Isaac Chen

On the Crossing Profile of Rectilinear Drawings of Complete Graphs

under the direction of Oriol Solé Pi

Abstract

Geometric graph theory is the study of various geometric representations of graphs. We are interested in how the *crossings* between edges of a *drawing* are distributed, extending previous work on the *crossing number* of various graphs. Let $\mathcal{D}_n$ denote a *rectilinear drawing* (all edges are drawn as line segments) of the complete graph K_n in $\mathbb{R}^2$. We prove tight lower and upper bounds on the number of edges in $\mathcal{D}_n$ that are crossed at most k times by other edges, a quantity denoted by $S_k(\mathcal{D}_n)$. We also consider the number of edges in $\mathcal{D}_n$ that are crossed exactly k times, denoted by $e_k(\mathcal{D}_n)$, and show a non-trivial lower bound for this quantity. Some additional minor results are also presented throughout the paper. Our results greatly expand on prior knowledge about the number of edges in $\mathcal{D}_n$ involved in 0 crossings and other work on the number of edges in a drawing of K_n with at most k crossings.

Nikola Gyulev

On Torus Orbits of Symplectic Leaves from Upper Cluster Algebras with Poisson Structure

under the direction of Hamilton Ji Wan

Abstract

Fomin and Zelevinsky introduced the upper cluster algebras, defined as an intersection of Laurent polynomial rings generated by the frozen variables, exchangeable variables, and the inverses of the exchangeable ones. In suitable cases, they can be geometrically interpreted as the coordinate ring of an affine algebraic variety called a cluster variety. Gekhtman, Shapiro, and Vainshtein (GSV) initiated the study of Poisson geometry on upper cluster algebras and defined a compatible torus action. There has been interest in the corresponding *symplectic leaves* on these cluster varieties. It has been conjectured that the cluster variety corresponding to any acyclic seed, equipped with the mentioned GSV Poisson structure (when it exists), has only finitely many torus orbits of symplectic leaves. Working with a class of type A seeds, we prove the conjecture when all frozen variables are invertible. On the other hand, we disprove the general conjecture by presenting an infinite family of counterexamples arising from the case of two exchangeable variables. We also classified the singular points in that case and showed that every such point is a symplectic point, enabling us to construct the counterexamples.

Sophia Liao

Orbits of the Braid Group Action on Rank m $\mathbb{F}_p$-Local Systems Over the n-Punctured Riemann Sphere

under the direction of Yonghwan Kim

Abstract

A local system describes the behavior of a multi-valued function around n punctures of a sphere. We can represent a rank m $\mathbb{F}_p$-local system as an n-tuple of matrices in $\mathrm{GL}_m(\mathbb{F}_p)$, but this representation is not unique. In particular, we can relate the representations of these local systems under conjugation and the action of the braid group. In this paper, we explore the orbits of the braid group action on the conjugacy classes n-tupl1es of matrices in $\mathrm{GL}_m(\mathbb{F}_p)$. Specifically, we find lower and upper bounds for the number of orbits, and we provide a description for the possible sizes of orbits when $n = 2$ and $n = 3$. In addition, we find explicit formulas for the number of orbits when $n = 2$, and we explore the nature of the orbits for particular cases when $n = 3$.

Austin Luo

Chip-Firing on Directed and Binary Trees

under the direction of Ryota Inagaki

Abstract

Chip-firing is a combinatorial game played on a graph $G = (V, E)$ in which we place chips on vertices. We first study a variant of chip-firing which is played on a directed graph $D = (V, E)$. In an infinite rooted directed k-ary tree, we place k^ℓ chips on the root for some positive integer ℓ and we say a vertex v can fire if it has at least outdegree(v) chips and it fires by dispersing 1 chip to each outneighbor. Once every vertex has less than outdegree(v) chips, we call this a stable configuration since no vertex can fire. When the chips are indistinguishable, we determine the number of times each vertex fires and where the chips end up upon reaching the stable configuration. When the chips are distinguishable, we determine the exact number of final configurations. We then study chip-firing on an undirected graph $G = (V, E)$. In an infinite rooted binary tree, we say a vertex can fire if it has at least $\deg(v)$ chips and it fires by dispersing 1 chip to each neighbor. We add a self loop to the root to ensure each vertex has $\deg(v) = 3$. When we place $2^\ell - 1$ labeled chips at the root, we give an upper bound for the number of final configurations. When starting with m chips at the root where m is a positive integer, we determine the number of times each vertex fires for when m can be expressed as the sum of 3 or 4 not necessarily distinct positive integers in the form of $2^\ell - 1$.

Tatiana Medved

Dynamical Torsion for Surfaces of Constant Negative Curvature

under the direction of Alain Kangabire

Abstract

The Ruelle zeta function, originating from the field of dynamical systems and related to differential geometry and spectral theory, is the focus of our study on compact oriented Riemannian surfaces Σ of constant negative curvature. Using a previous result by Chaubet and Dang that links the dynamical torsion to the normalized coefficient of the Ruelle zeta function at zero, we compute the torsion of finite-dimensional resonant states. Specifically, we determine the torsion to be $\det(M)h$, where the automorphism M acts on a basis of the de Rham cohomology group $H^1(\Sigma)$ and h is a fixed element of $\det(H^\bullet)$, independent of surface perturbations. This computation provides a rigorous expression for the torsion in terms of the geometric and dynamical data of the surface, informing future research for surfaces of variable negative curvature.

Christopher Qiu

Slice Genus Bounds for Knots using Grid Diagrams

under the direction of Yonghwan Kim

Abstract

A major question in knot theory is how to determine whether two knot representations depict distinct knots. In particular, knot invariants like the grid index $g(K)$ and the maximal Thurston–Bennequin number $\overline{\mathrm{tb}}(K)$ are crucial in distinguishing certain types of knots. It has been conjectured that $g(K)$ and $\overline{\mathrm{tb}}(K)$ detect roughly the same information. Generally, $g(K)$ is hard to determine, so we study $\overline{\mathrm{tb}}(K)$ instead. In this paper, we establish a connection between $g(K)$ and $\overline{\mathrm{tb}}(K)$ and give an explanation as to why the Thurston–Bennequin number gives a better approximation of the grid index when compared to other knot invariants and strategies. We also find an infinite sequence of potentially topologically slice knots with slice genus approaching infinity. As such knots are topologically slice but not smoothly slice, they could potentially give rise to counterexamples to the smooth four-dimensional generalization of the Poincaré Conjecture.

Akilan Sankaran

From Walking to Tunneling: A Generalized Pilot-Wave System

under the direction of Diego Israel Chavez

Abstract

We investigate the ability of millimetric walking droplets to tunnel between cavities. By synthesizing experimental and theoretical analysis, we provide a framework for droplet tunneling mechanics in three spatial dimensions. We define a generalized Dirichlet-to-Neumann operator that allows us to explicitly characterize droplet and wave-field dynamics under highly intricate variable-topography systems, allowing for numerical simulations of droplet tunneling probabilities and macroscopic dynamical evolution to a far greater degree of accuracy than existing models. Moreover, we demonstrate experimental droplet tunneling in complex cavity geometries, and discuss many-droplet coupling and interactions in the context of tunneling observations.

Abzal Tangsykbay

Bound of Fractal Uncertanity Exponent for Discrete Cantor Sets

under the direction of Alain Kangabire

Abstract

We bounded the case when $y = 0$ in the sum $\sum_{j=0}^{L} \sup_{y \in [0,\delta]} \left| \sum_{l=0}^{L} e^{2\pi i(\alpha j^2 + y)l^2} \right|$. These sums are important to analyze function waves over Discrete Cantor sets, and are closely related to the fractal uncertainty principle. We used previous well-known theorems and approximated the irrational α to a rational number using analytical number theory. We got that the sum is less than a number in the order of $N^{2-\frac{1}{30}}$. This bound is much better than the intuitive bound we had before the research, N^2. Further, this result can be used in the analysis of quantum open baker's maps.

Eric Wang

An Isoperimetric Problem for the First Neumann Eigenvalue on the Sphere

under the direction of Shrey Aryan

Abstract

The eigenvalue problem for the Laplacian has gained a lot of attention from its application to physics such as in the wave equation and heat equation. While eigenvalues are hard to compute for general regions, they can be bounded by geometric quantities of the region. Bounding eigenvalues for volume constraint has already been extensively explored, both in the plane and on the sphere, as well as in higher dimensions. However, bounds of eigenvalues by geometric quantities other than volumes have not been deeply studied.

In this paper, we prove an upper bound on the first nonzero Neumann eigenvalue of the Laplacian in terms of width and area for domains on a sphere. We also pose a conjecture for an upper bound of the first nonzero Neumann eigenvalue under perimeter constraint for convex domains on a sphere. Our results are spherical analogs of results of Henrot, Lemenant, Lucardesi (2022) for domains on the plane.

Davido Zhang

Lipschitz Optimal Transport Maps

under the direction of Shrey Aryan

Abstract

In optimal transport, Caffarelli famously constructed a Lipschitz map between two log-concave densities $f = e^{-V}$ and $g = e^{-W}$ using the quadratic cost $c(x, y) = -xy$. We considered variations of this problem. Firstly, we built foundations for extending this result to the transport between unequal dimensions $T : \mathbb{R}^n \to \mathbb{R}$. We found an analog to the Monge-Ampère Equation and conditions that guarantee continuity to its unique solution. Secondly, we recovered Caffarelli's result for the perturbed cost function $c(x, y) = -xy - \varepsilon|x|^2|y|^2$, which suggests the possibility of constructing contraction maps for a greater variety of costs.

Ashley Zhu

Curve Shortening Flow on n-Loop Curves

under the direction of Michael Law

Abstract

In this paper, we explore the properties of n-loop curves under Curve Shortening Flow. In differential geometry, n-loop curves, which are essentially generalized figure-8 curves, are of particular interest because of their dramatically different properties under Curve Shortening Flow from those of simple closed curves. We study specific families of n-loop curves and their parameter spaces. In particular, we divide the parameter spaces of these families into categories based on order of lobe disappearance. Through our work, we expand on recent results regarding the parameter space of a particular family of 3-loop curves. We additionally run numerical studies to approximate the parameter space of a family of 5-loop curves. From the results of our studies, we formulate a conjecture regarding the parameter space of n-loop curves for odd n. Additionally, we identify and prove several properties of the parameter space of 5-loop curves. For both the cases of $n = 3$ and $n = 5$, our results serve to partially generalize the Avoidance Principle beyond simple closed curves. From here, we make progress towards proving the n-loop conjecture for the case of $n = 5$.