Examples for the Robust Qualitative Uncertainty Principle

Tair Satubaldin

Under the direction of

Alexander Ortiz
PhD Student
Massachusetts Institute of Technology

Research Science Institute
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Abstract

In this paper, we explore the sharpness of the qualitative robust uncertainty principle by inspecting the examples of different functions that could possibly extremize the uncertainty principle. We look at the examples of one Gaussian, sum of Gaussians arranged in a finite arithmetic progression and the blurred and truncated Dirac comb. For each example, we approximate the product of measures of two smallest sets, where the function and its Fourier transform are $\frac{1}{10}$-concentrated. In the course of this research, we get that in all listed examples, the value of the explored product is approximately the same as in case of a Gaussian function.

Summary

In Fourier analysis, uncertainty principles are inequalities that show that both the function and its Fourier transform cannot both be concentrated near some points. In this paper, we study the sharpness of one of such uncertainty principle which is related to concentrations of the function and its Fourier transform on a measurable set. We inspect several examples where the inequality could be sharp to form conjectures concerning the nature of this uncertainty principle.

1 Introduction

The uncertainty principle is an inequality that shows that a function and its Fourier transform cannot both be localized. The most well-known example of this is Heisenberg's uncertainty principle. It states that for an L^2-normalized function $f : \mathbb{R} \to \mathbb{C}$ the following inequality is true

$$(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx)(\int_{-\infty}^{\infty} \xi^2 |f(\xi)|^2 d\xi) \geq \frac{1}{16\pi^2}.$$

This shows that the function and its Fourier transform cannot both have its L^2-masses too concentrated near the point $x = 0$. In this paper, we only focus on the qualitative robust uncertainty principle, which is the robust version of the qualitative uncertainty principle from Folland and Sitaram's survey [1]. The definition of robustness is taken from the study by Candes and Tao [2], and the uncertainty principle is considered robust if it still holds under small perturbations of the function. Robustness of the uncertainty principle ensures that the uncertainty principle can be applied to physical phenomena, since it will still hold after small measurement errors.

Theorem 1.1 (Robust qualitative uncertainty principle [3])**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be an L^2-normalized function, such that f is ϵ-concentrated on a measurable set T and $\hat{f}$ is δ-concentrated on a measurable set W, then*

$$|T||W| \geq (1 - \delta - \epsilon)^2,$$

where $|\cdot|$ denotes Lebesgue measure.

This means that f and $\hat{f}$ cannot both be concentrated on small measurable sets.

Definition 1.1. For a function $f : \mathbb{R} \to \mathbb{C}$ normalized in L^2-space, $E(f)$ is a set such that $\int_{E(f)} |f(x)^2| dx \geq 99/100$ and $|E(f)|$ is minimal. Analogously, $\Sigma(f)$ is a set such that $\int_{\Sigma(f)} |\hat{f}(\xi)|^2 d\xi \geq 99/100$ and $|\Sigma(f)|$ is minimal.

The constant $\frac{99}{100}$ was chosen to be big enough so that the uncertainty principle holds, but also so that $\sqrt{1 - \frac{99}{100}}$ was rational and we could write this statement equivalently as

"f $\frac{1}{10}$-concentrated on the set E". It follows from Theorem 1.1 that since f and $\hat{f}$ are $\frac{1}{10}$-concentrated on $E(f)$ and $\Sigma(f)$ respectively, $|E(f)||\Sigma(f)| \geq (1-(\frac{1}{10}+\frac{1}{10}))^2 = \frac{16}{25}$. The goal of this project is to try to compute this product for different functions f using different approximations and make conclusions about the relation between the function's structure and the value of $|E(f)||\Sigma(f)|$. This would build up our intuition of how to construct a function that extremizes the uncertainty principle. Finding such functions is important because it is not yet known what the greatest lower bound for the left-hand side product in Theorem 1.1 is. We are particularly interested in calculating the value of this product for sums of Gaussian functions and for functions concentrated near the points of delta functions in Fourier quasicrystals, since we can construct such functions using convolutions of functions and these quasicrystals. Studying the uncertainty principle is important because it is closely related to signal recovery, which has various applications [2].

2 Preliminaries

Definition 2.1. The Fourier Transform of a function $f : \mathbb{R} \to \mathbb{C}$ is denoted by $\hat{f}(\xi)$ and is defined by the formula

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)dx$$

for all real numbers ξ.

Definition 2.2. We define the convolution $f * g$ of two functions f and g by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Definition 2.3. A Dirac delta function at point x_0 is a measure δ_{x_0} that is defined for the subsets of the set of real numbers as

$$\delta_{x_0}(A) = \begin{cases} \delta_{x_0} = 1 & x_0 \in A \\ \delta_{x_0} = 0 & x_0 \notin A \end{cases}.$$

Corollary 1. *It can be seen from the definition of a Dirac delta function that for any continuous function $f : \mathbb{R} \to \mathbb{C}$, the following relation holds*

$$\int_{-\infty}^{\infty} f(x) d\delta_{x_0} = f(x_0).$$

Definition 2.4. A Fourier quasicrystal μ is a measure of a form $\sum_{i=0}^{\infty} a_i \delta_{b_i}$ with a Fourier transform of a form $\sum_{i=0}^{\infty} c_i \delta_{d_i}$, where $\{a_i\}$ and $\{c_i\}$ are sets of complex numbers, $\{b_i\}$ and $\{d_i\}$ are discrete sets of real numbers.

Theorem 2.1 (Parseval's identity)**.** *For two functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ such that there exist some constants c and α for which the inequalities $|f(x)| < \frac{c}{1+x^\alpha}$ and $|g(x)| < \frac{c}{1+x^\alpha}$ holds for every x, the following relation is true:*

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\xi)d\xi.$$

Definition 2.5. The Fourier transform of a measure μ is the distribution $\hat{\mu}(\xi)$ such that the relation

$$\int_{-\infty}^{\infty} f(x)d\mu = \int_{-\infty}^{\infty} \hat{f}(\xi)d\hat{\mu}(\xi)$$

holds for all test functions f, similar to Parseval's identity for functions.

Theorem 2.2 (Poisson summation formula)**.** *For a function $f : \mathbb{R} \to \mathbb{C}$ such that there exist some constants c and α for which the inequality $|f(x)| < \frac{c}{1+x^\alpha}$ holds for every x. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Corollary 2. *The Dirac comb with the formula $\sum_{n\in\mathbb{Z}} \delta_n$ is the most well-known example of a Fourier quasicrystal that arises from Poisson summation formula.*

Definition 2.6. The L^2-norm of a complex-valued function f is defined as the value of $(\int_{-\infty}^{\infty} |f(x)|^2 dx)^{1/2}$. The collection of all functions with finite L^2-norm is the L^2-space. We call the function L^2-normalized if its norm in L^2-space is 1.

Definition 2.7. The function f is ϵ-concentrated on a measurable set T if $||f - 1_T f|| \leq \epsilon$.

Definition 2.8. For each $\epsilon > 0$ we define

$$K_\epsilon(x) = \frac{2^{1/4}}{\epsilon^{1/2}} e^{-\pi x^2/\epsilon^2}.$$

The normalization of K_ϵ is chosen so that it is L^2-normalized. It also satisfies the relation $\hat{K}_\epsilon = K_{1/\epsilon}$.

3 General Observations

In this section, we write the general observations about the product $|E(f)||\Sigma(f)|$ for a general function f.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be such continuous function that there exists such k that $|f(x)|$ takes every value less than k times and is bounded. Define $S_c := \{x \in \mathbb{R} : |f(x)| > c\}$. Then $E(f)$ exists and can be chosen to be S_c for some c.*

Proof. Note that since $|f|$ is continuous and takes each value less than k times, $|S_c|$ is a decreasing continuous function of c that is defined for all positive numbers c. Take c such that $\int_{S_c} |f(x)|^2 dx = \frac{99}{100}$. Such c exists because the value of $\int_{S_c} |f(x)|^2 dx$ changes continuously with c (this follows from the fact that $|S_c|$ changes continuously with c) and approaches 0 for big c. Assume that there is a measurable set E such that $|E| < |S_c|$ and $\int_E |f(x)|^2 dx \geq \frac{99}{100}$. Let c' be such real number that $|S_{c'}| = |E|$. Then $\int_{S_{c'}} |f(x)|^2 dx = \int_{\mathbb{R}} 1_{S_{c'}} |f(x)|^2 dx = \int_0^\infty |\{x \in S_{c'} : |f(x)|^2 > t\}| dt$. Similarly, $\int_E |f(x)|^2 dx = \int_0^\infty |\{x \in E : |f(x)|^2 > t\}| dt$. For $t > \sqrt{c'}$,

$$|\{x \in S_{c'} : |f(x)|^2 > t\}| = |\{x : |f(x)|^2 > t\}| \geq |\{x \in E : |f(x)|^2 > t\}|$$

and for $t \leq \sqrt{c'}$,

$$|\{x \in S_{c'} : |f(x)|^2 > t\}| = |S_{c'}| = |E| \leq |\{x \in E : |f(x)|^2 > t\}|.$$

It follows that the first integral is greater than or equal to the second integral. Thus, $\int_{S_{c'}} |f(x)|^2 dx \geq \int_E |f(x)|^2 dx \geq \int_{S_c} |f(x)|^2 dx$, therefore $c' \geq c$. But $|S_{c'}| = |E| < |S_c|$, therefore $c' < c$. Contradiction, therefore the statement of the lemma is true. $\square$

Corollary 3. *This lemma can be applied to* $\hat{f}$ *and* $\Sigma(f)$ *too. This lemma makes it easier to determine* $|E(f)|$ *and* $|\Sigma(f)|$ *for many functions.*

Remark. *It is also easy to see that modulation and translation of* f *do not change the value of* $|E(f)||\Sigma(f)|$*, same for the* L^2*-normalized time scaled* f*.*

4 One Gaussian

In this section, we evaluate $|E(f)||\Sigma(f)|$ in the case when $f(x) = K_\epsilon(x)$. The function K_ϵ is interesting for us because it and its Fourier transform are quite concentrated near the point $x = 0$ and it extremizes the Heisenberg's uncertainty principle, therefore it would be interesting to see if that is the case for the robust qualitative uncertainty principle.

Example 4.1. *Firstly, we can compute the value of* $|E(f)||\Sigma(f)|$ *when* f *is a Gaussian function. It is sufficient to compute it only for* K_1 *because we can get any Gaussian from time scaling and translating* K_1 *and then* L^2*-normalizing the function we got, therefore the value of* $|E(f)||\Sigma(f)|$ *for all Gaussians is the same. We can understand that* $E(K_1) = \Sigma(K_1)$ *because* $\hat{K}_1 = K_1$*. We can deduce from Lemma 3.1 that* $E(K_1) = (-t, t)$ *for some positive number* t*. Then* $\int_{-t}^{t} 2^{1/2} e^{-2\pi x^2} dx = 99/100$*, therefore we can get that* $\int_{-t\sqrt{2\pi}}^{t\sqrt{2\pi}} e^{-x^2} dx = \frac{99}{100}\sqrt{\pi}$ *and* $t = \frac{\mathrm{erf}^{-1}(99/100)}{\sqrt{2\pi}} \approx 0.7266$*, where* erf^{-1} *denotes the inverse error function. Then* $|E(f)||\Sigma(f)| = 4t^2 \approx 2.1119$.

It can be seen that the difference between the value of the product $|E(K_\epsilon)||\Sigma(K_\epsilon)|$ and its lower bound from Theorem 1.1, which is equal to $\frac{16}{25}$, is quite large. This gives us a reason to look for other functions where this product might be smaller.

5 Sum of Gaussians

Sums of Gaussians is an interesting family of functions that can be used to approximate quite many functions. They are also convenient to work with because they vanish outside a certain interval and it is easy to compute the Fourier transform of a Gaussian. Below, we try to estimate the value of $|E(f)||\Sigma(f)|$ in the case when f is a finite sum of Gaussian functions.

Example 5.1. *Let f be an L^2-normalized multiple of $\sum_{j=1}^{n} A_j e^{-\pi\frac{(x-c_j)^2}{v_j^2}}$. Let's also assume that numbers c_j are sufficiently spread apart so that we can approximate the function at each point by the value of only one Gaussian from the sum that has the largest absolute value at that point. From Lemma 3.1 we can understand that $E(f)$ is the set of such numbers x that $|f(x)| \geq c$ for some constant c. It follows that $E(f)$ can be approximated as a union of n disjoint intervals with centers at points c_j, even though some of these intervals may have length zero. From solving the equation $A_j e^{-\pi\frac{(x-c_j)^2}{{v_j}^2}} = c$ we get that an interval with the center at point c_j has length $2\sqrt{-\frac{{v_j}^2 \ln \frac{c}{A_j}}{\pi}}$ for $A_j \geq c$ and length 0 for $A_j < c$. The Fourier transform has the formula $\hat{f}(\xi) = \sum_{j=1}^{n} A_j v_j e^{-2\pi i c_j \xi} e^{-\pi x^2 {v_j}^2}$. It is difficult to say anything about $\Sigma(f)$ in this case.*

Gaussians Centered at the Points of an Arithmetic Progression

In this subsection, we look at the case when Gaussians in the sum are an equal distance apart. This case is important because the L^2-mass of the function is concentrated near the integer points, similar to the case of the Dirac comb. This suggests that the Fourier transform of the function may be concentrated near the integer points too, since the Dirac comb is its own Fourier transform. In this case, it is reasonable to assume that the product $|E(f)||\Sigma(f)|$ is quite small.

Example 5.2. *Let $g(x) = \sum_{j=0}^{k-1} K_\epsilon(x-j)$. Then $\hat{g}(\xi) = \sum_{j=0}^{k-1} e^{-2\pi i j\xi} K_{1/\epsilon}(\xi) = \frac{e^{-2\pi i k\xi}-1}{e^{-2\pi i\xi}-1} K_{1/\epsilon}(\xi)$.*

In this case, for small enough ϵ, $|E(g)|$ is approximately the same as $k|E(K_\epsilon)|$ because if we approximate the function on k intervals as the value of only one Gaussian that is biggest on the interval, then $E(g)$ contains the portion of $|E(K_\epsilon)|$ of every Gaussian in the sum. For sufficiently large k, we can approximate $|\hat{g}(\xi)| = |\frac{e^{-2\pi ik\xi}-1}{e^{-2\pi i\xi}-1}K_{1/\epsilon}(\xi)|$ as $k|K_{1/\epsilon}(\xi)|$ near integer points and 0 far from integer points. Let's assume that $|\hat{g}(\xi)| = k|K_{1/\epsilon}(\xi)|$ for $x \in (n - \frac{1}{2k}, n + \frac{1}{2k})$ for all integer n. Then $\Sigma(g)$ is approximately $\bigcup_{j=-m}^{m}(j - \frac{1}{2k}, j + \frac{1}{2k})$ for the smallest m such that $\sum_{j=-m}^{m} K_{1/\epsilon}(j)^2 \geq \frac{99}{100}(\sum_{j=-\infty}^{\infty} K_{1/\epsilon}(j)^2)$, where both sides of the inequality can be approximated by integrals and the condition can be replaced by $\int_{-m}^{m} K_{1/\epsilon}(x)^2 dx \geq \frac{99}{100}(\int_{-\infty}^{\infty} K_{1/\epsilon}(x)^2 dx)$, then we can see that $|\Sigma(g)| \approx \frac{|\Sigma(K_{1/\epsilon})|}{k}$, therefore $|E(g)||\Sigma(g)| \approx |E(K_1)||\Sigma(K_1)|$.

6 Blurred and Truncated Dirac Comb

In this section, we construct a function that approaches the Dirac comb but vanishes outside some interval. It also has the property that it is very similar to its own Fourier transform and can be approximated very closely as the sum of Gaussians, making the evaluation of $|E(f)||\Sigma(f)|$ a lot easier.

Example 6.1. *Let $\mu(x)$ be $\sum_{n\in\mathbb{Z}} \delta_0(x-n)$, a Dirac comb. In this example, we construct a function that approaches the structure of a Dirac comb that nearly vanishes outside of some interval. Let $f = K_{1/\epsilon} * (\mu \cdot K_\epsilon)$ so that $f(x) = \sum_{n\in\mathbb{Z}} K_{1/\epsilon}(x-n)K_\epsilon(n)$. Then $\hat{f} = \hat{K}_{1/\epsilon} * (\hat{\mu} \cdot \hat{K}_\epsilon) = K_\epsilon \cdot (\mu * K_{1/\epsilon})$ and $\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} K_\epsilon(\xi)K_{1/\epsilon}(\xi - n)$. Let's notice that in this case, for small ϵ, both f and $\hat{f}$ at each point x can be approximated very closely by the value of $K_{1/\epsilon}(x-n)K_\epsilon(n)$ where n is the closest integer to x. If we approximate the value of $K_{1/\epsilon}(x-n)$ as $\epsilon^{1/2}$ in the interval $(n - \frac{\epsilon}{2}, n + \frac{\epsilon}{2})$ we can see that $\int_{-\infty}^{\infty} |f(x)|^2 dx \approx \sum_{j=-\infty}^{\infty} \int_{j-\frac{\epsilon}{2}}^{j+\frac{\epsilon}{2}} \epsilon K_\epsilon(j)^2 dx = \sum_{j=-\infty}^{\infty} \epsilon^2 K_\epsilon(j)^2 \approx \epsilon^2 \int_{-\infty}^{\infty} K_\epsilon(x)^2 dx$. Then if we L^2-normalize f, $E(f)$ will become approximately $\bigcup_{j=-t}^{t}(j - \frac{\epsilon}{2}, j + \frac{\epsilon}{2})$ such that $\epsilon^2 \int_{-t}^{t} K_\epsilon(x)^2 dx = \frac{99}{100}\epsilon^2 \int_{-\infty}^{\infty} K_\epsilon(x)^2 dx$. Then $2t \approx |E(K_\epsilon)|$*

and $|E(f)| \approx 2t\epsilon \approx \epsilon|E(K_\epsilon)| = |E(K_1)|$. *Analogous argument can be made for the* $\hat{f}$ *and* $\Sigma(f)$, *therefore we get that* $|E(f)||\Sigma(f)| \approx |E(K_1)||\Sigma(K_1)|$.

7 Discussion and Future Work

In this project, we have investigated the value of $|E(f)||\Sigma(f)|$ for the functions that could potentially extremize the robust qualitative uncertainty principle. By our approximations, we have discovered that the product is approximately the same for the functions that we chose, but we cannot make any rigorous conclusions from this because the approximations are very rough. The investigation of these examples did not give any definite results concerning for which functions f the value of $|E(f)||\Sigma(f)|$ is small or big, or what this value would be for a random function, but it suggests that probably the value of $|E(f)||\Sigma(f)|$ for the case when f is concentrated near the points of delta functions of a Fourier quasicrystal is not too big. This could sign that these "quasicrystal-like" functions could extremize the uncertainty principle in some cases, but more evidence is needed.

In the future, we plan to research the topic further. A good option would be to find or make computer software that would compute the value of the product $|E(f)||\Sigma(f)|$ for a given function f. This would give us some valuable insight into the behaviour of the inequality for different functions. With this addition, it would be possible to make some conjectures about the structure of the functions that are close to extremizing the uncertainty principle.

Another good option for our future work would be to come up with a way to approximate more precisely these functions from the paper. This would make it easier to make solid conclusions and form conjectures about the nature of the investigated product $|E(f)||\Sigma(f)|$.

The question of what functions extremize the uncertainty principle is closely related to our research, and we would want to take it into our focus in the future research. As all of the functions that extremize the discrete-time analogue of our uncertainty principle have been proven to be the Dirac comb and sequences that can be reduced to it by scalar multiplication and cyclic permutation in the time domain and in the frequency domain [3], it may be possible to classify all functions that extremize the uncertainty principle in our case too. Even though this problem sounds very difficult, the examples that we looked at could be useful in analyzing it to form conjectures and proving them. It would also be very interesting to look at the generalization of the problem for functions on higher-dimensional spaces.

In this project, we have mostly focused on functions that are sums of Gaussian functions, and for the future research a viable option would be to try changing the model functions. This may shed light on the problem from a different angle and make the computations and approximations of the value of $|E(f)||\Sigma(f)|$ easier.

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