Homological Equivalence and Forman Equivalence of Discrete Morse Functions on Graphs

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Abstract

Discrete Morse theory is a relatively recent technique which has proven to be a useful tool in diverse areas such as topology, computer science, and data analysis/denoising. Due to its discrete formulation, it is possible to study the set of all discrete Morse functions on a simplicial complex up to different notions of equivalence. The notion of homological equivalence was introduced by Ayala et. al, who obtained a count for the number of homological equivalence classes of discrete Morse functions on a given finite graph. The relationship between homological and Forman equivalence for discrete Morse functions on trees was described recently. In our paper, we generalize their study of the relationship between these notions of equivalence to all graphs. To do so, we develop the concepts of critical graphs and critical matrices, which describe the CW decomposition of a graph given by a specific discrete Morse function. We use these results to obtain a new proof of Ayala et al.'s characterization of possible homological equivalence classes of Morse functions on a given graph.

Summary

We consider a problem in graph theory in which we take a collection of points with lines connecting them and assign a number to each part of the graph to represent its elevation. We assign these elevations such that each vertex is beneath all but at most one of its edges and each edge is above all but at most one of its vertices. The importance of these assignments is that it allows us to figure out topological properties of the graph by only looking at the edges and vertices without exceptions, i.e. vertices beneath all their edges and edges above both of their vertices. This property makes these functions very useful to consider. In this paper, we discuss different ways to categorize these functions, and when we can make different types of these functions within each category. We specifically look at interplay between two specific types of categories, called homological equivalence, which involves looking at how the graph is built up as you look at pieces increasing in elevation, and Forman equivalence, which looks at which pieces of the graph are aforementioned special ones.

1 Introduction

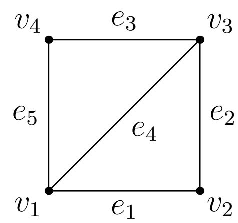
In 1998, Forman developed discrete Morse theory [1] as a discrete analogue to the already well established smooth theory, originally introduced by Morse [2] and further developed by Milnor [3]. A discrete Morse function assigns a discrete value to simplices in a simplicial complex in such a way that lower dimensional simplices are usually assigned lower values, and higher dimensional simplices are usually assigned higher values. These restrictions on the function allow us to "read off" the topology of the complex from its function values. Discrete Morse theory has a diverse set of applications, from topological data analysis to graph theory and computer science [4, 5].

In this paper we study differing natural notions of equivalence for discrete Morse functions on graphs. Our results build off Rand and Scoville [6], who showed that on trees, a fixed gradient vector field (Forman equivalence class) can be used to build a Morse function with any given homological sequence. An obvious follow-up question to their work is whether their results hold in a more general setting of finite graphs; this is the starting point for our paper.

The paper is structured as follows. In Section 2, we review relevant definitions from the literature. In Section 3, we develop the notion of the critical graph, and explain its relevance to studying homological sequences of Morse functions on graphs. In Section 4, we discuss the adjacency matrix of the critical graph, as well as the restrictions it imposes on the homological sequence with a fixed critical edge set. Theorem 4.2 allows us to generate pairs of homological sequences and Forman equivalence classes on a graph which cannot both be realized by a single discrete Morse function, see Example 7.1. Given a fixed Forman equivalence class, Theorem 4.3 gives a sufficient condition for a homological sequence to be realized in that class. In Section 5, we use our results from the previous sections to give a new proof of Ayala et. al.'s description of the possible homological equivalence classes of Morse functions on a finite graph in [7].

2 Preliminaries

A graph $G = (V_G, E_G)$ consists of a vertex set V_G, and an edge set E_G containing pairs of distinct vertices. We will denote an edge e connecting vertices u and v by (u, v) or (v, u). The vertices u and v in an edge $e = (u, v)$ are called endpoints of edge e.



$$V_G = \{v_1, v_2, v_3, v_4\}$$

$$E_G = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_1, v_3\}, e_5 = \{v_1, v_4\}\}$$

Figure 1: The diamond graph and its vertex and edge sets

We define a **path** $v_0 v_1 \cdots v_n$ on a graph $G = (V_G, E_G)$ to be a list of vertices with the property that $(v_{i-1}, v_i) \in E_G$ for all $i \in \{1 \ldots n\}$. A path has length n if it has $n+1$ vertices. The **distance** between two vertices u and v, denoted $\mathrm{dist}(u, v)$, is the path with the least length going from u to v.

To understand the topology of a graph, we can look at a discrete Morse function on it.

Definition 2.1. A **discrete Morse function** on a graph G is a function $f : G \to \mathbb{R}$ such that for each vertex $v \in V_G$, there is at most one edge $e \in E_G$ such that $f(v) \geq f(e)$. Conversely, for each edge $e \in E_G$, there exists at most one vertex $v \in V_G$ such that $f(v) \geq f(e)$.

Given a discrete Morse function f on a graph G, the **critical vertices** $C_v \subseteq V_G$ are the vertices $v \in V_G$ such that for each $e \in E_G$ such that v is an endpoint of e, $f(e) > f(v)$. The **critical edges** $C_e \subseteq E_G$ are the edges $(u, v) \in E_G$ such that $f(e) > f(v)$ and $f(e) > f(u)$.

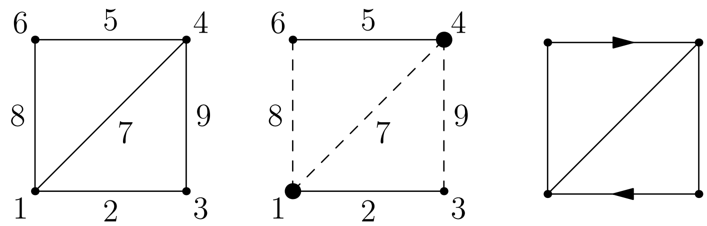


Figure 2: A discrete Morse function on the diamond graph with its critical simplices and induced gradient vector field

We define the critical values to be the set $\{f(v)|v \in C_v\} \cup \{f(e)|e \in C_e\}$. We define the **critical sequence** c_i to be the i-th least critical value. For example, in Figure 2, the critical sequence is $1, 4, 7, 8, 9$. A discrete Morse function is **excellent** when all of the critical values are distinct. From now on, all discrete Morse functions are assumed to be excellent.

A subgraph $\hat{G}$ of a graph G is defined as a graph $\hat{G} = (V_{\hat{G}}, E_{\hat{G}})$ such that $V_{\hat{G}} \subseteq V_G$, $E_{\hat{G}} \subseteq E_G$, and for each $(u,v) \in E_{\hat{G}}$, $u, v \in V_{\hat{G}}$. We write $G\ \hat{G} \subseteq G$ to mean that $\hat{G}$ is a subgraph of G.

One useful property of discrete Morse functions comes from the way they build up a graph. This works by looking at the pieces of a graph ("sublevel sets") beneath a certain value, and incrementing this value.

Definition 2.2. An **n-th level subgraph** G_n of G is the subgraph G_n with edge set

$$E_{G_n} = \{e \in E_G | f(e) \leq n\},$$

and vertex set

$$V_{G_n} = \{v \in V_G | e = (u,v) \in E_{G_n}\}$$

The level subgraphs of the diamond graph with f from Figure 2 are shown in Figure 3.

The **0-th Betti number** of a graph G, denoted $b_0(G)$, is the number of connected components of the graph. The **1-st Betti number** of a graph G, denoted $b_1(G)$, is $|E_G| - |V_G| + 1$; heuristically, it counts the number of "loops" in G. [8].

Definition 2.3. The **homological sequences** of a discrete Morse function f on graph G

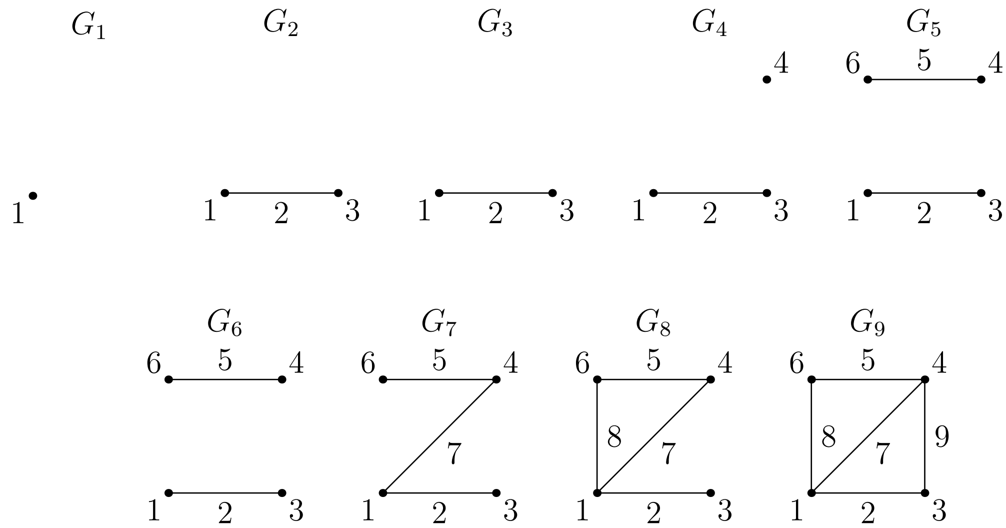


Figure 3: Level subgraphs of the diamond graph

are sequences $B_0(i)$ and $B_1(i)$ such that $B_0(i) = b_0(G_{c_i})$ and $B_1(i) = b_1(G_{c_i})$.

Example 2.1. *The homological sequences of the graph from Figure 2 are*

$$B_0: \quad 1, \quad 2, \quad 1, \quad 1, \quad 1, \quad 1$$

$$B_1: \quad 0, \quad 0, \quad 0, \quad 1, \quad 2, \quad 3.$$

Ayala *et al.*[9] proved a set of restrictions of which homological sequences are possible for discrete Morse functions on a graph.

Theorem 2.1 (Ayala et al. [9])**.** *The homological sequences of a connected graph under a function with m critical simplices follow the restrictions:*

$$B_0(0) = B_0(m-1) = b_0 = 1 \tag{1}$$

$$B_0(i) > 0 \tag{2}$$

$$|B_0(i+1) - B_0(i)| = 0 \text{ or } 1 \tag{3}$$

$$B_1(0) = 0 \tag{4}$$

$$B_1(m-1) = b_1 \tag{5}$$

$$B_1(i+1) - B_1(i) = 0 \text{ or } 1. \tag{6}$$

Homological sequences induce a nice notion of equivalence for discrete Morse functions.

Definition 2.4. We say two discrete Morse functions on a graph G are **homologically equivalent** when they have the same homological sequence.

This is one of several distinct notions of equivalence for Morse functions on graphs. The following alternate notion of equivalence is described by Rand and Scoville[6].

Definition 2.5. We say two discrete Morse functions on a graph G are **Forman equivalent** when they have the same critical simplices.

Remark 2.1. As mentioned in Corollary 2.13 by Rand and Scoville [6], two discrete Morse functions are equivalent if they induce the same gradient vector fields.

The following theorem is a corollary of Algorithm 2 by Rand and Scoville[6] and explains the relationships between Forman equivalence and homological equivalence on trees.

Theorem 2.2 (Rand and Scoville [6])**.** *Given two discrete Morse functions f and g on tree G with the same number of critical vertices and edges, one can create a new discrete Morse function h on G such that h that is Forman equivalent to f and homologically equivalent to g.*

3 Critical Graphs

Given a discrete Morse function f on a graph G, we can construct a new graph $C(G, C_e)$ by contracting all non-critical edges. This new graph, which we call the **critical graph**, encodes important Morse-theoretic information about the function f.

Pick C_v to be the set of all critical vertices of G under f, and pick C_e to be the set of all critical edges of G under f.

The **rooted tree** at critical vertex $v \in C_v$ is the graph $T_v \subseteq G$ whose vertex set is all the simplices which are contained in a path of non-critical edges to the root v.

Remark 3.1. The weak Morse inequalities, proven in [10], show that T_v is a tree.

The next theorem explains what happens after the edge contractions.

Theorem 3.1 (Ayala et al [9]). *Each vertex of a discrete Morse function $f : G \to \mathbb{R}$ is contained in a unique rooted tree.*

Armed with the notion of a rooted tree, we now introduce a new definition.

Definition 3.1. The induced **critical graph** of a graph G with discrete Morse function f and critical edges C_e, denoted

$$C(G, C_e),$$

is formed by contracting all of the non-critical edges of the graph.

These contractions result in every rooted tree being contracted to a point, with a vertex in $C(G, C_e)$ for each critical vertex.

The following lemma is useful soon in setting up the relationship between a critical vertex and the tree rooted at it.

Lemma 3.2. *A path $uv_1 \ldots v_{n-1}v$ between any two vertices u and v contains a critical edge if $f(e_1) > f(u)$ and $f(e_n) > f(v)$, where $e_1 = (u, v_1)$ and $e_n = (v_n, v)$.*

Proof. We induct on path length. For the base case, we assume we have a path of length 1. This means that $f(u) < f(e_1) = f(e_2) > f(v)$, so e_1 is critical.

Now assume any path of length n with $f(u) < f(e_1)$ and $f(v) < f(e_n)$ has a critical edge. Let $uv_1 \ldots v_n v$ be a path with $f(u) < f(e_1)$ and $f(v) < f(e_{n+1})$. There are two cases, either e_1 is critical or not. If it is critical, then obviously the path has a critical edge. Otherwise, if the edge is non-critical, then

$$f(u) < f(e_1) \leq f(v_1).$$

Because $f(v_1) \geq f(e_1)$, for any edge $e \in E_G$ containing v_1 with $e \neq e_1$, $f(v_1) < f(e)$, because f is a discrete Morse function. Thus, $f(v_1) < f(e_2)$. This means we have a path of length

n with $f(v_1) < f(e_2)$ and $f(v) < f(e_{n+1})$. By induction, $uv_1 \dots v_n v$ must have a critical edge. □

Corollary 1. *Any path $uv_1 \dots v_{n-1}v$ containing only non-critical edges must have either $f(e_1) \leq f(u)$ or $f(e_2) \leq f(v)$, where $e_1 = (u, v_1)$ and $e_n = (v_n, v)$.*

The next lemma explains the relationship between the rooted tree and its root.

Lemma 3.3. *Given a discrete Morse function $f : G \to \mathbb{R}$, the critical vertex in any the rooted tree is the global minimum on that tree.*

Proof of this theorem is in in the appendixProof 7

The following definition is useful when talking about the importance of the critical graph.

Definition 3.2. A **lazy** discrete Morse function f on a graph G is a Morse function such that every simplex is critical.

The following theorem displays the main importance of the critical graph: it keeps the homological sequence of the Morse function f, and generally does not lose any information about the topology of G under f.

Theorem 3.4. *For any graph G, equipped with discrete Morse function f that has Forman equivalence class (C_e, C_v) and a homological sequence (B_0, B_1), there exists a lazy discrete Morse function f on $C(G, C_e)$ with the same homological sequence.*

Proof. Let f be a discrete Morse function with critical simplices (C_e, C_v) and homological sequence (B_0, B_1). Note that there is a bijection between critical vertices and vertices in the critical graph. Similarly, there is a bijection between critical edges in the original graph and all edges in the critical graph. Call this bijection $\mathcal{A} : C(G, C_e) \to G$. Thus, for any f, we can simply define a lazy morse function such that $f(s) = f(\mathcal{A}(s))$.

We claim that this function $f : C(G, C_e) \to \mathbb{R}$ is an excellent lazy discrete Morse function. Note that our function is injective since the original Morse function was injective when restricted to critical simplices.

We now show that the function f on the critical graph is Morse. Let $e = (u, v) \in C_e$. This means $f(v), f(u) < f(e)$. If v and u are rooted in $v_i \in C_v$ and $v_j \in C_v$ respectively (not necessarily with $i \neq j$), then from Lemma 3.3, $f(v) \geq f(v_i)$ and $f(u) \geq f(v_j)$. Thus, for each edge $e = (v_i, v_j)$ in the critical graph, $f(v_i), f(v_j) < f(e)$.

Note that f has the same homological sequence as f' because the critical simplices have the same values as those in the original graph and are added to level subgraph in the same order. At each step, G_i is homotopy equivalent to $C(G, C_e)_i$, as we know from the main theorems of discrete Morse theory, by Forman [10]. Thus, the homological sequences are the same. $\square$

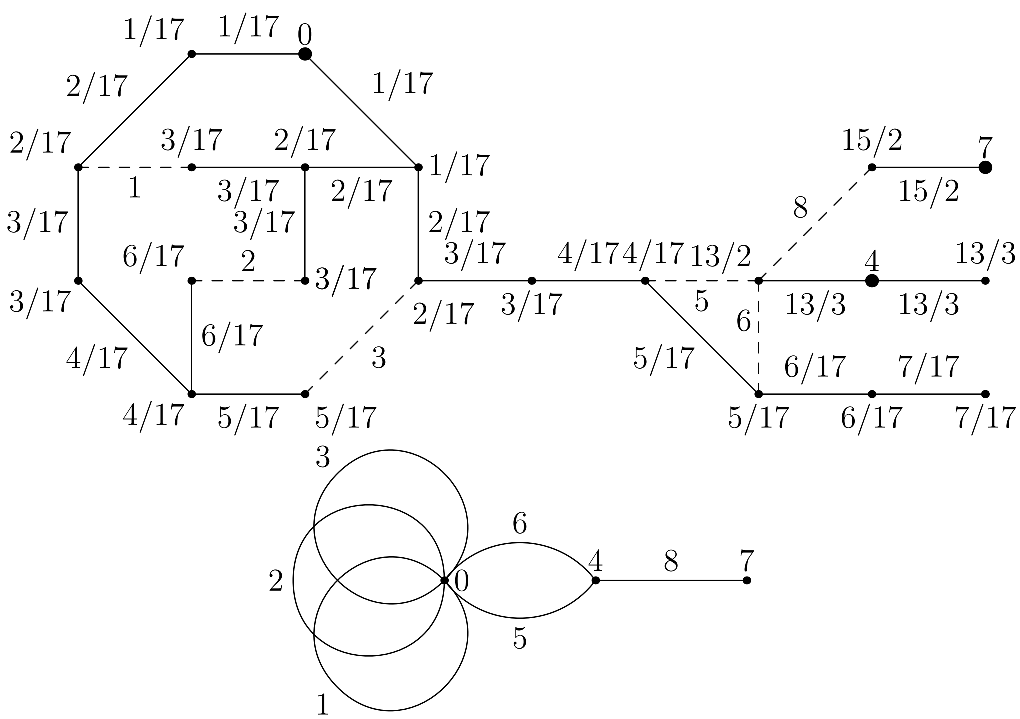


Figure 4: Critical graph

Example 3.1. *Figure 4 displays a discrete Morse function on a graph, along with its corresponding critical graph with corresponding excellent lazy Morse function. The bold vertices and dashed edges are critical. We see that these two functions have the exact same homological sequence.*

4 Critical Matrices and the Relationship Between Homological and Forman Equivalence

This section introduces the notion of the critical matrix, which gives limits on the admissible homological sequences of a graph within a given Forman equivalence class, generalizing Scoville and Rand [6].

Definition 4.1. Let f be a discrete Morse function with critical vertex set C_e. Let I be the set of orderings of C_e. The **critical matrix set** $\mathcal{C}$ of f is the set of adjacency matrices

$$\{M(G_i)\}_{i\in I},$$

where G_i are copies of the critical graph with the vertex ordering $i \in I$.

For a given homological sequence (B_0, B_1), define sequences

$$g_v(i) = |\{j \in \mathbb{N} | j \leq i, B_0(j) - B_0(j-1) = 1\}| + 1;$$

$$g_e(i) = |\{j \in \mathbb{N} | j \leq i, B_0(j) - B_0(j-1) = -1 \text{ or } B_1(j) - B_1(j-1) = 1\}|.$$

Fix an excellent Morse function f and let $\{\sigma_i\}$ be the set of critical simplices ordered so that $f(\sigma_i) < f(\sigma_{i+1})$ for all i. The function f gives homological sequences B_0, B_1 which in turn give sequences g_v, g_e.

The next lemma illustrates the relationship between g_v and the critical vertices in a level subcomplex of a graph.

Lemma 4.1. *Let f be a Morse function on a graph inducing gradient vector field (C_v, C_e)*

and homological sequence B_0, B_1 giving rise to sequence $g_v(i)$. For any i,

$$g_v(i) = |\{\sigma_j | j \leq i, \sigma_j \in C_v\}|.$$

Proof. We can prove this by induction. If $B_0(i) - B_0(i+1) = 1$, then the unique critical simplex added between steps i and $i+1$ must be a vertex, because adding a critical edge never increases B_0. $\square$

This gives the following relationship between $g_e(i)$ and the critical simplices.

Corollary 2. *Fix $i \in \mathbb{N}$. Then*

$$g_e(i) = |\{\sigma_j | j \leq i, \sigma_j \in C_e\}|.$$

The next theorem describes a necessary condition for a fixed homological sequence (B_0, B_1) to correspond to a Morse function whose gradient vector field is (C_v, C_e). Example 7.1 demonstrates how this necessary condition can fail and thus not allow a homological sequence.

Theorem 4.2. *Let G be a graph with a given gradient vector field (C_v, C_e). Then G admits an excellent discrete Morse function with gradient vector field (C_v, C_e), giving rise to homological sequence (B_0, B_1), giving rise to sequences $g_v(i)$ and $g_e(i)$ only if there exists some $M(G_i)$ such that the sum of the upper triangular portion of the upper left square $g_v(i) \times g_v(i)$ matrix must be at least $g_e(i)$ for all $1 \leq i \leq |\{c_i\}|$.*

Proof. Let f be a discrete Morse function on G with gradient vector field (C_v, C_e). Order the critical vertex set C_v so that $f(v_i) < f(v_{i+1})$, and define e_i similarly. Now we will consider an $M(G_i)$, the critical matrix corresponding to this ordering of critical vertices.

Fix $i \in \mathbb{N}$. The upper $g_v(i) \times g_v(i)$ left square of the matrix is the adjacency matrix of the maximal subgraph G' of the critical graph of G with the vertices corresponding to v_1 through $v_{g_v(i)}$. Say some edge e_j with $j \leq g_e(t)$ has vertices $u \in T_{v_a}$ and $v \in T_{v_b}$. Lemma 3.3 gives us $f(u) \geq f(v_a)$, and $f(v) \geq f(v_b)$, so it must be the case that $f(v_a), f(v_b) < f(e_j)$, so $a, b \leq g_v(i)$. Thus, every edge in G' contributes a 1 to the upper left square of $M(G_i)$. $\square$

The following definition is useful in characterizing which homological sequences are attainable by the algorithm in Theorem 4.3.

Definition 4.2. The **flatline number** of a homological sequence (B_0, B_1) is the index m_0 such that $b_0(i) = 1$ for all $i \geq m_0$.

The next theorem presents a partial converse to Theorem 4.2: it provides a sufficient condition for a gradient Morse function (C_v, C_e) to have a given homological sequence. Note that these conditions are not strictly necessary. However, the sufficient conditions provided here will later be crucial to our results in Section 5, where we present a new proof of Ayala's description of all classes of homological sequences that may realized by discrete Morse functions on a finite graph. Example 7.2 demonstrates the algorithm used in its proof.

Theorem 4.3. *Let G be a connected graph with a given gradient vector field (C_v, C_e). Then G admits a given homological sequence (B_0, B_1) following all of Ayala's restrictions from Theorem 2.1 giving rise to sequence $g_v(i)$ if there exists some $M(G_i)$ such that*

1. *The number of critical vertices is $g_v(m_0)$ and the number of critical edges is $\max_{i \in \mathbb{N}} g_v(i)$.*
2. *The sum of the first $g_v(i)$ diagonal entries is least $B_1(i)$ for all $i \leq m_0$.*
3. *If $g_v(i) < |C_v|$, then $B_0(i)$ is bounded below by the number of columns in the upper left $g_v(i) \times g_v(i)$ matrix that vanish above the diagonal.*

Proof. The intuition behind this theorem is that our gradient vector field (C_v, C_e) effectively the process of adding loops (increasing B_1) from the process of adding and subtracting components, so that we may build a Morse function with the desired homological sequence.

Pick v_i to be the ordering of the critical vertices of the graph G_i. For every k such that the kth column of $M(G_i)$ has non-zero upper triangular portion, we can pick one edge called e_i for each $i > 1$ such that $e_i = (v_i, v_j)$ where $j < i$.

Define the ordered set $\{l_i\}$ to be those $l_i = (u, v) \in C_e$ such that there exists some critical vertex v_k such that $u, v \in V_{T_{v_k}}$, and if $i > j$ and l_i and l_j connect vertices in T_{v_a} and T_{v_b} respectively, then $a \geq b$.

Now we build our Morse function f with gradient vector field (C_v, C_e) and homological sequence (B_0, B_1) in stages. At the ith stage, we define the ith critical simplex and possibly other noncritical simplices.

Before $g_v(i) = |C_v|$, we do the following algorithm at the ith step.

- $B_0(i) - B_0(i-1) = 1$: Define $f(v) = i$, where the vertex v corresponds to the $g_v(i)$th row and column of $M(G_i)$. For each $a \in V_{T_v}$, define $f(a) = i + \frac{\text{dist}(a,v)}{|V_{T_v}|}$. For each $e = (a, b) \in E_{T_v}$, define $f(e) = \frac{f(a)+f(b)}{2}$. This is essentially the same construction from Rand and Scoville in [6].

- $B_0(i) - B_0(i-1) = -1$: Pick j to be the least possible number such that e_j exists but $f(e_j)$ has not yet been defined. Define $f(e_j) = i$.

- $B_1(i) - B_1(i-1) = 1$: Define $f(l_j) = i$, where $f(l_{j-1})$ has been defined and $f(l_j)$ has not, or $j = 1$ and $f(l_1)$ had not been defined.

We claim the algorithm above defines a discrete Morse function on some subgraph of G containing all of the vertices of G. We will just check that we may always carry out the ith step in the algorithm; that the function defined by the algorithm is an excellent discrete Morse function is routine verification.

- When $B_0(i) - B_0(i-1) = -1$, such an e_i will exist as long as $B_0(i)$ is at least the number of columns with a 0 above diagonal column sum, because there are exactly $g_v(i)$ minus that number e_j with $j < g_v(i)$. This is exactly the restriction posed on the homological sequence, so we can always go through with the algorithm when $B_0(i) - B_0(i-1) = -1$.

- When $B_0(i) - B_0(i-1) = 1$, we can simply always add the next vertex.

- When $B_1(i) - B_1(i-1) = 1$, we can definitely add at least one new loop as long as not all l_i have not been added yet. This is exactly the restriction here as well, because it must be the case that the number of loops (and hence $B_1(i)$ until $g_v(i) = |V_c|$, evident by induction) is always at most the sum of the diagonal up until $g_v(i)$.

Now we proceed to assign values to the remaining edges in our graph. The only edges that remain unassigned are critical, since for every critical vertex we defined the function on its entire rooted tree. Pick j_0 to be the least i with $g_v(i) = |C_v|$. Note that at this point, we have a Morse function defined on a subgraph of G containing all of the vertices and some of the edges. This subgraph consists of $B_0(j_0)$ disconnected components. Since (C_v, C_e) describes a valid Morse function on the graph G, we can find $B_0(j_0) - 1$ in $C_e \setminus \{l_i\}$, such that the addition of these edges will create a connected graph. For every i where $B_0(i)$ decreases, we will add one of these $B_0(j_0) - 1$ edges. Every time $B_1(i)$ increases until the flatline point, we can assign value to a critical edge in the set $\{l_i\}$ by condition (2). Past the flatline point, we increase $B_1(i)$ by adding the remaining critical edges in any order. □

5 Characterizing All Homological Sequences Arising from Discrete Morse Functions on Graphs

The results in the previous section yield a new proof of Ayala's classification in [7] of the possible homological equivalence classes which may be realized on a discrete Morse function on a connected finite graph.

Throughout this section, we adopt the conventions of [7]: if G is a graph, we call G' a subdivision of G if G' is obtained from G by subdividing vertices. We say that G admits a homological sequence (B_0, B_1) if there is some subdivision G' of G and Morse function f on G' so that the homological sequence of f is (B_0, B_1).

In a graph G, a **cut edge** is an edge $e \in E_G$ such that $b_0(G) = 1$, but $b_0(G - e) = 2$.

The following fact was remarked in [7]; however, we present a proof for completeness.

Lemma 5.1. *If f is a discrete Morse function with m critical simplices and homological sequence (B_0, B_1) on a graph G with no cut edges, then*

$$B_1(m) - B_1(m-1) = 1.$$

Proof. Note that there is a unique global maximum critical edge e. Thus, $G - e = G_{f(e)-\varepsilon}$. Assume for the sake of contradiction that $B_0(m) - B_0(m-1) = -1$. This would mean that $B_0(m-1) = 2$. Thus, $b_0(G - e) = 2$. However, e cannot be a cut edge, so we have a contradiction. □

We now proceed to use using critical matrices to show Ayala's results on the admissible homological sequences on any graph G with no cut edges. Example 7.4 displays how to create the given discrete Morse function.

Theorem 5.2 (Theorem 4.3.2 in [7])**.** *On any graph G, we can construct a discrete Morse function f with any homological sequence (B_0, B_1) with m critical simplices, provided the sequence satisfies Ayala's restrictions in Theorem 2.1 as well as the restriction that $B_1(m) - B_1(m-1) = 1$.*

Proof. Pick a spanning tree of the graph T_v. Pick one edge $e = (u, v) \notin E_{T_v}$ and subdivide it into edges $e_1, \ldots, e_k$ and vertices $v, v_1, \ldots, v_k$ such that $e_i = (v_{i-1}, v_i)$, except $e_1 = (v, v_1)$.

Pick the critical vertices to be $C_v = \{v, v_1, \ldots, v_{k-1}\}$, and pick critical edges to be $C_e = \{e_1, \ldots e_k\} \cup E_G - E_{T_v}$, where $E_G - E_{T_v} - e$ are the edges not in the spanning tree that are not e. Note that each edge in $E_G - E_{T_v}$ connects a vertex in the spanning tree to another vertex in the spanning tree.

Note that T_{v_i} is just the vertex v_i, as v_i is only an endpoint of e_{i+1} and e_i, both of which are critical.

Thus, the resulting critical matrix is

$$M = \begin{cases} b_1(G) - 1 & i = j = 1 \\ 1 & |i-j| = 1 \text{ or } |i-j| = n-1 \\ 0 & \text{else.} \end{cases}$$

From Theorem 4.3 we can construct any Morse function satisfying $B_1(i) \leq b_1(G) - 1$, which is equivalent to the restriction that $B_0(m) - B_0(m-1) = 1$. □

Now we consider the possible homological sequences on graphs with a cut edge.

Definition 5.1. The **optimal cut edge** of a graph G is the cut edge e which separates G into two connected components R and L such that $|b_1(R) - b_1(L)|$ is maximized. The tuple $(\max(b_1(R), b_1(L)), \min(b_1(R), b_1(L)),)$ is defined as the **optimal split**.

The following lemma is can be found in Theorem 4.3 in [7].

Lemma 5.3 (Ayala et al. [7]). *If a graph G has an optimal split of (a, b), then all homological sequences corresponding to Morse functions with m critical simplices have either $B_1(i) > a \implies B_0(i) \geq 2$ until $i = m$, or $B_1(m) - B_1(m-1) = 1$.*

Now we show how to use critical matrices to prove Ayala et al's results for graphs with a cut edge. Example 7.5 demonstrates this construction.

Theorem 5.4. *If a graph G has an optimal split of $(b_1(R), b_1(L))$, then any homological sequence with the property that either $B_1(i) > a \implies B_0(i) \geq 2$ until the final step, or $B_1(i) - B_1(i-1) = 1$ at the final step is admissible.*

Proof. Consider the case where $B_1(i) > b_1(R) \implies B_0(i) \geq 2$ until the last step.

Define j_1 such that $B_1(j_1) = b_1(R) + 1$ and $B_1(j_1 - 1) = b_1(R)$ and define $j_0 < j_1$ such that $B_0(j_0) - B_0(j_0 - 1) = 1$ and $B_0(i) - B_0(i-1) < 1$ for all $i \in (j_0, j_1)$.

Subdivide the optimal cut edge $e = (u, v)$ into $k := g_v(m_0) - 1$ edges and make all of the critical vertices the original endpoints and the subdivided vertices. Label the edges $\{e_i\}_{i=1}^{k}$

such that $i = \max(\mathrm{dist}(s, v_1), \mathrm{dist}(t, v_1))$ where $e_i = (s, t)$. Label the vertices $\{v_i\}_{i=1}^{k+1}$ such that $i = \mathrm{dist}(v_i, v) + 1$ when $\mathrm{dist}(v_i, v_1) < g_v(j_0) - 1$, and label them $i = \mathrm{dist}(v_i, v) + 2$ when $g_v(j_0) - 1 \leq \mathrm{dist}(v_i, v_1) < k$ and pick $f(u) = g_v(j_0)$.

We pick spanning trees T_R for R and T_L for L. Define the critical vertices to be $\{v_i\}_{i=1}^{k+1}$, and the critical edge set to be

$$(E_R \setminus E_{T_R}) \cup (E_L \setminus E_{T_L}) \cup \{e_i\}_{i=1}^{k}.$$

This gives the critical matrix

$$M(G_i) = \begin{cases} b_1(R) & i = j = 1 \\ b_1(L) & i = j = g_v(j_0) \\ 1 & |i - j| = 1 \text{ and } i, j \neq g_v(j_0) \\ 1 & i = g_v(j_0) - 1, j = g_v(j_0) + 1 \text{ or } i = g_v(j_0) + 1, j = g_v(j_0) - 1 \\ 1 & i = g_v(j_0), j = |C_v| \text{ or } i = |C_v|, j = g_v(j_0) \\ 0 & \text{else.} \end{cases}$$

This is the critical matrix because each v_i connects only to v_{i+1} and v_{i-1} unless $|i - g_v(j_0)| \leq 0$ or $i = 1$. If $i = 1$, then it connects to itself $b_1(R)$ times, because that is the number of edges not in the spanning tree, as shown in [11].

From Theorem 4.3, we can construct any homological sequence with the restriction that the last $b_1(G) - (b_1(R) + b_1(L)) = 0$ steps, as well as for all $j_0 \leq i \leq m_0$ (meaning $g_v(i) \geq j_0$) we have $B_0(i) > 1$. □

Remark 5.1. On graphs with a vertex of degree one, the optimal split is $(b_1(G), 0)$. This means that, because $B_1(i) \leq b_1(G) = b_1(R)$, any homological sequence is admissible, as it is never the case that $B_1(i) > b_1(R)$, so there is no restriction on the possible homological sequences of the graph beyond Ayala's initial restrictions in Theorem 2.1.

We see that by picking critical simplices in the specific ways mentioned, we can create critical graphs with critical matrices allowing for the homological equivalence classes found by Ayala [7].

6 Future Work

Going forward, we plan to further investigate the gap between the necessary and sufficient conditions posed in section 4. Our goal would be a complete description of when we may find a Morse function that admits a given pair (homological sequence, gradient vector field).

Also, it would be interesting to see if there is some nice way to compare the possible homological sequences admissible by multiple Forman equivalence classes, so that we could see when one Forman equivalence class can admit a homological sequence from another Forman equivalence class.

Finally, we plan on applying this notion of the critical graph and critical matrix, inspired by the main theorems of discrete Morse theory, to investigate other Morse theoretic notions of equivalence. For example, it would be interesting to use this technique to look at merge tree equivalence, developed by Johnson and Scoville [12].

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Appendix

Proof of Lemma 3.3

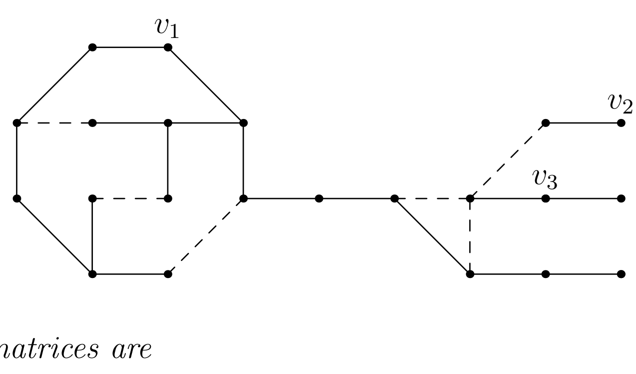
Proof. Let $v \in C_v$ be a critical vertex. We will prove that any $u \in V_{T_v}$ has $f(u) \geq f(v)$. Because $u \in V_{T_v}$, there is a path of non-critical edges from v to u. We will prove that $f(u) \geq f(v)$ by inducting on the path length. When the path length is 0, we have $u = v$, so $f(u) = f(v)$.

Assume, from the inductive hypothesis, that any vertex $s \in V_{T_v}$ with a path of n or fewer non-critical edges to v has $f(s) \geq f(v)$. Let u be a vertex with a path length of $n+1$. Pick the first edge $e_0 = (u, s)$.

Note that if the path ends with $e_n = (v, t)$, it must be the case that $f(e_n) > f(v)$, because v is critical. From Corollary 1, we have $f(e_0) < f(u)$, so $f(u) > f(s)$ because e_0 is non-critical. When $f(u) > f(s)$, we have $f(u) > f(s) \geq f(v)$. $\square$

The following is an example of a gradient vector field (C_v, C_e) and homological sequence (B_0, B_1) that do not satisfy the conditions of Theorem 4.2; thus, we may not find a Morse function with gradient vector field (C_v, C_e) and homological sequence (B_0, B_1).

Example 7.1. *Consider the Forman equivalence class (C_v, C_e) below. The dotted lines are critical edges, and the labeled vertices are critical.*



The possible critical matrices are

$$\begin{array}{c} v_1\\ v_2\\ v_3\end{array}\begin{pmatrix}3&2&0\\2&0&1\\0&1&0\end{pmatrix}\quad \begin{array}{c} v_1\\ v_3\\ v_2\end{array}\begin{pmatrix}3&0&2\\0&0&1\\2&1&0\end{pmatrix}\quad \begin{array}{c} v_2\\ v_1\\ v_3\end{array}\begin{pmatrix}0&2&1\\2&3&0\\1&0&0\end{pmatrix}\quad \begin{array}{c} v_2\\ v_3\\ v_1\end{array}\begin{pmatrix}0&1&2\\1&0&0\\2&0&3\end{pmatrix}\quad \begin{array}{c} v_3\\ v_1\\ v_2\end{array}\begin{pmatrix}0&0&1\\0&3&2\\1&2&0\end{pmatrix}\quad \begin{array}{c} v_3\\ v_2\\ v_1\end{array}\begin{pmatrix}0&1&0\\1&0&2\\0&2&3\end{pmatrix}.$$

We will show that this graph cannot admit the homological sequence

$$\begin{array}{rccccccccc} i: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ B_0: & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 2 & 1 \\ B_1: & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4. \end{array}$$

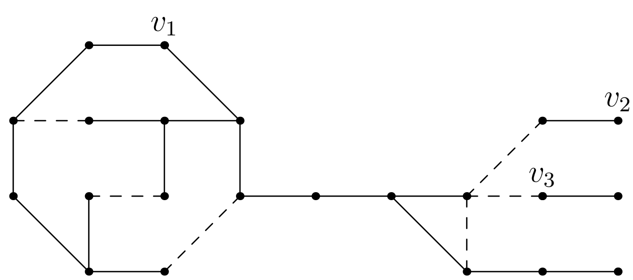
When $i = 4$, we have $g_v(4) = 1$, and $g_e(4) = 4$. This means that there would have to be some $M(G_i)$ with the upper left 1×1 square having an upper triangular sum of at least 4, meaning $M(G_i)_{1,1} = 4$. Note that all of the matrices have $M(G_i)_{1,1}$ as 3 or 0, so we cannot have this homological sequence with this Forman equivalence class.

Next is an example of a pair (C_v, C_e) and (B_0, B_1) satisfying the hypotheses of Theorem 4.3, and an application of the algorithm producing a Morse function with gradient vector field (C_v, C_e) and homological sequence (B_0, B_1).

Example 7.2. *We will demonstrate that it is possible to have the homological sequence*

$$\begin{array}{rccccccccc} i: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ B_0: & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 2 & 1 \\ B_1: & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4. \end{array}$$

on the graph with Forman equivalence class (C_v, C_e) below. The dashed edges and labeled vertices are critical.

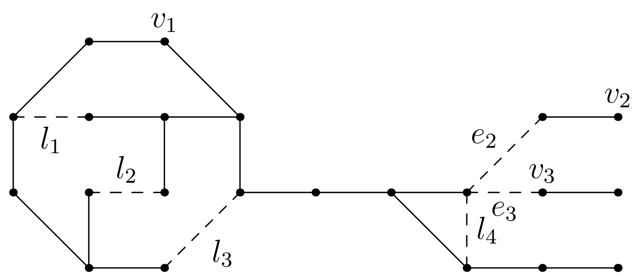


If we add the vertices in the order v_1, v_2, and v_3, we get the critical matrix

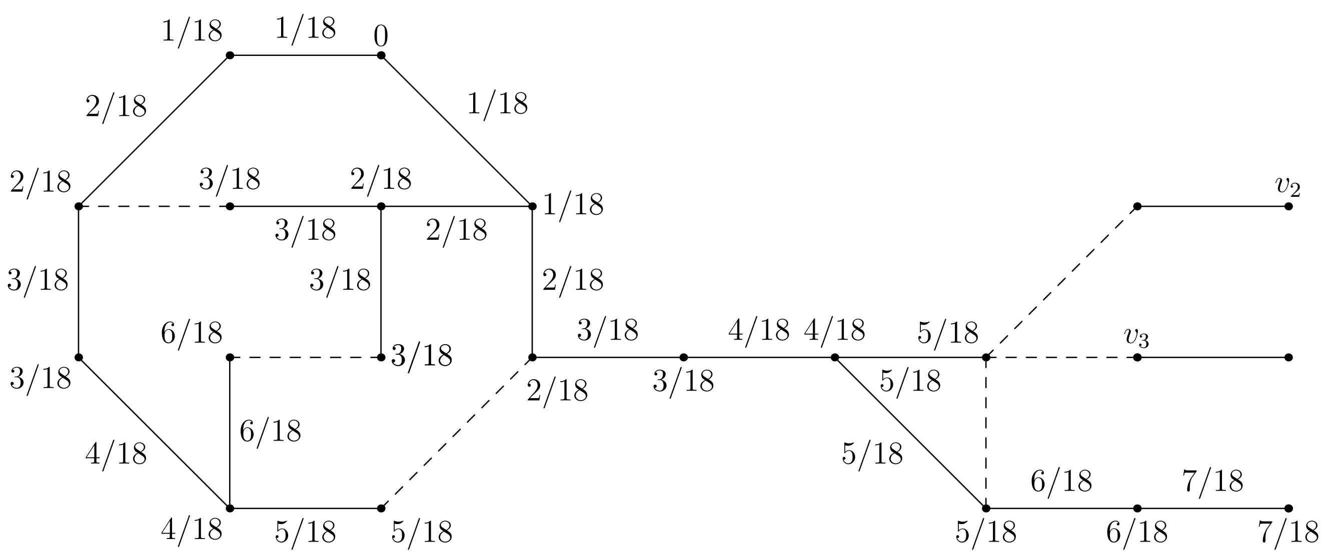
$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which should allow this homological sequence because it satisfies the conditions of Theorem 4.3.

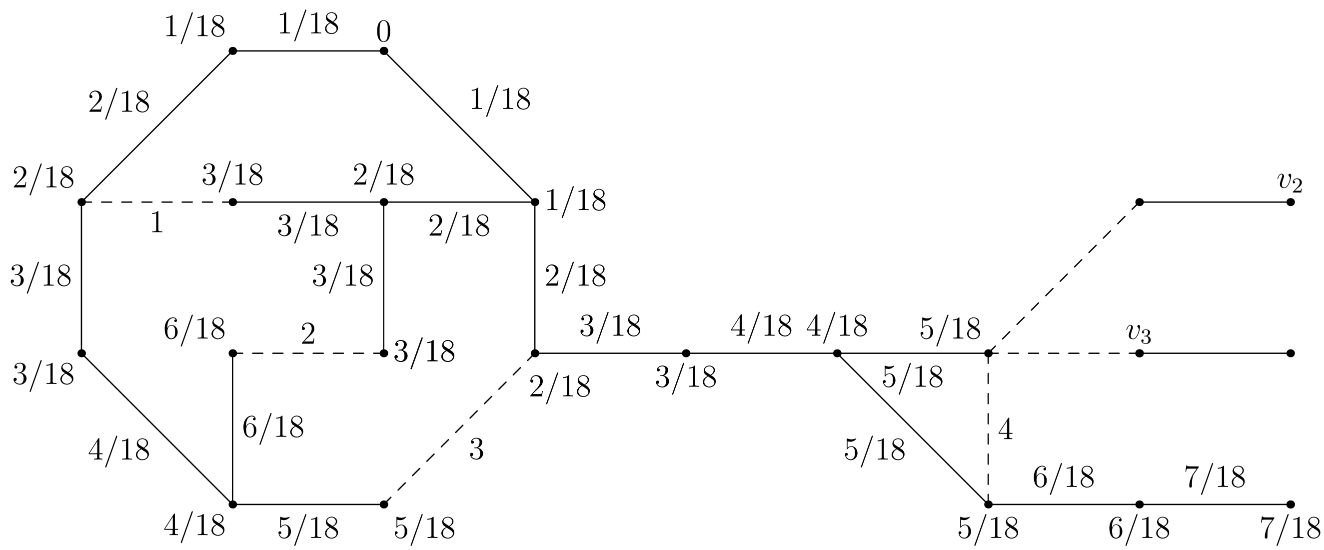
Now lets set an ordering on the edges which connect trees to itself. First we arbitrarily number the edges l_1 though l_4 connecting T_{v_1} to T_{v_1}. Note that there are neither more edges connecting T_{v_2} to T_{v_2} nor connecting T_{v_3} to T_{v_3}. We also pick e_2 to be the critical edge connecting T_{v_2} to T_{v_1}, and we pick e_3 to be the critical edge connecting T_{v_3} to T_{v_1}.



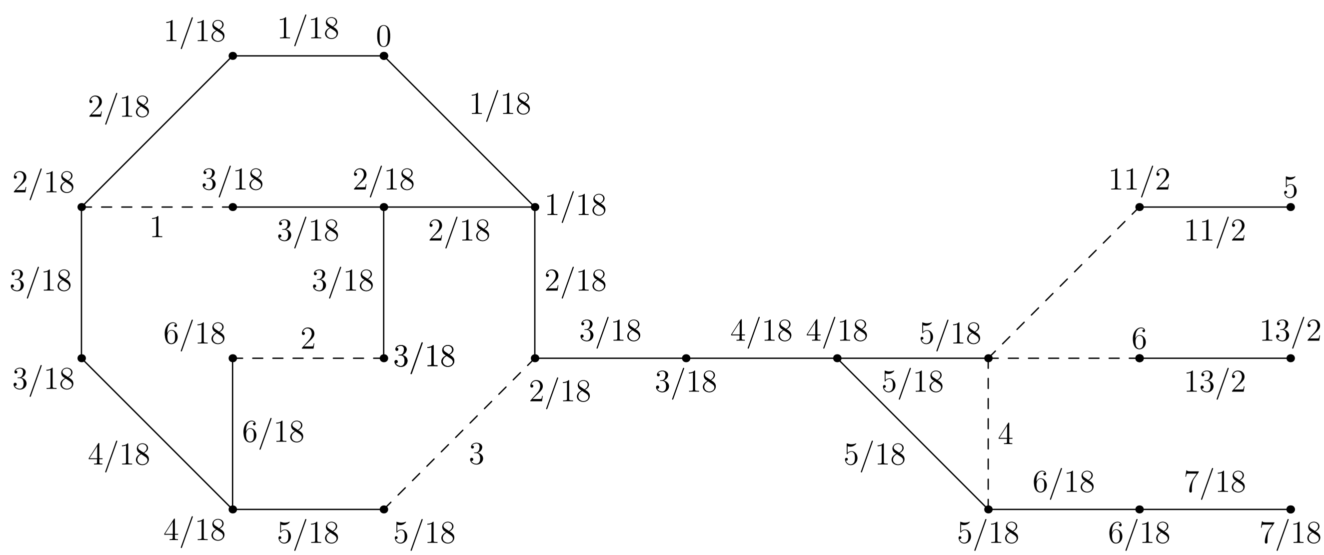
Now we go through the homological sequence and figure out what to add. First we define f on T_{v_1} with v_1 as the only critical vertex.



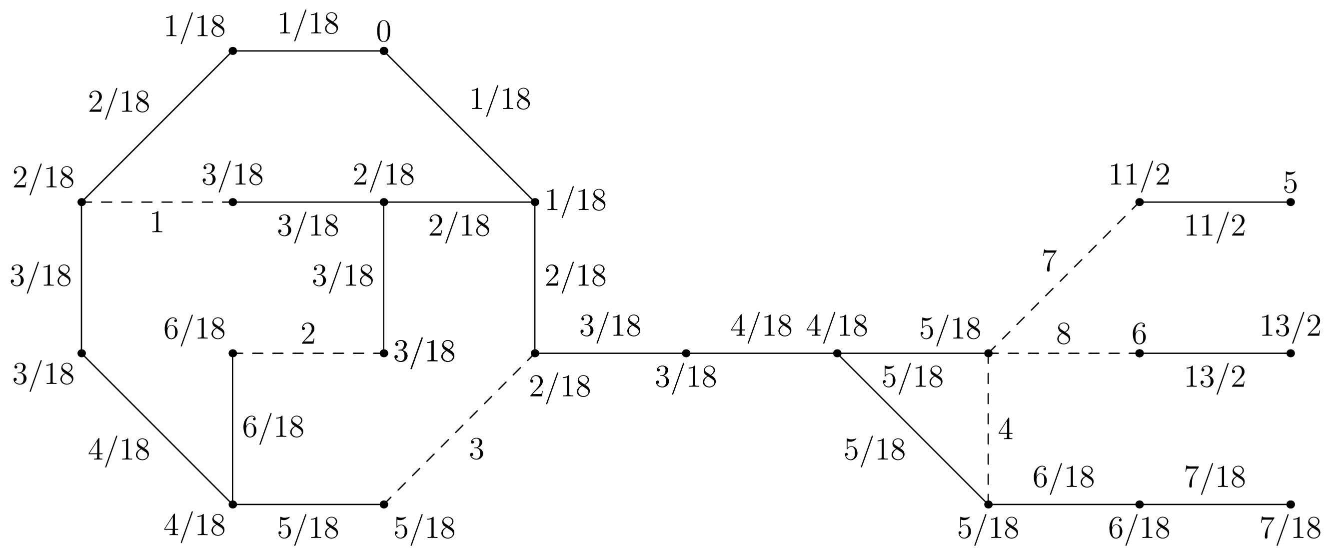
Because $B_1(i) - B_1(i-1) = 1$ for all i from 1 through 4, we define l_1 through l_4 to be 1 through 4 respectively.



Because $B_0(i) - B_0(i-1) = 1$ *for* $i = 5$ *and* 6, *we define* $f(v_2) = 5$ *and* $f(v_3) = 6$, *and define* f *on* T_{v_2} *and* T_{v_3} *as well.*



Because $B_0(i) - B_0(i-1) = -1$ *for* $i = 7$ *and* 8, *we define* $f(v_2) = 7$ *and* $f(v_3) = 8$.



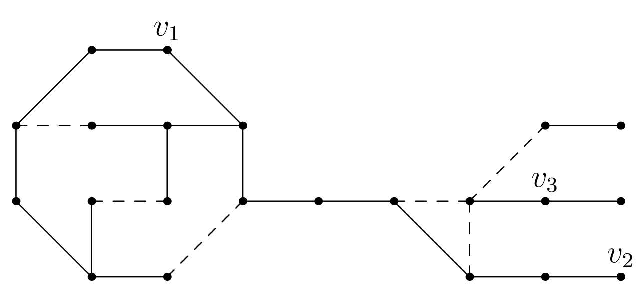
This discrete Morse function has the desired homological sequence.

Example 7.3. *Note that the sufficient condition of Theorem 4.3 is far stronger than the necessary condition in Theorem 4.2. We give an example of a graph and Forman equivalence class with a homological sequence which does not satisfy the hypotheses of Theorem 4.3; however, it is possible to find a Morse function that realizes them simultaneously.*

Let's try to get the homological sequence

$$\begin{array}{rccccccccc} i: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ B_0: & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ B_1: & 0 & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 4. \end{array}$$

on the graph with Forman equivalence class below with critical vertices labelled and critical edges dashed.

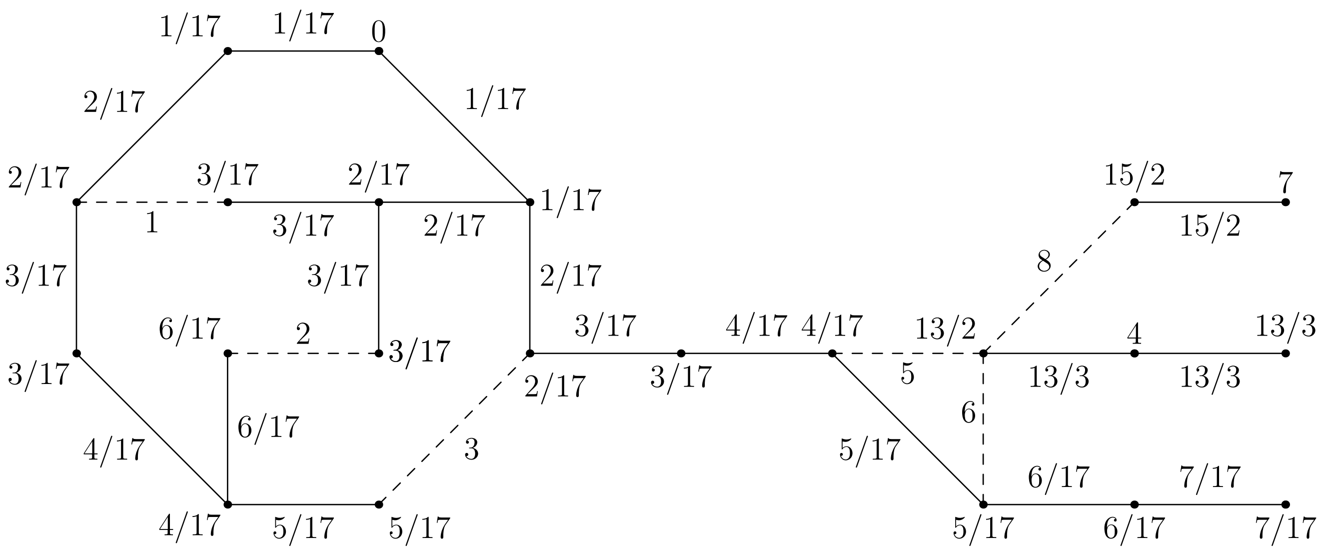


The existence of the matrix

$$M(G_i) = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

means that this is not explicitly prohibited, as it is the case that $g_e(i) \leq 3$ *for all* i *with* $g_v(i) = 1$*, as well as* $g_e(i) \leq 5$ *for all* i *with* $g_v(i) = 2$*, and* $g_e(i) \leq 6$ *for all* i *with* $g_v(i) = 3$*.*

However, this matrix does not explicitly allow this either, as the sufficient condition does not hold, because $B_1(6) > 3$*. We see, however, that the discrete Morse function below has this homological sequence.*

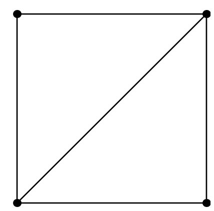


Now we see how Theorem 5.2 makes any homological sequence with a final step which increases B_1.

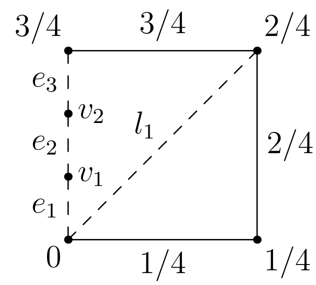
Example 7.4. *We will show how to get homological sequence*

$$\begin{array}{rccccccc} i: & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ B_0: & 1 & 2 & 1 & 1 & 2 & 1 & 1 \\ B_1: & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{array}$$

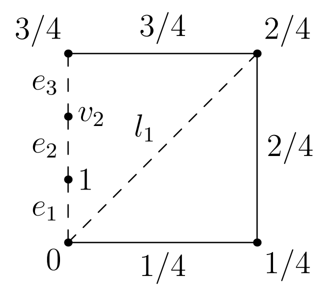
on the graph below.



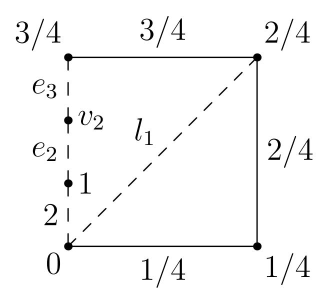
Start by picking a spanning tree and defining f on that spanning tree according to Rand and Scoville's method in [6], and subdivide one of the edges into $g_v(6) - 1$ vertices.



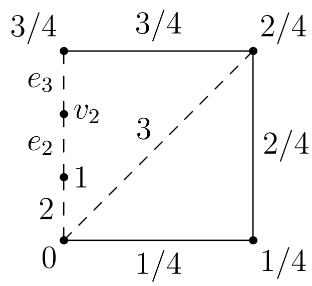
We have that $B_0(1) - B_0(0) = 1$, so $f(v_1) = 1$.



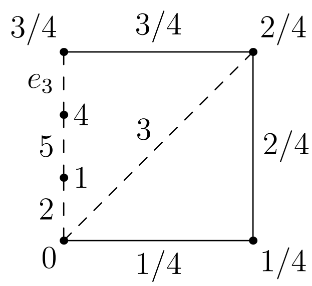
We have that $B_0(1) - B_0(0) = -1$, so $f(e_1) = 2$.



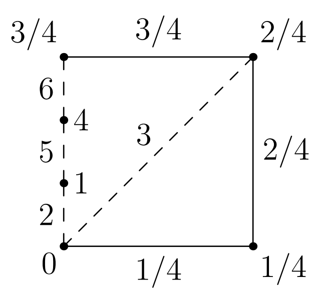
This time, $B_1(3) - B_1(2) = 1$, so $f(l_1) = 3$.



Again we have $B_0(4) - B_0(3) = 1$, then $B_0(5) - B_0(4) = -1$, so we define $f(v_2) = 4$ and $f(e_2) = 5$.



Now as the final step with $B_1(6) - B_1(5) = 1$, we add e_3.



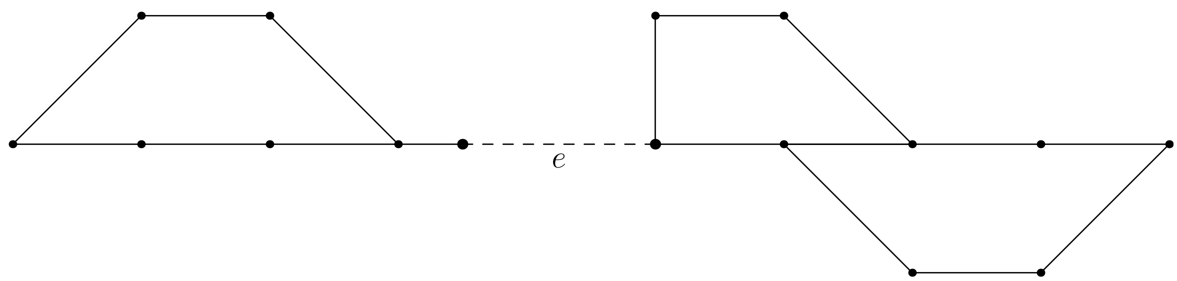
We see that this has the desired homological sequence.

The next example shows how to form any homological sequence on a graph with a cut edge with the restriction that $B_1(i) > a \implies B_0(i) \geq 2$ until the final step.

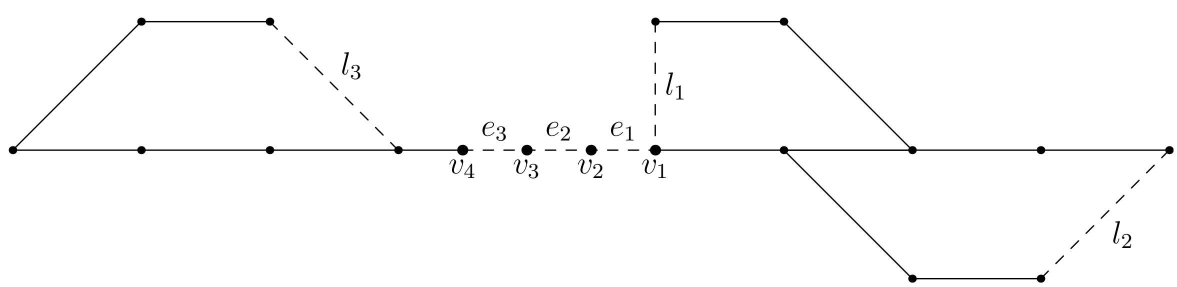
Example 7.5. *We will try to make the homological sequence*

i:	0	1	2	3	4	5	6	7	8	9
B_0:	1	1	2	3	2	2	2	3	2	1
B_1:	0	1	1	1	1	2	3	3	3	3

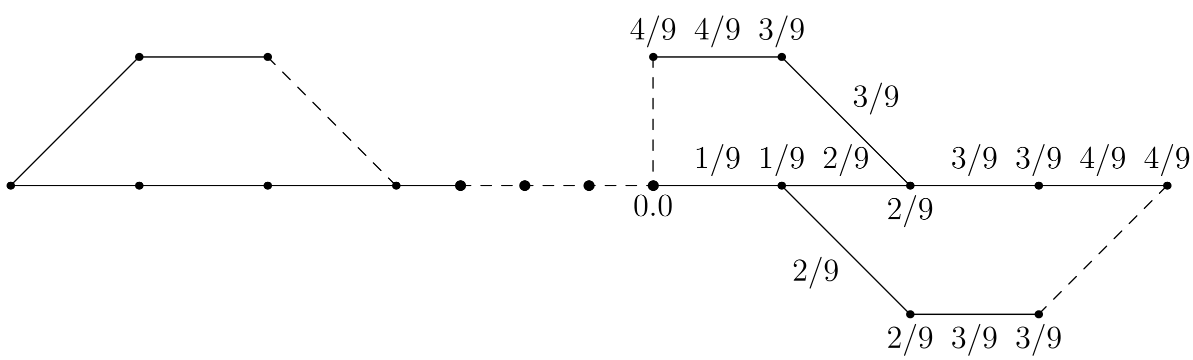
When a graph has a cut edge, start by picking the optimal cut edge and assigning its endpoints to be critical vertices.



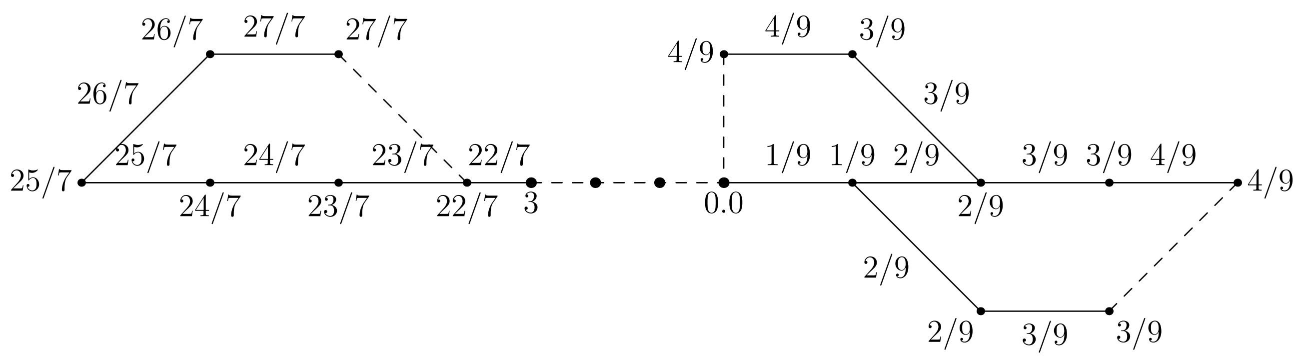
Pick $k = g_v(m_0) - 1$, *so one less than the number of critical vertices. Then we subdivide that edge* e *into* $k-1$ *edges labelled* e_1 *through* e_k *where* k *is , making all of the subdivided vertices* v_1 *through* v_{k+1} *critical, as well as the endpoints. Pick a spanning tree of* R *where* R *is the connected component of* $G - e$ *with* v_1. *Label the edges not in the spanning tree* l_1 *through* l_a, *where* $a = b_1(R)$. *Then pick a spanning tree of* L *where* L *is the connected component of* $G - e$ *with* v_k. *Label the edges not in the spanning tree* l_{a+1} *through* l_{a+b}, *where* $b = b_1(L)$.



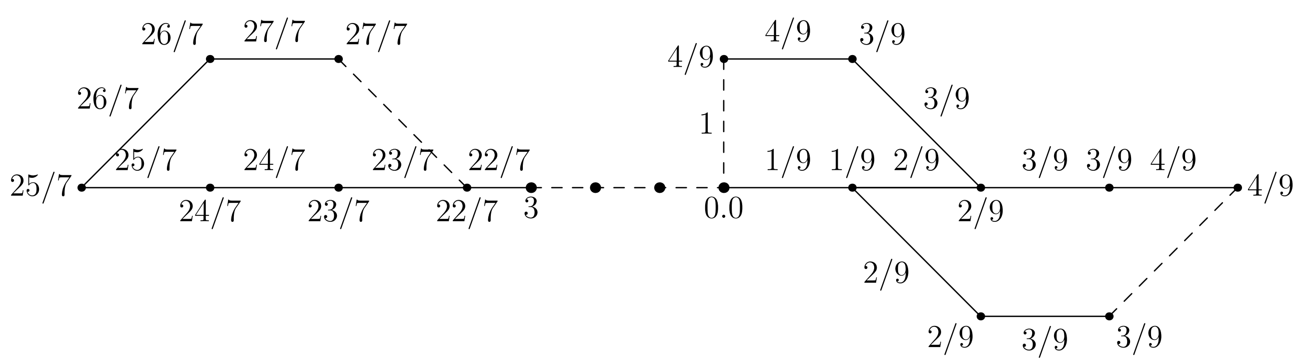
Start by defining f *on the spanning tree of* R *rooted in* v_1 *by using the algorithm from Rand and Scoville in [6].*



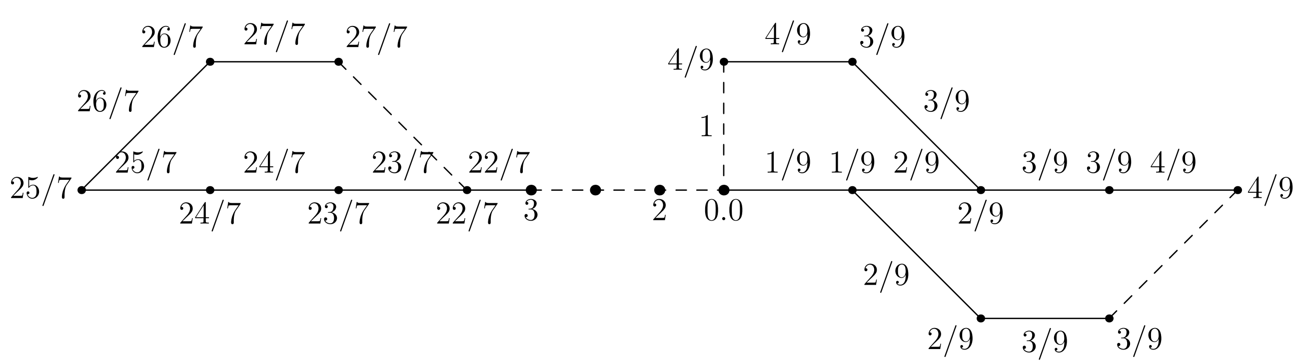
Now we must find j_0 so what to assign to $f(v_k)$. The first instance where $B_1(i) > b_1(R) = 2$ is $B_1(6)$, so $j_1 = 6$. The indices $j < j_1$ with $B_0(j) - B_0(j-1) = 1$ are 2 and 3, so $j_0 = 3$.



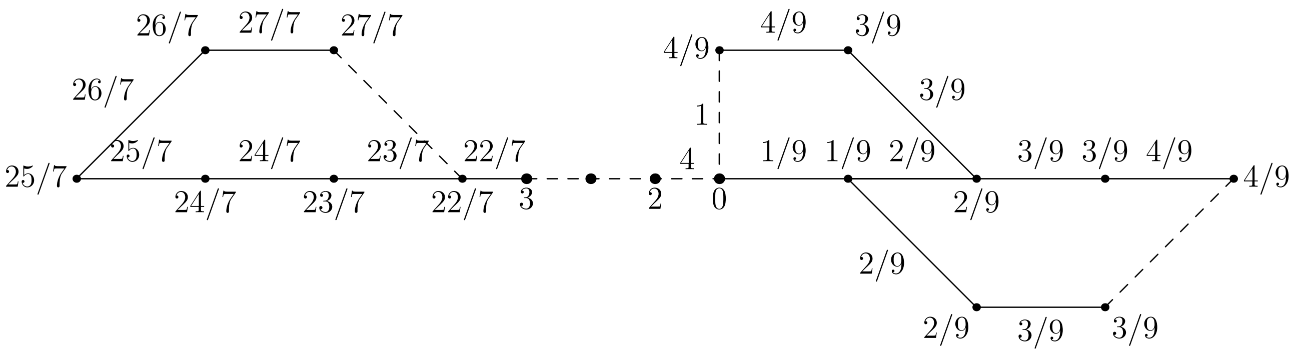
Next we look at what changes when $i = 1$. Because $B_1(1) - B_1(0) = 1$, we define the next l_j to be i, so $f(l_1) = 1$.



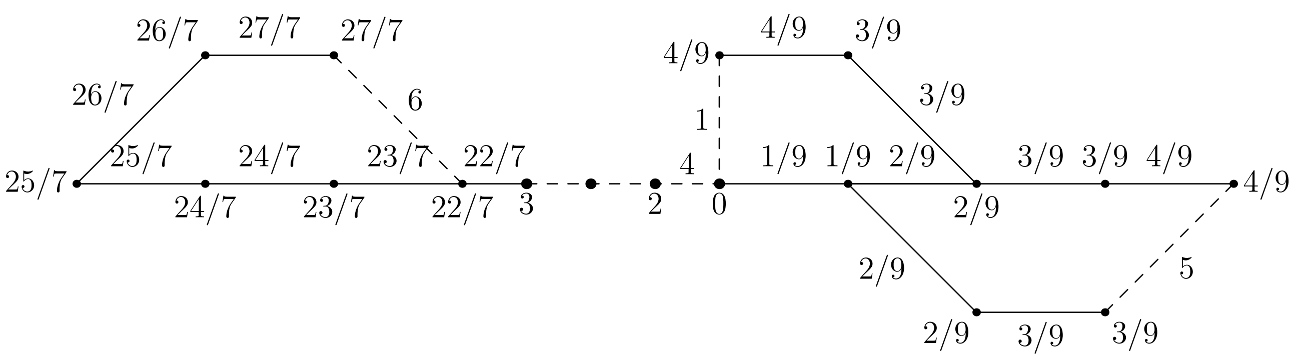
Because $B_0(2) - B_0(1) = 1$, we define the next v_j to be i, so $f(v_2) = 2$.



Now, however, we have $i = j_0$, so we have already taken care of this index, and we move on to $i = 4$. Now we have $B_0(4) - B_0(3) = -1$, so $f(e_1) = 4$.



Then $B_1(5) - B_1(4) = 1$ and $B_1(6) - B_1(5) = 1$, so we label $f(l_2) = 5$ and $f(l_3) = 6$.



For the final steps we have $B_0(7) - B_0(6) = 1$, and then $B_0(8) - B_0(7) = -1$ and $B_0(9) - B_0(8) = -1$, so $f(v_3) = 7$, $f(e_2) = 8$, and $f(e_3) = 9$.