Counting the Maximum Number of 3-rich Curves in Configurations of n Points in $\mathbb{P}^2_{\mathbb{F}_q}$

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Abstract

In this paper, we study the problem of counting 3-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$. A 3-rich conic is a conic that passes through exactly three points in a given set of n points. We consider points in general position, i.e., no three points are collinear and no six points lie on a conic. We explore the maximum and minimum number of 3-rich conics that can be drawn through the configuration of n points in $\mathbb{P}^2_{\mathbb{F}_q}$. First, we establish a lower bound for the number of 3-rich conics, and find an upper bound using combinatorial arguments. We then generalize the upper bound for 3-rich curves of any degree. The total number of 3-rich conics is obtained by subtracting the non-smooth conics (formed by the union of two lines) from the total number of conics. We provide explicit formulas for the number of smooth 3-rich conics generated by the set of n points in general position.

Summary

The orchard planting problem dates back to the 1820s and aims to find an optimal arrangement for trees in an orchard. Mathematically, we can translate this problem to points in a plane, where points and a plane represent trees and an orchard respectively. We look at a particular structure of this problem, where the orchard is a projective plane over a finite field. In this structure, we find an optimal arrangement for any set of n points.

1 Introduction

The **orchard planting problem** aims to find an optimal arrangement of n trees in an orchard that maximizes the number of 3-point lines that can be formed from the set of n points. Two papers have made significant contributions to solving this problem [1, 2].

In 2013, Green and Tao [1] laid the foundation for investigating the configurations of points in the plane and analyzing the number of ordinary lines that can be formed by connecting these points. They provided valuable insight into the behavior and distribution of ordinary lines within different arrangements of points. They defined *3-rich lines* as lines that pass through exactly three points from a set of given points and identified an upper bound for the number of 3-rich lines that can be drawn given n points.

Theorem 1.1 (Green-Tao [1])**.** *Suppose that P is a finite set of n points in the plane. Suppose that $n \geq n_0$ for some sufficiently large absolute constant n_0. Then there are no more than $\lfloor n(n-3)/6 \rfloor + 1$ lines that are 3-rich, that is they contain precisely 3 points of P.*

Building upon the Green-Tao work, Padmanabhan and Shukla [2], in 2020, delved into the specific application of sets defining few ordinary lines in the context of elliptic curves over finite fields and established a bound for the number of 3-rich lines that can be drawn given n points in a projective plane over a finite field.

Theorem 1.2 (Padmanabhan-Shukla [2])**.** *Let $N = q + 1 + 2\sqrt{q}$ with either q odd or $q = 2^n$ with n even. There exist point-line arrangements (N, t) in the projective plane over the finite field $\mathbb{F}_q$ with*

$$\mathcal{O}r(N) \geq \left\{ \begin{array}{ll} \lfloor \frac{N(N-3)}{6} \rfloor + 3, & \textit{if } 3 \mid N \\ \lfloor \frac{N(N-3)}{6} \rfloor + 1, & \textit{if } 3 \nmid N \end{array} \right\},$$

where $\mathcal{O}r(N)$ denotes the number of 3-rich lines in a projective plane over a finite field.

We define *degree d curves* as curves whose equations have highest power d. Degree d curves form a projective plane $\mathbb{P}^N$, where N is the dimension of the space of homogeneous polynomials in variables x, y, and z. We can think of the 3-rich lines discussed before as degree one curves. In this paper, we focus on degree two curves. We define a *3-rich curve* as a curve that passes through exactly three points. Given points $p_1, p_2, \ldots, p_n \in \mathbb{P}^2_{\mathbb{F}_q}$, we want to find the maximum number of 3-rich curves of degree two. Specifically, we investigate the case where no three of the given points are collinear and no six points lie on a degree two curve.

The remainder of this paper is divided into seven sections. In Section 2 we present a series of definitions necessary to understand the problem. In Sections 3 and 4 we determine a lower and upper bound for the number of 3-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$ and in Section 5, we establish a generalization of the upper bound to 3-rich curves of any degree d. In Section 6, we count the exact number of 3-rich conics and in Section 7 we determine the number of smooth 3-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$. Last, in Section 8, we summarize the conclusions of our main results and describe possible ways to continue the research by exploring 6-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$.

2 Preliminaries

To better understand the problem, we provide a list of definitions. In this paper we work in a projective plane over a finite field $\mathbb{P}^2_{\mathbb{F}_q}$.

Definition 2.1. A *Projective Plane over a Finite Field* $\mathbb{P}^n_{\mathbb{F}_q}$ is a parameter space of lines in a vector space of dimension $n+1$.

We explore finding the number of 3-rich conics in a configuration of n points.

Definition 2.2. *Conics* are degree two curves in the projective plane $\mathbb{P}^2_{\mathbb{F}_q}$.

Conics can be written in the general form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0,$$

where a, b, c, d, e, f are constants. Conics in $\mathbb{P}^2_{\mathbb{F}_q}$ can be formed by a double line, a union of two distinct lines, or a smooth (irreducible) curve.

We also include a definition of points in *general position*.

Definition 2.3. The set of n points in $\mathbb{P}^2_{\mathbb{F}_q}$ is in *general position* if no three points are collinear and no six points lie on a conic.

Using this definition, we avoid counting conics in the form of a double line and restrict the number of correction terms that exist when finding the total number of 3-rich conics in the set of n points in $\mathbb{P}^2_{\mathbb{F}_q}$.

Finally, we introduce $f_d(n)$ as the exact number of 3-rich curves of degree d that can be drawn given a configuration of n points in general position in $\mathbb{P}^2_{\mathbb{F}_q}$. In this paper we mostly discuss $f_2(n)$, the number of conics that can be drawn given a configuration of n points in general position in $\mathbb{P}^2_{\mathbb{F}_q}$.

3 Establishing a Lower Bound

We begin by establishing a lower bound for the number of 3-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$.

Theorem 3.1. *Given a set of n points in a projective plane over a finite field $\mathbb{P}^2_{\mathbb{F}_q}$, the minimum number of 3-rich conics is given by*

$$f_2(n) \geq \left\lfloor \frac{\binom{n}{3}\binom{q-2}{2} - \binom{n}{4}\binom{q-3}{1}}{q-1} \right\rfloor .$$

Proof. A conic is determined by 5 points. There are $\binom{n}{3}$ ways to select three points out of n points. We need two more points to determine the conic. A conic can be a double line, a union of two distinct lines, or a smooth curve. A double line has $q+1$ points, a union of two distinct lines has $2q+1$ points, and the number of points on a smooth curve in $\mathbb{P}^2_{\mathbb{F}_q}$ varies, but is always greater than $q+1$. The double line has the least number of points. Therefore, there are

$$\binom{q+1-3}{2} = \binom{q-2}{2}$$

ways to choose two choose the two remaining points to determine the conic. Consequently, there are

$$\binom{n}{3}\binom{q-2}{2} \tag{1}$$

conics that pass through at least three points.

However, this number includes the conics that pass through at least four of the n points. There are $\binom{n}{4}$ ways to select four points and

$$\binom{q+1-4}{1} = \binom{q-3}{1}$$

ways to select one point from the remaining $q+1$ points on the curve.

The number of conics that pass through at least four points is given by

$$\binom{n}{4}\binom{q-3}{1}. \tag{2}$$

By subtracting term 2 from term 1, we find

$$\binom{n}{3}\binom{q-2}{2} - \binom{n}{4}\binom{q-3}{1} \tag{3}$$

conics that pass through three points.

Since a conic can be written in the form $ax^2+by^2+cz^2+dxy+exz+fyz=0$, multiplying this conic by a scalar k yields the same conic. There are $q-1$ non-zero scalars k in $\mathbb{P}^2_{\mathbb{F}_q}$. By dividing term 3 by $q-1$, we determine the minimum number of 3-rich conics generated by n points in $\mathbb{P}^2_{\mathbb{F}_q}$ as

$$f_2(n) \geq \left\lfloor \frac{\binom{n}{3}\binom{q-2}{2} - \binom{n}{4}\binom{q-3}{1}}{q-1} \right\rfloor .$$

□

4 Establishing an Upper Bound

We continue with establishing an upper bound for the number of 3-rich conics in $\mathbb{P}^2_{\mathbb{F}_q}$.

Theorem 4.1. *Given a set of n points in a general position in $\mathbb{P}^2_{\mathbb{F}_q}$, the maximum number of 3-rich conics is given by*

$$f_2(n) \leq \binom{n}{3}(q^2+q+1).$$

Proof. Given n points in the set $P = \{p_1, p_2, \ldots, p_n\}$, we select three points. There are $\binom{n}{3}$ ways to arbitrarily select $p_i, p_j, p_k \in P$. Let H_{p_i}, H_{p_j}, and H_{p_k} be the set of curves C passing through p_i, p_j, and p_k respectively. Finding the number of conics that pass through p_i, p_j, and p_k is equivalent to determining the intersection of H_{p_i}, H_{p_j}, and H_{p_k}.

The intersection of these hyperplanes are the homogeneous points in $\mathbb{P}^2_{\mathbb{F}_q}$. Determining the number of such points in the projective plane $\mathbb{P}^2_{\mathbb{F}_q}$ gives us $H_{p_i} \cap H_{p_j} \cap H_{p_k}$ and consequently, the number of conics that pass through three points. The number of homogeneous points in $\mathbb{P}^2_{\mathbb{F}_q}$ is $q^2 + q + 1$ [3]. Thus,

$$f_2(n) \leq \binom{n}{3}(q^2 + q + 1)$$

yields the maximum number of 3-rich conics in the set of n points in $\mathbb{P}^2_{\mathbb{F}_q}$. □

5 Generalizing the Upper Bound

The bound we found above for 3-rich degree two curves can be generalized to curves of any degree d.

Theorem 5.1. *Given a set of n points in a general position in $\mathbb{P}^2_{\mathbb{F}_q}$, the maximum number of 3-rich curves of degree d is given by*

$$\binom{n}{3}\left(q^{\binom{d+1}{2}-4} + q^{\binom{d+1}{2}-5} + q^{\binom{d+1}{2}-6} + \cdots + q^2 + q + 1\right).$$

Proof. A curve of degree d is determined by

$$\binom{d+1}{2} - 1$$

points. Curves of degree d form a projective plane $\mathbb{P}^{\binom{d+1}{2}-1}$. Given n points in a projective plane over a finite field, we select

$$\binom{d+1}{2} - 1$$

points to determine the curve. We need three of these points to be in the set of n points. There are $\binom{n}{3}$ ways to choose the points p_i, p_j, p_k. An additional

$$\binom{d+1}{2} - 1 - 3 = \binom{d+1}{2} - 4$$

points are needed to determine the curve. Similarly to Theorem 5.1, we find the number of curves that pass through the three selected points by determining the intersection of H_{p_i}, H_{p_j}, and H_{p_k}. This is equivalent to finding the number of points in $\mathbb{P}^{\binom{d+1}{2}-1-3}$. There are $q^{\binom{d+1}{2}-4} + q^{\binom{d+1}{2}-5} + q^{\binom{d+1}{2}-6} + \cdots + q^2 + q + 1$ points in $\mathbb{P}^{\binom{d+1}{2}-1-3}$. Thus,

$$f_d(n) \leq \binom{n}{3}\left(q^{\binom{d+1}{2}-4} + q^{\binom{d+1}{2}-5} + q^{\binom{d+1}{2}-6} + \cdots + q^2 + q + 1\right)$$

gives the maximum number of degree d curves that pass through three points. □

6 Counting the Number of 3-rich Conics

Theorem 6.1. *Given a set of n points in a general position in $\mathbb{P}^2_{\mathbb{F}_q}$, the number of 3-rich conics generated is given by*

$$f_2(n) = \binom{n}{3}(q^2+q+1) - \binom{4}{3}\binom{n}{4}(q+1) + \binom{5}{3}\binom{n}{5}.$$

Proof. From Theorem 4.1, we know that

$$\binom{n}{3}(q^2+q+1)$$

conics that pass through at least three of n points. To find the number of conics that pass through exactly three of the n points, we need to add correction terms to above term to account for the curves that pass through four and five of the n points.

We begin with the correction term for the curves that pass through four points. There are $\binom{4}{3}$ ways to select three out of four points in $P = \{p_1, p_2, \ldots, p_n\}$, where n is the number of given points. There are $\binom{n}{4}$ ways to choose four of the n points. Similarly to Theorem 5.1, this is equivalent to determining $H_{p_i} \cap H_{p_j} \cap H_{p_k} \cap H_{p_l}$. The intersection of these hyperplanes is $\mathbb{P}^2_{\mathbb{F}_q}$ and the number of points in $\mathbb{P}^2_{\mathbb{F}_q}$ is $H_{p_i} \cap H_{p_j} \cap H_{p_k} \cap H_{p_l}$. Since there are $q+1$ homogeneous points in $\mathbb{P}_{\mathbb{F}_q}$,

$$\binom{4}{3}\binom{n}{4}(q+1)$$

is the first correction term.

We continue with the correction term for the curves that pass through five points. There are $\binom{5}{3}$ ways to select three points in P. There are $\binom{n}{5}$ ways to choose five of the points in P. Counting the number of curves that pass through five points is equivalent to determining $H_{p_i} \cap H_{p_j} \cap H_{p_k} \cap H_{p_l} \cap H_{p_m}$. Since only one curve passes through all points,

$$\binom{5}{3}\binom{n}{5}$$

gives the second correction term. Thus,

$$\binom{n}{3}\left(q^2+q+1\right) - \binom{4}{3}\binom{n}{4}(q+1) + \binom{5}{3}\binom{n}{5}$$

yields the number of 3-rich conics generated by the set of n points in general position in $\mathbb{P}^2_{\mathbb{F}_q}$. □

7 Counting Smooth 3-rich Conics

Before counting the smooth 3-rich conics, we need to introduce three new definitions. A *Type 1 conic* is a conic that is composed of two line ϕ_1 and ϕ_2, where line ϕ_1 passes through two of the n points and line ϕ_2 passes through exactly one of the points on ϕ_1 and exactly one other point from the n points.

A *Type 2 conic* is a conic composed of two line ϕ_1 and ϕ_2, where line ϕ_1 passes through two of the n points and line ϕ_2 passes through exactly one of n the points not on line ϕ_1.

A *Type 3 conic* is a conic of the form $ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ and is a smooth (irreducible) curve.

A conic can be formed by the union of two lines that intersect at one of the n points (Type 1 conic), a union of two lines that do not intersect at one of the n points (Type 2 conic), a double line, or a smooth curve (Type 3 conic). A set of n points in general position does not have three collinear points. The only way for a conic in the form of a double line to be 3-rich is for three points to be collinear. Thus, the conics of this form are not considered 3-rich conics.

In order to determine the number of Type 3 conics, we subtract the number of Types 1 and 2 conics from the total number of conics.

Theorem 7.1. *Given n points in general position in $\mathbb{P}^2_{\mathbb{F}_q}$, the number of smooth 3-rich conics generated is given by*

$$\binom{n}{3}\left(q^2+q+1\right) - \binom{4}{3}\binom{n}{4}(q+1) + \binom{5}{3}\binom{n}{5} - \binom{n}{2}(n-2)(q+4-n).$$

Proof. We first count the number of Type 1 conics. We choose two points from the set of n points to determine ϕ_1. There are $\binom{n}{2}$ ways to do this. We also choose two points to determine the second line ϕ_2. The first point is one of the two points on ϕ_1. The second point that determines ϕ_2 can be any point from the set of n points not on ϕ_1. There are $n-2$ such points. Thus, there are

$$\binom{n}{2}(2)(n-2)$$

Type 1 conics.

Counting the number of Type 2 conics is similar. There are $\binom{n}{2}$ ways to select ϕ_1. The second line, ϕ_2, must pass through exactly one of the n points not on ϕ_1. There are $n-2$ ways to select this point. In $\mathbb{P}^2_{\mathbb{F}_q}$, there are $q+1$ lines that pass through any given point. However, line ϕ_2 can not pass through any other of the n points. There are $n-1$ lines that pass through the selected point and one more of the n points. So, there are $q+1-(n-1) = q+2-n$ lines that pass through the given point and no other of the n points. Thus, there are

$$\binom{n}{2}(n-2)(q+2-n)$$

Type 2 conics.

Combining the number of Type 1 and Type 2 conics yields

$$\binom{n}{2}(2)(n-2) + \binom{n}{2}(n-2)(q+2-n) = \binom{n}{2}(n-2)(q+4-n)$$

non-smooth curves. Theorem 6.1 gives us the total number of 3-rich conics generated by n points in $\mathbb{P}^2_{\mathbb{F}_q}$. Subtracting the number of Type 1 and Type 2 conics from the total number of conics gives

$$\binom{n}{3}\left(q^2+q+1\right)-\binom{4}{3}\binom{n}{4}(q+1)+\binom{5}{3}\binom{n}{5}-\binom{n}{2}(n-2)(q+4-n),$$

which is the number of smooth 3-rich conics.

□

8 Conclusion and Future Work

In this paper, we explored a variation of the orchard planting problem of finding the optimal arrangement of n trees in an orchard to maximize the number of 3-rich lines that can be formed. Our analysis was based on the works of Green-Tao and Padmanabhan-Shukla.

In the paper we established both lower and upper bounds for the number of 3-rich conics in a projective plane over a finite field. We determined the lower bound to be

$$\left\lfloor \frac{\binom{n}{3}\binom{q-2}{2}-\binom{n}{4}\binom{q-3}{1}}{q-1} \right\rfloor$$

and the upper bound

$$\binom{n}{3}\left(q^2+q+1\right).$$

We then generalized the upper bound to curves of any degree d by

$$\binom{n}{3}\left(q^{\binom{d+1}{2}-4}+q^{\binom{d+1}{2}-5}+\cdots+q^2+q+1\right).$$

In addition, we determined the exact count for the number of 3-rich conics as

$$\binom{n}{3}\left(q^2+q+1\right)-\binom{4}{3}\binom{n}{4}(q+1)+\binom{5}{3}\binom{n}{5}.$$

Finally, we concluded that the number of smooth 3-rich conics could be determined as

$$\binom{n}{3}\left(q^2+q+1\right)-\binom{4}{3}\binom{n}{4}(q+1)+\binom{5}{3}\binom{n}{5}-\binom{n}{2}(n-2)(q+4-n).$$

In geometry, two points determine a line, prompting us to explore 3-rich lines. Similarly, five points determine a conic. Consequently, we should consider the question of determining the number of 6-rich conics. This question can be expressed geometrically as the intersection of a elliptic curve in six points, each with multiplicity one. When this occurs, the set of closed points forms an abelian group under addition. Solving for the number of solutions to $x_1+x_2+\cdots+x_6=0$, where $x_1, x_2, \ldots, x_6$ are the intersection points consequently gives the number of 6-rich curves for n points in a general position.

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References

[1] B. Green and T. Tao. On sets defining few ordinary lines. *Discrete Comput Geom*, 50:pp. 409–468, 2013.

[2] R. Padmanabhan and A. Shukla. Orchards in elliptic curves over finite fields. *Finite Fields and Their Applications*, 68:p. 101,756, 2020.

[3] T. Forbes. Theorem of the day. URL `https://theoremoftheday.org/`.