the random surface defined by* $H_t := \sqrt{\pi}(h_t - \mathbb{E}h_t)$ *under* Ω *converges to* $GFF(\mathbb{H})$ (*see* [*11*](*proposition 2.13*))*.*

Denote the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions by $\mathcal{G}(\cdot, \cdot)$, where

$$\mathcal{G}(z, w) = -\frac{1}{2\pi} \ln\left|\frac{z - w}{z - \bar{w}}\right|. \tag{3.35}$$

As mentioned in Section 1.3 of [7], an alternative description of the Gaussian free field is that it is the unique Gaussian process which satisfies

$$\mathbb{E}(F(x_1), F(x_2)) = \mathcal{G}(x_1, x_2). \tag{3.36}$$

By a general result about Gaussian free field, mentioned in [11], and the first chapter of [17], we get the correlations along the space-like path.

Corollary 3.1. *For any $N = 1, 2, \cdots$, let $\varkappa_j = (\nu'_j, \eta'_j) \in \mathcal{D}$ be any distinct N points. Denote the critical points of different $\varkappa_j$ by $\Omega_j = \Omega(\nu'_j, \eta'_j)$. Then*

$$\lim_{L\to\infty} \mathbb{E}(\prod_{j=1}^{N} H_t(\varkappa_j)) = \begin{cases} 0, & N \text{ is odd,} \\ \sum_{\sigma\in\mathcal{F}_N} \prod_{j=1}^{N/2} \mathcal{G}\left(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}\right), & N \text{ is even,} \end{cases}$$

where $\mathcal{F}_N$ denotes the set of $(2n-1)!!$ pairings of the indices.

Remark 3.5. *Note that theorem 1.3 in [1] is more general in the sense that it can describe the correlations of height functions at different time.*

Actually, we can deal with the correlations of height functions evaluated at different time using Duits' method [13]. Since Duits' arguments are based on the determinantal structure of the correlation functions, and we have the determinantal structure by theorem 3.1 already.

We have a discretization form of inner product similar to (3.33): for $\phi \in C_c^2(\mathbb{H})$,

$$\langle F, \phi\rangle := \langle F, \phi\rangle_\nabla = -\frac{\sqrt{\pi}}{L^2} \sum_{(x,m,t)\in L\mathcal{D}} F(x, m, t)\Delta\phi(\Omega(x/L, m/L, t/L))J(x/L, m/L, t/L), \tag{3.37}$$

where $J = J_\Omega$ is the Jacobian of $\Omega : \mathcal{D} \to \mathbb{R}_+^2$, $F : \mathbb{Z}\times\mathbb{N}\times\mathbb{Z}_{\geq 0} \to \mathbb{R}$ and $\phi \in C_c^2(\mathbb{H})$.

We have theorems similar to theorem 3.3 and corollary 3.1.

Theorem 3.4. *Let $\phi \in C_c^2(\mathbb{H})$, and h be the height function defined in Section 2. Then*

$$\lim_{L\to\infty} \mathbb{E}[\exp i\xi\langle h - \mathbb{E}h, \phi\rangle] = e^{-\frac{\|\phi\|_\nabla^2 \xi^2}{2}}. \tag{3.38}$$

As $L \to \infty$, the pushforward of the random surface defined by $H_L(\nu, \eta, \tau) :=$ $\sqrt{\pi}(h([\nu L], [\eta L], [\tau L]) - \mathbb{E}h([\nu L], [\eta L], [\tau L]))$ under Ω converges to $GFF(\mathbb{H})$, with

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N, \qquad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_N.$$

In particular, at a given point, H_L is centered Gaussian with $\mathcal{O}(\ln L)$ fluctuations.

As a corollary, we have a counterpart for theorem 1.3 in [1]

Theorem 3.5. *For any $N = 1, 2, \cdots$, let $\varkappa_j = (\nu_j, \eta_j, \tau_j) \in \mathcal{D}$ be any distinct N triples such that*

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N, \qquad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_N, \tag{3.39}$$

and denote

$$\Omega_j = \Omega(\nu_j, \eta_j, \tau_j).$$

Then

$$\lim_{L\to\infty} \mathbb{E}(\prod_{j=1}^{N} H_L(\varkappa_j)) = \begin{cases} \sum_{\sigma\in\mathcal{F}_N} \prod_{j=1}^{N/2} \mathcal{G}\left(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}\right), & N \text{ is even} \\ 0, & N \text{ is odd} \end{cases} \tag{3.40}$$

where $\mathcal{F}_N$ denotes the set of all pairings on $[N]$.

Notice that condition (3.39) is sufficient for the determinantal structure (3.2) to hold, and that theorem 3.3 and 3.4 has the same proof except that theorem 3.3 is dealing with kernel defined on two-dimensional spaces and that we need to be more careful when proving theorem 3.4.

3.3 Proof of theorem 3.4

Now let's start to prove theorem 3.4 using the methods in [13]. To do this, we may consider the limit behavior of $\operatorname{Var}\langle h - \mathbb{E}h, \phi\rangle$. And we need the following constructions, which are the modifications of the sketch of proof in [13].

Definition 3.2. *For $f : \mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0 \to \mathbb{R}$ with finite support, define the* ***linear statistics*** *(random variables) X_f by*

$$X_f = \sum_{(x,m,t)\in\mathcal{C}} f(x,m,t), \tag{3.41}$$

where $\mathcal{C}$ is a random configuration. With kernel K given in (3.4)*, by definition 3,*

$$\mathbb{E}X_f = \sum_{(x,m,t)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0} f(x,m,t)K(x,m,t;x,m,t) \tag{3.42}$$

$$\operatorname{Var} X_f = \sum_{(x,m,t)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0} f(x,m,t)^2 K(x,m,t;x,m,t) -$$

$$\sum_{\substack{(x_1,m_1,t_1)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0,\\(x_2,m_2,t_2)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0}} f(x_1,m_1,t_1)f(x_2,m_2,t_2)K(x_1,m_1,t_1;x_2,m_2,t_2)K(x_2,m_2,t_2;x_1,m_1,t_1). \tag{3.43}$$

Lemma 3.1.

$$\sum_{x_2\in\mathbb{Z}} K(x_1,m_1,t_1;x_2,m_2,t_2)K(x_2,m_2,t_2;x_1,m_1,t_1) = K(x_1,m_1,t_1)\delta_{m_1,m_2}\delta_{t_1,t_2}, \tag{3.44}$$

for $(x_1,m_1,t_1)\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0$, and $(m_2,t_2)\in\mathbb{N}\times\mathbb{N}_0$, and either $(m_1,t_1),(m_2,t_2)$ comparable under relation $\prec$ (see (3.1)*), or $(m_1,t_1)=(m_2,t_2)$.*

Remark 3.6. *Before proving lemma 3.1, we note that this lemma does not deal with all triples in $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}_0$, while a similar lemma in [13] deal with all doublets in $\mathbb{Z}\times\mathbb{N}$. Nevertheless, all our future tasks deal with either the variance of H_L at a point* ($\textit{corresponding to coinciding}$ $(m_1,t_1),(m_2,t_2)$) *or covariance of H_L's at multiples points satisfying the relation* (3.39) (*corresponding to the case when (m_i,t_i)'s are comparable under $\prec$*)*, lemma 3.1 works for all cases we consider.*

Proof. Write kernel K as (3.4) with $F_t \equiv F$ for one of the four types we discussed earlier, and $\alpha_l \equiv 1$.

(I) If $m_1 \geq m_2$, $t_1 \leq t_2$, we deform Γ_1 so that it contains Γ_0. Due to $(w-z)^{-1}$, we pick up a residue that gives a single integral over Γ_0. In this case, the integrand of the single integral has no pole inside Γ_0, so it vanishes and we are left with the double integral only.

(II) If $m_1 < m_2$, $t_1 \geq t_2$, or if $m_1 = m_2$, $t_1 > t_2$, we deform Γ_0 so that it contains Γ_1, which is exactly the form (3.4) for $(m_1, t_1) \prec (m_2, t_2)$.

So we can rewrite (3.4) for this case as (with the contours further extended)

$$K(x_1, m_1, t_1; x_2, m_2, t_2) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_1'} dz \oint_{\Gamma_0'} dw \frac{(F(\frac{1}{1-z}))^{t_1}}{(F(\frac{1}{1-w}))^{t_2}} \frac{z^{m_1}}{w^{m_2}} \frac{(1-w)^{x_2}}{(1-z)^{x_1}} \frac{1}{(1-z)(w-z)}, \tag{3.45}$$

where Γ_1' contains Γ_0', and Γ_0' also contains Γ_1 if $m_1 > m_2$, $t_1 \leq t_2$ or if $m_1 = m_2$, $t_1 \leq t_2$; Γ_0' contains Γ_1', and Γ_1' also contains Γ_0 if $m_1 < m_2$, $t_1 \geq t_2$, or if $m_1 = m_2$, $t_1 > t_2$.

The sum is over terms $(\frac{1-w}{1-z'})^{x_2}$, where (w, z), (w', z') are the coordinate systems of the kernels $K(x_1, m_1, t_1; x_2, m_2, t_2)$, $K(x_2, m_2, t_2; x_1, m_1, t_1)$ respectively, and we have

$$\sum_{x_2 \in \mathbb{Z}} \frac{1}{(2\pi i)^2} \oint \oint \frac{F(\frac{1}{1-z'})^{t_2}}{F(\frac{1}{1-w})^{t_2}} \frac{z'^{m_2}}{w^{m_2}} \frac{(1-w)^{x_2}}{(1-z')^{x_2+1}} \frac{dw dz'}{(w-z)(w'-z')} = \frac{1}{2\pi i} \oint_{\Gamma_0'} \frac{dw}{(w-z)(w'-w)}. \tag{3.46}$$

The r.h.s vanishes if w' and z are on the same side with respect to contour Γ_0', that is precisely when $m_1 \neq m_2$ or $m_1 = m_2$ with $t_1 \neq t_2$. Thus we prove (3.44) for the case when $\delta_{m_1, m_2} \delta_{t_1, t_2} = 0$. Finally, if $m_1 = m_2$, $t_1 = t_2$, w' is in the region enclosed by Γ_0', and hence the l.h.s of (3.44), by using (3.46) and taking the residue at $w = w'$, which is

$$\begin{aligned} &\frac{1}{(2\pi i)^3} \oint_{\Gamma_0} dw' \oint_{\Gamma_1'} dz \oint_{\Gamma_0'} dw \frac{F(\frac{1}{1-z})^{t_1}}{F(\frac{1}{1-w'})^{t_1}} \frac{z^{m_1}}{w'^{m_1}} \frac{(1-w')^{x_1}}{(1-z)^{x_1+1}} \frac{1}{(w-z)(w'-w)} \\ =&\frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw' \oint_{\Gamma_1'} dz \frac{F(\frac{1}{1-z})^{t_1}}{F(\frac{1}{1-w'})^{t_1}} \frac{z^{m_1}}{w'^{m_1}} \frac{(1-w')^{x_1}}{(1-z)^{x_1}} \frac{1}{(1-z)(w'-z)}, \end{aligned}$$

where Γ_0 contains Γ_1', which is $K(x_1, m_1, t_1; x_1, m_1, t_1)$ by (3.4). □

As the diagonal of K^2 agree with K, we can rewrite the variance of linear statistic as follows

Proposition 3.1. *Denote by D the difference operator such that $Df(x, m) = f(x, m) - f(x-1, m)$ for real-valued function f defined on $\mathbb{Z} \times \mathbb{N}$. Then*

$$\operatorname{Var} X_f = \sum_{\substack{(x_1, m_1, t_1) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}_0, \\ (x_2, m_2, t_2) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}_0}} Df(x_1, m_1, t_1) Df(x_2, m_2, t_2) R(x_1, m_1, t_1; x_2, m_2, t_2), \tag{3.47}$$

where $R(y_1, n_1, s_1; y_2, n_2, s_2)$

$$= \begin{cases} \sum_{x_1 \geq y_1} \sum_{x_2 < y_2} K(x_1, n_1, s_1; x_2, n_2, s_2) K(x_2, n_2, s_2; x_1, n_1, s_1), & y_1 \geq y_2, \\ \sum_{x_1 < y_1} \sum_{x_2 \geq y_2} K(x_1, n_1, s_1; x_2, n_2, s_2) K(x_2, n_2, s_2; x_1, n_1, s_1), & y_1 < y_2. \end{cases} \tag{3.48}$$

Using the definition of the h_t in the comments under definition 5, and letting $\phi \in C_c^2(\mathbb{H})$, we can rewrite $\langle h, \phi \rangle$ as X_f, where

$$f(x,m,t) = -\frac{\sqrt{\pi}}{L^2} \sum_{\substack{y \le x \\ (y,m,t) \in L\mathcal{D}}} \Delta\phi\left(\Omega(y/L, m/L, t/L)\right) J(y/L, m/L, t/L). \tag{3.49}$$

To describe $\operatorname{Var}\langle h, \phi \rangle$, we need one further lemma from [13], where we need to check its validity since our R is defined for different kernel K, see remark at the end of the section.

Lemma 3.2. *Fix $\delta > 0$, and set $x_i = [L\nu_i]$, $m_i = [L\eta_i]$, $t_i = [L\tau_i]$, $i = 1, 2$. Then for (ν_i, η_i, τ_i)'s lie in some compact subset of $\mathcal{D}$ and $\|(\nu_1, \eta_1, \tau_1) - (\nu_2, \eta_2, \tau_2)\| \ge L^{-\frac{1}{2}+\delta}$,*

$$\lim_{L\to\infty} R(x_1, m_1, t_1; x_2, m_2, t_2) = \frac{1}{\pi}\mathcal{G}\left(\Omega(\nu_1, \eta_1, \tau_1), \Omega(\nu_2, \eta_2, \tau_2)\right) \sim \mathcal{O}(\ln L)$$

Combining proposition 3.1 with lemma 3.2, we obtain the limit of $\operatorname{Var}\langle h - \mathbb{E}h, \phi\rangle$:

Proposition 3.2. *Let h be the height function, $\phi \in C_c^2(\mathbb{H})$, and $\langle \cdot, \cdot \rangle$ defined as* (3.37)*, and let $\| \cdot \|_\nabla$ denote the Sobolev norm induced by* (3.31)*. Then*

$$\lim_{L\to\infty} \operatorname{Var}\langle h - \mathbb{E}h, \phi\rangle = \|\phi\|_\nabla^2. \tag{3.50}$$

Proof. (Slightly modified from [13]) Writing $\langle h, \phi \rangle$ as X_f. By (3.49), we have $Df = -\frac{\sqrt{\pi}}{L^2}\Delta\phi$. And using proposition 3.1 and lemma 3.2, we obtain

$$\begin{aligned}
&\lim_{L\to\infty} \operatorname{Var}\langle h - \mathbb{E}h, \phi\rangle = \lim_{L\to\infty} \operatorname{Var} X_f \\
&= -\lim_{L\to\infty} \frac{\pi}{L^4} \sum\sum \Delta\phi(\Omega_1)\Delta\phi(\Omega_2) R(\nu_1, \eta_1, \tau_1; \nu_2, \eta_2, \tau_2) J(\nu_1, \eta_1, \tau_1) J(\nu_2, \eta_2, \tau_2) \\
&= -\int \cdots \int \Delta\phi(\Omega_1)\Delta\phi(\Omega_2)\mathcal{G}(\Omega_1, \Omega_2) J(\nu_1, \eta_1, \tau_1) J(\nu_2, \eta_2, \tau_2) d\nu_1 d\eta_1 d\tau_1 d\nu_2 d\eta_2 d\tau_2 \\
&= -\iint_{\mathbb{H}\times\mathbb{H}} \Delta\phi(\Omega_1)\Delta\phi(\Omega_2)\mathcal{G}(\Omega_1, \Omega_2) dm(\Omega_1) dm(\Omega_2) \\
&= -\int_{\mathbb{H}} \phi(\Omega)\Delta\phi(\Omega) dm(\Omega) = \|\phi\|_\nabla^2.
\end{aligned}$$

where dm denotes the planar Lebesgue measure, and the second last equality follows from the fact that $\mathcal{G}$ is the Green's function for the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions. □

The strategy of proving theorem 3.4 is to use cutoff to reduce the case to functions with bounded support. Writing $\langle h, \phi \rangle$ as X_f with f in the form (3.49), we observe that $\operatorname{supp}(f)$ is unbounded. Following [13], we split the function as

$$f = f_1 + f_2, \text{ with } f_2 = f\chi_{\mathcal{D}_\epsilon}, \tag{3.51}$$

where χ for characteristic function, $\mathcal{D}_\epsilon = \{(\nu,\eta,\tau) \in \mathcal{D} | \operatorname{Im}\Omega(\nu,\eta,\tau) > \epsilon\}$, so that f_2 has bounded support, and ϵ small enough so that $\mathcal{D}_\epsilon$ contains $\operatorname{supp}(\Delta\phi \circ \Omega)$.

By [13], we have the following two results: variance of X_{f_1} tends to zero as $\epsilon \to 0$; $\lim_{L\to\infty}\left(\mathbb{E}\left(e^{i\xi(X_{f_2}-\mathbb{E}X_{f_2})}\right) - e^{-\frac{1}{2}\xi^2 \operatorname{Var} X_{f_2}}\right) = 0$ uniformly for ξ in compact subsets of $\mathbb{C}$. We take the first result for grant since it does not depend on the actual expression of the kernel K nor the corresponding R:

Lemma 3.3. *For $\epsilon > 0$, and f_1 defined as* (3.51)*. There exists a positive function g_ϕ with $\lim_{\epsilon\searrow 0} g_\phi(\epsilon) = 0$, such that*

$$\limsup_{L\to\infty} \operatorname{Var} X_{f_1} \leq (g_\phi(\epsilon))^2.$$

The second result is restated below:

Proposition 3.3.

$$\lim_{L\to\infty}\left(\mathbb{E}\left(e^{i\xi(X_{f_2}-\mathbb{E}X_{f_2})}\right) - e^{-\frac{1}{2}\xi^2 \operatorname{Var} X_{f_2}}\right) = 0$$

uniformly for ξ in compact subsets of $\mathbb{C}$.

To prove theorem 3.4, using (3.49), it suffices to show that

$$\lim_{L\to\infty} \mathbb{E}\exp(i\xi(X_f - \mathbb{E}X_f)) = \exp\left(-\frac{\xi^2\|\phi\|_\nabla^2}{2}\right), \quad t \in \mathbb{R}.$$

Split $f = f_1 + f_2$, and write $Y_{f_i} := X_{f_i} - \mathbb{E}X_{f_i}$, it suffices to show that

$$|\mathbb{E}\exp(i\xi Y_f) - \mathbb{E}\exp(i\xi Y_{f_2})| \longrightarrow 0, \tag{3.52}$$

$$\left|\mathbb{E}\exp(i\xi Y_{f_2}) - \exp(-\frac{1}{2}\xi^2 \operatorname{Var} Y_{f_2})\right| \longrightarrow 0, \tag{3.53}$$

$$\left|\exp(-\frac{1}{2}\xi^2 \operatorname{Var} Y_{f_2}) - \exp\left(-\frac{\xi^2\|\phi\|_\nabla^2}{2}\right)\right| \longrightarrow 0. \tag{3.54}$$

(3.52) follows from the l.h.s is bounded by $\xi\sqrt{\operatorname{Var} Y_{f_1}}$, which by lemma 3.3, tends to 0, when $\epsilon \to 0$; (3.53) follows from proposition 3.3; (3.54) is a result of proposition 3.2, since $\limsup_{L\to\infty} |\operatorname{Var} Y_{f_2} - \operatorname{Var} Y_f| \xrightarrow{\epsilon\searrow 0} 0$ using Cauchy-Schwartz inequality, and then use $\lim_{L\to\infty} \operatorname{Var} Y_f = \|\phi\|_\nabla^2$. Hence we prove theorem 3.4.

Remark 3.7. *Note that lemma 3.2 and proposition 3.3 are based on some further asymptotic analysis of kernel K and the corresponding R. Writing $(F(w))^t$ in* (3.4) *as $e^{t\ln(F(w))}$, the analysis in [13] actually works for our case as well. The basic idea is to rewrite* (3.4) *as two parts $I_1 + I_2$ similar to* (3.5) *by taking the L limit, with S defined as* (3.7)*:*

$$I_1 = \frac{1}{2\pi i}\int_{\bar{\zeta}}^{\zeta} \exp(L(S_1(w) - S_2(w)))\frac{dw}{1-w}$$

$$I_2 = \frac{1}{(2\pi i)^2}\oint_{\Gamma_1}\oint_{\Gamma_0} \exp(L(S_1(w) - S_2(z)))\frac{dw}{(1-z)(w-z)}$$

where ζ and ζ' are the two intersection points of the steepest descent path Γ_0 and Γ_1 (meaning $Re(S_1(\zeta) - S_1(\Omega_1))$−$Re(S_2(\zeta) - S_2(\Omega_2))$), and the direction of the integration contour of I_1 is given in (3.6)*.*

Note that the function "S" in our paper has the same utility as function "F" in [13]; S is well behaved in $\Omega(\mathcal{D})$ (e.g., $S''(\Omega)$ is bounded from below), and the expressions of S and Duits' F are quite similar.

To prove lemma 3.2, first write R as a quadruple integral, and apply the technique in the proof of lemma 3.1 to discuss the contours. Next rewrite the quadruple integral in scaling limit. For details, see [13].

4 Continuous case

The continuous-time anisotropic 2-dimensional growth model can be described as follows: consider a Markov chain on the state space of interlacing variables

$$\mathcal{S}^{(n)} = \left\{ \{x_k^m\}_{\substack{m=1,\cdots,n \\ k=1,\cdots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} | x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1} \right\}, \qquad n \in \mathbb{Z}^+,$$

where the double-indexed x_k^m (which also refers to the particle itself) can be interpreted as the position of particle with label (k, m). The initial condition we consider is fully-packed, that is $x_k^m(0) = k - m - 1$ for all k, m.

The particles evolve according to the following dynamics. Each of the particles x_k^m has an independent exponential clock of rate one. When the x_k^m-clock rings, the particle attempts to jump to the right by one: if $x_k^m = x_k^{m-1} - 1$, then the jump is blocked; else we find the largest $c \geq 1$ such that $x_k^m = x_{k+1}^{m+1} = \cdots = x_{k+c-1}^{m+c-1}$ and all these c particles jump to the right by one.

Similarly to the discrete case, for a triplet $\varkappa = (\nu, \eta, \tau)$ in the (2+1)-dimensional space-time in the continuous model, we consider $h([(\nu - \eta)L] + \frac{1}{2}, [\eta L], \tau L)$, where the brackets stand for the floor function. Define the $x-$density as the local average number of particles on unit length in the $x-$direction. Then for large L, one expects the density to be $L^{-1}\partial_\nu h$. And it is known that the domain $\mathcal{D} \in \mathbb{R}_+^3$, where the $x-$density of our system is asymptotically in $(0, 1)$ is given by $|\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}$, that is we have map our $\varkappa$ to the upper complex plane by $\Omega : \mathcal{D} \to \mathbb{H}$ such that $|\Omega(\nu, \eta, \tau)| = \sqrt{\eta/\tau}, |1 - \Omega(\nu, \eta, \tau)| = \sqrt{\nu/\eta}$. The preimage of any point in $\Omega(\mathcal{D})$ is a ray in $\mathcal{D}$ with constant ratios $(\nu : \eta : \tau)$. The limit shape $\mathbf{h}(\varkappa)$ satisfies $\mathbf{h}(\alpha\varkappa) = \alpha\mathbf{h}(\varkappa)$ for any $\alpha > 0$, that is, the height function grows linearly in time along the array, where

$$\mathbf{h}(\varkappa) := \lim_{L\to\infty} \frac{\mathbb{E}h([(\nu - \eta)L] + \frac{1}{2}, [\eta L], \tau L)}{L}.$$

The velocity of surface growth, $\partial_\tau\mathbf{h} =: \mathbf{v}$ depends on the two macroscopic slopes $\mathbf{h}_\nu := \partial_\nu\mathbf{h}$, $\mathbf{h}_\eta := \partial_\eta\mathbf{h}$. The anisotropy comes from the fact that the Hessian of $\mathbf{v} = \mathbf{v}(\mathbf{h}_\nu, \mathbf{h}_\eta)$ is strictly negative for $(\nu, \eta, \tau) \in \mathcal{D}$.

Continuous Case as the Limit of the Discrete Case

As mentioned in [1], the above continuous process can be realized as a limit of the discrete-time analog (B), (D) by taking $\beta^-, \gamma^- \to 0$ for the Toeplitz $F_t(z) = (1+\beta^-/z)$ or $(1-\gamma^-/z)^{-1}$ respectively, and then the transition probability of the Markov chain generated by $T_m(\alpha_1, \cdots, \alpha_m; 1+\beta^-/z)$ converge to (similar for the case in which $F_t(z) = (1-\gamma^-/z)^{-1}$)

$$\begin{aligned}&\lim_{\beta^-\to 0}(T_m(\alpha_1,\cdots,\alpha_m;1+\beta^-/z))^{[t/\beta^-]}(X_m,Y_m) && \text{by (2.3)}\\ =&\frac{\det(\alpha_i^{y_j})_{i,j=1}^n}{\det(\alpha_i^{x_j})_{i,j=1}^n}\frac{\det\big(t^{y_i-x_j}\mathbb{1}(y_i-x_j\geq 0)/(y_i-x_j)!\big)_{i,j=1}^m}{\exp(t\sum_{i=1}^m\alpha_i)}.\end{aligned}$$

Remark 4.1. *In the view of our formalized discrete-analog setting, the continuous case is equivalent to extending the domain of t to be $\mathbb{R}_{\geq 0}$, and $(F(z))^t = \exp(t/z)$, that is, $F_t(z) \equiv \exp(1/z)$. Factoring out $e^{t_1-t_2}$ in* (3.4)*, we have an alternative form for* (3.7)

$$S(w) = -\eta\ln w + \nu\ln(1-w) + \tau w,$$

then the corresponding domain, critical points, expressions for (3.25)*, and* (3.21) *are*

$$\mathcal{D}: |\sqrt{\eta}-\sqrt{\tau}| < \sqrt{\nu} < \sqrt{\eta}+\sqrt{\tau}, \tag{4.1}$$

$$|\Omega(\nu,\eta,\tau)| = \sqrt{\eta/\tau},\ |1-\Omega(\nu,\eta,\tau)| = \sqrt{\nu/\tau}, \tag{4.2}$$

$$\mathbf{v} = \text{Im}\ \frac{1}{\pi}\int_\nu^\infty -\Omega_\nu d\nu = \frac{1}{\pi}\text{Im}\ \Omega, \tag{4.3}$$

$$\mathbf{h}(\nu,\eta,\tau) = \frac{1}{\pi}\big(-\nu\theta_2 + \eta(\pi-\theta_1) + \tau\text{Im}\ \Omega\big). \tag{4.4}$$

retrieving the result about the limit shape given in [1].

Remark 4.2. *There is another kind of continuous-time growth model corresponding to (A), (C) in which the particles jump to the left instead of to the right, and when the x_k^m-clock rings, the particle is blocked if $x_{k-1}^{m-1} = x_k^m$; else find the largest $c \geq 1$ such that $x_k^m = x_k^{m+1} + 1 = \cdots = x_k^{m+c-1} + (c-1)$ and move all the c particles to the left by 1. And the limiting $F_t(z) = \exp(z)$. Then the corresponding expression for* (3.7) *is*

$$S(w) = -\eta\ln w + \nu\ln(1-w) - \tau\frac{1}{1-w},$$

then the corresponding domain, critical points, expressions for (3.25)*, and* (3.21) *are*

$$\mathcal{D}: \nu+\tau < 2\sqrt{\eta\tau}, \tag{4.5}$$

$$|\Omega(\nu,\eta,\tau)| = \sqrt{\frac{\eta}{\eta-\nu}},\ |1-\Omega(\nu,\eta,\tau)| = \sqrt{\frac{\tau}{\eta-\nu}}, \tag{4.6}$$

$$\mathbf{v} = \text{Im}\ \frac{1}{\pi}\int_\nu^\infty \frac{\Omega_\nu}{(1-\Omega)^2}d\nu = \frac{1}{\pi}\text{Im}\ \frac{1}{1-\Omega} = \frac{1}{\pi}\frac{\text{Im}\ \Omega}{|1-\Omega|^2}, \tag{4.7}$$

$$\mathbf{h}(\nu,\eta,\tau) = \frac{1}{\pi}\left(-\nu\theta_2 + \eta(\pi-\theta_1) + \tau\frac{\text{Im}\ \Omega}{|1-\Omega|^2}\right). \tag{4.8}$$

Remark 4.3. *Note that the corresponding analog of the processes discussed in remark 3.3, and comment under remark 3.4 can be described as follows (similar to the construction in [5]): We can also consider a continuous-time growth model in which the particles jump to both left and right. The base discrete analog has F_t as*

$$F_{2s}(z) = 1 + \beta^+ z, \quad F_{2s+1}(z) = 1 + \beta^-/z. \tag{4.9}$$

(We can talk about alternatively choosing the four base F functions as well, but to keep things concise, we only consider two of them here.)
We consider $\alpha_k = 1$, $k \in \mathbb{Z}$ and some smooth positive increasing functions $a(t), b(t)$ with $a(0) = b(0) = 0$. Then, let $\beta_t^+ = \dot{a}(t)$, $\beta_t^- = \dot{b}(t)$, then the left jump rate of particle k is $\beta_t^+ \alpha_k = \dot{a}(t)$, while the right jump rate is $\beta_t^-/\alpha_k = \dot{b}(t)$. Then slightly modifying the arguments in [5], and combine with those in remarks 4.1 and 4.2, by factoring out $e^{a(t)-b(t)}$ in kernel K, see (3.4), and using the geometric averaging mentioned in remark 3.3, we will obtain the corresponding expression for (3.7),

$$\begin{aligned} S(w) &= -\eta \ln w + \nu \ln(1-w) + \frac{a(t)}{L} w - \frac{b(t)}{L(1-w)} \\ &= -\eta \ln w + \nu \ln(1-w) + \tilde{a}(\tau) w - \tilde{b}(\tau) \frac{1}{1-w}. \end{aligned} \tag{4.10}$$

We assume the increments of a (resp. of b) for partitions of $\mathbb{R}_+$ are mutually independent, which can be provided by any one-dimensional Lévy process. As $L \to \infty$, the convergence of $\tilde{a}(\tau)$, and $\tilde{b}(\tau)$ are tail event, so by Kolmogorov's zero-one law, it either converges almost surely (for example, if we choose them to be Lévy processes) or diverges almost surely. We assume the geometric averages may converge to some $\tilde{a}$, $\tilde{b}$. Then we can compute the domain, and the critical points. With these we obtain the expressions for (3.21) as follows:

$$\mathbf{h}(\nu, \eta, \tau) = \frac{1}{\pi} \left(-\nu \theta_2 + \eta(\pi - \theta_1) + \tau \ \mathrm{Im}\ \Omega \left(\tilde{a} + \frac{\tilde{b}}{|1-\Omega|^2} \right) \right). \tag{4.11}$$

Another construction is discussed in [1]: for any two continuous functions $a(\tau)$, $b(\tau)$ on $\mathbb{R}_+$ with $a(0) = b(0) = 0$, consider the limit as $\epsilon \to 0$ of the Markov chain with alternating F_t as in (4.9),

$$\beta^+(t) = \epsilon a(\epsilon t), \quad \beta^-(t) = \epsilon b(\epsilon t).$$

Then the Markov chain converges to a continuous time Markov chain, whose generator at time s is $a(s)$ times the generator of the left jumping Markov chain plus $b(s)$ times he generator of the right jumping Markov chain. And we will get similar results as (4.11).

Remark 4.4. *Notice that in (4.3), (4.7), and τ's coefficient in (4.11), we can rewrite the imaginary part of Ω as $(\sin\theta_1 \sin\theta_2)/(\sin\theta_3)$. Hence we can directly compute the Hessian of $\mathbf{v} = \mathbf{v}(\mathbf{h}_\nu, \mathbf{h}_\eta)$, and it is of signature $(-1, 1)$; by [16], both of the continuous growth models mentioned above are anisotropic in KPZ class.*

Remark 4.5. *It is shown in [1] that the* $h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)$ *has a mean-zero normal distribution with variance growing as* $\ln(L)$. *And this can be seen as a corollary of theorem 3.4 using remark 4.1. We have a similar assertion about the left jumping and alternating jumping cases.*

The final main result in [1] is that Ω is a map from $\mathcal{D}$ to $\mathbb{H}$ such that on space-like submanifolds, the multi-point (with order assumptions about $\eta,\tau-$components) fluctuations of the height function are asymptotically equal to those of the pullback of the Gaussian free (massless) field on $\mathbb{H}$:

Theorem 4.1. *For any* $N=1,2,\cdots$, *let* $\varkappa_j=(\nu_j,\eta_j,\tau_j)\in\mathcal{D}$ *be any distinct* N *triples such that* τ,η *satisfies condition* (3.39)*:*

$$\tau_1\le\tau_2\le\cdots\le\tau_N,\qquad \eta_1\ge\eta_2\ge\cdots\ge\eta_N.$$

Denote

$$H_L(\nu,\eta,\tau):=\sqrt{\pi}(h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)-\mathbb{E}h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L))$$

$$\Omega_j=\Omega(\nu_j,\eta_j,\tau_j),\quad \textit{and}\quad \mathcal{G}(z,w)=-\frac{1}{2\pi}\ln\left|\frac{z-w}{z-\bar{w}}\right|.$$

Then

$$\lim_{L\to\infty}\mathbb{E}(\prod_{j=1}^{N}H_L(\varkappa_j))=\begin{cases}\sum_{\sigma\in\mathcal{F}_N}\prod_{j=1}^{N/2}\mathcal{G}\left(\Omega_{\sigma(2j-1)},\Omega_{\sigma(2j)}\right), & N \textit{ is even}\\ 0, & N \textit{ is odd}\end{cases}\tag{4.12}$$

where $\mathcal{F}_N$ *denotes the set of all pairings on* $[N]$.

Since by Duits' argument, theorem 4.1 may hold whenever the process has a determinantal kernel, and it is possible that the process has such a kernel without assumption (3.39). We are interested in whether such a kernel exists without any restriction, or equivalently, the above theorem holds without assumption (3.39), we run simulations to test if the following conjecture:

Conjecture 4.1. *Without assumption* (3.39), *theorem 4.1, in particular* (4.12), *still holds.*

5 Simulation results

Denote by $\mathcal{M}^{(n)}(t)$ the measure for the evolution on $\mathcal{S}^{(n)}$. Obviously $\mathcal{S}^{(n)}\subset\mathcal{S}^{(n')}$ for $n\le n'$, and by the definition of the evolution, $\mathcal{M}^{(n)}(t)$ is a marginal of $\mathcal{M}^{(n')}(t)$ for any $t\ge0$. So we can think of $\mathcal{M}^{(n)}$'s as marginals of the measure $\mathcal{M}=\varprojlim\mathcal{M}^{(n)}$ on $\mathcal{S}=\varprojlim\mathcal{S}^{(n)}$.

5.1 Simulation for the continuous case

ALGORITHM

Inspired by the flash demonstration in [2], we design the simulation algorithm as follows:

Given $\varkappa_1, \cdots, \varkappa_N$ and L, set $T = L \cdot \max(\tau_j)_{j\in[N]}$ and $n = L \cdot \max(\eta_j)_{j\in[N]}$ as the time upper bound and the size of marginal for our simulation.

• Approximate the distribution of the exponential clocks' rings by generating a sequence of exponential random variables $s_1, \cdots, s_r$ of rate N which sums up to less than or equal to T such that including one more such random variable will make the sum greater than T, and let $t_i = \sum_{l=1}^{i} s_l$ for $i \in [r]$, and assign t_i to some x_m^k as a time of its clock ring independently, uniformly at random.

• Simulate the process according to the rings with the time-increasing order and the dynamic given in Section 1. Evaluate h for each $\varkappa$ and the given L by counting at the given time τL on the given row $[\eta L]$. For the run of simulation, save a evaluation set $\big(h([(\nu_1-\eta_1)L]+\frac{1}{2},[\eta_1 L],\tau_1 L),\cdots,(h([(\nu_N-\eta_N)L]+\frac{1}{2},[\eta_N L],\tau_N L)\big)$.

• Run the above procedures for many times and get the estimated expectation of $\mathbb{E}h([(\nu_j-\eta_j)L]+\frac{1}{2},[\eta_j L],\tau_j L)$, and demean by these expectations each evaluation set $\big(h([(\nu_1-\eta_1)L]+\frac{1}{2},[\eta_1 L],\tau_1 L),\ \cdots,(h([(\nu_N-\eta_N)L]+\frac{1}{2},[\eta_N L],\tau_N L)\big)$, gaining sets of the form $(H_L(\varkappa_1),\cdots,H_L(\varkappa_N))$, each for one specific run of simulation, and gain $\mathbb{E}(\prod_{j=1}^{N} H_L(\varkappa_j))$.

5.2 Numerical results

The numerical results from our simulation agree with our conjecture:

• For N to be odd, we take $N = 3$, $(\varkappa_1;\varkappa_2;\varkappa_3) = \frac{1}{100}((4,1,5);(2,4,6);(8,7,10))$ (with the η,τ-components violating assumption (3.39)), when $L \geq 2000$, and the estimated expectation taking over 10000 runs, the resulting $\mathbb{E}(\prod_{j=1}^{3} H_L(\varkappa_j))$ is very close to 0 (with deviation within $\pm$ 0.01).

• For N to be even, we take $N = 2$ and $N = 4$, and we tave $(\varkappa_1;\varkappa_2;\varkappa_3;\varkappa_4) = \frac{1}{100}((4,1,5);(2,4,6);(8,7,10);(6;3;8))$ such that all η,τ-components are pair wisely violating assumption (3.39). Then we have

$$\begin{aligned}
\mathcal{G}(\Omega_1,\Omega_2) &= 0.11762, \quad \mathcal{G}(\Omega_3,\Omega_4) = 0.27125,\\
\mathcal{G}(\Omega_1,\Omega_3) &= 0.16789, \quad \mathcal{G}(\Omega_2,\Omega_4) = 0.17242,\\
\mathcal{G}(\Omega_1,\Omega_4) &= 0.27163, \quad \mathcal{G}(\Omega_2,\Omega_3) = 0.21047.
\end{aligned}$$

We do simulations for $(\varkappa_1;\varkappa_2)$, $(\varkappa_3;\varkappa_4)$, and $(\varkappa_1;\varkappa_2;\varkappa_3;\varkappa_4)$. The only pairing for $\{1,2\}$, $\{3,4\}$, and $\{2,4\}$ are $\{1,2\}$, $\{3,4\}$, and $\{2,4\}$ respectively; the pairings for $\{1,2,3,4\}$ are $\{(1,2),(3,4)\}, \{(1,3),(2,4)\}$, and $\{(1,4),(2,3)\}$. We can easily get the corresponding values of the r.h.s of (4.12) for these two cases are 0.11762, 0.27125, 0.17242, and 0.11832, respectively.

We obtain the expected product by running more runs for larger L to achieve enough

accuracy in the limit.
For $N = 2$, take the pair $(\varkappa_1; \varkappa_2) = \frac{1}{100}((4, 1, 5); (2, 4, 6))$.
Taking $L = 200$, the expected product is 0.165 ± 0.03;
taking $L = 500$, the expected product is 0.160 ± 0.02;
taking $L = 2000$, the expected product is 0.150 ± 0.001;
taking $L = 5000$, the expected product is 0.142 ± 0.004;
taking $L = 6000$, the expected product is 0.136 ± 0.01.
Then take the pair $(\varkappa_3, \varkappa_4) = \frac{1}{100}((8, 7, 10); (6, 3, 8))$. We obtain expected product 0.341 ± 0.03 at $L = 1000$; 0.270 ± 0.015 at $L = 2000$.
Next we take the pair $(\varkappa_2, \varkappa_4) = \frac{1}{100}((2, 4, 6); (6, 3, 8))$. We obtain expected product 0.185 ± 0.01 at $L = 1000$, and 0.189 ± 0.01 at $L = 2000$.

For $N = 4$, our evaluation set is $(\varkappa_1; \varkappa_2; \varkappa_3; \varkappa_4) = \frac{1}{100}((4, 1, 5); (2, 4, 6); (8, 7, 10); (6; 3; 8))$.
Taking $L = 200$, the expected product is 0.342 ± 0.003;
taking $L = 500$, the expected product is 0.28 ± 0.01;
taking $L = 1000$, the expected product is 0.14 ± 0.01;
taking $L = 2000$, the expected product is 0.23 ± 0.02.

So we observe that $\mathbb{E}(H_L(\varkappa_3)(H_L(\varkappa_4)))$ converges to 0.271, $\mathbb{E}(H_L(\varkappa_1)(H_L(\varkappa_2)))$ should converge to some value near 0.118, and $\mathbb{E}((H_L(\varkappa_1))(H_L(\varkappa_2))(H_L(\varkappa_3))(H_L(\varkappa_4)))$ is likely to converge to some value in the interval $[0.08, 0.18]$ as L grows large.

We have another set of trials dealing with the same $\varkappa$ set with $\varkappa_1$ replaced by $\varkappa_1' = (4, 2, 5)$, and the new results are not so satisfactory.
For $N = 2$, $(\varkappa_1', \varkappa_2) = \frac{1}{100}((4, 2, 5); (2, 4, 6))$, we have $\mathcal{G}(\Omega_1', \Omega_2) = 0.16943$.
When $L = 500$, the expected product is 0.241 ± 0.005;
when $L = 2000$, the expected product is 0.229 ± 0.006;
when $L = 10000$, we have one estimated value for the product, 0.220 (over 60000 runs).
We observe that $\mathbb{E}(H_L(\varkappa_1')H_L(\varkappa_2))$ is likely to converge to some value less than 0.2.

For $N = 4$, the r.h.s of (4.12) for $(\varkappa_1', \varkappa_2, \varkappa_3, \varkappa_4)$ is 0.21306.
When $L = 200$, the expected product converges to 0.174 ± 0.001;
when $L = 500$, the expected product converges to 0.184 ± 0.001;
when $L = 2000$, the expected product converges to 0.19 ± 0.02.
So we observe that $\mathbb{E}(H_L(\varkappa_1') \prod_{j=2}^4 H_L(\varkappa_j))$ converges to 0.21.

But something weird happens when we increase L further. We have trials for $L = 5000$ and $L = 10000$ with $\sim 10^5$ runs. When $L = 10000$, then $\{([\nu L], [\eta L], [\tau L])\}$ equals to $\{(400, 200, 500), (200, 400, 600), (800, 700, 1000), (600, 300, 800)\}$; we obtain an estimated mean of the expected product as 0.410. By the simulation, the estimated average heights are 134.274, 350.525, 469.645, 210.166, and it is not rare ($\sim 0.5\%$) that in some simulations, all four heights deviate from the average heights by more than 1, e.g. (136, 352, 471, 212). Since H_L is scaled by constant $\sqrt{\pi}$, the product is scaled by π^2, so the example will result in a product 20.096. The standard deviation of this 10^5 products is huge: 4.973, and the standard deviation barely decreases as we increase the number of runs from 100 to 10^5, which is not the case for relatively small L. We observe that when we remove the 40 extreme simulated products (mostly lying on the right tails) outside ± 10 standard deviations interval centered at the mean 0.410, the mean decreases to 0.320 , and the standard deviation to 3.094; when we remove the 500 products (with the newly removed locate relatively balanced on the two tails) outside ± 5 standard deviations interval centered at 0.410, the mean decreases to 0.234, and the standard deviation to 2.486. One thing we should mention is that although each H_L is Gaussian, we do not expect their product to be Gaussian as they are weakly correlated. The case when $L = 5000$ is less dramatic: the original mean and standard deviation from the simulation is 0.383, and 3.518. The results are similar: when we remove the 10 s.d. tails, mean decreases to 0.295 and s.d. to 2.559; the 6, 7 s.d. tails are quite balanced on the two sides in size.

Interpreting the Results

$N = 2$: Notice that the Gaussian free field is the unique Gaussian process which satisfies

$$\mathbb{E}(F(x_1), F(x_2)) = \mathcal{G}(x_1, x_2),$$

it suffices to check for $N = 2$, as the higher moments of Gaussian processes can always be written in terms of the moments of order 2. By the simulation result of $N = 2$, the conjecture is true asymptotically.

Comparing the convergence of the trials, we find that as L grows, the expected product of $(\varkappa_3, \varkappa_4)$ converges to the r.h.s of (4.12) more quickly. One needs to note that this can be explained by the norm of this pair is greater than the norms of the other pairs as the convergence depends on the size of $\varkappa$'s and L. We first compare the convergence of (4.12) for $(\varkappa_1, \varkappa_2)$ and $(\varkappa_1', \varkappa_2)$. We observe that the rays of $(\varkappa_1, \varkappa_2)$ are farther apart in the Euclidean space than $(\varkappa_1', \varkappa_2)$, and actually the former pair has faster convergence of (4.12) than the latter.

One thing we observe by taking different sets of $(\varkappa_\alpha, \varkappa_\beta)$'s with (where the $\varkappa_\alpha$ has similar Euclidean norms), is that fixing $\varkappa_\beta$, when the two rays of $\varkappa_\alpha$ and $\varkappa_\beta$ starting from the origin in $\mathcal{D}$ are farther apart in $\mathbb{R}^3_+$, then $\mathbb{E}(H_L(\varkappa_\alpha)(H_L(\varkappa_\beta))$ converges to $\mathcal{G}(\Omega_\alpha, \Omega_\beta)$ more quickly with L's growth. Same for the case when there are four $\varkappa$'s: if the rays of $\varkappa_1, \cdots, \varkappa_4$ (with $\varkappa_2, \varkappa_3, \varkappa_4$ fixed) are farther apart, then the expected product converges to $\sum_{\text{pairings}} \mathcal{G}(\Omega_{\sigma(1)}, \Omega_{\sigma(2)}) \mathcal{G}(\Omega_{\sigma(3)}, \Omega_{\sigma(4)})$ more quickly. Thus we form the following conjecture:

Conjecture 5.1. *For the non-coinciding $\varkappa_1, \cdots, \varkappa_n$, fix $\varkappa_2, \cdots, \varkappa_n$, and let $\overrightarrow{0\varkappa_1}$ of fixed length rotate in the domain. Then when the rays of $\varkappa_1$ are farther apart from the rays* *of* $\varkappa_2, \cdots, \varkappa_n$ *in* $\mathcal{D}$*, the convergence rate of*

$$\mathbb{E}(H_L(\varkappa_1)\cdots(H_L(\varkappa_n)) \xrightarrow{L\to\infty} \begin{cases} \sum_{\sigma \text{ pairings}} \prod_{j=1}^{N/2} \mathcal{G}\left(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}\right) & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases} \tag{5.1}$$

is faster.

The statement **farther apart** should be made more clearly, for example, in terms of weighted coulomb potential energy, weighted norm, or sum of weighted squares of differences in the three coordinates.

Another thing that we are interested in is the convergence of products of H_L to its expected value. We want to see if there is a there is a relationship between the angles of rays through origin, on which lie the points where the H_L's are evaluated, and the convergence rate of sample mean of the product (over runs of simulations) to its expected value. So we do lots of tests about the more complicated case when $N = 4$.

$N = 4$: We capture a new phenomenon which is not significant in $N = 2$ case, that is, the large tails and large standard deviation of the distribution. And it is quite possible that using other large L or other $\varkappa$'s will result in deviation of the estimated expected product of H_L's to values smaller than the result given by r.h.s of (4.12). Since the standard deviation is quite large when the sample size is $\sim 10^5$, we might need increase the size to $\sim 10^7$ or even larger in order to observe its convergence to the predicted value. The results of $(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$ and $(\varkappa_1', \varkappa_2, \varkappa_3, \varkappa_4)$ supports the conjecture:

Conjecture 5.2. *When the rays of* $\varkappa_1, \cdots, \varkappa_n$ *are farther apart, the convergence of the sample mean of* $\prod_{i=1}^n H_L(\varkappa_i)$ *to its expected value is faster.*

5.3 Simulation for the discrete case

We do similar simulations on the discrete case to test the counterpart of conjecture 4.1 for the discrete case where $F_t(z) = 1 - \beta^-/z$. The algorithm is different from that for the continuous case in the following sense: the jumping time (which we assume to be in $\mathbb{N}$ for the discrete case); there is no such clock-ring; each round of update goes from row 1 to row n thoroughly.

We take the same evaluation points as before, and choose $F_t(z) = 1 + 0.5z^{-1}$; our results so far are listed below.

$N = 2$: Our first evaluation set is $(\varkappa_1; \varkappa_2)$.
When $L = 500$, the expected product is 0.132 ± 0.01;
when $L = 1000$, the expected product is 0.129 ± 0.007;
when $L = 2000$, the expected product is 0.131 ± 0.01.
Our second evaluation set is $(\varkappa_3; \varkappa_4)$, where we only run the simulation for $L = 1000$, and obtain the expected product as 0.29 ± 0.01.

$N = 4$: We take the evaluation set to be $(\varkappa_1; \varkappa_2; \varkappa_3; \varkappa_4)$. When $L = 1000$, the expected product is 0.138 ± 0.02; when $L = 2000$, the expected product is 0.134 ± 0.01.

Possible future research topics

Some further interesting topics for simulation might be testing the conjecture 5.1, 5.2 and quantitatively examine the following fact: as the two points move away from each other along the characteristic ray, the correlation of their H_L's should eventually become finite. The variance part (for two coinciding points) is given in theorem 3.4.

Note that a similar study has been carried out by Prähofer and Spohn in [18], obtaining

$$\lim_{t\to\infty} \mathrm{Var}\,(h(\mathbf{x}, t) - h(\mathbf{x'}, t)) \sim \ln|\mathbf{x} - \mathbf{x'}|,\ (\mathbf{x} \text{ and } \mathbf{x'} \text{ are in } \mathbb{R}^2),$$

for large $|\mathbf{x} - \mathbf{x'}| \to \infty$, but not growing with t. But we are interested in the problem for more general comparisons.

It may also be worthwhile to study about what happens to covariance on space-time rays that are preimages under Ω of the same point in $\mathbb{H}$. Another topic is the convergence of the discrete growth model with alternating F_t's.

Remark 5.1. *We recently realize* (*thanks to Professor Borodin*) *that the determinantal structure of the correlations functions holds for a much more general class of two-dimensional growth models than the one we focus on. Recent results in* [*19*] *show that the determinantal structure holds for less strict assumption than* (3.39). *Since our proof in this paper using Duits' arguments just require the existence of the determinantal structure, the scaled fluctuation of the random surface should converge to some* $\Omega-$*pullback of the Gaussian free field on* $\mathbb{H}$ *for a family of two-dimensional growth models with less restrictions. By our simulation result, we do expect that the same result holds without any assumption on the evaluation points* (*in space-time*).

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