Width of Graph Mappings

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July 27, 2020

Abstract

The *width* $|X/Y|$ of a topological space X over a topological space Y is the smallest integer k for which some continuous mapping $F : X \to Y$ has no point in Y with preimage greater than k. When X and Y are graphs, little is known about lower or upper bounds on the width. We investigate the width of complete graphs K_N over $m \times n$ grids $R_{m,n}$, as well as graphs with ε proportion of edges over general graphs G. First, we show the general lower bound $|X/Y| \geq (h(X)|V|)/(2|X/L|)$, where L is a line segment, $h(X)$ is the Cheeger constant of graph X, and $|V|$ is the number of vertices in X. Then, with a linear path projection, we show that $|K_N/R_{m,n}| = \Theta(N^2/\min(m,n))$, with the stricter bound $|K_N/R_{m,1}| \sim N^2/8$ for $m \times 1$ grids. Finally, we show that the width of a graph with N vertices and ε proportion of edges over a graph with V vertices is $\Omega(\varepsilon^2 N^2)$ for $\varepsilon \leq 2/V$ and $\Omega(\varepsilon^2 N^2/V)$ otherwise.

Summary

A *circuit* is a layout of line segments on a flat surface, and *nodes* are the points where these line segments intersect. We place *components* at points on these segments, and place *wires* along line segments to connect some pairs of components. An interesting problem is adjusting this component-wire placement to minimize wire overlap. We show that when the circuit is a grid and wires connect all pairs of components, this minimal wire overlap is roughly the number of wires divided by the side length of the grid. We also prove that when the number of wires is small relative to the number of components, the minimal width is roughly the number of wires squared, but when larger, is inversely proportional to the number of nodes.

1 Introduction

A graph is a set of *vertices* and *edges* that connect pairs of distinct vertices. We consider the topological space arising from a graph by replacing vertices with points and edges with copies of the closed unit interval. Thus, for graphs G, X, we can define *mappings* $F : G \to X$ as continuous functions that send points on G to points on X. Two example mappings between graphs are shown in Figure 1. The *width* of F is the maximum number of points from G sent to a single point in X, while the *width* of G over X is the minimal width of a mapping from G to X. An interesting problem is to compute the width of G over X for classes of graphs G, X, such as trees and complete graphs.

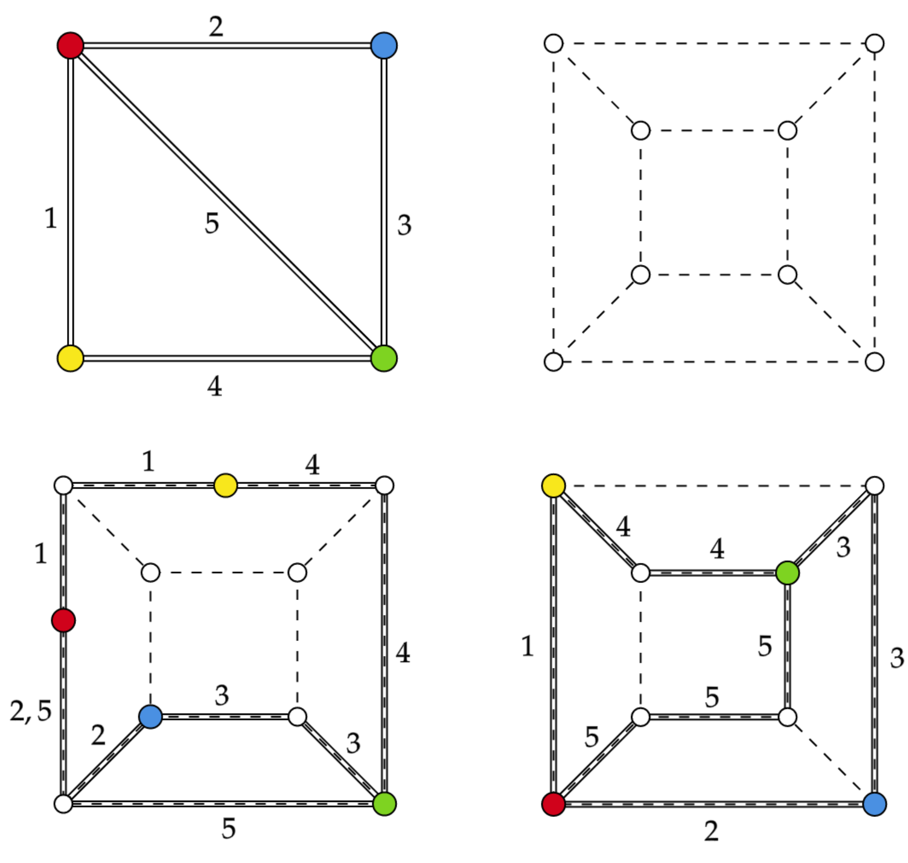

Figure 1: Two possible mappings from the top left graph into the top right are shown in the bottom two diagrams. Colors indicate where the vertices map, and numbers indicate where the edges map.

Although no general formula is known for the width of a graph over a line segment L, its value has been computed or bounded in special cases. Gromov [1] showed that the width of a graph with N vertices and $\varepsilon\binom{N}{2}$ edges over L is $O(\varepsilon^2 N^2)$. Lengauer [2] proved that the depth of a k-ary tree with depth d over L is $kO(d)$. Thilikos and Yannakakis [3, 4] created algorithms that determine if the width of a given tree over L is less than k and construct a mapping of minimal width.

Because edges can be mapped in many ways onto general graphs X, it is more difficult to compute the width of a graph over X. Simpler cases and variants have been tackled, though. Chavez and Trapp [5] showed that the width of a tree T over a cycle is the same as the width of T over L. Bezrukov [6] showed that the edge congestion, a variant of width where mappings must send vertices to vertices, of an n-hypercube over a rectangular grid with 2^n vertices is $O(n^2)$. Hruska [7] calculated the edge congestion of trees and spanning trees over grids and bipartite graphs.

It is of interest to consider the case where G is a grid, which has application to circuit layouts in very large-scale integration design (VLSI). Bhatt and Leighton [8] showed that the minimal area of layout of a circuit is at least the square of the width.

In Section 2, we introduce preliminary notation and formalize the notion of width. In Section 3, we define the Cheeger constant and prove a general lower bound for widths of graphs over graphs. In Section 4, we show that the width of a complete graph with N vertices over an $m \times m$ grid is $\Theta(N^2/m)$. In Section 5, we show that the width of a graph with N vertices and $\varepsilon\binom{N}{2}$ edges over an graph G with V vertices is $\Omega(\varepsilon^2 N^2)$ for $\varepsilon < 2/V$, and $\Omega(\varepsilon N^2/V)$ when $\varepsilon \geq 2/V$.

2 Preliminaries

2.1 Graph Terminology

A *graph* G is a pair of sets (V, E), where V is the set of points called *vertices*, and $E(G)$ is the set of *edges* connecting vertices. We only consider *simple graphs*, which have no more than one edge between a pair of vertices and no loops.

In this paper, we focus on three families of graphs: complete graphs, ε-complete graphs, and grids.

The *complete graph* K_N is a graph with N vertices and $\binom{N}{2}$ edges connecting all pairs of distinct vertices.

For a real number $\varepsilon > 0$, we say a graph G is ε*-complete* if and only if $|E| \geq \varepsilon\binom{|V|}{2}$, where $|E|$ is the number of edges and $|V|$ is the number of vertices.

The $m \times n$ *grid* $R_{m,n}$ is a graph whose vertices are $V = \{(i,j) \mid 0 \leq i \leq m, 0 \leq j \leq n, i, j \in \mathbb{Z}\}$, with edges between vertices a distance one apart in $\mathbb{R}^2$. These vertices are called *gridpoints*, and the edges are called *unit segments*. The *perimeter* of $R_{m,n}$ is the set of points that lie on the convex hull of the vertices, consisting of two vertical and two horizontal segments, which are called *sides* of the grid.

2.2 Formalization of Width

When defining width, we consider graphs as topological spaces by replacing vertices with points and edges with copies of the closed unit interval $[0, 1]$, so that 0 corresponds to one endpoint of the edge and 1 to the other. We consider *mappings* from graph X onto graph Y where the vertices of X are injectively sent to points on Y, and edges of X are continuously sent to paths on Y.

Definition 2.1. For graphs X, Y, the *width* of a continuous mapping $F : X \to Y$ is given by

$$w(F) = \max_{y \in Y} |F^{-1}(y)|,$$

where $F^{-1}(y)$ is the set of $x \in X$ for which $F(x) = y$.

Definition 2.2. For graphs X, Y, let $\mathcal{F}$ be the set of mappings $F : X \to Y$. The *width* of X over Y is given by

$$|X/Y| := \min_{F \in \mathcal{F}} w(F).$$

2.3 Asymptotic Notation

Because we investigate asymptotic bounds when computing width, we formally define the following asymptotic notation.

Definition 2.3 (Asymptotic Notation)**.** Consider functions $f, g : \mathbb{N} \to \mathbb{R}$. Then:

1. $f \sim g$ if and only if $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = 1$,
2. $f \gtrsim g$ if and only if $f \geq h$ for some function $h : \mathbb{N} \to \mathbb{R}$ satisfying $h \sim g$,
3. $f \lesssim g$ if and only if $f \leq h$ for some function $h : \mathbb{N} \to \mathbb{R}$ satisfying $h \sim g$,
4. $g = O(f)$ if and only if $n > n_0$ implies $g(n) \leq Cf(n)$ for some constants C, n_0,
5. $g = \Omega(f)$ if and only if $n > n_0$ implies $g(n) \geq Cf(n)$ for some constants C, n_0,
6. $g = \Theta(f)$ if and only if $g = O(f)$ and $g = \Omega(f)$.

Note that $f \gtrsim g$ and $f \lesssim g$ imply $f \sim g$.

3 Lower Bounds Using the Cheeger Constant

We begin by defining the boundary of a subset of vertices of a graph. We compute its size when the graph is a complete graph.

Definition 3.1. Consider a graph G with vertices V. The *boundary* $\partial\overrightarrow{S}$ of a subset of vertices $S \subseteq V$ is the set of edges between a vertex inside S and a vertex outside S.

Example 3.1. Suppose we have a complete graph K_N, and let S be a subset with k vertices. All pairs of vertices must be connected, so $\partial\overrightarrow{S}$ consists of the $k(N-k)$ edges between vertices inside and outside S.

The Cheeger constant defines how large the boundary of any subset must be relative to the size of the subset. We calculate the Cheeger constant of a complete graph.

Definition 3.2. Let G be a graph with vertices V, and define $\mathcal{S}$ to be the power set of V. The *Cheeger constant* of G is given by

$$h(G) := \min_{S \in \mathcal{S}} \frac{|\partial\overrightarrow{S}|}{\min(|S|, |V| - |S|)}.$$

Example 3.2. We compute the Cheeger constant of a complete graph K_N when N is even. Suppose we choose a subset S with k vertices, where $k \leq \frac{N}{2}$. From Example 3.1, $|\partial\overrightarrow{S}| = k(N-k)$, so

$$\frac{|\partial\overrightarrow{S}|}{\min(|S|, |V| - |S|)} = \frac{k(N-k)}{k} = N - k \geq \frac{N}{2}.$$

Similarly, when $k \geq \frac{N}{2}$ we find

$$\frac{|\partial\overrightarrow{S}|}{\min(|S|, |V| - |S|)} = \frac{k(N-k)}{N-k} = k \geq \frac{N}{2}.$$

Equality can be achieved by letting S be any subset of K_N with exactly $\frac{N}{2}$ vertices, so $h(K_N) = \frac{N}{2}$.

Now we prove a more general lower bound for the width of a graph over a graph, using both the Cheeger constant and the width.

Theorem 3.1 (Cheeger Bound)**.** *Consider a graph G with vertices V. The width of G over a graph X satisfies*

$$|G/X| \geq \left\lfloor \frac{|V|}{2} \right\rfloor \cdot \frac{h(G)}{|X/L|}.$$

Proof. Send $G \to X$ and then $X \to L$ through mappings F_2, F_1.

If $F_1(F_2(v))$ is injective on vertices $v \in V(G)$, there must exist $x \in L$ so that $F_1(F_2(v))$ lies to the left of x for exactly $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices v.

If $F_1(F_2(v))$ is not injective, we must have $F_1(F_2(v_1)) = F_1(F_2(v_2))$ for two vertices $v_1, v_2 \in V(G)$. But as $F_2(v_1) \neq F_2(v_2)$, and F_1 maps vertices of X injectively, at least one of $F_2(v_1)$, $F_2(v_2)$ is not a vertex of X. This point must lie on an edge of X, so we can alter the mapping F_2 by moving the point infinitesimally along the edge. Now $F_1(F_2(v_1)) \neq F_1(F_2(v_2))$, even though the width of F_2 has not changed. Thus, we can repeat this process until $F_1(F_2(v))$ is indeed injective on vertices $v \in V(G)$, and then assume the existence of $x \in L$ for which $\left\lfloor \frac{|V|}{2} \right\rfloor$ vertices $v \in V(G)$ have $F_1(F_2(v))$ lie left of x.

Take S to be the set of such v. The size of the boundary of S is

$$|\partial \overrightarrow{S}| \geq \min(|S|, |V| - |S|) \cdot h(G) = \left\lfloor \frac{|V|}{2} \right\rfloor \cdot h(G).$$

Also, x has a preimage of size at most $|X/L|$. As each edge in the $\partial \overrightarrow{S}$ must pass through one of the points in this preimage, there must be at least $\left\lfloor \frac{|V|}{2} \right\rfloor \cdot \frac{h(G)}{|X/L|}$ crossings at one of these points, and this gives a lower bound on $|G/X|$, as desired. □

Corollary 1. *If $X = L$, then $|X/L| = 1$ and the Cheeger bound becomes $|G/L| \geq \left\lfloor \frac{|V|}{2} \right\rfloor \cdot h(G)$.*

4 Width of Complete Graphs over Grids

In this section, we compute the asymptotic behavior of the width of a complete graph K_N over a $m \times 1$ grid $R_{m,1}$. We also calculate the width of K_N over a $m \times n$ grid $R_{m,n}$ up to a constant factor.

4.1 Width of Complete Graph over $m \times 1$ grid

We begin by using bounding techniques similar to Theorem 3.1 and a cyclic construction to compute the width of K_N over a 1×1 grid.

Theorem 4.1 (Complete Graph to 1×1 grid)**.** *We have* $|K_N/R_{1,1}| \sim \frac{N^2}{8}$.

Proof. Note that $|K_N/R_{1,1}| = |K_N/C|$, where C is a circle. Mappings $F : K_N \to C$ send vertices of K_N to points $P_1, P_2, .., P_N$ on C and edges of K_N to arcs on C between all pairs of points P_i, P_j.

First we show the lower bound $|K_N/C| \gtrsim \frac{N^2}{8}$. Draw a line ℓ intersecting C at points X, Y so that at least $\lfloor \frac{N}{2} \rfloor$ points P_i lie on each side of ℓ. There are at least $\lfloor \frac{N}{2} \rfloor^2$ arcs between points on different sides of ℓ. Each arc must cross go through X or Y, so at least $\frac{1}{2} \lfloor \frac{N}{2} \rfloor^2$ paths must cross one of these points. Then the width is $|K_N/C| \geq \frac{1}{2} \lfloor \frac{N}{2} \rfloor^2 \sim \frac{N^2}{8}$.

Now we show the upper bound $|K_N/C| \lesssim \frac{N^2}{8}$. It suffices to construct a mapping F with width $\sim \frac{N^2}{8}$. Let P_i be vertices of a regular N-gon, and connect all pairs of points P_i, P_j that are not diametrically opposite by the minor arc between them, as shown in Figure 2. If P_i, P_j are diametrically opposite, connect them by one of the two arcs between them.

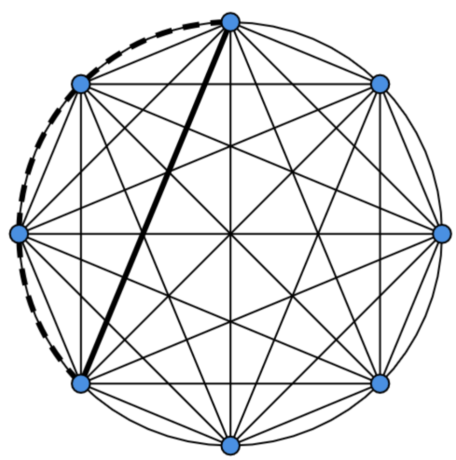

Figure 2: Connecting the shortest arc between points on a regular N-gon shows the upper bound for the width of a complete graph over a circle.

Any point $a \in C$ lies in the interior of at most $k+1$ arcs that contain k points in their interior, where $0 \leq k \leq \lfloor \frac{N}{2} \rfloor - 1$. Because each element of $F^{-1}(a)$ corresponds to one of these arcs (and possibly a itself),

$$|F^{-1}(a)| \leq 1 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} k = 1 + \frac{\lfloor \frac{N}{2} \rfloor \left(\lfloor \frac{N}{2} \rfloor + 1 \right)}{2} \sim \frac{N^2}{8}.$$

This shows $|K_N/C| \lesssim \frac{N^2}{8}$ and completes our proof. $\square$

With the construction in Theorem 4.1 and bounding techniques from Theorem 3.1, we show that that the width of K_N over a 1×2 grid is approximately the width of K_N over a 1×1 grid.

Theorem 4.2 (Complete Graph to 2×1 Grid)**.** *We have* $|K_N/R_{2,1}| \sim \frac{N^2}{8}$.

Proof. First, to show the upper bound $|K_N/R_{2,1}| \lesssim \frac{N^2}{8}$, we construct a mapping $F : K_N \to R_{2,1}$ with width $\sim \frac{N^2}{8}$. Map all vertices of K_N to a 1×1 grid contained within $R_{1,2}$, and map all edges onto the 1×1 grid with the construction in Theorem 4.1. This shows $|K_N/C| \lesssim \frac{N^2}{8}$.

Thus, the only thing that remains is to show $|K_N/C| \gtrsim \frac{N^2}{8}$. Divide $R_{1,2}$ into sections A, B, C that all meet at points X, Y, as shown in Figure 3. Let the number of points on these three sections be a, b, c, respectively, so that $a + b + c + N$.

If any one of a, b, c is at least $\frac{N}{2}$, we can draw a loop around $\lfloor \frac{N}{2} \rfloor$ vertices in the corresponding section that passes through two points on the grid, as shown in Figure 3. All paths between vertices inside and outside the loop must pass through these points, so we can use the lower bound from Theorem 4.1 to show $|K_N/R_{1,2}| \sim \frac{N^2}{8}$.

Otherwise, we have $a, b, c < \frac{N}{2}$. Without loss of generality, suppose $a \geq b \geq c$. Note that there are $ab + bc + ac$ paths between vertices in different sections, which all pass through X or Y, so

$$\max\left(|F^{-1}(X)|, |F^{-1}(Y)|\right) \geq \frac{ab + bc + ac}{2} = \frac{a(N-a)}{2} + \frac{bc}{2} \geq \frac{a(2N-3a)}{2}.$$

The last step follows from $N \geq 2a$ and the fact that bc is minimized for a given a when $b = a$, $c = N - 2a$. As $\frac{a(2N-3a)}{2}$ is decreasing for $a \geq \frac{N}{3}$, its minimum is $\frac{N^2}{8}$, occurring at $a = \frac{N}{2}$. Therefore $|K_N/R_{1,2}| \gtrsim \frac{N^2}{8}$, completing the lower bound and the proof. $\square$

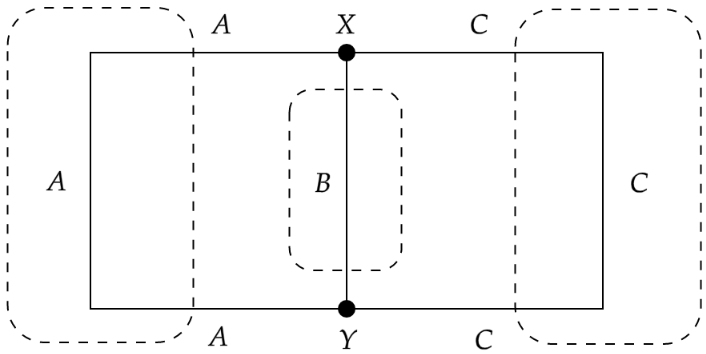

Figure 3: If one of the regions A, B, C contains more than half the vertices, we can draw a loop around $\frac{N}{2}$ vertices in that region, which passes through two points on the grid.

To show that the width of K_N over a $m \times 1$ grid is also asymptotically $\frac{N^2}{8}$, we use casework and combine the techniques in Theorem 3.1, Theorem 4.1, and Theorem 4.2.

Theorem 4.3 (Complete Graph to $m \times 1$ Grid)**.** *We have* $|K_N/R_{m,1}| \sim \frac{N^2}{8}$*.*

Proof. The proofs for $m = 1, 2$ follow from Theorem 4.1 and Theorem 4.2. Thus, assume $m \geq 3$. Place $R_{m,1}$ on the coordinate plane so that its vertices are $\{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq 1\}$ and vertices are connected if and only if their distance is 1. Suppose a mapping $F : K_N \to R_{m,1}$ sends vertices of K_N to points $P_1, P_2, ..., P_N \in R_{m,1}$. Let $P(a)$ be the number of P_i whose x-coordinates are less than or equal to a, and define $a_0 := \inf\{a : P(a) \geq \frac{N}{2}\}$.

If $a_0 \in \{1, ..., m-2\}$, then let X and Y be $(a_0, 0)$ and $(a_0, 1)$, and denote A, B, C to be the sets of points with x-coordinates less than, equal to, and greater than a_0. By definition of a_0, both A, C contain at most $\frac{N}{2}$ points P_i. If B contains more than $\frac{N}{2}$ points P_i, we can draw a loop around $\frac{N}{2}$ of them, intersecting B twice, and apply the lower bound argument from Theorem 4.1 to show $|K_N/R_{m,1}| \gtrsim \frac{N^2}{8}$. If B contains at most $\frac{N}{2}$ points P_i, we can directly apply the lower bound argument for a 2×1 grid from Theorem 4.2 to show $|K_N/R_{m,1}| \gtrsim \frac{N^2}{8}$.

If $a_0 = 0$ or $a_0 = m-1$, either the leftmost vertical or rightmost vertical segment of the grid contains at least $\frac{N}{2}$ vertices. We draw a circle around some $\frac{N}{2}$ vertices that intersects this segment twice, and use the lower bound from Theorem 4.1 to show $|K_N/R_{m,1}| \gtrsim \frac{N^2}{8}$.

If a_0 is not an integer, we draw the line $x = a_0$, so that at least $\frac{N}{2} - 2$ points are on both sides. We again use the lower bound argument from Theorem 4.1 to show $|K_N/R_{m,1}| \gtrsim \frac{N^2}{8}$.

For all possible values of a_0, $|K_N/R_{m,1}| \gtrsim \frac{N^2}{8}$. Furthermore, we can place P_i around the perimeter of $R_{m,1}$, deform it into a circle, and use the construction from Theorem 4.1 to show the upper bound $|K_N/R_{m,1}| \lesssim \frac{N^2}{8}$. Thus, $|K_N/R_{m,1}| \sim \frac{N^2}{8}$. □

4.2 Width of Complete Graph over $m \times n$ grid

We show that the width of a complete graph with N vertices over an $m \times n$ grid is asymptotically $\frac{N^2}{\min(m,n)}$. We begin by proving a key lemma regarding the number of line segments between points on the perimeter of the grid that pass through a unit square.

Lemma 4.4. *Suppose N points are distributed uniformly around the perimeter of grid $R_{m,m}$, and segments are drawn between every pair of points on different sides. The number of line segments passing through any unit square in the interior of the grid is $\sim \frac{3N^2}{8m}$.*

Proof. Consider a particular unit square S within the $m \times m$ grid. Clearly there are $\binom{4}{2}\left(\frac{N}{4}\right)^2 = \frac{3N^2}{8}$ ways to connect points on different sides of the perimeter. So, as $N \to \infty$, the number of line segments passing through S approaches $\frac{3pN^2}{8}$, where p is the probability that for two points randomly chosen on different sides of a grid, the line between them passes through S.

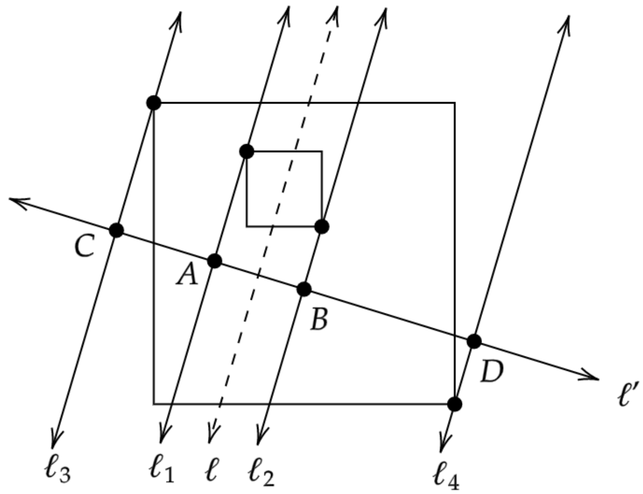

Figure 4: A dilation of size m sends the lines through S to lines through grid.

The only thing that remains is to compute p. For any line ℓ, let ℓ_1 and ℓ_2 be the two lines parallel to ℓ that pass through opposite corners of S and are farthest apart, and let ℓ_3 and ℓ_4 be the two lines parallel to ℓ that pass through opposite corners of the grid and are farthest apart. These lines are shown in Figure 4. Note that ℓ passes through S if and only if it lies between ℓ_1 and ℓ_2 and passes through the grid if and only if it lies between ℓ_3 and ℓ_4. Furthermore, by similarity, the distance between ℓ_1 and ℓ_2 is $\frac{1}{m}$ times the distance between ℓ_3 and ℓ_4. Therefore, if we fix the slope of a line, the probability that it passes through S given that it passes through the grid is $\frac{1}{m}$. Because there is a bijection between pairs of points on different sides of the grid and lines that pass through the grid, we have $p = \frac{1}{m}$, and our answer is $\sim \frac{3N^2}{8m}$, as desired. □

Theorem 4.5. *Let $m \leq n$ be positive integers. The width of a complete graph with N vertices over an $m \times n$ grid satisfies $|K_N/R_{m,n}| = \Theta\left(N^2/m\right)$.*

Proof. First, we show the lower bound $|K_N/R_{m,n}| \gtrsim \frac{N^2}{4(m+1)}$. The proof is similar to that of Theorem 4.2. Take a vertical line passing through on the leftmost segment of $R_{m,n}$ and continuously move it right. Consider the first time at which more than half of the N mapped points are on the line or to its left. If this occurs when the line does not lie on a segment of the grid, there are $\sim \left(\frac{N}{2}\right)^2 = \frac{N^2}{4}$ paths between points on the left and right of the line. As there are $m+1$ intersection points of the line with the grid, and each of the $\sim \frac{N^2}{4}$ paths must go through at least one of these points, at least $\sim \frac{N^2}{4(m+1)}$ paths will cross a point. If the line passes through a segment of the grid, we use the argument from Theorem 4.2 to show that the width is $\gtrsim \frac{N^2}{4(m+1)}$. These cases are shown in Figure 5.

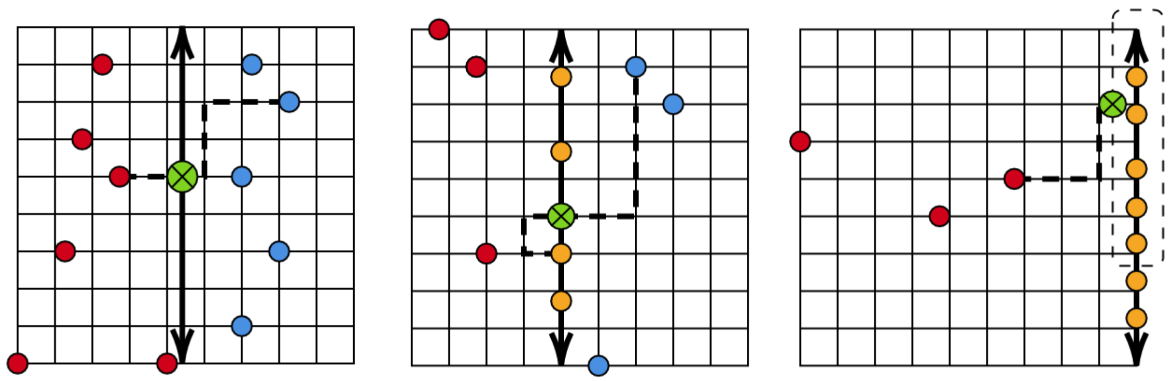

Figure 5: There are two cases to determine the width of a complete graph over a grid: one where the dividing line does not lie on grid segments, in the leftmost mapping; and one where it does, in the other two mappings.

Now, we show the upper bound $|K_N/R_{m,n}| = O(N^2/m)$. It suffices to find a mapping from K_N onto $R_{m,m}$, which is a subgraph of $R_{m,n}$, so that $O(N^2/m)$ paths pass through every point. First, uniformly distribute the N points along the perimeter of $R_{m,m}$. If two points lie on different sides of the grid, draw a path between them that only touches edges of unit squares that the segment between the points passes through, called an *approximately linear path*. If two vertices lie on the same side of the grid, draw the shortest path from one vertex to the other side of the grid, where it ends at a point called its *reflection*, and then draw an approximately linear path back to the other vertex. Example of such paths are shown in Figure 6.

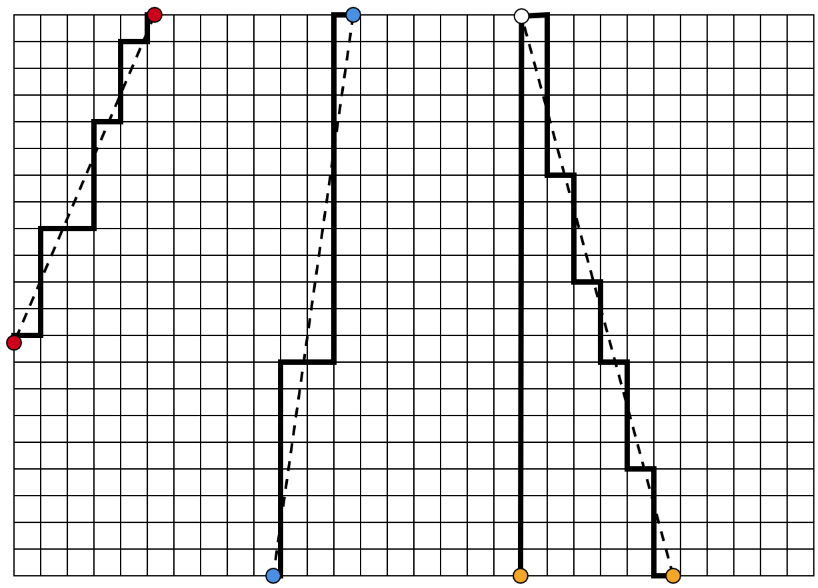

Figure 6: Paths are drawn between vertices on opposite, adjacent, and the same sides.

From Lemma 4.4, there are $\sim \frac{3N^2}{8m}$ segments between points on different sides of the grid that pass through a unit square. Similarly, there are at most $\sim \frac{3N^2}{8m}$ segments between the reflection of a point and a point on the same side of the grid that pass through a unit square. As each point on the grid belongs to at most four unit squares, from the definition of approximately linear paths, we have at most $\sim \frac{3N^2}{m}$, or $O(N^2/m)$ paths passing through every point. Our final case, paths between a point and its reflection, only add $O(N^2/m^2)$ paths passing through each point. Thus, there is an upper bound of $O(N^2/m)$ on width, as desired. $\square$

5 Width of ε-complete graphs over general graphs

In this section, we prove a lower bound for the width of ε-complete graphs over general graphs. We begin by demonstrating a lower bound for the width of an ε-complete graph over a circle C.

Theorem 5.1 (Proportion-ε edges onto circle)**.** *For any ε-complete graph G', we have $|G'/C| \gtrsim \frac{\varepsilon^2 N^2}{8}$.*

Proof. Without loss of generality, suppose the vertices of G' map to distinct vertices of a regular N-gon inscribed in C, with the arc distance between any two consecutive vertices is 1. Let a_i be the number of arcs of length i between vertices of the N-gon, for $1 \leq i \leq N-1$. Note the following:

$$\sum_{i=1}^{N-1} a_i = \varepsilon \binom{N}{2}, \quad 0 \leq a_i \leq N.$$

Now consider the quantity $\sum\limits_{i=1}^{N-1} i a_i$, which is the sum of the lengths of all arcs. For any integer $1 \leq k \leq N$,

$$\begin{aligned} \sum_{i=1}^{N-1} i a_i &= \sum_{i=1}^{N-1} \sum_{j=i}^{N-1} a_j \geq \sum_{i=1}^{k} \sum_{j=i}^{N-1} a_j \\ &= \sum_{i=1}^{k} \left(\varepsilon \binom{N}{2} - \sum_{j=1}^{i-1} a_j \right) \geq \sum_{i=1}^{k} \left(\varepsilon \binom{N}{2} - N(i-1) \right) \\ &= k\varepsilon \binom{N}{2} - N \binom{k}{2} = \frac{Nk}{2}((N-1)\varepsilon + 1 - k). \end{aligned}$$

Because this quadratic in k is maximized at $k_0 = \frac{(N-1)\varepsilon+1}{2}$, let k be the closest integer to k_0. As $|k - k_0| \leq \frac{1}{2}$,

$$\begin{aligned} \sum_{i=1}^{N-1} i a_i &\geq \frac{Nk}{2}((N-1)\varepsilon + 1 - k) \\ &\geq \frac{N\left(\frac{(N-1)\varepsilon}{2}\right)}{2} \left((N-1)\varepsilon + 1 - \frac{(N-1)\varepsilon}{2} \right) \\ &= \frac{N(N-1)\varepsilon}{4} \left(\frac{(N-1)\varepsilon}{2} + 1 \right). \end{aligned}$$

Thus, there exists some interval between consecutive vertices with at least

$$\frac{1}{N} \sum_{i=1}^{N-1} i a_i \geq \frac{(N-1)^2 \varepsilon^2}{8} + \frac{(N-1)\varepsilon}{4} \sim \frac{\varepsilon^2 N^2}{8}$$

arcs passing through it, so $|G'/C| \gtrsim \frac{\varepsilon^2 N^2}{8}$, as desired. □

Corollary 2. *If G is an ε-complete graph, $|G/L| \gtrsim \frac{\varepsilon^2}{4}\binom{N}{2}$ because L is a subgraph of C.*

Now, we use the lower bound from Theorem 5.1 to obtain a lower bound for the width of an ε-complete graph over any graph G.

Theorem 5.2 (Proportion-ε edges onto graph)**.** *Fix a real number $\varepsilon > 0$ and a graph G with V vertices. Let G' be any ε-complete graph with N vertices. If $\varepsilon \leq \frac{2}{V}$, then $|G'/G| = \Omega(\varepsilon^2 N^2)$. Otherwise, the width is $|G'/G| = \Omega(\varepsilon N^2/V)$.*

Proof. Label the edges of G with distinct integers from $1, 2, \ldots, T$, where T is the number of edges in G. Define V_i to be the number of vertices from G' that are mapped onto edge i, so that $\sum_i V_i = N$. Similarly, define E_i to be the number of edges from G' mapped onto the interior of edge i, and denote $E := \sum_i E_i$.

There are $\varepsilon\binom{N}{2} - E$ edges in G' that are not mapped onto the interior of an edge of G. These edges pass through at least one of the V vertices in G. Thus, some vertex has at least $\frac{\varepsilon\binom{N}{2}-E}{V}$ edges crossing through it, so this expression is a lower bound for the width.

Also, note that $E_j \geq \frac{E}{N}V_j$ for some j, as otherwise $\sum_i E_i < \frac{E}{N}\sum_i V_i \leq E$. Consider the subgraph lying entirely on edge j, which is essentially an ε_0-complete graph with V_j vertices for $\varepsilon_0 = \frac{E_j}{\binom{V_j}{2}}$. From Corollary 2, the width of this subgraph over L is at least $\frac{\varepsilon_0^2}{4}\binom{V_j}{2} = \frac{E_j^2 V_j(V_j-1)}{4V_j^2(V_j-1)^2} \geq \frac{E_j^2}{4V_j^2} \geq \frac{E^2}{4N^2}$. Thus, $|G'/G| \geq \frac{E^2}{4N^2}$ as well.

Combining our two lower bounds on width and varying E, we have

$$|G'/G| \geq \min_{E\in\left[0,\varepsilon\binom{N}{2}\right]} \max\left(\frac{\varepsilon\binom{N}{2} - E}{V}, \frac{E^2}{4N^2}\right).$$

The maximum of these two quantities is minimized when they are equal, or

$$\begin{aligned}
\frac{E^2}{4N^2} &= \frac{\varepsilon\binom{N}{2} - E}{V}, \\
0 &= VE^2 + 4N^2E - 2\varepsilon N^3(N-1), \\
E &= \frac{-4N^2 + \sqrt{16N^4 + 8\varepsilon N^3(N-1)V}}{2V}, \\
E &= \frac{2N^2}{V}\left(-1 + \sqrt{1 + \frac{\varepsilon}{2}\left(1 - \frac{1}{N}\right)V}\right), \\
\frac{E^2}{4N^2} &\sim \frac{N^2}{V^2}\left(-1 + \sqrt{1 + \frac{\varepsilon V}{2}}\right)^2.
\end{aligned}$$

Taylor expansion yields $(-1+\sqrt{1+x})^2 = 2+x-2\sqrt{1+x} \geq \frac{x^2}{4} - \frac{x^3}{8} \geq \frac{x^2}{8}$ for $x \leq 1$. For $x > 1$, we have $(-1+\sqrt{x+1})^2 \geq ((\sqrt{2}-1)\sqrt{x}+1)^2 \geq (3-2\sqrt{2})x$. Now let $x = \frac{\varepsilon V}{2}$. When $\varepsilon \leq \frac{2}{V}$, we have

$$\frac{E^2}{4N^2} \gtrsim \frac{N^2}{64V^2}\left(\frac{\varepsilon V}{2}\right)^2 = \frac{\varepsilon^2 N^2}{256}.$$

When $\varepsilon > \frac{2}{V}$, we have

$$\frac{E^2}{4N^2} \gtrsim \frac{N^2}{V^2}\left(\frac{\varepsilon V(3-2\sqrt{2})}{2}\right) = \frac{(3-2\sqrt{2})\varepsilon N^2}{V}.$$

Thus, $|G'/G|$ is $\Omega(\varepsilon^2 N^2)$ for $\varepsilon \leq \frac{2}{V}$ and $\Omega(\varepsilon N^2/V)$ otherwise. $\square$

6 Acknowledgements

I would like to thank my mentor Elia Portnoy, a graduate student in the MIT Math Department, for proposing the idea for this project and guiding me throughout the research process. I would like to thank head mentor Dr. Tanya Khovanova of the MIT Math Department for the valuable suggestions on my paper and the direction of my research. I would like to thank Dr. John Rickert of the Rose-Hulman Institute of Technology for the important advice on my presentation and paper during the research period. Thank you to Rupert Li for proofreading my paper and providing helpful feedback. Thanks also to the first-week TAs Albert Wang and Shloka Janapaty for teaching LaTeX tips and formatting. I would like to sincerely thank the MIT Math Department, RSI and CEE for giving me the opportunity to conduct this research. Finally, I am very grateful to the Department of Defense for making this research possible.

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